Dynamics of exoplanets Assignment 2.

Q4 Fourier expansions

a) (angular momentum/unit mass:

$$h = r^2 \dot{f} = \sqrt{G m_{12} a(1-e^2)}\ ) \Rightarrow \dot{f} = \nu \frac{(1+e\cos f)^2}{(1-e^2)^{3/2}}$$

$$X_{n'}^{-(l+1),m}(e_0) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-imf_0}}{(R/a_0)^{l+1}} e^{in'M_0} dM_0.$$

for $n'=0$.

So $$X_0^{-(l+1),m}(e_0) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-imf_0}}{(R/a_0)^{l+1}} e^0 dM_0.$$

$$\frac{R}{a_0} = \frac{1-e_0^2}{1+e_0\cos f_0}$$

$$X_0^{-(l+1),m}(e_0) = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{1-e_0^2}{1+e_0\cos f_0}\right)^{-(l+1)} e^{-imf_0} dM_0$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (1-e_0^2)^{-(l+1)} (1+e_0\cos f_0)^{(l+1)} e^{-imf_0} dM_0.$$

and, $M_0 = \nu_0(t-T_p) \Rightarrow dM_0 = \nu_0 dt,$

and, $\dot{f} \Rightarrow df_0 = \nu_0 \frac{(1+e_0\cos f_0)^2}{(1-e_0^2)^{3/2}} dt,$

$$\Rightarrow dM_0 = \frac{\nu_0}{\nu_0} \frac{(1-e_0^2)^{3/2}}{(1+e_0\cos f_0)^2} df_0$$

So $X_0^{-(l+1),m}(e_0) = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1-e_0^2}{1+e_0\cos f_0}\right)^{-(l+1)} e^{-imf_0}\frac{(1-e_0^2)^{3/2}}{(1+e_0\cos(f_0))^2}\,df_0.$

$$= \frac{1}{2\pi}\int_0^{2\pi}\frac{(1-e_0^2)^{3/2}}{(1-e_0^2)^{l+1}}\frac{(1+e_0\cos f_0)^{l+1}}{(1+e_0\cos f_0)^2}e^{-imf_0}\,df_0.$$

$$= \frac{1}{2\pi}(1-e_0^2)^{-(l-1/2)}\int_0^{2\pi}(1+e_0\cos f_0)^{l-1}e^{-imf_0}\,df_0.$$

b). Looking at $X_0^{-(l+1),m}(e_0)$

We realise that the real part in the intergral $(1+e_0\cos f_0)^{l-1}$ is an even function while the imaginery part, e^{-imf_0} is an odd function. So that when integrating from $(0, 2\pi)$ the over all intergrals will be real.

c) Show that, $X_0^{-3,2}(e_0) = X_0^{-4,2} = 0$,

now,
$$X_0^{-3,2}(e_0) = X_0^{-(2+1),2} = \frac{1}{2\pi(1-e_0^2)^{3/2}} \int_0^{2\pi} (1+e_0\cos f_0)^1 e^{-2if_0}\, df_0.$$

$$= \frac{1}{2\pi(1-e_0^2)^{3/2}} \underbrace{\int_0^{2\pi} \left(e^{-2if_0} + e^{-2if_0} e_0 \cos f_0\right) df_0}_{=0 \text{ via mathematica}}$$

and,
$$X_0^{-4,3}(e_0) = X_0^{-(3+1),3} = \underbrace{\frac{1}{2\pi(1-e_0^2)^{5/2}}}_{K} \int_0^{2\pi} (1+e_0\cos f_0)^2 e^{-3if_0}\, df_0.$$

$$= K \underbrace{\int_0^{2\pi} \left(1+2e_0\cos f_0 + e_0^2\cos^2 f_0\right) e^{-3if_0}\, df_0}_{=0 \text{ via mathematica}}$$

$$X_0^{-3,0}(e_0) = X_0^{-(2+1),0} = \frac{1}{2\pi(1-e_0^2)^{3/2}} \int_0^{2\pi} (1+e_0\cos f_0)^1\, df_0.$$

$$= \frac{1}{2\pi(1-e_0^2)^{3/2}} \Big[f_0 + e_0 \sin(f_0) \Big]_0^{2\pi} = \frac{2\pi}{2\pi(1-e_0^2)^{3/2}} = (1-e_0^2)^{-3/2}$$

Taylor expansion yields (about 0)

$$= f(0) + \frac{3\times 0}{(1-0)^{5/2}} e_0 + \left(\frac{15\times 0^2}{(1-0^2)^{7/2}} + \frac{3}{(1-0^2)^{5/2}}\right)\frac{e_0^2}{2} + O(e_0^4)$$

$$= 1 + \frac{3}{2}e_0^2 + O(e_0^4)$$

$$X_0^{-4,1}(e_0) = X_0^{-(3+1),1}(e_0)$$

$$= \frac{1}{2\pi(1-e_0^2)^{5/2}} \underbrace{\int_0^{2\pi} (1+e_0\cos f_0)^2 e^{\cdot i f_0} \, df_0}$$

$$= \frac{1}{2\pi(1-e_0^2)^{5/2}} \times 2e\pi \quad // \text{ via mathematica,}$$

$$= \frac{e}{(1-e_0^2)^{5/2}}$$

Taylor expansion aroun 0 gives,

$$= f(0) + f'(0)e_0 + f''(0)\frac{e_0^2}{2} + f'''(0)\frac{e_0^3}{6}$$

$$= 0 + \left(\frac{5x0}{(1-0)^{7/2}} + \frac{1}{(1-0)^{5/2}}\right)e_0 + O(e^3)$$

$$= e_0 + O(e^3)$$

f) Begining with.

