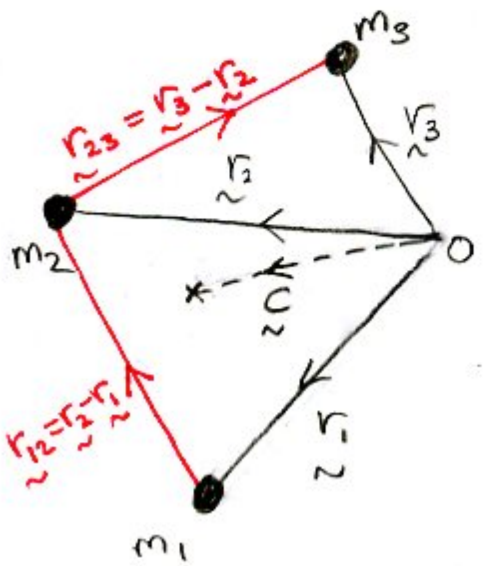
Dynamics of Exoplanets Problem Set SOLUTIONS

1. Show that the position vector $\mathbf{C}$ of the centre of mass of a system of N gravitating bodies moves with constant velocity, where $\mathbf{C}$ is defined by

$$\mathbf{C} = \frac{\sum_{i=1}^{N} m_i \mathbf{r}_i}{\sum_{i=1}^{N} m_i},$$

m_i is the mass of the ith body and $\mathbf{r}_i$ is its position vector relative to an inertial frame.



The equations of motion for N bodies ($N=3$ in the picture) are:

$$m_1 \ddot{\underset{\sim}{r}}_1 = \frac{G m_1 m_2}{r_{12}^2} \hat{\underset{\sim}{r}}_{12} + \frac{G m_1 m_3}{r_{13}^2} \hat{\underset{\sim}{r}}_{13} + \cdots + \frac{G m_1 m_N}{r_{1N}^2} \hat{\underset{\sim}{r}}_{1N}$$

$$m_2 \ddot{\underset{\sim}{r}}_2 = \frac{-G m_1 m_2}{r_{12}^2} \hat{\underset{\sim}{r}}_{12} + \frac{G m_2 m_3}{r_{23}^2} \hat{\underset{\sim}{r}}_{23} + \cdots + \frac{G m_2 m_N}{r_{2N}^2} \hat{\underset{\sim}{r}}_{2N}$$

$$\vdots$$

$$m_N \ddot{\underset{\sim}{r}}_N = -\frac{G m_1 m_N}{r_{1N}^2} \hat{\underset{\sim}{r}}_{1N} - \frac{G m_2 m_N}{r_{2N}^2} \hat{\underset{\sim}{r}}_{2N} - \cdots - \frac{G m_{N-1} m_N}{r_{N-1,N}^2} \hat{\underset{\sim}{r}}_{N-1,N}$$

where $\underset{\sim}{r}_{ij} \equiv \underset{\sim}{r}_j - \underset{\sim}{r}_i$

Adding these equations together gives

$$m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 + \cdots + m_N\ddot{\mathbf{r}}_N = \mathbf{0}$$

Integrating once with respect to t gives

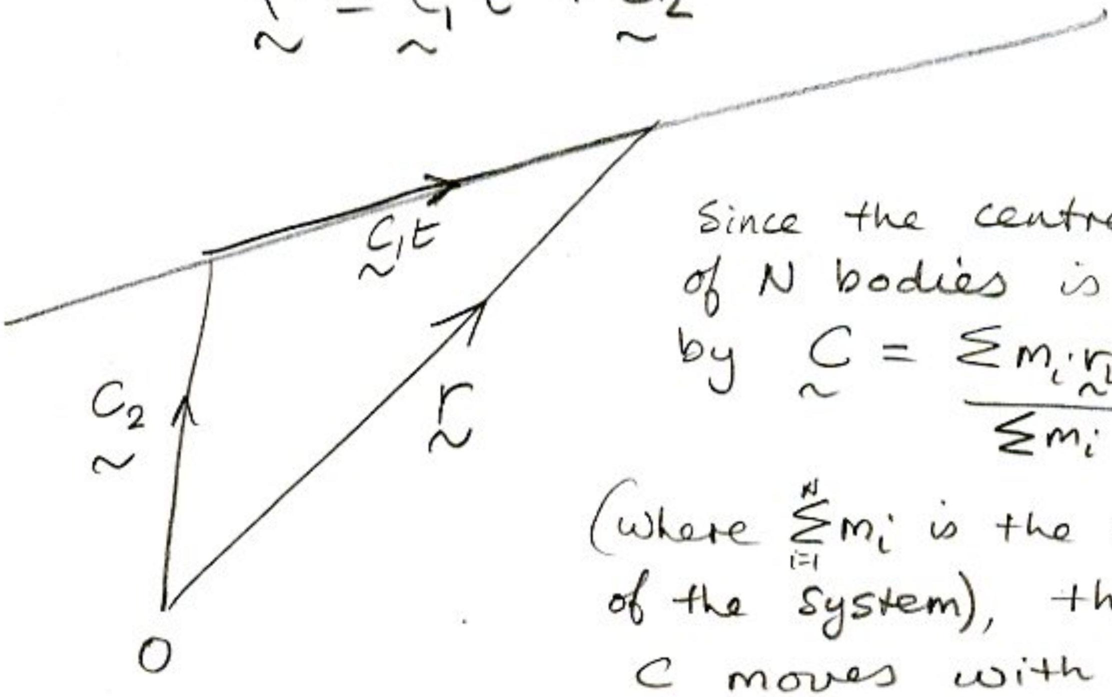
$$m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2 + \cdots + m_N\dot{\mathbf{r}}_N = \mathbf{C}_1 \quad \text{(a constant vector)}$$

Integrating again gives

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \cdots + m_N\mathbf{r}_N = \mathbf{C}_1 t + \underset{\substack{\uparrow \\ \text{const. vector.}}}{\mathbf{C}_2}$$

A point moving at constant velocity can be described by the position vector

$$\mathbf{r} = \mathbf{C}_1 t + \mathbf{C}_2$$



Since the centre of mass of N bodies is given by $\mathbf{C} = \dfrac{\sum m_i \mathbf{r}_i}{\sum m_i} = \dfrac{\mathbf{C}_1 t + \mathbf{C}_2}{\sum m_i}$ (where $\sum_{i=1}^{N} m_i$ is the total mass of the system), then $\mathbf{C}$ moves with constant velocity.

2. The total angular momentum, $\mathbf{J}$, about the centre of mass of a system of N particles is defined as

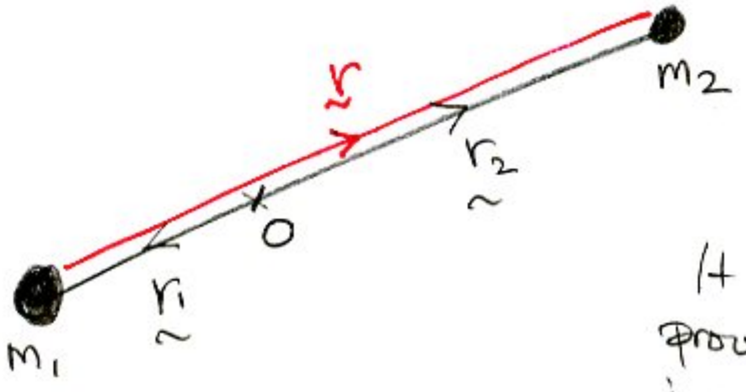
$$\mathbf{J} = \sum_{i=1}^{N} m_i \, \mathbf{r}_i \times \dot{\mathbf{r}}_i.$$

Show that the total angular momentum of a binary composed of masses m_1 and m_2 is given by

$$\mathbf{J} = \mu \, r^2 \dot{\varphi} \, \hat{\mathbf{h}},$$

where $\mu = m_1 m_2/(m_1 + m_2)$ is the *reduced mass* of the binary, $\hat{\mathbf{h}}$ is a vector perpendicular to the motion and $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$.

