Dynamics Assignment.

a) $\underset{\sim}{v}_{orb} = \underset{\sim}{A}\, \underset{\sim}{v}_{ref}$.

from figure 4, we rotate firstly around the x axis. by the angle I, such that,

$$\underset{\sim}{r}' = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos I & \sin I \\ 0 & -\sin I & \cos I \end{pmatrix}}_{= A_I} \underset{\sim}{r},$$

Then it is rotated around the new z-axis with an angle ω.
Such that.

$$\underset{\sim}{r}'' = \overbrace{\begin{pmatrix} \cos\omega & \sin\omega & 0 \\ -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix}}^{= A_\omega} \underset{\sim}{r}',$$

So $\underset{\sim}{r}'' = \begin{pmatrix} \cos\omega & \sin\omega & 0 \\ -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos I & \sin I \\ 0 & -\sin I & \cos I \end{pmatrix} \underset{\sim}{r}.$

$$= \underbrace{\begin{pmatrix} \cos\omega & \sin\omega\cos I & \sin\omega\sin I, \\ -\sin\omega & \cos\omega\cos I & \cos\omega\sin I \\ 0 & -\sin I & \cos I \end{pmatrix}}_{\underset{\sim}{A} = A_\omega A_I} \underset{\sim}{r}.$$

b),   Let $U_{ref} = \{x_1, y_1, z_1\}$,

so $|U_{ref}|^2 = x_1^2 + y_2^2 + z_1^2$.

now $U_{orb} = A\,U_{ref}$,

$= \{x_1 \cos\omega + y_1 \cos i \sin\omega + z_1 \sin i \sin\omega,$

$y_1 \cos i \cos\omega + z_1 \cos\omega \sin i - x_1 \sin\omega,$

$z_1 \cos i - y_1 \sin i\}$

now $|U_{orb}|^2$.

$= x_1^2 (\cos^2(\omega)) + y_1^2 \cos^2 i \sin^2\omega + z_1^2 \sin^2 i \sin^2\omega.$

$+ 2x_1 y_1 \cos\omega \cos i \sin\omega + 2x_1 z_1 \cos\omega \sin i \sin\omega$
$+ 2y_1 z_1 \cos i \sin i \sin^2\omega.$

$+ y_1^2 \cos^2 i \cos^2\omega + z_1^2 \cos^2\omega \sin^2 i + x_1^2 \sin^2\omega$
$+ 2y_1 z_1 \cos i \sin i \cos^2\omega - 2x_1 z_1 \cos\omega \sin i \sin\omega$
$- 2x_1 y_1 \cos i \cos\omega \sin\omega.$

$+ z_1^2 \cos^2 i + y_1^2 \sin^2 i - 2y_1 z_1 \cos i \sin i$

now group w.r.t $x_1^2, y_1^2, z_1^2, z_1 x_1, z_1 y_1, x_1 y_1$

$= x_1^2(\cos^2\omega + \sin^2\omega) + y_1^2(\cos^2 i \sin^2\omega + \cos^2 i \cos^2\omega + \sin^2 i)$
$+ z_1^2(\sin^2 i \sin^2\omega + \cos^2\omega \sin^2 i + \cos^2 i)$
$+ 2x_1 y_1 (0) + 2x_1 z_1 (0) + 2y_1 z_1 (\cos i \sin i \sin^2\omega$
$+ \cos i \sin i \cos^2\omega$
$- \cos i \sin i)$

$$= x_1^2 + y_1^2\left(\cos^2 i \sin^2\omega + \cos^2 i \cos^2\omega + \sin^2 i\right)$$

$$+ z_1^2\left(\sin^2 i \sin^2\omega + \cos^2\omega \sin^2 i + \cos^2 i\right)$$

$$+ 2y_1 z_1 \underbrace{\left(\cos i \sin i \sin^2\omega + \cos i \sin i \cos^2\omega - \cos i \sin i\right)}$$

$$= \cos i \sin i\left(\sin^2\omega + \cos^2\omega\right) - \cos i \sin i = 0$$

$$\cos^2 i \sin^2\omega + \cos^2 i \cos^2\omega + \sin^2 i.$$

$$= \cos^2 i\left(\sin^2\omega + \cos^2\omega + \frac{\sin^2 i}{\cos^2 i}\right) = \cos^2 i\left(1 + \frac{\sin^2 i}{\cos^2 i}\right)$$