$$\mathcal{R} = \frac{G\mathcal{M}_i m_3}{a_i} \sum_{l=2}^{\infty} \sum_{m=-l,2}^{l} \frac{1}{2} C_{lm}^2 \mathcal{M}_l \left(\frac{a_i}{a_o}\right)^{l+1} e^{+im(\varpi_i - \varpi_o)} \left[\left(r/a_i\right)^l e^{imf_i}\right]\left[\frac{e^{-imf_o}}{(R/a_o)^{l+1}}\right]$$

We can sobsitute,

$$\left(\frac{r}{a_i}\right)^l e^{imf_i} = \sum_{n=-\infty}^{\infty} X_n^{l,m}(e_i) e^{inM_i}$$

and

$$\left(\frac{e^{-imf_o}}{(R/a_o)^{l+1}}\right) = \sum_{n=-\infty}^{\infty} X_n^{-(l+1),m}(e_o) e^{-in'M_o}$$

to get,

$$\mathcal{R} = \frac{G\mathcal{M}_i m_3}{a_i} \sum_{l=2}^{\infty} \sum_{m=-l,2}^{l} \sum_{n'=-\infty}^{\infty} \sum_{n=\infty}^{\infty} \frac{1}{2} C_{lm}^2 \mathcal{M}_l \left(\frac{a_i}{a_o}\right)^{l+1} X_n^{l,m}(e_i) X_{n'}^{-(l+1),m}(e_o) \underbrace{e^{im(\varpi_i - \varpi_o)} e^{inM_i} e^{-in'M_o}}_{e^{i(m(\varpi_i - \varpi_o) + nM_i - n'M_o)} = e^{i\phi_{mnn'}}}$$

where $\phi_{mnn'} = nM_i - n'M_o + m(\varpi_i - \varpi_o)$

So

$$\mathcal{R} = \frac{G\mathcal{M}_i m_3}{a_i} \sum_{l=2}^{\infty} \sum_{m=-l,2}^{l} \sum_{n'=-\infty}^{\infty} \sum_{n=\infty}^{\infty} \frac{1}{2} C_{lm}^2 \mathcal{M}_l \left(\frac{a_i}{a_o}\right)^{l+1} X_n^{l,m}(e_i) X_{n'}^{-(l+1),m}(e_o) \times e^{i\phi_{mnn'}}$$

Since $e^{i\phi_{mnn'}} = \cos(\phi_{mnn'}) + i\sin(\phi_{mnn'})$

and we are summing from $n = n' = -\infty$ to $n = n' = \infty$ the sin function is odd and would hence cancle out leaving us only with $\cos(\phi_{mnn'})$

We can also generalise our expression too.

$$\mathcal{R} = \frac{G\nu_i m_3}{a_i} \sum_{\ell=2}^{\infty} \sum_{m=m_{min},2}^{\ell} \sum_{n'=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \xi_m C_{\ell m}^2 \mathcal{M}_\ell \left(\frac{a_i}{a_0}\right)^{\ell+1} \times X_n^{\ell,m}(e_i) X_{n'}^{-(\ell+1),m}(e_0) \cos(\phi_{mnn'})$$

where,

$$\xi_m = \begin{cases} 1/2, & m=0 \\ 1, & \text{else} \end{cases} \quad \text{and} \quad m_{min} = \begin{cases} 0, & \ell = \text{even} \\ 1, & \ell = \text{odd} \end{cases}$$

and since,

$$\sum_{\ell=2}^{\infty} \sum_{m=m_{min},2}^{\ell} T_{\ell m} = \sum_{m=0}^{\infty} \sum_{\ell=\ell_{min},2}^{\infty} T_{\ell m} \quad \text{where } \ell_{min} = \begin{cases} 2, & m=0 \\ 3, & m=1 \\ m, & m \geq 2 \end{cases}$$

We get,

$$\mathcal{R} = \frac{G\nu_i m_3}{a_i} \sum_{\ell=\ell_{min},2}^{\infty} \sum_{m=0}^{\infty} \sum_{n'=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \xi_m C_{\ell m}^2 \mathcal{M}_\ell \left(\frac{a_i}{a_0}\right)^{\ell+1} X_n^{\ell,m}(e_i) X_{n'}^{-(\ell+1),m}(e_0) \times \cos(\phi_{mnn'})$$

by letting,

$$\sum_{l=l_{min},2}^{\infty} \zeta_m c_{ln}^{2} M_l \left(\frac{a_i}{a_o}\right)^{l+1} X_n^{l,n}(e_i) X_{n'}^{-(l+1),m}(e_o)$$

$$= \left(\frac{m_{12}}{m_3}\right) R_{mnn'}$$

We can subsitute it back in to get.

