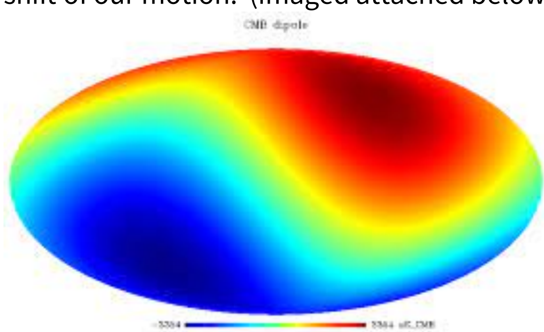
2 RG&TC-Code

2 Maximally symmetric space time

A) The three truly maximally symmetric space-times are:
Minkowski spacetime (flat space and in this workshop equivalent to k = 0)
de Sitter space (for when k>0 equivalent to a closed universe) and
anti de Sitter space (for when k<0 shape like a hyperboloid)

B) The defining feature of a comoving coordinate (u^i) is such that for an observer where u^i is constant will be considered as comoving so that the universe will remain isotropic to them.

C) An observer on earth is not well described with a fixed comoving coordinated because our velocity (u^i) changes as we go around the sun. A great example of this is the CMB dipole caused by Doppler shift of our motion. (imaged attached below)



As you can clearly see the universe does not appear isotropic due to our circular motion around the sun.

D) The Ricci tensor can be calculated by contracting the 1st and 3rd indices of the Riemann tensor therefore,

$R_{jl} = \gamma^{ik} R_{ijkl} = k\,\gamma^{ik}(\gamma_{ik}\,\gamma_{jl} - \gamma_{il}\,\gamma_{jk}) =$

$k\left(\gamma^{ik}\,\gamma_{ik}\,\gamma_{jl} - \gamma^{ik}\,\gamma_{il}\,\gamma_{jk}\right) = k(3\,\gamma_{jl} - \delta^{k}{}_{l}\,\gamma_{jk}) = k(3\,\gamma_{jl} - \gamma_{jl}) = 2\,k\,\gamma_{jl}$

E)

In[328]:=
```
xCoord = {r, θ, φ};
g = {
   {Exp[2 β[r]], 0, 0},
   {0, r^2, 0},
   {0, 0, r^2 Sin[θ]^2}
  };
RGtensors[g, xCoord];
```

In[266]:=
```
Rdd // MatrixForm
R
```

Out[266]//MatrixForm=

$$\begin{pmatrix} \frac{2\beta'[r]}{r} & 0 & 0 \\ 0 & e^{-2\beta[r]}\left(-1+e^{2\beta[r]}+r\,\beta'[r]\right) & 0 \\ 0 & 0 & e^{-2\beta[r]}\,\mathrm{Sin}[\theta]^2\left(-1+e^{2\beta[r]}+r\,\beta'[r]\right) \end{pmatrix}$$

Out[267]=

$$\frac{2\,e^{-2\beta[r]}\left(-1+e^{2\beta[r]}+2\,r\,\beta'[r]\right)}{r^2}$$

We have

$R_{11} = \frac{2\beta'[r]}{r} = 2\,k\,\gamma_{11} = 2\,ke^{2\beta[r]} \Rightarrow \beta'[r] = kre^{2\beta[r]}$

and

$R_{22} = e^{-2\beta[r]}\left(-1+e^{2\beta[r]}+r\,\beta'[r]\right) = 2\,k\,\gamma_{22} = 2\,kr^2$

Substituting our expression for $\beta'[r]$

$e^{-2\beta[r]}\left(-1+e^{2\beta[r]}+kr^2\,e^{2\beta[r]}\right) = -e^{-2\beta[r]}+1+kr^2 = 2\,kr^2$

$-e^{-2\beta[r]} = kr^2-1$

$-2\,\beta[r] = \mathrm{Ln}\left[1-kr^2\right]$

$\beta[r] = -\frac{1}{2}\,\mathrm{Ln}\left[1-kr^2\right]$

F) (Side note I don't not use the $\bar{r}$ convention cause it got a little messy so you will just have to use your imagination at the end of part three we replace $\bar{r}$ with a dimensionless distance. I hope you follow)

When substituting back the result $\beta[r] = -\frac{1}{2}\,\mathrm{Ln}(1-kr^2) \Rightarrow \left(e^{2\beta[r]} = \frac{1}{1-kr^2}\right)$ we get