$$= \cos^2 i + \sin^2 i = 1$$

$$\left(\sin^2 i \sin^2\omega + \cos^2\omega \sin^2 i + \cos^2 i\right)$$

$$= \sin^2 i + \cos^2 i = 1$$

$$\therefore \; |v_{orb}|^2 = x_1^2 + y_2^2 + z_2^2.$$

6) ii). $v_{ref} = x_1, y_1, \cancel{z_2}$ $\qquad w_{ref} = (x_2, y_2, z_2)$

So. $\quad v_{ref} \cdot w_{ref} = x_1 x_2 + y_1 y_2 + z_1 z_2$

$$w_{ref} = \{x_2\cos\omega + y_2\cos i \sin\omega + z_2 \sin i \sin\omega,$$

$$y_2 \cos i \cos\omega + z_2 \cos\omega \sin i - x_2 \sin\omega,$$

$$z_2 \cos i - y_2 \sin i\}$$

now

$U_{orb} \cdot W_{orb}$

$= (z_1 z_2 \cos^2 i + x_1 x_2 \cos^2 \omega + y_1 y_2 \cos^2 i \cos^2 \omega$

$- y_2 z_1 \cos i \sin i - y_1 z_2 \cos i \sin i + y_2 z_1 \cos i \cos^2 \omega \sin i$
$+ y_1 z_2 \cos i \cos^2 \omega \sin i + y_1 y_2 \sin^2 i + z_1 z_2 \cos \omega^2 \sin^2 i$
$+ x_1 x_2 \sin^2 \omega + y_1 y_2 \cos^2 i \sin^2 \omega + y_2 z_1 \cos i \sin i \sin^2 \omega$
$+ y_1 z_2 \cos i \sin i \sin^2 \omega + z_1 z_2 \sin^2 i \sin^2 \omega$

group all $x_1 x_2$ ...... to get ..

$$x_1 x_2 \overbrace{(\cos^2 \omega + \sin^2 \omega)}^{=1} + z_1 z_2 \overbrace{(\cos^2 i + \cos \omega^2 \sin^2 i + \sin^2 i \sin^2 \omega)}^{=1}$$
$$+ y_1 y_2 \underbrace{(\cos^2 i \cos^2 \omega + \sin^2 i + \cos^2 i \sin^2 \omega)}_{=1}$$
$$+ y_2 z_1 \underbrace{(\cos i \cos^2 \omega \sin i - \cos i \sin i + \cos i \sin i \sin^2 \omega)}_{=0}$$
$$+ y_1 z_2 \underbrace{(-\cos i \sin i + \cos i \cos^2 \omega \sin i + \cos i \sin i \sin^2 \omega)}_{=0}$$

Where, $\cos^2 i + \cos \omega^2 \sin^2 i + \sin^2 i \sin^2 \omega$

$= \cos^2 i + \sin^2 i (1) = 1,$

simmilarly for $\cos^2 i \cos^2 \omega + \sin^2 i + \cos^2 i \sin^2 \omega,$

and. $\cos i \cos^2 \omega \sin i + \cos i \sin i \sin^2 \omega - \cos i \sin i$
$= \cos i \sin i (\cos^2 \omega + \sin^2 \omega) - \cos i \sin i = 0.$

So $U_{orb} \cdot W_{orb} = x_1 x_2 + y_1 y_2 + z_1 z_2 = U_{ref} \cdot W_{ref}.$

also since $U_{orb} \cdot W_{orb} = |U_{orb}||W_{orb}| \cos(\theta) = |U_{ref}||W_{ref}| \cos(\theta).$

implies that the angle is conserved. between vectors since $|U_{orb}| = |U_{ref}|$

c) $r = \dfrac{a(1-e^2)}{1+e\cos f} = a(1-e^2)(1+e\cos f)^{-1}$

So $\dfrac{d}{dt} r = \dot{r} = a(1-e^2)\dfrac{d}{dt}(1+e\cos f)^{-1}$

$= a(1-e^2)\, e\sin(f)\,(1+e\cos f)^{-2}\,\dot{f}$

$= \dfrac{a(1-e^2)\, e\sin(f)}{(1+e\cos f)^2}\,\dot{f}$

now $r^2\dot{f} = a^2 v\sqrt{1-e^2} \Rightarrow \dot{f} = \dfrac{a^2 v\sqrt{1-e^2}}{r^2}$

and $r^2 = \dfrac{a^2(1-e^2)^2}{(1+e\cos f)^2}$

So $\dot{f} = \cancel{a^2} v\sqrt{1-e^2}\left(\dfrac{(1+e\cos f)^2}{\cancel{a^2}(1-e^2)^2}\right) = \dfrac{v(1+e\cos f)^2\sqrt{1-e^2}}{(1-e^2)^2}$

now $\dfrac{\sqrt{1-e^2}}{(1-e^2)^2} = (1-e^2)^{1/2}(1-e^2)^{-2} = (1-e^2)^{-3/2}$