$$\mathcal{R} = \frac{G \mu_i m_3}{a_i}\left(\frac{m_{12}}{m_3}\right)\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty} R_{mnn'} \cos(\phi_{mnn'})$$

where $\mu_i = \frac{m_1 m_2}{m_{12}}$

$$\mathcal{R} = \frac{G m_1 m_2}{a_i}\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty} R_{mnn'} \cos(\phi_{mnn'})$$

Q 5, Setting $n = n' = 0$ we get

$$\mathcal{R} = \frac{G m_1 m_2}{a_i} \sum_{m=0}^{\infty} R_{m00} \cos \phi_{m00}$$

Where

$$R_{m00} = \left(\frac{m_3}{m_1+m_2}\right) \sum_{l=l_{min},2}^{\infty} \zeta_m C_{lm}^2 M_l \left(\frac{a_i}{a_o}\right)^{l+1} X_n^{l,m}(e_i) X_{n'}^{-(l+1),m}(e_o)$$

and

$$\phi_{m00} = m(\bar{\omega}_i - \bar{\omega}_o), \quad \zeta_m = \begin{cases} 1/2, & m=0 \\ 1, & m>0 \end{cases} \quad \text{and} \quad l_{min} = \begin{cases} 2 & m=0 \\ 3 & m=1 \\ m, & m \geq 2 \end{cases}$$

We want to solve $\mathcal{R}$ for the octopole ($l=3$) beginning at $m=0$, $l_{min}=2$, so our 1st term in the sum (up to quadropole) is $m=0$, $l=2$, now for $m=1$, $l_{min}=3$ and our 2nd term is for $m=1$, $l=3$ so we get,

$$\mathcal{R} = \frac{G m_1 m_2}{a_i}\left(R_{000} \cos \phi_{000} + R_{100} \cos \phi_{100} \right), \quad \phi_{000} = 0 \quad \phi_{100} = \bar{\omega}_i - \bar{\omega}_o$$

$$= \frac{G m_1 m_2}{a_i}\left(\frac{m_3}{m_1+m_2}\right)\left(\zeta_0 C_{20}^2 M_2 \left(\frac{a_i}{a_o}\right)^3 X_0^{2,0}(e_i) X_0^{-3,0}(e_o) \cos(0) \right.$$

$$\left. + \zeta_1 C_{31}^2 M_3 \left(\frac{a_i}{a_o}\right)^4 X_0^{3,1}(e_i) X_0^{-4,1}(e_o) \cos(\bar{\omega}_i - \bar{\omega}_o) \right)$$

now $\xi_0 = \frac{1}{2}$ $C_{20}^2 = (-1/\sqrt{2}) = \frac{1}{2}$, $\mathcal{M}_2 = \left(\frac{m_0 + m_2}{m_{12}}\right) = 1$

$$X_0^{2,0}(e_i) = 1 + \frac{3}{2}e_i^2 \;,\; X_0^{-3,0}(e_o) = 1 + \frac{3}{2}e_o^2 + O(e^4)$$

$$\xi_1 = 1 \quad C_{31}^2 = \frac{3}{8} \;,\; \mathcal{M}_3 = \left(\frac{m_1^2 - m_2^2}{m_{12}^2}\right)$$

$$X_0^{3,1}(e_i) = -\frac{5}{8}e_i(4 + 3e_i^2) = -\frac{5}{2}e_i + O(e_i^3)$$

$$X_0^{-4,1}(e_o) = \frac{e_o}{(1-e_o^2)^{5/2}} = e_o + O(e_o^3)$$

Subsituting this we get,

$$\mathcal{R} = \frac{Gm_1m_2m_3}{a_i(m_1+m_2)}\left[\frac{1}{2}\times\frac{1}{2}\left(\frac{a_i}{a_o}\right)^3\left(1+\frac{3}{2}e_i^2\right)\left(1+\frac{3}{2}e_o^2\right)\right.$$
$$\left. + 1\times\frac{3}{8}\left(\frac{m_1^2-m_2^2}{m_{12}^2}\right)\left(-\frac{5}{2}e_i\right)(e_o)\cos(\varpi_i - \varpi_o)\left(\frac{a_i}{a_o}\right)^4\right]$$

ignoring higer order terms $O(e^3)$ + and let $a = \frac{a_i}{a_o}$

$$\mathcal{R} = \frac{Gm_1m_2m_3}{a_i(m_1+m_2)}\left[\frac{1}{4}\left(1+\frac{3}{2}e_i^2+\frac{3}{2}e_o^2\right)a^3 - \frac{15}{16}\left(\frac{(m_1+m_2)(m_1-m_2)}{(m_1+m_2)(m_1+m_2)}\right)e_ie_o a^4\cos(\varpi_i - \varpi_o)\right]$$

Where $\nu = \frac{m_1m_2}{m_1+m_2}$ is the redeced mass

$$\mathcal{R} = \frac{G\nu m_3}{a_i}\left[\frac{1}{4}\left(1+\frac{3}{2}e_i^2+\frac{3}{2}e_o^2\right)a^3 - \frac{15}{16}\frac{m_1-m_2}{m_1+m_2}e_ie_oa^4\cos(\varpi_i - \varpi_o)\right]$$