$d\sigma^2 = \left[\frac{1}{1-kr^2}\,dr^2 + r^2\,d\Omega^2\right]$ and therefore the space time metric would be

$ds^2 = -dt^2 + R^2(t)\left[\frac{1}{1-kr^2}\,dr^2 + r^2\,d\Omega^2\right]$

G) Since k = R/6 We can conclude that the value of k sets the curvature and size of the spatial surfaces and can take on any value. However it is possible to normalize k such that it can only have the values (0, ±1) by containing that information into the scale factor of R(t). This does not change the form of the metric but implicitly redefines the scale factor in a way.

Where k = 0 corresponds to no curvature (flat)

k = -1 corresponds to negative curvature (Open) and

k = +1 corresponds to positive curvature (Closed).

3 The geometry of spacetime

A)

For the case where k = 0

$d\chi = dr$

WE can simply integrate this to get

$\chi = r$

therefore

$r(\chi) = \chi$

For k = 1

$d\chi = \left[1 - r^2\right]^{-\frac{1}{2}} dr$

In[268]:=
```
Integrate[(1 - r^2)^(-1/2), r]
```

Out[268]=
```
ArcSin[r]
```

$\chi = \mathrm{Sin}^{-1}[r]$

therefore

$r(\chi) = \mathrm{Sin}[\chi]$

For k = -1

$d\chi = \left[1 + r^2\right]^{-\frac{1}{2}} dr$

In[269]:=
```
Integrate[(1 + r^2)^(-1/2), r]
```

Out[269]=
```
ArcSinh[r]
```

$\chi = \mathrm{Sinh}^{-1}[r]$

therefore

$r(\chi) = \mathrm{Sinh}[\chi]$

B)

For K = 0 working in 2d since we need to show the nature of the lines

the metric I will use is $ds^2 = dr^2 + r^2\, d\theta^2$

The Geodesics for this metric is

```
r''[λ] - r[λ] * θ'[λ]^2 == 0
```

and

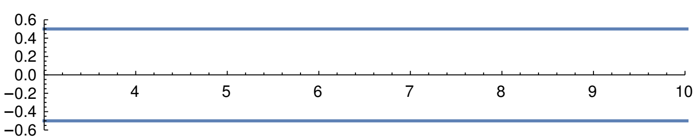
```
θ''[λ] + 2/r[λ] r'[λ] * θ'[λ] == 0
```

```
xCoord = {r, θ};
g = {
   {1, 0},
   {0, r^2}
  };
RGtensors[g, xCoord]
```

In[324]:=
```
s1 = NDSolve[{r''[λ] - r[λ] * θ'[λ]^2 == 0, θ''[λ] + 2/r[λ] r'[λ] * θ'[λ] == 0,
    r[0] == 0.7, r'[0] == 1, θ[0] == π / 2, θ'[0] == -1}, {r, θ}, {λ, 1, 10}];
s2 = NDSolve[{r''[λ] - r[λ] * θ'[λ]^2 == 0, θ''[λ] + 2/r[λ] r'[λ] * θ'[λ] == 0,
    r[0] == 0.7, r'[0] == 1, θ[0] == π / 2, θ'[0] == 1}, {r, θ}, {λ, 1, 10}];
x[λ_] = r[λ] * Sin[θ[λ]];
y[λ_] = 0;


ParametricPlot[{Evaluate[{x[λ], y[λ]} /. s1] - 0.5, Evaluate[{x[λ], y[λ]} /. s2] + 0.5} // Flatten,
 {λ, 1, 10}, PlotRange → {{3, 10}, {-0.6, 0.6}}]
```

Out[259]=



For K = 1

Similarly to flat space the metric is $ds^2 = \frac{1}{1-r^2} dr^2 + r^2 d\theta^2$

The geodesic equation for this case is

$r''[\lambda] - (r[\lambda]/((-1 + r[\lambda])(1 + r[\lambda]))) * r'[\lambda]^2 + (-1 + r[\lambda]) * r[\lambda] * (1 + r[\lambda]) * \theta'[\lambda]^2 == 0$