So $\dot{r} = \dfrac{a(1-e^2)\, e\sin(f)}{(1+e\cos f)^2}\, v(1+e\cos f)^2\,(1-e^2)^{-3/2}$

$= av(1-e^2)^{-1/2} e\sin(f) = \dfrac{ave\sin(f)}{\sqrt{1-e^2}}$

recall $\dot{f} = \nu (1+e\cos f)^2 (1-e^2)^{-3/2}$

and $r = \dfrac{a(1-e^2)}{1+e\cos f}$

so

$$r\dot{f} = \frac{a(1-e^2)}{\cancel{1+e\cos f}} \times \left(\nu (1+e\cos f)^2 (1-e^2)^{-3/2} \right)$$

$$= \frac{a\nu(1+e\cos f)}{\sqrt{1-e^2}}$$

D) The velocity of the star is defined as

$\vec{V}_* = -\frac{M_p}{M_p + M_*} \vec{v}$ where $\vec{v} = \dot{r} e_r + r\dot{f} e_f$.

$\therefore \vec{v} = (\dot{r}\cos(f) - r\dot{f}\sin(f)) i'' + (\dot{r}\sin(f) + r\dot{f}\cos(f)) j''$

$+ (0) k''$

Since the orbit rotates in the x''-y'' Plane,

in order to calculate the radial velocity,

we need to change our coordinates to that of the observer with the predefined Matrix A,

Since $\vec{U}_{orb} = A \vec{U}_{ref}$

$A^T \vec{U}_{orb} = A^T A \vec{U}_{ref} = \vec{U}_{ref}$

$$\therefore \vec{V}_{ref} = \begin{pmatrix} \cos(\omega) & -\sin(\omega) & 0 \\ \cos(i)\sin(\omega) & \cos(i)\cos(\omega) & -\sin(i) \\ \sin(i)\sin(\omega) & \cos(\omega)\sin(i) & \cos(i) \end{pmatrix} \begin{pmatrix} \dot{r}\cos f - r\dot{f}\sin(f) \\ \dot{r}\sin(f) + r\dot{f}\cos(f) \\ 0 \end{pmatrix}$$

Since the radial velocity is along the Z axis of the observer's Frame:

$$\vec{V}_r = \sin(i)\sin(\omega)(\dot{r}\cos f - r\dot{f}\sin(f)) + \cos(\omega)\sin(i)(\dot{r}\sin(f) + r\dot{f}\cos(f)) + 0$$

where $\dot{r} = a\nu \frac{e\sin f}{\sqrt{1-e^2}}$ and $r\dot{f} = \frac{a\nu(1+e\cos f)}{\sqrt{1-e^2}}$

now $V_{*r} = -\frac{M_P}{M_{*P}}\Big(\dot{r}\big(\cos f \sin(i)\sin(\omega) + \cos(\omega)\sin(i)\sin(f)\big)$

$+ r\dot{f}\big(\sin(i)\cos(f)\cos(\omega) - \sin(i)\sin(\omega)\sin(f)\big)$

$= -\frac{M_P}{M_{*P}}\Big(\frac{a v e \sin(f)}{\sqrt{1-e^2}} \sin(i)\big(\cos f \sin\omega + \cos(\omega)\sin(f)\big)$

$+ \frac{a v (1+e\cos f)}{\sqrt{1-e^2}} \sin(i)\big(\cos(f)\cos(\omega) - \sin(\omega)\sin(f)\big)\Big)$

$= \underbrace{-\frac{M_P}{M_{*P}} \frac{a v \sin(i)}{\sqrt{1-e^2}}}_{= K}\Big(e\sin(f)\big(\cos f \sin(\omega) + \cos(\omega)\sin(f)\big)$