For a binary, $\mathbf{J} = m_1 \mathbf{r}_1 \times \dot{\mathbf{r}}_1 + m_2 \mathbf{r}_2 \times \dot{\mathbf{r}}_2$



Here we have placed the inertial origin at the centre of mass. It is also possible to prove this with arbitrary inertial origin, but messier.

$$\mathbf{r}_1 = -\frac{m_2}{m_1+m_2} \mathbf{r}$$

$$\mathbf{r}_2 = \frac{m_1}{m_1+m_2} \mathbf{r}$$

$$\mathbf{J} = m_1 \left(-\frac{m_2}{m_{12}} \mathbf{r} \right) \times \left(-\frac{m_2}{m_{12}} \dot{\mathbf{r}} \right) + m_2 \left(\frac{m_1}{m_{12}} \mathbf{r} \right) \times \left(\frac{m_1}{m_{12}} \dot{\mathbf{r}} \right)$$

where we have used the shorthand $m_{12} = m_1 + m_2$

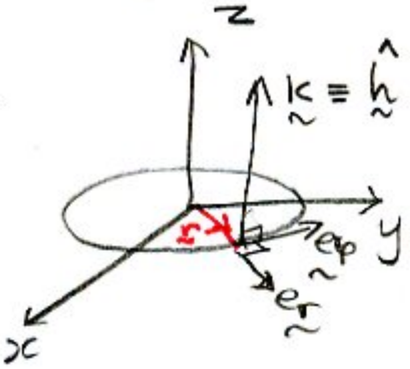
Thus

$$\mathbf{J} = \left[m_1 \left(\frac{m_2}{m_{12}} \right)^2 + m_2 \left(\frac{m_1}{m_{12}} \right)^2 \right] (\mathbf{r} \times \dot{\mathbf{r}})$$

$$= \frac{m_1 m_2 (m_1 + m_2)}{m_{12}^2} \, \mathbf{r} \times \dot{\mathbf{r}} = \mu \, \mathbf{r} \times \dot{\mathbf{r}}$$

Putting $\underset{\sim}{r} = r\underset{\sim}{e_r}$, $\dot{\underset{\sim}{r}} = \dot{r}\underset{\sim}{e_r} + r\dot{\varphi}\underset{\sim}{e_\varphi}$,

$$\underset{\sim}{r} \times \dot{\underset{\sim}{r}} = r\underset{\sim}{e_r} \times (\dot{r}\underset{\sim}{e_r} + r\dot{\varphi}\underset{\sim}{e_\varphi})$$
$$= \underset{\sim}{0} + r^2\dot{\varphi}\underset{\sim}{k} \quad (\text{or } r^2\dot{\varphi}\hat{\underset{\sim}{h}})$$



3. Starting with

- $r = \dfrac{a(1-e^2)}{1+e\cos(\varphi-\omega)} \equiv \dfrac{a(1-e^2)}{1+e\cos f}$ (see diagram)
- $h = r^2\dot{\varphi} = \sqrt{Gm_{12}a(1-e^2)}$ (angular momentum per unit mass),
- $\nu = \sqrt{\dfrac{Gm_{12}}{a^3}}$, (orbit frequency or *mean motion*=2π/[orbital period])

where $m_{12} \equiv m_1 + m_2$, a is the semimajor axis of the orbit and G is the universal gravitational constant, show that

(a) $h = a^2\nu\sqrt{1-e^2}$

$$h = \sqrt{Gm_{12}a(1-e^2)} = \sqrt{\frac{Gm_{12}}{a^3}\cdot a^4(1-e^2)}$$
$$= \nu a^2\sqrt{1-e^2}$$

(b) $\dot{\varphi} = \nu\dfrac{(1+e\cos f)^2}{(1-e^2)^{3/2}}$

$$\dot{\varphi} = \frac{h}{r^2} = a^2\nu\sqrt{1-e^2}\cdot\frac{(1+e\cos f)^2}{a^2(1-e^2)^2}$$
$$= \nu\frac{(1+e\cos f)^2}{(1-e^2)^{3/2}}$$

(c) $r\dot{\varphi} = a\nu\dfrac{(1+e\cos f)}{\sqrt{1-e^2}}$

obvious!

(d) $\dot{r} = a\nu \dfrac{e}{\sqrt{1-e^2}} \sin f$

$$r = \frac{a(1-e^2)}{1+e\cos f} = \frac{\alpha}{1+e\cos f}$$

$$\to \dot{r} = \frac{-\alpha}{(1+e\cos f)^2} \times (-e\cos f) \times \dot{f}$$

$$= +\frac{1}{\alpha} r^2 (e\cos f)\, \dot{f}$$

$$= \frac{h}{\alpha} e \cos f$$

$$= \frac{a^2\nu\sqrt{1-e^2}}{a(1-e^2)} e\cos f$$

$$= \frac{a\nu e\cos f}{\sqrt{1-e^2}}$$

(e) the orbital speed at pericentre (or periastron or perihelion or perigee or perigalacticon) is $v_p = a\nu\sqrt{\dfrac{1+e}{1-e}}$

$f = 0$ at peri

$$\underset{\sim}{v}_p = 0\, \underset{\sim}{e}_r + (r\dot{\varphi})_p\, \underset{\sim}{e}_\varphi \equiv v_p\, \underset{\sim}{e}_\varphi$$

$$v_p = a\frac{\nu(1+e)}{\sqrt{1-e^2}} = a\nu\sqrt{\frac{1+e}{1-e}}$$

(f) and the total energy or *binding energy*, E, of a binary is given by $E = -\dfrac{1}{2}\dfrac{Gm_1m_2}{a}$. Deduce from this that the semimajor axis is constant.

$$E = \frac{\mu}{2}\, \underset{\sim}{\dot{r}} \cdot \underset{\sim}{\dot{r}} - \frac{G m_1 m_2}{r} = \frac{\mu}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - G M_{12} \cdot \mu \cdot \left(\frac{1+e\cos f}{a(1-e^2)}\right)$$

$$= \frac{\mu}{2} \cdot a^2\nu^2\left(\frac{e^2}{1-e^2}\sin^2 f + \frac{(1+e\cos f)^2}{1-e^2}\right) - \mu\nu^2a^2\,\frac{(1+e\cos f)}{1-e^2}$$

$$= \frac{\mu}{2}\frac{a^2\nu^2}{1-e^2}\left[e^2\sin^2 f + 1 + 2e\cos f + e^2\cos^2 f - 2 - 2e\cos f\right]$$

$$= \frac{\mu}{2}\frac{a^2\nu^2}{1-e^2}\left[e^2 - 1\right] = -\frac{1}{2}\mu a^2 \nu^2$$

$$= -\frac{1}{2}\frac{M_1 M_2}{M_{12}} a^2 \cdot \frac{G M_{12}}{a^3}$$

$$= -\frac{1}{2}\frac{G M_1 M_2}{a}$$

(g) Is the orbital speed at pericentre infinite when $e = 1$ (ie, when the orbit is parabolic)? If not, why not?

NO! Because $a = r_p/(1-e)$, r_p = periastron distance

and $\nu^2 = \dfrac{GM_{12}}{a^3} = \dfrac{GM_{12}}{r_p^3}(1-e)^3$

So $v_p = \sqrt{\dfrac{GM_{12}}{r_p}} \cdot \sqrt{1+e} \to \sqrt{\dfrac{2GM_{12}}{r_p}} \neq \infty$ (or 0!).

Harder problems:

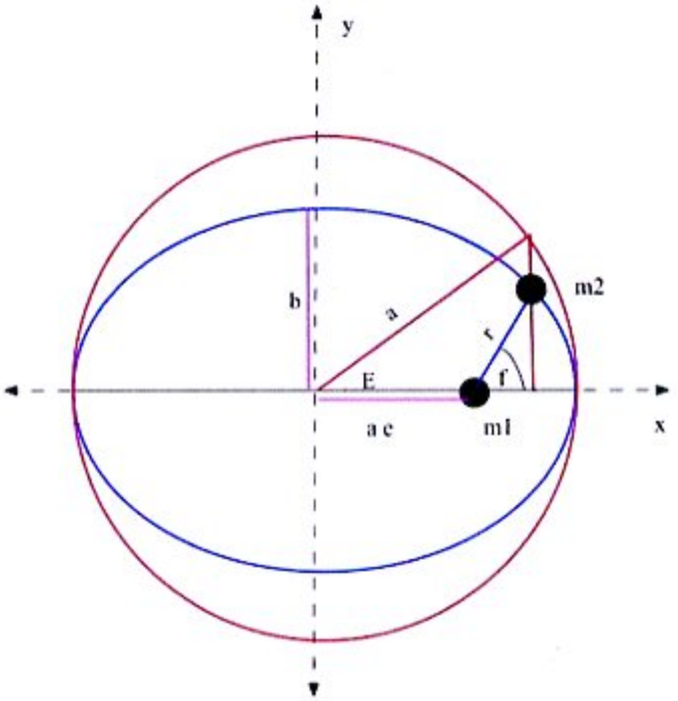
4. Show that the orbital period, T, of a binary is given by

$$T = 2\pi \left[\frac{a^3}{G(m_1 + m_2)} \right]^{1/2}.$$

To do this, use start with

$$T = \int_0^T dt = \int_0^{2\pi} \frac{dt}{df} df \tag{1}$$

then write in terms of an angle called the **eccentric anomoly** E as follows. This is defined geometrically in the following figure:



The eccentric anomoly is defined relative to an origin placed at the *centre* or the ellipse rather than at the focus (relative to which r and f are defined), and relative to a circle which *circumscribes* the ellipse. The equation of the ellipse relative to the $x-y$ coordinate systems is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{2}$$

where a is the semimajor axis and $b = a\sqrt{1-e^2}$ is the semiminor axis. The position of m_2 (a point on the ellipse) is therefore given by

$$x = a \cos E \tag{3}$$

$$y = b\sqrt{1 - x^2/a^2} = b \sin E = a\sqrt{1-e^2} \sin E. \tag{4}$$

The distance from the origin to the focus (the position of m_1) is $a\,e$. From the figure we have the following relationships:

$$ae + r\cos f = x = a\cos E, \qquad r \sin f = y = a\sqrt{1-e^2}\sin E, \tag{5}$$

so that

$$r = \frac{a(1-e^2)}{1 + e\cos f} = a(1 - \cos E) \tag{6}$$

and

$$\cos f = \frac{\cos E - e}{1 - e\cos E}, \qquad \sin f = \frac{\sqrt{1-e^2}\sin E}{1 - e\cos E}. \tag{7}$$

Another relation which can be derived from this is

$$\tan\frac{f}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}. \tag{8}$$

Hint: Begin with equation (6) and substitute into equation (1).

$$T = \int_0^T dt = \int_0^{2\pi} \frac{dt}{df}\,df = \int_0^{2\pi} \frac{df}{\dot f} = \frac{a^2(1-e^2)^2}{h}\int_0^{2\pi} \frac{df}{(1+e\cos f)^2}$$

where we have used $r^2\dot f = h$.