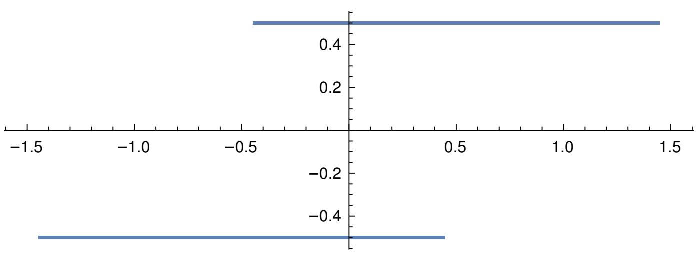
and

```
θ''[λ] + 2/r[λ] r'[λ] * θ'[λ] == 0
```

```
xCoord = {r, θ};
g = {
   {(1 - r^2)^-1, 0},
   {0, r^2}
  };
RGtensors[g, xCoord];
```

In[319]:=
```
s1 =
  NDSolve[{r''[λ] - (r[λ] / ((-1 + r[λ]) (1 + r[λ]))) * r'[λ]^2 + (-1 + r[λ]) * r[λ] * (1 + r[λ]) * θ'[λ]^2 == 0,
    θ''[λ] + 2/r[λ] r'[λ] * θ'[λ] == 0, r[0] == 0.7, r'[0] == 1,
    θ[0] == π / 2, θ'[0] == -1}, {r, θ}, {λ, 1, 10}];
s2 = NDSolve[{r''[λ] - (r[λ] / ((-1 + r[λ]) (1 + r[λ]))) * r'[λ]^2 + (-1 + r[λ]) * r[λ] * (1 + r[λ]) * θ'[λ]^2 ==
     0, θ''[λ] + 2/r[λ] r'[λ] * θ'[λ] == 0, r[0] == 0.7,
    r'[0] == 1, θ[0] == π / 2, θ'[0] == 1}, {r, θ}, {λ, 1, 10}];
x[λ_] = r[λ] * Sin[θ[λ]];
y[λ_] = 0;
ParametricPlot[
 {Evaluate[{x[λ], y[λ]} /. s1] - 0.5, Evaluate[{x[λ], y[λ]} /. s2] + 0.5} // Flatten, {λ, 1, 10}]
```

Out[323]=



For K = -1

$ds^2 = \frac{1}{1+r^2} dr^2 + r^2 d\theta^2$

the geodesic for this metric will be

$r''[\lambda] - \frac{r[\lambda]}{1+r[\lambda]^2} * r'[\lambda]^2 + -r[\lambda]\left(1 + r[\lambda]^2\right) * \theta'[\lambda]^2 == 0$

and

$\theta''[\lambda] + \frac{2}{r[\lambda]} r'[\lambda] * \theta'[\lambda] == 0$

In[285]:=
```
xCoord = {r, θ};
g = {
   {(1 + r^2)^-1, 0},
   {0, r^2}
  };
RGtensors[g, xCoord]
GUdd // MatrixForm
```

In[331]:=
```
s1 = NDSolve[
  {r''[λ] - r[λ]/(1 + r[λ]^2) * r'[λ]^2 + -r[λ] (1 + r[λ]^2) * θ'[λ]^2 == 0, θ''[λ] + 2/r[λ] r'[λ] * θ'[λ] == 0,
   r[0] == 0.7, r'[0] == 1, θ[0] == π / 2, θ'[0] == -1}, {r, θ}, {λ, 1, 10}];
s2 = N NDSolve[{r''[λ] - r[λ]/(1 + r[λ]^2) * r'[λ]^2 + -r[λ] (1 + r[λ]^2) * θ'[λ]^2 == 0,
    θ''[λ] + 2/r[λ] r'[λ] * θ'[λ] == 0, r[0] == 0.7, r'[0] == 1,
    θ[0] == π / 2, θ'[0] == 1}, {r, θ}, {λ, 1, 10}];
x[λ_] = r[λ] * Sin[θ[λ]];
y[λ_] = 0;
ParametricPlot[
 {Evaluate[{x[λ], y[λ]} /. s1] - 0.5, Evaluate[{x[λ], y[λ]} /. s2] + 0.5} // Flatten, {λ, 1, 10}]
```