Q6 Lagranges Planetary equations.

$$\mathcal{R} = \frac{G\mathcal{M}_0 m_3}{a_i}\left[\frac{1}{4}\left(1+\frac{3}{2}e_i^2+\frac{3}{2}e_o^2\right)\alpha^2 - \frac{15}{16}\left(\frac{m_1-m_2}{m_1+m_2}\right)e_i e_o \alpha^4 \cos(\bar{\omega}_i-\bar{\omega}_o)\right]$$

where $\alpha = \left(\frac{a_o}{a_i}\right)$

$$\frac{da_b}{dt} = \frac{2}{m_b V_b a_b}\frac{\partial\mathcal{R}}{\partial\lambda_b}$$

where i = inner = b

o = outer = c

$$\frac{de_b}{dt} = -\frac{1}{m_b V_b a_b^2 e_b}\frac{\partial\mathcal{R}}{\partial\bar{\omega}_b}$$

$$\frac{d\bar{\omega}_b}{dt} = \frac{1}{m_b V_b a_b^2 e_b}\frac{\partial\mathcal{R}}{\partial e_b}$$

These approximations are good for planetary systems that are close to circular orbits ($m_c, m_b \ll m_*$ and $e_b, e_c \ll 1$)

Since $\mathcal{R}$ is independent of the mean longitudes (λ)

$\dot{a}_b = \dot{a}_c = 0$,

and,

$$\dot{e}_b = -(m_b V_b a_b^2 e_b)^{-1}\frac{G\mathcal{M}_b m_c}{a_b}\left(+\frac{15}{16}\underbrace{\left(\frac{m_*-m_a}{m_*+m_a}\right)}_{\approx 1} e_b e_c \alpha^4 \sin(\bar{\omega}_b-\bar{\omega}_c)\right)$$

$$= -\frac{G\mathcal{M}_b m_c}{m_b V_b a_b^3}\frac{15}{16}e_c\alpha^4\sin(\bar{\omega}_b-\bar{\omega}_c)$$

so

$$\nu_G = \sqrt{\frac{GM_{12}}{a_G^3}} = \sqrt{\frac{GM_*}{a_G^3}} \qquad N_G = \frac{m_* m_G}{m_* + m_G} \simeq M_G.$$

so $$\frac{G N_G m_c}{m_G \nu_G a_G^3} = \frac{G N_G m_c}{m_G \nu_G^2 a_G^3} \nu_G$$

$$= \frac{G m_G m_c a_G^3}{m_G G M_* a_G^3} \nu_G = \frac{m_c}{m_*} \nu_G.$$

$\therefore$

$$\dot{e}_G = -\frac{15}{16} \frac{m_c}{m_*} \nu_G e_c \left(\frac{a_G}{a_c}\right)^4 \sin(\bar{\omega}_G - \bar{\omega}_c)$$

$$\dot{\bar{\omega}}_G = (\cancel{m_G} \nu_G a_G^2 e_G)^{-1} \times \frac{G N_G m_c}{a_G} \left[\frac{1}{4}(3e_G) a^3 - \frac{15}{16} e_c a^4 \cos(\bar{\omega}_G - \bar{\omega}_c)\right]$$

$$= \frac{G m_c}{\nu_G a_G^3 e_G} \frac{3}{4} a^3 \left[e_G - \frac{5}{4} a e_c \cos(\bar{\omega}_G - \bar{\omega}_c)\right]$$

$$= \frac{G M_c}{\nu_G a_G^3} \frac{3}{4} a^3 \left[1 - \frac{5}{4} a \left(\frac{e_c}{e_G}\right) \cos(\bar{\omega}_G - \bar{\omega}_c)\right]$$

where $$\frac{G M_c}{\nu_G a_G^3} = \nu_G \left(\frac{m_c}{m_*}\right)$$

so $$\dot{\bar{\omega}}_G = \frac{3}{4} \nu_G \left(\frac{m_c}{m_*}\right) a^3 \left[1 - \frac{5}{4} a \left(\frac{e_c}{e_G}\right) \cos(\bar{\omega}_G - \bar{\omega}_c)\right]$$

now

$$\dot{e}_c = -(m_c V_c a_c^2 e_c)^{-1} \frac{\partial R}{\partial \varpi_c}$$

$$\dot{\varpi}_c = (m_c V_c a_c^2 e_c)^{-1} \frac{\partial R}{\partial e_c}$$

so $$\dot{e}_c = -(m_c V_c a_c^2 e_c)^{-1} \frac{G N_G m_c}{a_G}\left(-\frac{15}{16} e_G e_c \alpha^4 \sin(\bar{\varpi}_G - \bar{\varpi}_c)\right)$$

$$= \frac{G N_G}{V_c a_c^2 a_G} \frac{15}{16} e_G \alpha^4 \sin(\bar{\varpi}_G - \bar{\varpi}_c)$$

now $V_c \approx \sqrt{\frac{G M_*}{a_c^3}}$ $\quad N_G \approx M_G$.