From eqn (6),

$$\frac{-a(1-e^2)\cdot}{(1+e\cos f)^2}(-e\sin f)\,df = -ae(-\sin E)\,dE$$

so that

$$\frac{df}{(1+e\cos f)^2} = \frac{1}{1-e^2}\frac{\sin E}{\sin f}\,dE.$$

$$= \frac{1}{1-e^2}\cdot \sin E \cdot \frac{1-e\cos E}{\sqrt{1-e^2}\sin E}\,dE \quad (\text{from } (7))$$

$$= \frac{1}{(1-e^2)^{3/2}}(1-e\cos E)\,dE \qquad (*)$$

Thus

$$T = \frac{a^2(1-e^2)^2}{\sqrt{GM_{12}a(1-e^2)}}\cdot\frac{1}{(1-e^2)^{3/2}}\int_0^{2\pi}(1-e\cos E)\,dE$$

$$= 2\pi\sqrt{\frac{a^3}{G(M_1+M_2)}} = \frac{2\pi}{\nu}$$

Note that when $f=0$, $E=0$, & when $f=2\pi$, $E=2\pi$.

5. Use the the analysis from the previous question to show that the time dependence of the true anomoly, f, (the orbital angle measured from periastron) is given implicitly by

$$\nu t = 2\,\mathrm{Tan}^{-1}\left[\sqrt{\frac{1-e}{1+e}}\tan\frac{f}{2}\right] - \frac{e\sqrt{1-e^2}\sin f}{1+e\cos f} + \text{const}, \tag{9}$$

where $\nu = 2\pi/T$ is the orbital frequency (mean motion) of the binary. Show that this reduces to

$$f = \nu t + \text{const}$$

for circular orbits.

This time

$$t = \int dt = \int \frac{df}{\dot{f}} = \frac{(1-e^2)^{3/2}}{\nu} \int \frac{df}{(1+e\cos f)^2}$$

where we have used Q3b because we have now estabished the form of ν. Also note there are no limits.

Now from (*) on previous page we have

$$\nu t = (1-e^2)^{3/2} \cdot \frac{1}{(1-e^2)^{3/2}} \int (1-e\cos E)\, dE$$

$$= E - e\sin E + \text{const.}$$

$\boxed{\nu t = E - e\sin E}$ is called "Kepler's Equation".

From (8),

$$\frac{E}{2} = \mathrm{Tan}^{-1}\left[\sqrt{\frac{1-e}{1+e}}\tan\frac{f}{2}\right].$$

From (7) we get

$$\cos E = \frac{e + \cos f}{1 + e\cos f} \quad \& \quad \sin E = \frac{\sqrt{1-e^2}\sin f}{1+e\cos f}$$

So that

$$\nu t = 2\,\mathrm{Tan}^{-1}\left[\sqrt{\frac{1-e}{1+e}}\tan\frac{f}{2}\right] - \frac{e\sqrt{1-e^2}\sin f}{1+e\cos f}$$

$$+ \text{const.}$$

(3)

(5) $$\underset{\sim}{\ddot{r}} = -\frac{GM_{12}}{r^2}\hat{\underset{\sim}{r}} - \frac{\beta}{r^3}\hat{\underset{\sim}{r}} \qquad (\hat{\underset{\sim}{r}} \equiv \underset{\sim}{e_r})$$

From notes, $$\underset{\sim}{\ddot{r}} = (\ddot{r} - r\dot{\varphi}^2)\underset{\sim}{e_r} + (r\ddot{\varphi} + 2\dot{r}\dot{\varphi})\underset{\sim}{e_\varphi}$$ 1

Thus
$$\ddot{r} - r\dot{\varphi}^2 = -\frac{GM_{12}}{r^2} - \frac{\beta}{r^3} \qquad -(8)$$ 1

and $$r\ddot{\varphi} + 2\dot{r}\dot{\varphi} = 0 \qquad -(9)$$ 1

3 total

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(6) Eq^n (9) gives
$$\frac{d}{dt}(r^2\dot{\varphi}) = 0$$

So $r^2\dot{\varphi} =$ constant $\equiv h$ 1

Putting $\dot{\varphi} = \frac{h}{r^2}$ in (8) gives
$$\ddot{r} - \frac{h^2}{r^3} + \frac{GM_{12}}{r^2} + \frac{\beta}{r^3} = 0$$

or $$\ddot{r} - \frac{(h^2 - \beta)}{r^3} + \frac{GM_{12}}{r^2} = 0.$$ 1

Putting $r = \frac{1}{u}$, $$\frac{d}{dt} = \frac{d\varphi}{dt}\frac{d}{d\varphi} = \dot{\varphi}\frac{d}{d\varphi}$$
$$= \frac{h}{r^2}\frac{d}{d\varphi} = hu^2\frac{d}{d\varphi}$$

gives
$$\left(hu^2\frac{d}{d\varphi}\right)\left(hu^2\underbrace{\frac{d(1/u)}{d\varphi}}_{-\frac{1}{u^2}\frac{du}{d\varphi}}\right) - (h^2-\beta)u^3 + GM_{12}u^2 = 0$$ 3

or $$-h^2u^2\frac{d^2u}{d\varphi^2} - (h^2-\beta)u^3 + GM_{12}u^2 = 0.$$

4

Dividing by $-h^2u^2$ gives

$$\frac{d^2u}{d\varphi^2} + (1-\beta/h^2)u = \frac{GM_{12}}{h^2} \qquad -(10)$$ 1

Total 6

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7 Putting $\Omega^2 = 1-\beta/h^2$, the general solution to eqn (10) is

$$u(\varphi) = C_1\cos\Omega\varphi + C_2\sin\Omega\varphi + \underbrace{\frac{GM_{12}}{\Omega^2h^2}}_{\text{particular solution}},$$ 2

where C_1 & C_2 are arbitrary constants.

This can be written in alternative form as

$$u(\varphi) = \alpha\cos(\Omega\varphi-\omega) + \frac{GM_{12}}{\Omega^2h^2},$$ 2

where α and ω are arbitrary constants.

Since $u = 1/r$ we have

$$r = \frac{1}{\alpha\cos(\Omega\varphi-\omega) + \frac{GM_{12}}{\Omega^2h^2}}$$

$$= \frac{\Omega^2h^2/GM_{12}}{1+\frac{\alpha\Omega^2h^2}{GM_{12}}\cos(\Omega\varphi-\omega)}$$ 2

$$\equiv \frac{a(1-e^2)}{1+e\cos(\Omega\varphi-\omega)},$$

with the last step <u>defining</u> a and e such that

$$a(1-e^2) = \frac{\Omega^2 h^2}{GM_{12}} \qquad \text{—} (*)$$

and $\quad e = \dfrac{\alpha \Omega^2 h^2}{GM_{12}}$.

Since α is an arbitrary constant, so is e.

Also from $(*)$, the angular momentum per unit mass is

$$h = \frac{\sqrt{GM_{12}\, a(1-e^2)}}{\Omega}$$

$$= \frac{h_K}{\Omega} \qquad \text{from eq}^n\ (7).$$

Total 6

(6)

(8) Periastron corresponds to the minimum value of r. This occurs when $\cos(\Omega\varphi - \omega) = 1$, that is,

when $\Omega\varphi - \omega = 0, 2\pi, 4\pi, \ldots$

For example, for $\omega = 0$ we have

$$\Omega\varphi = 2n\pi, \qquad n = 0, 1, 2 \ldots$$

So that periastron passage occurs at different values of φ each orbit given by

$$\varphi_n = \frac{2n\pi}{\Omega}, \qquad n = 0, 1, 2 \ldots$$ 1 1 1

Putting $\Omega = \sqrt{1-\epsilon}$,

$$\varphi_n = \frac{2n\pi}{\sqrt{1-\epsilon}} = 2n\pi\left(1 + \tfrac{1}{2}\epsilon + \ldots\right)$$

$$\simeq 2n\pi + n\pi\epsilon$$ 2

So that

$$\varphi_n - \varphi_{n-1} = \pi\epsilon$$ (modulus 2 pi) 1

Total 6

(5) The orbit is not really closed because $\frac{1}{\sqrt{1-\epsilon}}$ is not a rational number when $\epsilon = 0.2$ or $\epsilon = 0.06$. 3

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(6) Apsidal motion will be in the opposite direction. 2

Exoplanets

Problem Set 2 Solutions

The following questions refer to the lecture notes
"Resonance, chaos and stability: The three-body problem in astrophysics".