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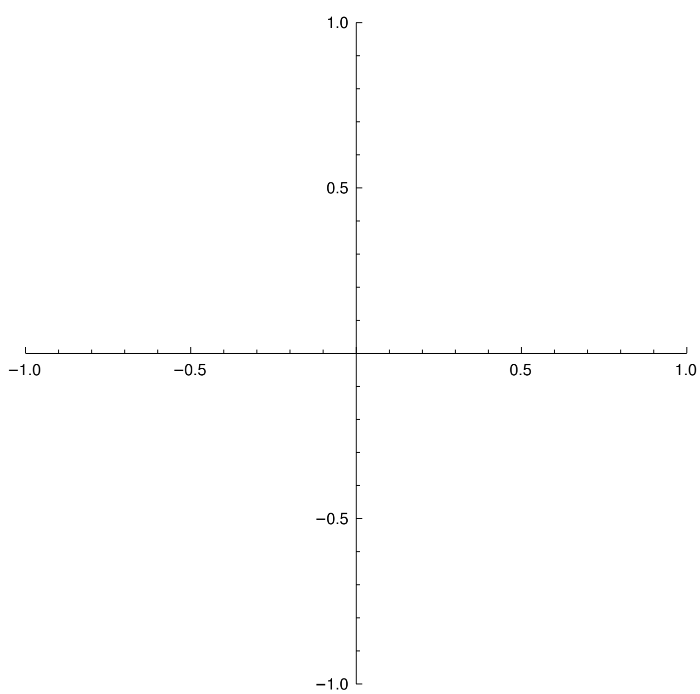
··· ReplaceAll: ≪1≫ is neither a list of replacement rules nor a valid dispatch table, and so cannot be used for replacing.

··· General: Further output of ReplaceAll::reps will be suppressed during this calculation.

Out[335]=



I wasn't too sure what to do and how to make the plots work but I was fairly confident in my calculations of the geodesic equations.

C) We can rewrite our metric using dimensionless scale factor:

a(t) = $\frac{R(t)}{R_0}$, ⇒ R(t) = $a(t)\, R_0$

Dimensionless distance

r = $R_0\, \bar{r} \Rightarrow \bar{r} = \frac{r}{R_0}$

$(\mathrm{dr} = R_0\, \bar{\mathrm{dr}}) \Rightarrow \bar{\mathrm{dr}}^2 = \frac{\mathrm{dr}^2}{R_0{}^2}$

and a curvature parameter with dimensions of inverse length squared:

$\kappa = \frac{k}{R_0{}^2} \Rightarrow \mathrm{k} = \kappa \mathrm{R}_0{}^2$

Starting with our metric (in this FLRW metric its really suppose to be $\bar{r}$ but it was pretty messy)

$\mathrm{ds}^2 = -\mathrm{dt}^2 + R^2(t)\left[\frac{1}{1-\mathrm{kr}^2}\,\mathrm{dr}^2 + r^2\, \mathrm{d}\Omega^2\right]$

Substituting our change of variables give:

$\mathrm{ds}^2 = -\mathrm{dt}^2 + a(t)^2\, R_0{}^2\left[\frac{1}{1-\kappa \mathrm{R}_0{}^2 * \left(\frac{r}{R_0}\right)^2}\left(\frac{\mathrm{dr}}{R_0}\right)^2 + \left(\frac{r}{R_0}\right)^2 \mathrm{d}\Omega^2\right]$

$\mathrm{ds}^2 = -\mathrm{dt}^2 + a(t)^2\, \frac{\cancel{R_0}^2}{R_0{}^2}\left[\frac{1}{1-\kappa r^2}\,(\mathrm{dr})^2 + (r)^2\, \mathrm{d}\Omega^2\right]$

$\mathrm{ds}^2 = -\mathrm{dt}^2 + a(t)^2\left[\frac{1}{1-\kappa r^2}\,(\mathrm{dr})^2 + (r)^2\, \mathrm{d}\Omega^2\right]$

In this change of coordinates κ can take any value not just 0, ±1

4 Perfect fluid

A)

In[134]:=
```
xCoord = {t, r, θ, φ};
g = {
   {-1, 0, 0, 0},
   {0, a[t]^2/(1 - κ * r^2), 0, 0},
   {0, 0, a[t]^2 * r^2, 0},
   {0, 0, 0, a[t]^2 * r^2 Sin[θ]^2}
  };
RGtensors[g, xCoord]
```