$+ (1+e\cos(f))\big(\cos(f)\cos(\omega) - \sin(\omega)\sin(f)\big)\Big)$

since $\frac{M_P}{M_{*P}} a = a_*$ and $K = \frac{a_* v \sin(i)}{\sqrt{1-e^2}}$

So $V_{*r} = -K\Big(e\big(\cancel{\sin(f)\cos f \sin\omega} + \cos(\omega)\sin^2(f) + \cos(\omega)\cos^2(f)$

$\cancel{- \sin(\omega)\sin(f)\cos(f)}\big)$

$+ \cos(f)\cos(\omega) - \sin(\omega)\sin(f)\Big)$

and $\cos(\omega)(\sin^2 f + \cos^2(f)) = \cos(\omega)$

$V_{*r} = -K\big(e\cos(\omega) + \cos(f)\cos(\omega) - \sin(\omega)\sin(f)\big)$

using the trig identity for Products of 2 angles.

$\sin(\omega)\sin(f) = \frac{1}{2}[\cos(\omega - f) - \cos(\omega + f)]$

$\cos(\omega)\cos(f) = \frac{1}{2}[\cos(\omega - f) + \cos(\omega + f)]$

So $\cos(\omega)\cos(f) - \sin(\omega)\sin(f) = \frac{1}{2}[2\cos(\omega + f)] = \cos(\omega + f)$

and,

$$V_* r = -K[e\cos(\omega) + \cos(\omega + f)].$$

now recall that $\omega = \omega_p = \omega_* - \pi$
and $f = f_p = f_*$.

we get,

$$V_* r = -K[e\cos(\omega_* - \pi) + \cos((\omega_* + f_*) - \pi)]$$

$$= -K[-e\cos(\omega_*) - \cos(\omega_* + f_*)]$$

$$= K[e\cos(\omega_*) + \cos(\omega_* + f_*)]$$

Dynamics of Exoplanets Assignment 1 Part 2

Lewis Picker

May 10, 2022

Abstract

The mission is to fit a two planet system to the extra-solar system GJ 876 using the open software Systemic Live. GJ 876 is know to have four confirmed planets orbiting a red dwarf with a mass of roughly a third of the Sun. There are two planets of concern that has a 2:1 resonance such that the inner planet orbits twice for each outer orbit. This implies that the point of conjunction will occur at the same location relative to the frame of the system and therefore will exchange energy which accumulates over many orbits.

1 Observations

A total of 171 velocities were measured in the time-span of about 8 years the 1990's by the Lick and Keck observatories where Keck made the most contribution with 155 measurements. For this task we will only consider that data sourced from the Keck observatory. The telescope offset for the Keck telescope was about 15.02 m/s

2 Fitting the First planet

Before we begin fitting planets to the radial velocity data we first need to make a reasonable guess for the parameters of the planet that is causing the majority of the shifts. From figure 1 there is a strong correlation with the period of about 61 days. We then allowed for the built in Levenberg-Marquardt algorithm to solve for the planets parameters which is outlined in table 1. The Resulting fit had a very large χ^2, and a therefore did not fit the data very well hinting at another planet in the system

3 Fitting the Second planet

After Fitting the first planet there was still a significant spike from the power spectrum at about 30 days see fig 2. Therefore we added a second planet with that period and adjusted the mass and mean anomaly in order to fit the data more precisely. Since the resonance of the system was roughly 2:1 this would imply that the orbits were somewhat circular and hence we left the eccentricity as 0. The fit was again optimise and allowed to consider a possible eccentricity for the orbit, after playing around with the parameters we managed to minimise χ^2 to 11.49. We were content with our fits and the resulting parameters are listed in table 1. Referring to figure 3 the fit was quite reasonable however since χ^2 was considerable large and some data points unaccounted for it was possible to fit another planet.

4 And what about a Third?

Although this was just for fun I noticed that there was yet again another spike in the power spectrum at around 15 days in figure 4. This made intuitive sense since that if it was indeed caused by a planet there would be a 1:2:4 ratio. After fitting a 3rd planet to the data and minimising χ^2 to 2.88 we resulted in the following parameters that are summarised in table 1.