1. **Pendulums**

(a) Show that a form for the pendulum "energy" which is zero on the separatrix is given by (equation (3.2))

$$E = \tfrac{1}{2}\dot{\phi}^2 - \omega_0^2(\cos\phi + 1). \tag{1}$$

Starting with the equation of motion

$$\ddot{\phi} + \omega_0^2 \sin\phi = 0,$$

multiply by $\dot{\phi}$ to get

$$\dot{\phi}\ddot{\phi} + \omega_0^2\dot{\phi}\sin\phi \;=\; \frac{d}{dt}\left(\tfrac{1}{2}\dot{\phi}^2\right) + \omega_0^2\,\frac{d}{dt}(-\cos\phi) = 0$$

so that

$$\tfrac{1}{2}\dot{\phi}^2 - \omega_0^2\,\cos\phi = \text{constant}.$$

Putting the constant equal to $E + C$ and associating the separatrix with zero energy (we are free to choose the constant however we like), we can determine C by insisting that the point $(\phi, \dot{\phi}) = (\pi, 0)$ (the right-hand hyperbolic point in Figure 3.2(b)) corresponds to $E = 0$. Thus

$$0 - \omega_0^2\cos\pi = 0 + C,$$

that is, $C = \omega_0^2$ and eqn (1) follows.

(b) Show that the equation for the two branches of the separatrix is

$$\dot{\phi} = \pm 2\,\omega_0\cos(\phi/2). \tag{2}$$

Since we have defined E to be such that it is zero on the separatrix, eqn (1) gives us

$$0 = \tfrac{1}{2}\dot{\phi}^2 - \omega_0^2(\cos\phi + 1)$$

so that

$$\begin{aligned}\dot{\phi} &= \pm\sqrt{2}\,\omega_0\sqrt{\cos\phi + 1}\\ &= \pm\sqrt{2}\,\omega_0\sqrt{2\cos(\phi/2) - 1 + 1}\\ &= \pm 2\,\omega_0\cos(\phi/2).\end{aligned}$$

(c) Without looking at the notes, derive an integral expression for the libration period in terms of the maximum pendulum excursion, ϕ_m and the small angle frequency, ω_0.

(d) Do the same for the circulation period T_{circ} in terms of $\dot{\phi}(0)$ and ω_0.

(e) Show that when $\phi_m \ll 1$, $T_{lib} \simeq 2\pi/\omega_0$.

Starting with eqn (3.4) in the lecture notes (and noting the typo)

$$T_{lib} = \int_0^{T_{lib}} dt = 4\int_0^{\phi_m} \frac{d\phi}{\dot{\phi}} = \frac{4\sqrt{2}}{\omega_0}\int_0^{\phi_m} \frac{d\phi}{\sqrt{\cos\phi - \cos\phi_m}},$$

we first note that since $\phi_m \ll 1$, ϕ is also small during a cycle (strictly we must also assume that $\dot{\phi}(0) \ll 1$) put $\cos\phi_m = 1-\frac{1}{2}\phi_m^2+\ldots$ and $\cos\phi = 1-\frac{1}{2}\phi^2+\ldots$ so that

$$\begin{aligned}
T_{lib} &= \frac{4}{\omega_0}\int_0^{\phi_m} \frac{d\phi}{\sqrt{\phi_m^2-\phi^2}} \\
&= \frac{4}{\phi_m\omega_0}\int_0^{\phi_m} \frac{d\phi}{\sqrt{1-(\phi/\phi_m)^2}} \\
&= \frac{4}{\omega_0}\int_0^{1} \frac{dx}{\sqrt{1-x^2}} \\
&= \frac{4}{\omega_0}\cdot \mathrm{Sin}^{-1}(1) \\
&= \frac{4}{\omega_0}\cdot\frac{\pi}{2} \\
&= \frac{2\pi}{\omega_0}.
\end{aligned}$$

(f) Show that when $\dot{\phi}(0) \gg 2\omega_0$, $T_{circ} \simeq 2\pi/\dot{\phi}(0)$.

This one is very simple. Putting $\dot{\phi}(0) \gg 2\omega_0$ in eqn (3.5) (and noting that $|\cos\phi - 1| < 2$),

$$T_{circ} = \frac{2}{\dot{\phi}_0}\int_0^{\pi} d\phi = \frac{2\pi}{\dot{\phi}_0}.$$

2. **Linear versus non-linear resonance**

(a) Show that the solution to

$$\ddot{\phi} + \omega^2\phi = A\sin\Omega t \tag{3}$$

which satisfies satisfying $\phi(0) = \dot{\phi}(0) = 0$ is

$$\phi(t) = \frac{A}{\Omega^2-\omega^2}\left[(\Omega/\omega)\sin\omega t - \sin\Omega t\right] \tag{4}$$

when $\Omega \neq \omega$, and

$$\phi(t) = \frac{A}{2\omega^2}\left[\sin\omega t - \omega t\cos\omega t\right] \tag{5}$$

when $\Omega = \omega$.

See MTH2032 notes :-).

(b) Show that the maximum value of ϕ attained when $\Omega \neq \omega$ is $(A/\omega)/|\Omega - \omega|$, while the envelope of (5) is given by $\pm At/2\omega$ when $\Omega = \omega$.

$$\dot{\phi} = \frac{A}{\Omega^2 - \omega^2} \left[\Omega(\cos \omega t - \cos \Omega t)\right] = 0$$

when

$$\cos \omega t = \cos \Omega t.$$

When this is true, so is

$$\sin \omega t = \pm \sin \Omega t.$$

When this is true,

$$\begin{aligned} \phi(t) &= \frac{A}{\Omega^2 - \omega^2} \left[(\Omega/\omega) \sin \omega t \pm \sin \omega t\right] \\ &= \frac{A}{\Omega^2 - \omega^2} \left[(\Omega/\omega) \pm 1\right] \sin \omega t. \end{aligned}$$

This is a maximum when $\sin \omega t = 1$ and we take the plus sign so that

$$\phi = \frac{A}{\omega} \frac{1}{|\Omega - \omega|}.$$

For the envelope of the resonant solution, first find the maxima:

$$\dot{\phi} = \frac{A}{2\omega^2} \left[\omega \cos \omega t - \omega \cos \omega t + \omega^2 t \sin \omega t\right] = 0$$

or

$$t \sin \omega t = 0.$$

Maxima and minima correspond to $\cos \omega t = \pm 1$ so that the envelope is given by

$$\phi_{max,min} = \frac{A}{2\omega^2} \left[0 - \omega t(\pm 1)\right] = \pm \frac{At}{2\omega}.$$

3. **The butterfly effect explained**

Discuss the origin of extreme sensitivity to initial conditions when a weakly interacting system is chaotic.

See lecture notes.

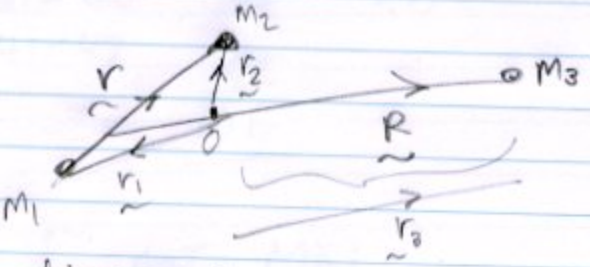
Disturbing

1(b) Show that

$$\underset{\sim}{J} = \mu_1 \underset{\sim}{r} \times \dot{\underset{\sim}{r}} + \mu_0 \underset{\sim}{R} \times \dot{\underset{\sim}{R}}$$

Total ang mom:

$$\underset{\sim}{J} = m_1 \underset{\sim}{r_1} \times \dot{\underset{\sim}{r_1}} + m_2 \underset{\sim}{r_2} \times \dot{\underset{\sim}{r_2}} + m_3 \underset{\sim}{r_3} \times \dot{\underset{\sim}{r_3}}$$



From diagram:

$$\underset{\sim}{r_1} = -\frac{m_3}{M_{123}} \underset{\sim}{R} - \frac{m_2}{M_{12}} \underset{\sim}{r} \equiv -\beta_3 \underset{\sim}{R} - \alpha_2 \underset{\sim}{r}$$

$$\underset{\sim}{r_2} = -\frac{m_3}{M_{123}} \underset{\sim}{R} + \frac{m_1}{M_{12}} \underset{\sim}{r} \equiv -\beta_3 \underset{\sim}{R} + \alpha_1 \underset{\sim}{r} \qquad (*)$$

$$\underset{\sim}{r_3} = \frac{m_{12}}{M_{123}} \underset{\sim}{R} \equiv \beta_{12} \underset{\sim}{R}$$