So $$\frac{G N_G}{V_c a_c^2 a_G} = \frac{G M_G}{V_c^2 a_c^2 a_G} V_c = \frac{G M_G a_c^3}{G M_* a_c^2 a_G} V_c = \frac{M_G a_c}{M_* a_G} V_c$$

so $$\dot{e}_c = \frac{M_G}{m_*} V_c \frac{a_c}{a_G} \frac{15}{16} e_G \left(\frac{a_G}{a_c}\right)^4 \sin(\bar{\varpi}_G - \bar{\varpi}_c) \frac{15}{16}$$

$$= \frac{15}{16} V_c \left(\frac{m_G}{m_*}\right)\left(\frac{a_G}{a_c}\right)^3 e_G \sin(\bar{\varpi}_G - \bar{\varpi}_c)$$

$$\dot{\varpi}_c = (m_c \nu_c a_c^2 e_c)^{-1} \frac{G m_b m_c}{a_b}\left[\frac{3}{4} e_c a^3 - \frac{15}{16} e_b a^4 \cos(\varpi_b - \varpi_c)\right]$$

$$= \frac{G m_b}{\nu_c a_b a_c^2 e_c} \frac{3}{4} e_c a^3 \left[1 - \frac{15}{16}\frac{e_b}{e_c} a \cos(\varpi_b - \varpi_c)\right]$$

$$= \frac{G m_b}{\nu_c a_b a_c^2} \frac{3}{4} a^3 \left[1 - \frac{15}{16}\frac{e_b}{e_c} a \cos(\varpi_b - \varpi_c)\right]$$

now

$$\frac{G m_b}{\nu_c a_b a_c^2} = \frac{\cancel{G} m_b a_c^{\cancel{3}}}{a_b \cancel{a_c^2} \cancel{G} m_*} \nu_c = \frac{a_c}{a_b}\frac{m_b}{m_*}\nu_c$$

So $$\dot{\varpi}_c = \frac{3}{4}\nu_c \frac{m_b}{m_*} a^2 \left[1 - \frac{15}{16}\frac{e_b}{e_c} a \cos(\varpi_b - \varpi_c)\right]$$

when $\frac{a_b}{a_c} \ll 1$, (Planet b is much closer. the rates of the longitudes of Periatron reduces too).

$$\dot{\varpi}_b = \frac{3}{4}\nu_b \left(\frac{m_c}{m_*}\right) a^3 \quad , \quad \dot{\varpi}_c = \frac{3}{4}\nu_c \frac{m_b}{m_*} a^2$$

b)

$$\dot{\varpi}_G = \frac{3}{4} V_G \left(\frac{m_c}{m_*}\right)\left(\frac{a_G}{a_c}\right)^3 \quad \text{and} \quad \dot{\varpi}_c = \frac{3}{4} V_c \left(\frac{m_G}{m_*}\right)\left(\frac{a_G}{a_c}\right)^2$$

The Period, of $\dot{\varpi}_G$ can be calculated by,

$$P_{\varpi} = \frac{2\pi}{\dot{\varpi}_G}$$

So $$P_{\varpi} = 2\pi\left(\frac{4}{3} V_G^{-1}\left(\frac{m_*}{m_c}\right)\left(\frac{a_c}{a_G}\right)^3\right)$$

Simmilarly the Period of motion is,

$$P_G = \frac{2\pi}{V_G},$$

So

$$P_{\varpi_G} = \frac{2\pi}{2\pi}\left(\frac{4}{3} P_G\left(\frac{m_*}{m_c}\right)\left(\frac{a_c}{a_G}\right)^3\right)$$

and since $V_c = \sqrt{\frac{G m_*}{a_c^3}}$, $V_G = \sqrt{\frac{G m_*}{a_G^3}}$

$$\Rightarrow \frac{V_G^2}{V_c^2} = \frac{a_c^3}{a_G^3} = \frac{(2\pi/P_G)^2}{(2\pi/P_c)^2} = \left(\frac{P_c}{P_G}\right)^2$$

So $$P_{\varpi_G} = \frac{4}{3}\left(\frac{m_*}{m_c}\right)\left(\frac{P_c}{P_G}\right)^2 P_G$$

c)

$$\dot{e}_b = -\frac{15}{16} V_b \left(\frac{m_c}{m_*}\right)\left(\frac{a_b}{a_c}\right)^4 e_c \sin(\bar{\omega}_b - \bar{\omega}_c)$$

$$\dot{e}_c = \frac{15}{16} V_c \left(\frac{m_b}{m_*}\right)\left(\frac{a_b}{a_c}\right)^3 e_b \sin(\bar{\omega}_b - \bar{\omega}_c)$$

$$\Rightarrow \dot{e}_b V_b^{-1} m_c^{-1} \frac{a_c}{a_b} e_c^{-1} = -\frac{15}{16} m_*^{-1} \left(\frac{a_b}{a_c}\right)^3 \sin(\bar{\omega}_b - \bar{\omega}_c)$$

$$\Rightarrow -\dot{e}_c V_c^{-1} m_b^{-1} e_b^{-1} = -\frac{15}{16} m_*^{-1} \left(\frac{a_b}{a_c}\right)^3 \sin(\bar{\omega}_b - \bar{\omega}_c).$$

such that we can equate the 2 expressions.