In[140]:=
```
Rdd // MatrixForm // FullSimplify
R
```

Out[140]//MatrixForm=

$$\begin{pmatrix} -\frac{3\,a''[t]}{a[t]} & 0 & 0 & 0 \\ 0 & \frac{2\left(\kappa+a'[t]^2\right)+a[t]\,a''[t]}{1-r^2\,\kappa} & 0 & 0 \\ 0 & 0 & r^2\left(2\left(\kappa+a'[t]^2\right)+a[t]\,a''[t]\right) & 0 \\ 0 & 0 & 0 & r^2\,\mathrm{Sin}[\theta]^2\left(2\left(\kappa+a'[t]^2\right)+a[t]\,a''[t]\right) \end{pmatrix}$$

Out[141]=

$$\frac{6\left(\kappa+a'[t]^2+a[t]\,a''[t]\right)}{a[t]^2}$$

In[157]:=
```
GUdd[[2, 2, 1]];
GUdd[[3, 3, 1]];
GUdd[[4, 4, 1]];
```

B) If we were to model the matter/energy in our universe as a perfect fluid, at rest $(U^i = 0)$ in our comoving coordinates. The 4-velocity would be $U^\mu = (1, 0, 0, 0)$. The way I like to justify this is to remind myself that the magnitude of a 4-velocity must be equal to 1 (in our case where we set the speed of light to be 1). Therefore an object must always be traveling either through space, or time and normally both with the exception of a photon. For example a photon traveling at the speed of light along the x-direction would have a 4-velocity of $U^\mu = (0, 1, 0, 0)$. Notice that it does not travel through time. As expected. Therefore a stationary object where $U^i = 0$, must be traveling through time at the speed of light. Hence $U^0 = 1$.

C) The stress energy tensor for a perfect fluid is

$T_{\mu\nu} = (\rho + p)\, U_\mu\, U_\nu + p g_{\mu\nu}$

And since its stationary

$T_{00} = (\rho + p) - p = \rho$

While the $U_i = 0$ so that the $T_{0\,i} = T_{i0} = 0$

And we are left with with p on the diagonal from g_{ij}

By raising the index of $T_{\mu\nu}$ is the same as multiplying by the Minkowski metric (therefore $T^0{}_0$ becomes negative) and thus.

$T^\mu{}_\nu = \mathrm{Diag}(-\rho, p, p, p)$

D) Conservation of energy implies that the covariant derivative of $T^\mu{}_0$ is 0 ($\nabla_\mu T^\mu{}_0 = 0$)

$\nabla_\mu T^\mu{}_0 = \partial_\mu T^\mu{}_0 + \Gamma^\mu{}_{\mu\lambda}\, T^\lambda{}_0 - \Gamma^\lambda{}_{\mu 0}\, T^\mu{}_\lambda = 0$

And since the Tensor is diagonal $\mu == \lambda$ such that

For $\mu = 0$ we get

$\partial_0 T^0{}_0 = -\partial_0 \rho$

For $\mu = 1$

$\Gamma^1{}_{1\lambda}\, T^\lambda{}_0 - \Gamma^\lambda{}_{10}\, T^1{}_\lambda = \Gamma^1{}_{10}\, T^0{}_0 + (-\Gamma^1{}_{10}\, T^1{}_1) = \frac{a'[t]}{a[t]}(-\rho) - \frac{a'[t]}{a[t]}(p) = -\frac{a'[t]}{a[t]}(\rho + p)$

For $\mu = 2$

$\Gamma^2{}_{2\lambda}\, T^\lambda{}_0 - \Gamma^\lambda{}_{20}\, T^2{}_\lambda = \Gamma^2{}_{20}\, T^0{}_0 + (-\Gamma^2{}_{20}\, T^2{}_2) = -\frac{a'[t]}{a[t]}(\rho + p)$

Similarly for $\mu = 3$

$\Gamma^3{}_{3\lambda}\, T^\lambda{}_0 - \Gamma^\lambda{}_{30}\, T^2{}_3 = \Gamma^3{}_{30}\, T^0{}_0 + (-\Gamma^3{}_{30}\, T^3{}_3) = -\frac{a'[t]}{a[t]}(\rho + p)$

Such that when we combine all of these we get

$-\partial_0 \rho \text{ -}3\frac{\dot{a}}{a}(\rho + p) = 0$

E) Assuming that our perfect fluid can be described by the equation of state:

$p = \omega\rho$

Where ω is a constant independent of time, we can rearrange our expression to get.