Thus $\underset{\sim}{r_1} \times \dot{\underset{\sim}{r_1}} = \left(-\beta_3 \underset{\sim}{R} - \alpha_2 \underset{\sim}{r}\right) \times \left(-\beta_3 \dot{\underset{\sim}{R}} - \alpha_2 \dot{\underset{\sim}{r}}\right)$

$$= \beta_3^2 \underset{\sim}{R} \times \dot{\underset{\sim}{R}} + \alpha_2 \beta_3 \left(\underset{\sim}{R} \times \dot{\underset{\sim}{r}} + \underset{\sim}{r} \times \dot{\underset{\sim}{R}}\right) + \alpha_2^2 \underset{\sim}{r} \times \dot{\underset{\sim}{r}}$$

$$\underset{\sim}{r_2} \times \dot{\underset{\sim}{r_2}} = \left(-\beta_3 \underset{\sim}{R} + \alpha_1 \underset{\sim}{r}\right) \times \left(-\beta_3 \dot{\underset{\sim}{R}} + \alpha_1 \dot{\underset{\sim}{r}}\right)$$

$$= \beta_3^2 \underset{\sim}{R} \times \dot{\underset{\sim}{R}} - \alpha_1 \beta_3 \left(\underset{\sim}{R} \times \dot{\underset{\sim}{r}} + \underset{\sim}{r} \times \dot{\underset{\sim}{R}}\right) + \alpha_1^2 \underset{\sim}{r} \times \dot{\underset{\sim}{r}}$$

$$\underset{\sim}{r_3} \times \dot{\underset{\sim}{r_3}} = \beta_{12}^2 \underset{\sim}{R} \times \dot{\underset{\sim}{R}}$$

Thus $\underset{\sim}{J} = \left[(m_1+m_2)\beta_3^2 + m_3 \beta_{12}^2\right] \underset{\sim}{R} \times \dot{\underset{\sim}{R}} + (m_1\alpha_2 - m_2\alpha_1)\beta_3 \left(\underset{\sim}{R} \times \dot{\underset{\sim}{r}} + \underset{\sim}{r} \times \dot{\underset{\sim}{R}}\right)$

$$+ \left(m_1 \alpha_2^2 + m_2 \alpha_1^2\right) \underset{\sim}{r} \times \dot{\underset{\sim}{r}}$$

$$m_{12}\beta_3^2 + m_3\beta_{12}^2 = \frac{m_{12}m_3^2 + m_3 m_{12}^2}{m_{123}^2}$$

$$= \frac{m_3 m_{12}}{m_{123}} \times 1 = \mu_o$$

$$m_1\alpha_2 - m_2\alpha_1 = \frac{m_1 m_2 - m_2 m_1}{m_{12}} = 0$$

$$m_1\alpha_2^2 + m_2\alpha_1^2 = \frac{m_1 m_2^2 + m_2 m_1^2}{m_{12}^2}$$

$$= \frac{m_1 m_2}{m_{12}} \times 1 = \mu_i$$

1(c) $$E = \tfrac{1}{2}m_1 \dot{\underset{\sim}{r}}_1\cdot\dot{\underset{\sim}{r}}_1 + \tfrac{1}{2}m_2 \dot{\underset{\sim}{r}}_2\cdot\dot{\underset{\sim}{r}}_2 + \tfrac{1}{2}m_3 \dot{\underset{\sim}{r}}_3\cdot\dot{\underset{\sim}{r}}_3 - \frac{Gm_1m_2}{|\underset{\sim}{r}_2-\underset{\sim}{r}_1|} - \frac{Gm_1m_3}{|\underset{\sim}{r}_3-\underset{\sim}{r}_1|} - \frac{Gm_2m_3}{|\underset{\sim}{r}_3-\underset{\sim}{r}_2|}$$

Easy to show (see second year probs)

$$E_i = \tfrac{1}{2}m_1 \dot{\underset{\sim}{r}}_1\cdot\dot{\underset{\sim}{r}}_1 + \tfrac{1}{2}m_2 \dot{\underset{\sim}{r}}_2\cdot\dot{\underset{\sim}{r}}_2 - \frac{Gm_1m_2}{|\underset{\sim}{r}_2-\underset{\sim}{r}_1|}$$

$$= \tfrac{1}{2}\mu_i \dot{\underset{\sim}{r}}\cdot\dot{\underset{\sim}{r}} - \frac{Gm_1m_2}{r}$$

and $$E_o + R = \tfrac{1}{2}m_3 \dot{\underset{\sim}{r}}_3\cdot\dot{\underset{\sim}{r}}_3 - \frac{Gm_1m_3}{|\underset{\sim}{r}_3-\underset{\sim}{r}_1|} - \frac{Gm_2m_3}{|\underset{\sim}{r}_3-\underset{\sim}{r}_2|}$$

$$= \underbrace{\left[\tfrac{1}{2}\mu_o \dot{\underset{\sim}{R}}\cdot\dot{\underset{\sim}{R}} - \frac{Gm_{12}m_3}{R}\right]}_{E_o} + \underbrace{\left[\frac{Gm_{12}m_3}{R} - \frac{Gm_1m_3}{|\underset{\sim}{R}+\alpha_2\underset{\sim}{r}|} - \frac{Gm_2m_3}{|\underset{\sim}{R}-\alpha_1\underset{\sim}{r}|}\right]}_{-R}$$

from (*)

1 (c) alternative solution:

Starting with

$$\mu_{12}\ddot{\underset{\sim}{r}} + \frac{Gm_1m_2}{r^2}\hat{\underset{\sim}{r}} = \frac{\partial R_0}{\partial \underset{\sim}{r}} \qquad — \text{①}$$

$$\mu_{123}\ddot{\underset{\sim}{R}} + G\frac{m_{12}m_3}{R^2}\hat{\underset{\sim}{R}} = \frac{\partial R_0}{\partial \underset{\sim}{R}} \qquad — \text{②}$$

First note that

$$\frac{Gm_1m_2}{r^2}\hat{\underset{\sim}{r}} = \frac{\partial}{\partial \underset{\sim}{r}}\left(\frac{-Gm_1m_2}{r}\right)$$

and $$\frac{Gm_{12}m_3}{R^2}\hat{\underset{\sim}{R}} = \frac{\partial}{\partial \underset{\sim}{R}}\left(-\frac{Gm_{12}m_3}{R}\right)$$

[Recall that $\nabla\phi(r) = \frac{d\phi}{dr}\hat{\underset{\sim}{r}}$]

To form an energy, take $\dot{\underset{\sim}{r}}\cdot$①, $\dot{\underset{\sim}{R}}\cdot$②:

$$\mu_{12}\dot{\underset{\sim}{r}}\cdot\ddot{\underset{\sim}{r}} - \dot{\underset{\sim}{r}}\cdot\frac{\partial}{\partial \underset{\sim}{r}}\left(\frac{Gm_1m_2}{r}\right) = \dot{\underset{\sim}{r}}\cdot\frac{\partial R_0}{\partial \underset{\sim}{r}}$$

$$\mu_{123}\dot{\underset{\sim}{R}}\cdot\ddot{\underset{\sim}{R}} - \dot{\underset{\sim}{R}}\cdot\frac{\partial}{\partial \underset{\sim}{R}}\left(\frac{Gm_{12}m_3}{R}\right) = \dot{\underset{\sim}{R}}\cdot\frac{\partial R_0}{\partial \underset{\sim}{R}}$$

Since there is no explicit time dependance,

$$\frac{d}{dt} = \frac{\partial}{\partial \underset{\sim}{r}} + \frac{\partial}{\partial \underset{\sim}{R}}.$$

Noting that

$$\frac{\partial}{\partial \underset{\sim}{r}}\left(\frac{1}{R}\right) = 0, \quad \frac{\partial}{\partial \underset{\sim}{R}}\left(\frac{1}{r}\right) = 0$$