$$\frac{\dot{e}_b}{V_b m_c e_c} \frac{a_c}{a_b} = \frac{-\dot{e}_c}{V_c m_b e_b}$$

$$\dot{e}_b = -\frac{V_b}{V_c} \frac{m_c}{m_b} \frac{e_c}{e_b} \frac{a_b}{a_c} \dot{e}_c \qquad \frac{V_b}{V_c} = \left(\frac{a_c^3}{a_b^3}\right)^{1/2} = \left(\frac{a_c}{a_b}\right)^{3/2}$$

$$= -\left(\frac{a_c}{a_b}\right)^{3/2} \left(\frac{a_c}{a_b}\right)^{-1} \left(\frac{m_c}{m_b}\right) \frac{e_c}{e_b} \dot{e}_c$$

$$= -\left(\frac{a_c}{a_b}\right)^{1/2} \left(\frac{m_c}{m_b}\right) \frac{e_c}{e_b} \dot{e}_c = -\gamma \frac{e_c}{e_b} \dot{e}_c$$

$$\frac{de_b}{dt} e_b = -\gamma e_c \frac{de_c}{dt},$$

now intergrate our equation from the epoch time to time t,

$$\int_{T_0}^{t} e_c \dot{e}_c \, dt = -\gamma \int_{T_0}^{t} e_c \dot{e}_c \, dt.$$

now, $\frac{d}{dt}\left(\frac{1}{2}e_c^2\right) = e_c \dot{e}_c$ Via chain rule.

So,

$$\frac{1}{2}e_c^2\Big|_{T_0}^{t} = -\gamma \frac{1}{2}e_c^2\Big|_{T_0}^{t} \equiv \int_{T_0}^{t} e_c\dot{e}_c dt = -\gamma\int_{T_0}^{t} e_c\dot{e}_c dt.$$

$$\Rightarrow \quad e_c^2(t) - e_c^2(T_0) = -\gamma e_c^2(t) + \gamma e_c^2(T_0).$$

$$\Rightarrow \quad e_c^2(t) + \gamma e_c^2(t) = e_c^2(T_0) + \gamma e_c^2(T_0).$$

d). $e_G(t) = \langle e_G \rangle + \delta e_G$, $\quad e_C(t) = \langle e_C \rangle + \delta e_C$

Substituting into eqn 62, 65.

$$\dot{e}_G = -\frac{15}{16}\nu_G\left(\frac{m_C}{m_*}\right)\left(\frac{a_G}{a_C}\right)^4\left(\langle e_C \rangle + \delta e_C\right)\sin(\varpi_G - \varpi_C)$$

$$\dot{e}_C = \frac{15}{16}\nu_C\left(\frac{m_G}{m_*}\right)\left(\frac{a_G}{a_C}\right)^3\left(\langle e_G \rangle + \delta e_G\right)\sin(\varpi_G - \varpi_C)$$

e) The secclar Period, of variation of the eccentricities would be,

$$P_\omega = \frac{2\pi}{\omega} = \frac{4}{3} \nu_j^{-1} \left(\frac{m_*}{m_s}\right)\left(\frac{a_s}{a_j}\right)^3 \frac{\gamma}{\gamma - 1}$$

where $\gamma = \left(\frac{m_s}{m_j}\right)\left(\frac{a_s}{a_j}\right)^{1/2}$

now $\nu_j = \frac{2\pi}{P_j} \Rightarrow \nu_j^{-1} = \frac{P_j}{2\pi}$,

$P_j = 11.87\,yr$

$\frac{m_s}{M_\odot} = 0.0003$, $\frac{m_j}{m_*} = 0.001$

$a_j = 5.2\,au$

$a_s = 9.54\,au$,

so $\gamma = \left(\frac{0.0003}{0.001}\right)\left(\frac{9.54}{5.2}\right)^{1/2} = 0.406$

$$P_\omega = \frac{4}{3} P_j \left(0.0003\right)^{-1}\left(\frac{9.54}{5.2}\right)^3 \frac{0.406}{0.406 - 1} = -222660\,yr$$

=

f) setting $\dot{e}_c = \dot{e}_G = 0$ we get

$$\dot{e}_G = 0 = \underbrace{-\frac{15}{16} \nu_G \left(\frac{m_c}{m_*}\right)\left(\frac{a_G}{a_c}\right)^4}_{\text{constant}} e_c \sin(\varpi_G - \varpi_c)$$