$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + \omega\rho) \Rightarrow \frac{\dot{\rho}}{\rho} = -3(1+\omega)\frac{\dot{a}}{a}$

Such that

$\frac{1}{p}\mathrm{d}\rho = -3(1+\omega)\frac{1}{a}\mathrm{da}$

We can now integrate both sides to get

$\mathrm{Ln}(\rho) = \text{-}3(1+\omega)\mathrm{Ln(a)} + \mathrm{C} = \mathrm{Ln}\left(a^{-3(1+\omega)}\right) + \mathrm{C}$

Exponentiating both sides gives

$\rho = C_1\, a^{-3(1+\omega)}$

therefore

$\rho \propto a^{-3(1+\omega)}$

F)We are told that fluids have different values of ω specifically matter (dust) $\omega = 0$

Radiation has $\omega = 1/3$ and a vacuum $\omega = \text{-}1$. And we therefore get the relations of density to be:

$\rho_m \propto a^{-3}$,

$\rho_r \propto a^{-3(4/3)} = a^{-4}$ and

$\rho_v \propto a^0 = 1$.

Since a is a scale factor of the universe it can be considered to have unit lengths [l].

Consider how density is calculated $\left(\rho = \frac{m}{[l]^3} = \mathrm{ml}^{-3}\right) \Rightarrow \rho \propto [l]^{-3}$. Since matter exists in 3 dimensions.

It makes sense physical sense to set $\omega = 0$ such that the relation can reduce to $\rho_m \propto a^{-3}$ justifying that matter takes up 3 dimensions in space. While a vacuum technically isn't a real physical thing and therefore takes up now space and hence will not be related to the scale factor so in that case $\omega = \text{-}1$ such that its density is constant.

In the current universe the density of radiation is about thousandth of matter $\left(\rho_r \simeq \frac{\rho_m}{10^3}\right)$. We see that the relationship of radiation density drops off at a rate of a^{-4} (which is quicker than the rate of which matter drops off) This more rapid decline is partly due to redshift and the loss of energy from that effect (proportional to a^{-1} since expands in a radial direction) . However when the universe was young it was also a lot smaller so at a certain point the universe would of been dominated by radiation (which makes sense due to the presence of the cosmic horizon). Where as the energy density of the vacuum is constant and would be negligible in the early universe. However in the distant future the energy density of mass and radiation will drop off significantly (as cosmologist expects that the universe will continue to expand indefinitely) And if the energy density of a vacuum is nonzero it will inevitable become the dominant source of energy density.

5 The Friedman Equations

A)

Starting with the Einstein equation in the form

$R_{\mu v} = 8\pi G\left(T_{\mu v} - \frac{1}{2} g_{\mu v} T\right)$

For the case where $\mu = v = 0$

$R_{00} = -3 \frac{a''[t]}{a[t]} = 8\pi G(\rho + 1/2(-\rho + 3p)) = 4\pi G(\rho + 3p)$

$\Rightarrow \frac{a''[t]}{a[t]} = -\frac{4}{3}\pi G(\rho + 3p)$ (second Friedmann equation)

And for when μv = ij Since the isotropy of the metric implies that the tensor is the same for the space coordinates.

$R_{ij} = \frac{a''[t]}{a[t]} + 2\left(\frac{a'[t]}{a}\right)^2 + 2\frac{\kappa}{a^2} = 8\pi G\left(p - \frac{1}{2}(-\rho + 3p)\right) = 4\pi G(2p + \rho - 3p) = 4\pi G(\rho - p)$

Substituting the expression for $\frac{a''[t]}{a[t]}$ to get

$-\frac{4}{3}\pi G(\rho + 3p) + 2\left(\frac{a'[t]}{a}\right)^2 + 2\frac{\kappa}{a^2} = 4\pi G(\rho - p)$

$2\left(\left(\frac{a'[t]}{a}\right)^2 + \frac{\kappa}{a^2}\right) = 4\pi G(\rho - p) + \frac{4}{3}\pi G(\rho + 3p) = 4\pi G\rho + \frac{4}{3}\pi G\rho = \frac{16}{3}\pi G\rho$

$\left(\left(\frac{a'[t]}{a}\right)^2 + \frac{\kappa}{a^2}\right) = \frac{8}{3}\pi G\rho$

$\left(\frac{a'[t]}{a}\right)^2 = \frac{8}{3}\pi G\rho - \frac{\kappa}{a^2} = H^2$ (First Friedmann equaiton)

B) From the Friedmann equations it is possible to derive a relation with the scale of the universe and time.