We can write

$$\frac{d}{dt}\left(\frac{\mu_{12}}{2}\dot{\underset{\sim}{r}} \cdot \dot{\underset{\sim}{r}}\right) - \dot{\underset{\sim}{r}} \cdot \frac{\partial}{\partial \underset{\sim}{r}}\left(\frac{Gm_1m_2}{r} + \frac{Gm_{12}m_3}{R} - \mathcal{R}\right) = 0$$

$$\frac{d}{dt}\left(\frac{\mu_{123}}{2}\dot{\underset{\sim}{R}} \cdot \dot{\underset{\sim}{R}}\right) - \dot{\underset{\sim}{R}} \cdot \frac{\partial}{\partial \underset{\sim}{R}}\left(\frac{Gm_1m_2}{r} + \frac{Gm_{12}m_3}{R} - \mathcal{R}\right) = 0$$

or adding,

$$\frac{d}{dt}\left[\frac{1}{2}\mu_{12}\dot{\underset{\sim}{r}} \cdot \dot{\underset{\sim}{r}} + \frac{1}{2}\mu_{123}\dot{\underset{\sim}{R}} \cdot \dot{\underset{\sim}{R}} - \frac{Gm_1m_2}{r} - \frac{Gm_{12}m_3}{R} - \mathcal{R}\right]$$

$$\equiv \frac{dE}{dt} = 0$$

So that $E = \text{const.}$

2(a) $n=0$

$$C_0 = \frac{1}{2}\int_{-1}^{1} \frac{1}{\sqrt{a^2+b^2-2abx}}\, dx$$

$$= \frac{1}{2\sqrt{a^2+b^2}}\int_{-1}^{1} \frac{dx}{\sqrt{1-\alpha x}}, \quad \alpha = \frac{2ab}{a^2+b^2}$$

$$= \left(\frac{-2}{\alpha}\right)\frac{1}{2\sqrt{a^2+b^2}}\left[\sqrt{1-\alpha x}\right]_{-1}^{1}$$

$$= -\left(\frac{a^2+b^2}{2ab}\right)\frac{1}{\sqrt{a^2+b^2}}\left[\sqrt{1-\alpha} - \sqrt{1+\alpha}\right]$$

$$1-\alpha = \frac{a^2+b^2-2ab}{a^2+b^2} = \frac{(a-b)^2}{a^2+b^2} = \frac{(b-a)^2}{a^2+b^2}$$

$$1+\alpha = \frac{(a+b)^2}{a^2+b^2}$$

$$C_0 = -\frac{\sqrt{a^2+b^2}}{2ab}\left(\frac{(b-a)-(b+a)}{\sqrt{a^2+b^2}}\right)$$

$$= \frac{1}{b}$$

Note that $\sqrt{(b-a)^2} = b-a$ when $b>a$

$= a-b$ when $b<a$.

$n = 1$

$$C_1 = \frac{3}{2}\int_{-1}^{1} \frac{x}{\sqrt{a^2+b^2-2abx}} \quad dx$$

$$= \frac{3}{2\sqrt{a^2+b^2}} \int_{-1}^{1} \frac{x\,dx}{\sqrt{1-\alpha x}}$$

Integrate by parts:

$$\int_{-1}^{1} \underset{u}{x}\ \underset{v'}{(1-\alpha x)^{-1/2}}\, dx$$

$$= \left[x \cdot \left(\frac{-2}{\alpha}\right)(1-\alpha x)^{1/2}\right]_{-1}^{1} - \left(\frac{-2}{\alpha}\right)\int_{-1}^{1} (1-\alpha x)^{1/2}\, dx$$

$$= \frac{-2}{\alpha}\left[(1-\alpha)^{1/2} + (1+\alpha)^{1/2}\right] + \frac{2}{\alpha}\left(\frac{-2}{3\alpha}\right)\left(1-\alpha x\right)^{3/2}\Big]_{-1}^{1}$$

$$= \quad '' \quad -\frac{4}{3\alpha^2}\left((1-\alpha)^{3/2} - (1+\alpha)^{3/2}\right)$$

$$= \frac{-2}{\alpha}\left[\frac{b-a+b+a}{\sqrt{a^2+b^2}}\right] - \frac{4}{3\alpha^2}\left[\frac{(b-a)^3-(b+a)^3}{(a^2+b^2)^{3/2}}\right]$$

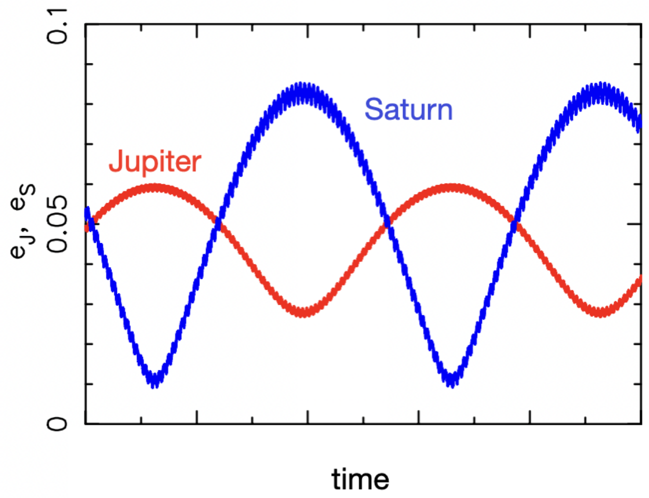
$$= -\frac{2(a^2+b^2)}{2ab}\,\frac{2b}{\sqrt{a^2+b^2}} - \frac{4}{3}\,\frac{(a^2+b^2)^2}{4a^2b^2}\left[\frac{-6b^2a-2a^3}{(a^2+b^2)^{3/2}}\right]$$

$$= \sqrt{a^2+b^2}\left[-\frac{2}{a} + \frac{2}{a} + \frac{2}{3}\frac{a}{b^2}\right]$$

so that $C_1 = \dfrac{a}{b^2}$

(e) Estimate the secular period of variation of the eccentricities of Jupiter and Saturn (see figure below which shows a full three-body integration in which the effect of neglected terms is evident). Give your answer in years.

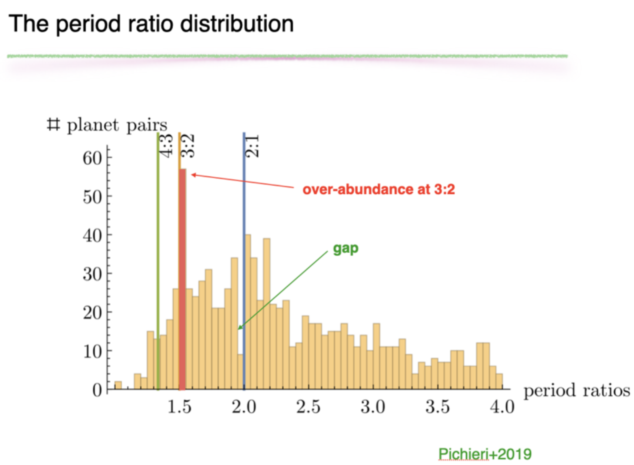
(f) Under what circumstances would we expect there to be no secular variation in the eccentricities? *Hint: Find the "fixed points" of the system by putting* $\dot{e}_b = 0$, $\dot{e}_c = 0$ *and* $\dot{\varpi}_b - \dot{\varpi}_c = 0$.

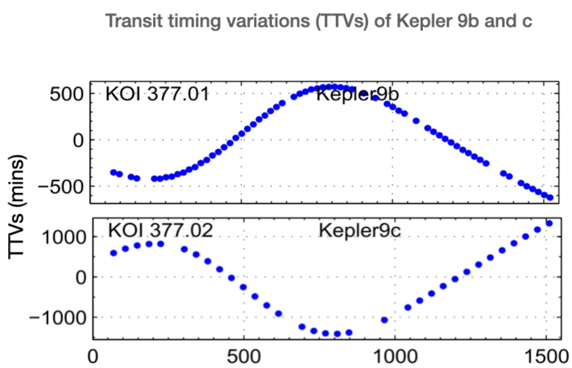


7. **Planetary systems near a first-order mean motion resonance**

It was mentioned in lectures that systems with period ratios near 2/1, 3/2, 4/3 etc are referred to as "first-order resonant" because the leading terms in their Fourier expansions are first-order in the eccentricities. Here we show why this is true.

Note that many first-order resonant systems have been discovered, these playing a particularly important role in the characterization of multi-transiting systems. This is because the gravitational interaction between the planets is often significant enough to produce *transit timing variations* (TTVs) which encode information about the planet-to-star mass ratios. The ability to measure TTVs becomes especially important when the system is too faint to make radial velocity observations practical.