$\therefore \quad \sin(\varpi_G - \varpi_c) = 0 \Rightarrow \varpi_G - \varpi_c = \pm\pi.$

and for no secular variations in eccentricity

$\dot{\varpi}_G - \dot{\varpi}_c = 0$

$$\Rightarrow \frac{3}{4}\left(\frac{a_G}{a_c}\right)^2 \frac{1}{m_*}\left(\nu_G m_c \frac{a_G}{a_c}\left[1 - \frac{5}{4}\left(\frac{a_G}{a_c}\right)\left(\frac{e_c}{e_G}\right)\cos(\varpi_G - \varpi_c)\right]\right.$$

$$\left. - \nu_c m_G \left[1 - \frac{5}{4}\left(\frac{a_G}{a_c}\right)\left(\frac{e_G}{e_c}\right)\cos(\varpi_G - \varpi_c)\right]\right) = 0$$

$$\Rightarrow \nu_G m_c \left(\frac{a_G}{a_c}\right)\left[1 - \frac{5}{4}\left(\frac{a_G}{a_c}\right)\left(\frac{e_c}{e_G}\right)\cos(\pm\pi)\right]$$

$$= \nu_c m_G \left[1 - \frac{5}{4}\left(\frac{a_G}{a_c}\right)\left(\frac{e_c}{e_G}\right)^{-1}\cos(\pm\pi)\right]$$

expand and rearange

$$\nu_G m_c \left(\frac{a_G}{a_c}\right) - \frac{5}{4}\left(\frac{a_G}{a_c}\right)^2\left(\frac{e_c}{e_G}\right)\nu_G m_c \cos(\pm\pi)$$

$$= \nu_c m_G - \frac{5}{4}\nu_c m_G \left(\frac{a_G}{a_c}\right)\left(\frac{e_c}{e_G}\right)^{-1}\cos(\pm\pi)$$

multinly by $\left(\frac{e_c}{e_G}\right)$ to get

$$\left(V_G m_C\left(\frac{a_G}{a_C}\right) - V_C m_G\right)\frac{e_C}{e_G} - \frac{5}{4}\left(\frac{a_G}{a_C}\right)^2\left(\frac{e_C}{e_G}\right)^2 V_G m_C \cos(\pm\pi)$$

$$+ \frac{5}{4} V_C m_G\left(\frac{a_G}{a_C}\right)\cos(\pm\pi) = 0$$

We notice that this is a quadratic equation for $\frac{e_C}{e_G}$

Where $A = -\frac{5}{4}\left(\frac{a_G}{a_C}\right)^2 V_G m_C \cos(\pm\pi)$

$B = V_G m_C\left(\frac{a_G}{a_C}\right) - V_C m_G$

$C = \frac{5}{4} V_C m_G\left(\frac{a_G}{a_C}\right)\cos(\pm\pi)$

such that,

$$\left(\frac{e_C}{e_G}\right) = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A},$$

for the case of possitive π $\cos(\pi) = 1$,

$$\left(\frac{e_C}{e_G}\right) = \frac{-\left(V_G m_C\left(\frac{a_G}{a_C}\right) - V_C m_G\right) \pm \sqrt{\left(V_G m_C\left(\frac{a_G}{a_C}\right) - V_C m_G\right)^2 + \frac{25}{4}\left(\frac{a_G}{a_C}\right)^3 V_C V_G m_C m_G}}{-\frac{5}{2}\left(\frac{a_G}{a_C}\right)^2 V_G m_C}$$

for the case where $\cos(-\pi) = -1$

$$\left(\frac{e_C}{e_G}\right) = \frac{-\left(V_G m_C\left(\frac{a_G}{a_C}\right) - V_C m_G\right)}{\frac{5}{2}\left(\frac{a_G}{a_C}\right)^2 V_G m_C}$$

$$\pm \frac{\sqrt{\left(V_G m_C\left(\frac{a_G}{a_C}\right) - V_C m_G\right)^2 + \frac{25}{4}\left(\frac{a_G}{a_C}\right)^3 V_G V_C m_G m_C}}{\frac{5}{2}\left(\frac{a_G}{a_C}\right)^2 V_G m_C}$$

This leaves us, with 4 equations/exceptions for the ratio of eccentricities where there is no secular evolution.

A2 Numerical solution

$$\frac{de_G}{dt} = -\frac{15}{16} \nu_G \left(\frac{m_c}{m_*}\right)\left(\frac{a_G}{a_c}\right)^4 e_c \sin(\bar{\omega}_G - \bar{\omega}_c)$$

$$\frac{d\bar{\omega}_G}{dt} = \frac{3}{4} \nu_G \left(\frac{m_c}{m_*}\right)\left(\frac{a_G}{a_c}\right)^3 \left[1 - \frac{5}{4}\left(\frac{a_G}{a_c}\right)\left(\frac{e_c}{e_G}\right) \cos(\bar{\omega}_G - \bar{\omega}_c)\right]$$

$$\frac{de_c}{dt} = \frac{15}{16} \nu_c \left(\frac{m_G}{m_*}\right)\left(\frac{a_G}{a_c}\right)^3 e_G \sin(\bar{\omega}_G - \bar{\omega}_c)$$

$$\frac{d\bar{\omega}_c}{dt} = \frac{3}{4} \nu_c \left(\frac{m_G}{m_*}\right)\left(\frac{a_G}{a_c}\right)^2 \left[1 - \frac{5}{4}\left(\frac{a_G}{a_c}\right)\left(\frac{e_G}{e_c}\right) \cos(\bar{\omega}_G - \bar{\omega}_c)\right]$$