5 Results

These are the results

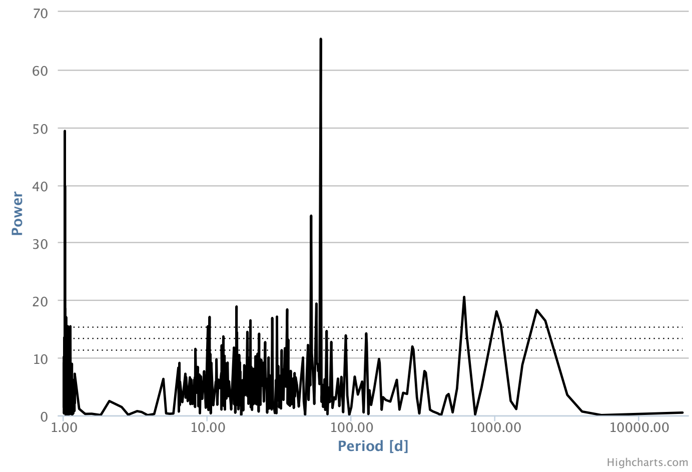

Figure 1: Power relation to the period(days) Implies there is a high likelihood that there is a planet with an orbital period of about 61 days

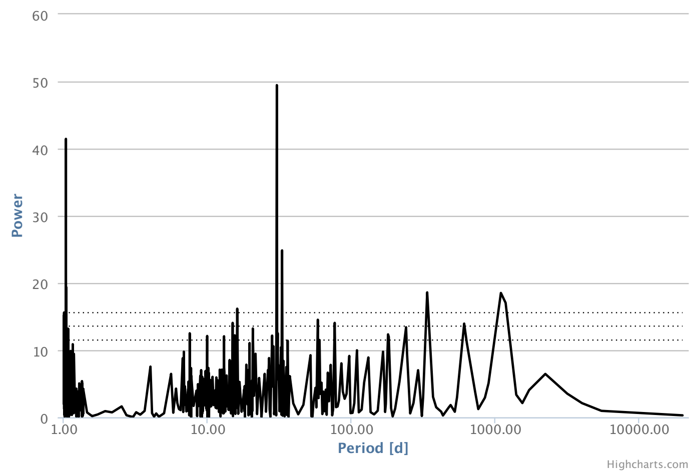

Figure 2: Power Spectrum after fitting first planet, indicating a high likelihood of another planet with a period of about 30 days

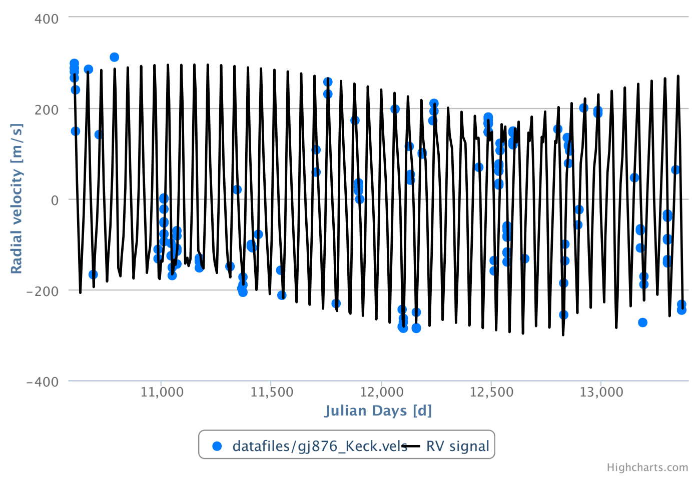

Figure 3: Radial Velocity fit of GJ 876 with 2 planets in resonance

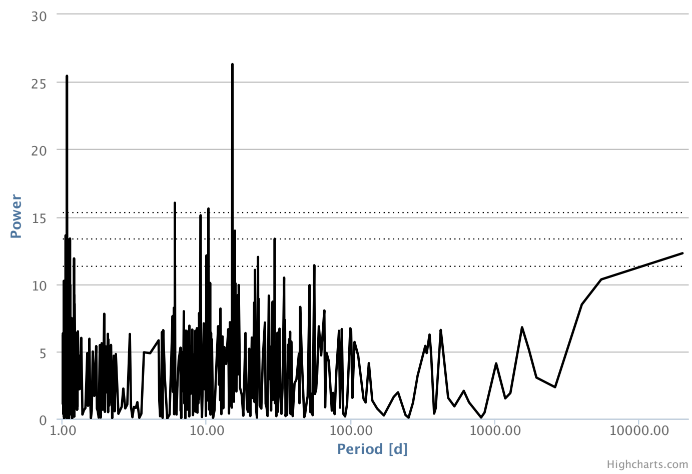

Figure 4: Power spectrum after two planets were fitted to the data, similarly it seems that it was highly likely for another planet to have an orbit of 15 days