By expressing f_b and M_b (f_i and M_i) in the definition of the Hansen coefficient (41) in terms of the eccentric anomaly E_b (and similarly for (43); see Appendix), it is possible to write the Hansen coefficients as Taylor series expanded in terms of e_b and e_c respectively. It turns out that the leading terms of these expansions are such that (Mardling 2013)

$$X_n^{l,m}(e) = x_n^{l,m} e^{|m-n|} + \mathcal{O}(e^{|m-n|+2}), \tag{73}$$

where $x_n^{l,m}$ are simple fractions, and the notation $\mathcal{O}$ means "terms of order".[7] For example, from (52), $x_0^{3,1} = -5/2$, and more generally one can show that $x_n^{l,n} = 1$ for all l and n, while $x_n^{l,n+1} = -\frac{1}{2}(l+2n+2)$ and $x_{n+1}^{-(l+1),n} = \frac{1}{2}(l+2n+2)$. Note for these last two that $|m-n| = 1$ so that

$$X_n^{l,n+1}(e_b) \simeq -\tfrac{1}{2}(l+2n+2)\, e_b \quad \text{and} \quad X_{n+1}^{-(l+1),n}(e_c) \simeq \tfrac{1}{2}(l+2n+2)\, e_c. \tag{74}$$

Now consider the disturbing function (53), with all terms discarded except those which are first-order in eccentricity, and in addition, are such that $n' = n+1$. The latter requirement ensures "small divisors" when the period ratio is such that $P_c/P_b \simeq (n+1)/n$ (see Question (7d) below). We will justify the neglect of zeroth-order terms *a posteriori*. Using the approximations (74) we then have

$$\mathcal{R} = \frac{Gm_* m_b}{a_b} \sum_{n=1}^{\infty} \left[\mathcal{R}_{n+1\,n\,n+1} \cos\phi_{n+1\,n\,n+1} + \mathcal{R}_{n\,n\,n+1} \cos\phi_{n\,n\,n+1} \right], \tag{75}$$

where the Fourier amplitudes are

$$\mathcal{R}_{n+1\,n\,n+1} = \left(\frac{m_c}{m_*}\right) \left\{ \sum_{l=n+1,2}^{\infty} \left(c_{l\,n+1}^2 x_n^{l,n+1} \right) \left(\frac{a_b}{a_c} \right)^{l+1} \right\} e_b \equiv \left(\frac{m_c}{m_*}\right) f_n^{(b)}(\alpha)\, e_b \tag{76}$$

and

$$\mathcal{R}_{n\,n\,n+1} = \left(\frac{m_c}{m_*}\right) \left\{ \sum_{l=l_{\min},2}^{\infty} \left(c_{ln}^2 x_{n+1}^{-(l+1),n} \right) \left(\frac{a_b}{a_c} \right)^{l+1} \right\} e_c \equiv \left(\frac{m_c}{m_*}\right) f_n^{(c)}(\alpha)\, e_c, \tag{77}$$

where again, $\alpha = a_b/a_c$ and

$$l_{\min} = \begin{cases} 3, & n = 1, \\ n, & n \geq 2, \end{cases} \tag{78}$$

while the Fourier angles are

$$\phi_{n+1\,n\,n+1} = n\lambda_b - (n+1)\lambda_c + \varpi_b \quad \text{and} \quad \phi_{n\,n\,n+1} = n\lambda_b - (n+1)\lambda_c + \varpi_c. \tag{79}$$

Note that for single-star planetary problems, the functions $f_n^{(b)}(\alpha)$ and $f_n^{(c)}(\alpha)$ are expressed in terms of *Laplace coefficients* (mentioned above) which express the infinite sum in terms of closed-form integrals (which nevertheless need to be evaluated numerically; see Mardling 2013). This is because the parameter $\alpha < 1$ tends to be significant for closely-packed systems.

On the other hand, for circumbinary planet problems it is sufficient to include only one or two terms in the summation over l because stable systems have small values of α by necessity.

Having said that, our main purpose here is to understand the dependence of the disturbing function on the orbital elements (rather than derive expressions which are capable of replacing N-body calculations), so for this purpose, we will include only the leading term in l (as we did for the

[7] Note that the next term in the series is $\mathcal{O}(e^2)$ higher so a leading order approximation is good for small eccentricities, although the associated coefficients tend to be larger for larger n.

secular problem above). The error incurred in making this approximation is demonstrated in the following table, were "lov" stands for "leading-order value". Not surprisingly, the error increases as the period ratio approaches unity, with the leading order term generally underestimating the true value.

resonance	α	$f_n^{(b)}(\alpha)$	lov	$f_n^{(c)}(\alpha)$	lov
2:1	0.63	−0.47	−0.56	0.27	0.21
3:2	0.76	−1.18	−0.95	1.90	1.33
4:3	0.83	−1.94	−1.26	2.71	1.60
5:4	0.86	−2.71	−1.51	3.52	1.82

(a) Convince yourself that the expressions in (76) and (77) are correct.

Since $m_2 \ll m_1$, replace m_3/m_{12} by m_3/m_1 and put $\mathcal{M}_l = 1$ in (54). Now we wish to keep only the terms which are first-order in the eccentricities. From (73), these are the terms for which $|m-n| = 1$ for e_b, and $|m-n'| = |m-(n+1)| = 1$ for e_c, that is, the terms with $m = n+1$, $m = n-1$ for for e_b, and $m = n+2$, $m = n$ for e_c.

First consider terms with $m = n+1$. To first-order in eccentricity, these are such that

$$X_n^{l,m}(e_b) = X_n^{l,n+1}(e_b) \simeq x_n^{l,n+1} e_b \quad \text{and} \quad X_{n'}^{-(l+1),m}(e_c) = X_{n+1}^{-(l+1),n+1} \simeq 1.$$

Thus $\mathcal{R}_{n+1\,n\,n+1}$ is first order in eccentricity. On the other hand, the term with $m = n-1$ is such that

$$X_{n'}^{-(l+1),m}(e_c) = X_{n+1}^{-(l+1),n-1} = \mathcal{O}(e_c^{|n-1-(n+1)|}) = \mathcal{O}(e_c^2),$$

making $\mathcal{R}_{n-1\,n\,n+1} \propto e_b e_c^2$, that is, *third order* in eccentricity, so we do not include it. Similarly, terms with $m = n$ are such that

$$X_n^{l,m}(e_b) = X_n^{l,n}(e_b) \simeq 1 \quad \text{and} \quad X_{n'}^{-(l+1),m}(e_c) = X_{n+1}^{-(l+1),n} \simeq x_{n+1}^{l,n} e_c$$

so that $\mathcal{R}_{n\,n\,n+1}$ is first order in eccentricity, while terms with $m = n+2$ are again third-order in eccentricity and are therefore not included.

Putting $m = n+1$ and $n' = n+1$ in (53) then reduces the triple sum to a single sum (only one term from the m-sum and one from the n'-sum are retained), and since $m = n+1 \geq 2$ for $n \geq 1$, $l_{\min} = n+1$ from (78). Note that since from (**??**) m is always positive or zero, the negative-n terms are not first-order in eccentricity (and the $n = 0$ term does not correspond to any resonant terms), so that sum starts at 1 over n.

Finally, $(m_c/m_*)e_b$ is a common factor for all terms with $m = n+1$ and $n' = n+1$ so we can gather together all the factors which depend only on a_b/a_c (the terms in the curly brackets in (76) and call it $f_n^{(b)}(\alpha)$.

Similar arguments can be made for terms with $m = n$ and $n' = n+1$, with l starting at n except when $m = 1$ in which case it starts at 3.

(b) Show that to leading order in a_b/a_c, the Fourier amplitudes and angles relevant to the "2:1 mean motion resonance" are

$$\mathcal{R}_{212} = -\frac{9}{4}\left(\frac{m_c}{m_*}\right)\left(\frac{a_b}{a_c}\right)^3 e_b, \qquad \phi_{212} = \lambda_b - 2\lambda_c + \varpi_b \qquad \text{(quadrupole)}, \tag{80}$$

$$\mathcal{R}_{112} = \frac{21}{16}\left(\frac{m_c}{m_*}\right)\left(\frac{a_b}{a_c}\right)^4 e_c, \qquad \phi_{112} = \lambda_b - 2\lambda_c + \varpi_c \qquad \text{(octupole)}, \tag{81}$$

while those for the "3:2 resonance" are

$$\mathcal{R}_{323} = -\frac{45}{16}\left(\frac{m_c}{m_*}\right)\left(\frac{a_b}{a_c}\right)^4 e_b, \qquad \phi_{323} = 2\lambda_b - 3\lambda_c + \varpi_b \qquad \text{(octupole)}, \tag{82}$$

$$\mathcal{R}_{223} = 3\left(\frac{m_c}{m_*}\right)\left(\frac{a_b}{a_c}\right)^3 e_c, \qquad \phi_{223} = 2\lambda_b - 3\lambda_c + \varpi_c \qquad \text{(quadrupole)}. \tag{83}$$

Note that while the amplitude associated with the outer orbit is generally one "multipole order" lower than that for the inner orbit (eg, for the 3:2 resonance these are octupole and quadrupole respectively), the single exception is the 2:1 resonance for which the amplitude associated with the outer orbit is one multipole order *higher*. This has important implications for multiplanets systems, for example, there appear to be more systems trapped near the 3:2 commensurability than near the 2:1 commensurability[8] (the latter being somewhat "easier" to move through).