$a \propto t^{2/n}$

Where n = 3 for matter n = 4 for radiation and n = 0 for a vacuum.

If we consider a matter dominated universe

$a \propto t^{2/3}$

and similarly for a radiation dominated universe

$a \propto t^{1/2}$

In both cases if we were to go backwards in time to t = 0 we will get a singularity a = 0

The physical interpretation of this initial state is the creation of the universe otherwise know as the big bang and the creation of space. Since the scale size of the universe was 0 at the origin of time it implies that matter wasn’t simply injected into the universe but energy and spacetime came to exist at the same time.

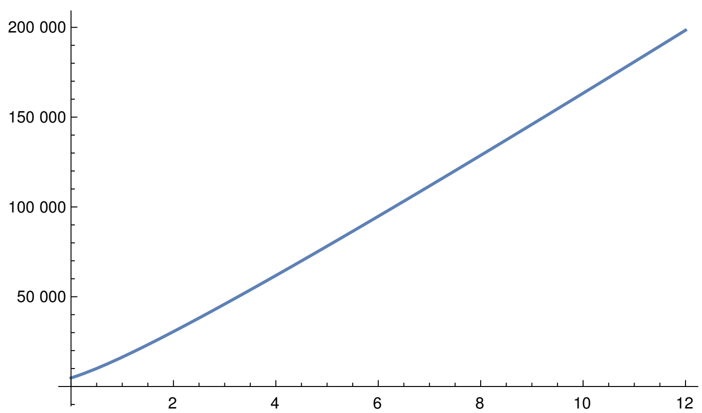
6 Distance in Cosmology

Hubble constant is 70km/s/mpc
Converting to speed of light ($3*10^5$) km/s
Distance will be in Mpc
ΩΛ = 0.6889 (from Wikipedia)

In[336]:=
```
Ωm = 0.25;
ΩΛ = 1 - Ωm;
H0 = 70/(3*10^5);
f[z_] = (Ωm (1 + z)^3 + ΩΛ)^(1/2);
dl[z_] = (1 + z)/H0 Integrate[f[z]^-1, z];
```

In[193]:=
```
Plot[dl[z], {z, 0, 12}]
```

Out[193]=



Some interesting
HD1 (galaxy) discovered in 2022 as the earliest and most distant know galaxy.
z = 13.27

J0313-1806 (Quasar)
z = 7.64

WHL0137-LS (most distant individual star march 2022)
z = 6.2

PJ352–15 quasar jet
z = 5.831

SN UDS10Wil (type Ia supernova)
z = 1.914

Cosmic Background radiation (cosmic decoupling)
z = 1000

Nearby objects
Andromeda galaxy
z = –0.001004

In[194]:=
```
dl[1.914]
```

Out[194]=
```
29 303.5
```

At what redshift does the contribution of radiation become important?
The current ratio of matter and radiation density is $\frac{\rho_r}{\rho_m} = 10^{-3}$ and we can relate this too the scale of the universe by $\frac{\rho_r}{\rho_m} = \frac{a^{-4}}{a^{-3}} = a^{-1}$ I think its up to the individual to decide what fraction radiation becomes significant but I would say that when radiation makes up a 10th of the density $\left(\frac{\rho_r}{\rho_m} = \frac{1}{10}\right) \Rightarrow a_t = 10$ and $a_0 = 1000$ (current scale of the universe) . Scaling such that $a_0 = 1$ we get $a_t = \frac{1}{100}$. Using Carroll's expression which relates the scale of the universe and the redshift of emitted light.
$a_{\text{em}} = \frac{1}{1+ z_{\text{em}}}$
We can estimate the value of the redshift for which the universe had a significant proportion of the radiation energy density.
$\frac{1}{100} = a_{\text{em}} = \frac{1}{1+z_{\text{em}}}$
$1 + z_{\text{em}} = 100 \Rightarrow z_{\text{em}} = 99$
(for when the ratio of $\frac{\rho_r}{\rho_m} = 1$ the redshift would be z_{em} = 999)
This makes intuitive sense since the Cosmic microwave background has a redshift of z = 1000 and that the furthest galaxy has a red shift of about z = 13.27