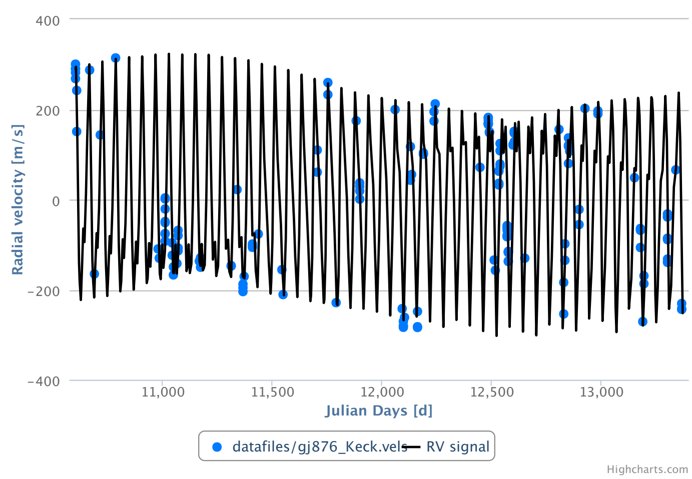

Figure 5: Radial velocity plot with a fitted 3rd planet

Parameters	Planet 1	Planet 2	Planet 3
Period(days)	61.0387	30.2228	15.0458
Mass(Mj)	1.9316	0.6452	0.1189
Mean Anomaly(deg)	84.51	353.73	202.68
Eccentricity	0.00115	0.0128	0.1177
Long. of peri.	292.03	345.90	77.08

Table 1: Tabulated data showing the results of a triple planetary system with a χ^2 value of 2.88

6 Conclusion

Comparing our parameters for our planets to the accepted values found in the Exoplanet Archive We see that the periods of the first 2 planets are accurate to at least 3 decimal places, and that the eccentricity of the first planet was close to 0 as we expected. Interestingly the eccentricity of the second planet was much larger than expected with a consensus of about e = 0.25 amongst the literature. However more recent results from Rosenthal et al. in 2021 yielded an eccentricity close to 0 which might (and should) be the true value if we consider them to be in resonance. Unfortunately the 3rd planet from our investigation was a false alarm since the accepted values of the 3rd and 4th planets have a period of 2 and 124 days respectively. Another interesting comparison is that the mass of our planets compared badly since the literature had a consensus of about 2.6Mj and 0.8Mj. This underestimation on our part might be caused by the fact that we didn't accurately predict the 3rd and 4th planet in the GJ 876 system. Screenshots from systemic live with our results can be found in the appendix.