Using the unit conversion,

$$\frac{d\tau}{dt} = \frac{3}{4} \nu_G \left(\frac{m_c}{m_*}\right)\left(\frac{a_G}{a_c}\right)^3$$

We can rewrite our expression for e_G, ... ect,

So $$\frac{de_G}{d\tau} = \frac{de_G}{dt}\frac{dt}{d\tau} = \frac{4}{3} \nu_G^{-1} \left(\frac{m_c}{m_*}\right)^{-1}\left(\frac{a_G}{a_c}\right)^{-3} \left(-\frac{15}{16} \nu_G \left(\frac{m_c}{m_*}\right)\left(\frac{a_G}{a_c}\right)^4 e_c \sin(\bar{\omega}_G - \bar{\omega}_c)\right)$$

$$= -\frac{5}{4}\left(\frac{a_G}{a_c}\right) e_c \sin(\bar{\omega}_G - \bar{\omega}_c)$$

$$\frac{d\bar{\omega}_G}{d\tau} = \frac{d\bar{\omega}_G}{dt}\frac{dt}{d\tau} = 1 - \frac{5}{4}\left(\frac{a_G}{a_c}\right)\left(\frac{e_c}{e_G}\right) \cos(\bar{\omega}_G - \bar{\omega}_c)$$

$$\frac{de_c}{d\tau} = \frac{de_c}{dt}\frac{dt}{d\tau} = \frac{4}{3}V_G^{-1}\left(\frac{m_c}{m_*}\right)^{-1}\left(\frac{a_G}{a_c}\right)^{-3}$$

$$\times \frac{15}{16}V_c\left(\frac{m_G}{m_*}\right)\left(\frac{a_G}{a_c}\right)^{3} e_G \sin(\varpi_G - \varpi_c)$$

$$= \frac{5}{4}\left(\frac{V_c}{V_G}\right)\left(\frac{m_G}{m_c}\right) e_G \sin(\varpi_G - \varpi_c)$$

now $V = \left(\frac{G m_{12}}{a^3}\right)^{1/2}$ so, $\frac{V_c}{V_G} = \frac{a_G^{3/2}}{a_c^{3/2}} = \left(\frac{a_G}{a_c}\right)^{3/2}$,

so $\left(\frac{V_c}{V_G}\right)\left(\frac{m_G}{m_c}\right) = \left(\frac{a_G}{a_c}\right)^{3/2}\left(\frac{m_G}{m_c}\right) = \left(\frac{a_G}{a_c}\right)\left(\frac{a_G}{a_c}\right)^{1/2}\left(\frac{m_G}{m_c}\right) = \gamma^{-1}\frac{a_G}{a_c}$.

where $\gamma = \frac{m_c}{m_G}\left(\frac{a_c}{a_G}\right)^{1/2}$

so

$$\frac{de_c}{d\tau} = \frac{5}{4}\gamma^{-1}\frac{a_G}{a_c} e_G \sin(\varpi_G - \varpi_c)$$

$$\frac{d\varpi_c}{d\tau} = \frac{d\varpi_c}{dt}\frac{dt}{d\tau} = \frac{4}{3}V_G^{-1}\left(\frac{m_c}{m_*}\right)^{-1}\left(\frac{a_G}{a_c}\right)^{-3}$$

$$\times \frac{3}{4}V_c\left(\frac{m_G}{m_*}\right)\left(\frac{a_G}{a_c}\right)^{2}\left[1 - \frac{5}{4}\left(\frac{a_G}{a_c}\right)\left(\frac{e_G}{e_c}\right)\cos(\varpi_G - \varpi_c)\right]$$

$$= \underbrace{\left(\frac{V_c}{V_G}\right)\left(\frac{m_G}{m_c}\right)\left(\frac{a_G}{a_c}\right)^{-1}}\left[1 - \frac{5}{4}\left(\frac{a_G}{a_c}\right)\left(\frac{e_G}{e_c}\right)\cos(\varpi_G - \varpi_c)\right]$$

$$\left(\frac{a_G}{a_c}\right)^{3/2}\left(\frac{m_G}{m_c}\right)\left(\frac{a_G}{a_c}\right)^{-1} = \left(\frac{a_G}{a_c}\right)^{\frac{1}{2}}\left(\frac{m_G}{m_c}\right) = \gamma^{-1}$$

so $$\frac{d\varpi_c}{d\tau} = \gamma^{-1}\left[1 - \frac{5}{4}\left(\frac{a_G}{a_c}\right)\left(\frac{e_G}{e_c}\right)\cos(\varpi_G - \varpi_c)\right]$$

6). from my plot,

$$P = (152875 - 28176)\text{ years} \approx 1.2 \times 10^5 \text{ years.}$$

Where as the 'true' secular period given from the plot in the assignment,

$$P \approx (7-1) \times 10^4 \approx 6 \times 10^4 \text{ years.}$$

This differs by a factor of 2.

a possible reason for this discrepancy is that we were only solving for the secular evolution without considering the minor oscilations (the squiggles) that arise, from other terms in the $\mathcal{R}$ function

In other words when going from problem 4 to 5 we picked out the $n = n' = 0$ sum. Perhaps if we considered the entire intergral the discrepancy would disapear.