7 Appendix

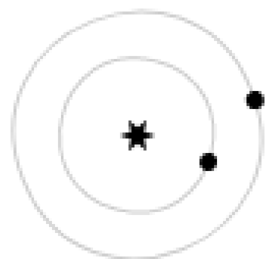

Figure 6: Diagram example of the 2 planet system

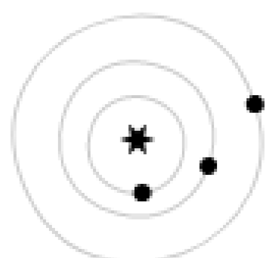

Figure 7: Diagram example of the 3 planet system

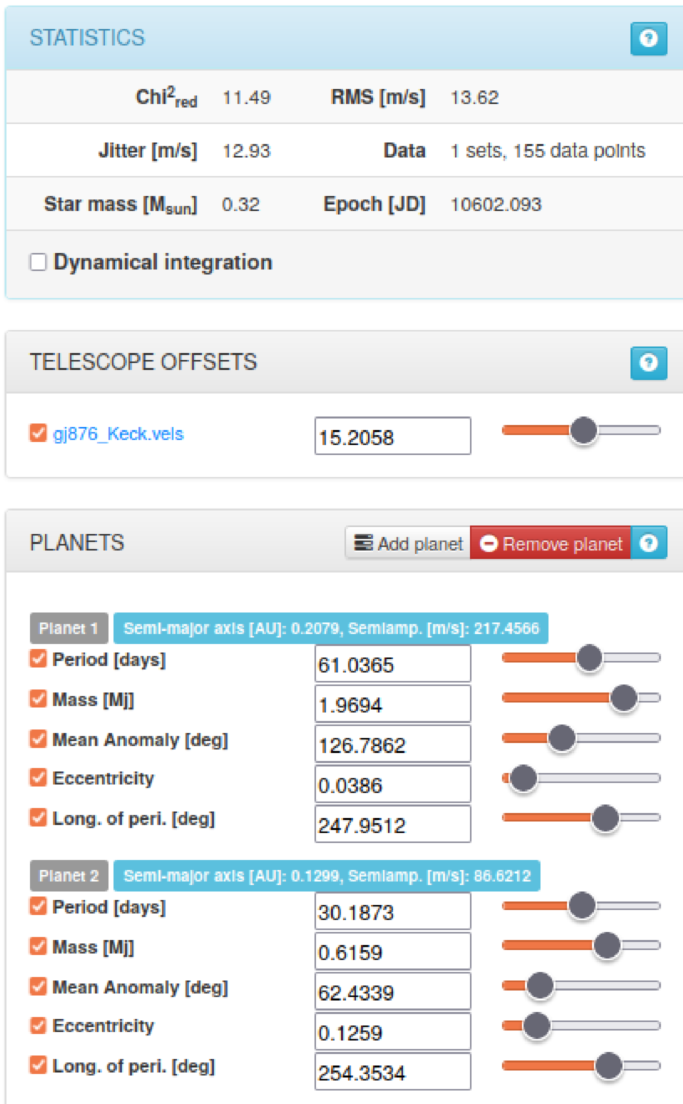

Figure 8: Statistics and parameters of a fitted double planetary system

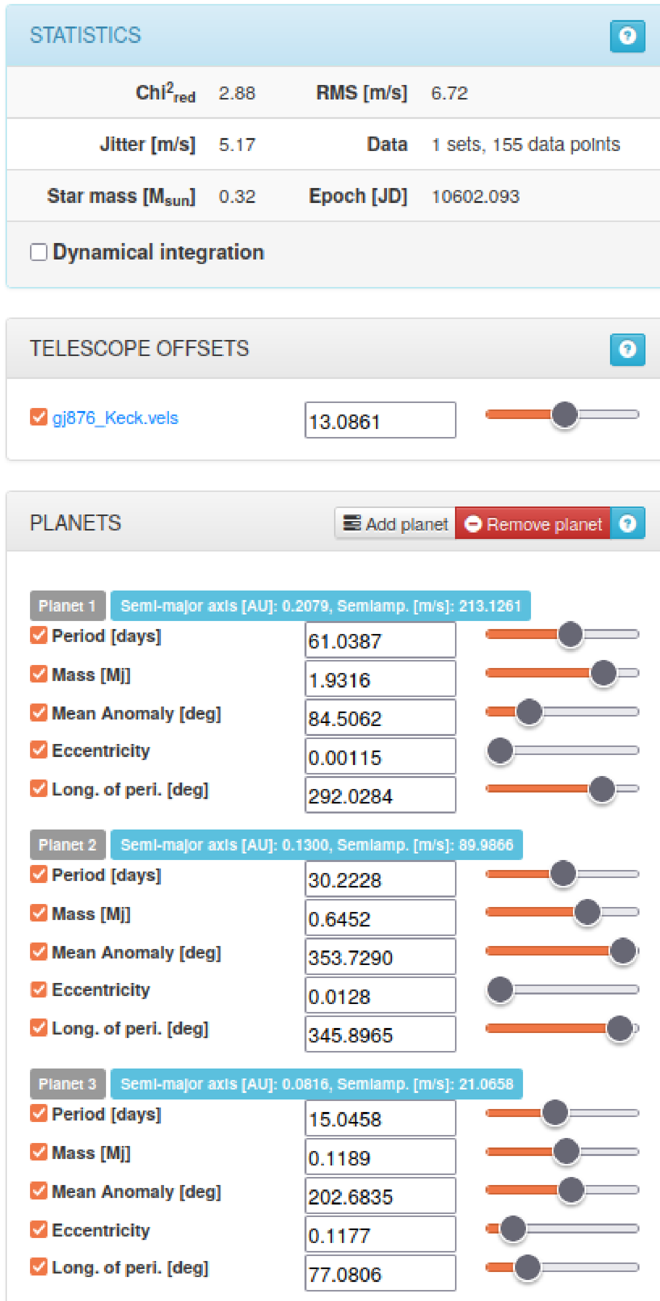

Figure 9: Statistics and parameters of a fitter triple planetary system