For 2:1 we have $n = 1$ so that the first term in the curly bracket in (76) corresponds to $l = 2$ so that from notes below (36) and equation (74),

$$f_1^{(b)}(\alpha) \simeq c_{22}^2 x_1^{2,2}\alpha^3 = (3/4)[-(1/2)(2+2+2)]\alpha^3 = -(9/4)\alpha^3$$

and

$$f_1^{(c)}(\alpha) \simeq c_{31}^2 x_2^{-4,1}\alpha^4 = (3/8)[(1/2)(3+2+2)]\alpha^4 = (21/16)\alpha^4.$$

For 3:2 we have

$$f_2^{(b)}(\alpha) \simeq c_{33}^2 x_2^{3,3}\alpha^4 = (5/8)[-(1/2)(3+4+2)]\alpha^4 = -(45/16)\alpha^4$$

and

$$f_2^{(c)}(\alpha) \simeq c_{22}^2 x_3^{-3,2}\alpha^3 = (3/4)[(1/2)(2+4+2)]\alpha^3 = 3\alpha^3.$$

We are now in the position to write down the rates of change of the elements for systems with small eccentricities. These can then be integrated analytically, and we will see why it is that we can ignore all but two terms in the disturbing function (75) when a system is near a particular first-order commensurability.

(c) Show that the rates of change of the inner orbital elements are given to first-order in eccentricity by

$$\begin{aligned}\frac{\dot a_b}{a_b} &= -2\nu_b\left(\frac{m_c}{m_*}\right)\sum_{n=1}^{\infty} n\,\Big\{e_b\,f_n^{(b)}(\alpha)\sin(n\lambda_b-(n+1)\lambda_c+\varpi_b)\\ &\qquad +e_c\,f_n^{(c)}(\alpha)\sin(n\lambda_b-(n+1)\lambda_c+\varpi_c)\Big\}\end{aligned} \tag{84}$$

$$\dot e_b = \nu_b\left(\frac{m_c}{m_*}\right)\sum_{n=1}^{\infty} f_n^{(b)}(\alpha)\sin(n\lambda_b-(n+1)\lambda_c+\varpi_b) \tag{85}$$

and

$$\dot\varpi_b = \nu_b\left(\frac{1}{e_b}\right)\left(\frac{m_c}{m_*}\right)\sum_{n=1}^{\infty} f_n^{(b)}(\alpha)\cos(n\lambda_b-(n+1)\lambda_c+\varpi_b), \tag{86}$$

[8] See Figure at the bottom of page 16.

while those of the outer orbit are

$$\begin{aligned}\frac{\dot{a}_c}{a_c} &= 2\nu_c\left(\frac{m_b}{m_*}\right)\sum_{n=1}^{\infty}(n+1)\left\{e_b\,(f_n^{(b)}(\alpha)/\alpha)\sin(n\lambda_b-(n+1)\lambda_c+\varpi_b)\right.\\ &\qquad\left.+e_c\,(f_n^{(c)}(\alpha)/\alpha)\sin(n\lambda_b-(n+1)\lambda_c+\varpi_c)\right\} \qquad (87)\end{aligned}$$

$$\dot{e}_c = \nu_c\left(\frac{m_b}{m_*}\right)\sum_{n=1}^{\infty}(f_n^{(c)}(\alpha)/\alpha)\sin(n\lambda_b-(n+1)\lambda_c+\varpi_c) \qquad (88)$$

and

$$\dot{\varpi}_c = \nu_c\left(\frac{1}{e_c}\right)\left(\frac{m_b}{m_*}\right)\sum_{n=1}^{\infty}(f_n^{(c)}(\alpha)/\alpha)\cos(n\lambda_b-(n+1)\lambda_c+\varpi_c). \qquad (89)$$

Things to notice:

- The rates of change of the inner and outer orbital elements are proportional to m_c/m_* and m_b/m_* respectively;
- The rates of change of the longitudes of periastron are large for small eccentricity.

First put

$$\frac{Gm_*m_b}{a_b} = \frac{Gm_*}{a_b^3}\cdot m_b a_b^2 = m_b\nu_b^2 a_b^2$$

in (75). Then from (58)-(60) with (75)-(79) we have

$$\begin{aligned}\frac{1}{a_b}\frac{da_b}{dt} &= \frac{2}{m_b\nu_b a_b^2}\frac{\partial\mathcal{R}}{\partial\lambda_b}\\ &= 2\nu_b\sum_{n=1}^{\infty}\left[\mathcal{R}_{n+1\,n\,n+1}\frac{\partial}{\partial\lambda_b}\cos\phi_{n+1\,n\,n+1}+\mathcal{R}_{n\,n\,n+1}\frac{\partial}{\partial\lambda_b}\cos\phi_{n\,n\,n+1}\right]\\ &= -2\nu_b\left(\frac{m_c}{m_*}\right)\sum_{n=1}^{\infty}n\cdot\left[e_b f_n^{(b)}(\alpha)\sin\phi_{n+1\,n\,n+1}+e_c f_n^{(c)}(\alpha)\sin\phi_{n\,n\,n+1}\right],\end{aligned}$$

$$\begin{aligned}\frac{de_b}{dt} &= -\frac{1}{m_b\nu_b a_b^2 e_b}\frac{\partial\mathcal{R}}{\partial\varpi_b}\\ &= -\nu_b\left(\frac{1}{e_b}\right)\sum_{n=1}^{\infty}\left[\mathcal{R}_{n+1\,n\,n+1}\frac{\partial}{\partial\varpi_b}\cos\phi_{n+1\,n\,n+1}+\mathcal{R}_{n\,n\,n+1}\frac{\partial}{\partial\varpi_b}\cos\phi_{n\,n\,n+1}\right]\\ &= \nu_b\left(\frac{m_c}{m_*}\right)\sum_{n=1}^{\infty}1\cdot f_n^{(b)}(\alpha)\sin\phi_{n+1\,n\,n+1},\end{aligned}$$

and

$$\begin{aligned}\frac{d\varpi_b}{dt} &= \frac{1}{m_b\nu_b a_b^2 e_b}\frac{\partial\mathcal{R}}{\partial e_b}\\ &= \nu_b\left(\frac{1}{e_b}\right)\sum_{n=1}^{\infty}\left[\frac{\partial}{\partial e_b}\left(\mathcal{R}_{n+1\,n\,n+1}\right)\cos\phi_{n+1\,n\,n+1}+\frac{\partial}{\partial e_b}\left(\mathcal{R}_{n\,n\,n+1}\right)\cos\phi_{n\,n\,n+1}\right]\\ &= \nu_b\left(\frac{1}{e_b}\right)\left(\frac{m_c}{m_*}\right)\sum_{n=1}^{\infty}f_n^{(b)}(\alpha)\cos\phi_{n+1\,n\,n+1}.\end{aligned}$$

For the outer orbital elements, this time put

$$\frac{Gm_* m_b}{a_b} = \frac{Gm_*}{a_c^3}\left(\frac{a_c}{a_b}\right)\left(\frac{m_b}{m_c}\right) m_c a_c^2 = m_c \nu_c^2 a_c^2 \left(\frac{a_c}{a_b}\right)\left(\frac{m_b}{m_c}\right)$$

in (75). Then

$$\begin{aligned}
\frac{1}{a_c}\frac{da_c}{dt} &= \frac{2}{m_c \nu_c a_c^2}\frac{\partial \mathcal{R}}{\partial \lambda_c} \\
&= 2\nu_c \left(\frac{a_c}{a_b}\right)\left(\frac{m_b}{m_c}\right)\sum_{n=1}^{\infty}\left[\mathcal{R}_{n+1\,n\,n+1}\frac{\partial}{\partial \lambda_c}\cos\phi_{n+1\,n\,n+1} + \mathcal{R}_{n\,n\,n+1}\frac{\partial}{\partial \lambda_c}\cos\phi_{n\,n\,n+1}\right] \\
&= 2\nu_c \left(\frac{m_b}{m_*}\right)\sum_{n=1}^{\infty}(n+1)\cdot\left[e_b(f_n^{(b)}(\alpha)/\alpha)\sin\phi_{n+1\,n\,n+1} + e_c(f_n^{(c)}(\alpha)/\alpha)\sin\phi_{n\,n\,n+1}\right],
\end{aligned}$$

$$\begin{aligned}
\frac{de_c}{dt} &= -\frac{1}{m_c \nu_c a_c^2 e_c}\frac{\partial \mathcal{R}}{\partial \varpi_c} \\
&= -\nu_c \left(\frac{1}{e_c}\right)\left(\frac{a_c}{a_b}\right)\left(\frac{m_b}{m_c}\right)\sum_{n=1}^{\infty}\left[\mathcal{R}_{n+1\,n\,n+1}\frac{\partial}{\partial \varpi_c}\cos\phi_{n+1\,n\,n+1} + \mathcal{R}_{n\,n\,n+1}\frac{\partial}{\partial \varpi_c}\cos\phi_{n\,n\,n+1}\right] \\
&= \nu_c \left(\frac{m_b}{m_*}\right)\sum_{n=1}^{\infty} 1\cdot(f_n^{(c)}(\alpha)/\alpha)\sin\phi_{n\,n\,n+1},
\end{aligned}$$

and

$$\begin{aligned}
\frac{d\varpi_c}{dt} &= \frac{1}{m_c \nu_c a_c^2 e_c}\frac{\partial \mathcal{R}}{\partial e_c} \\
&= \nu_c \left(\frac{1}{e_c}\right)\left(\frac{a_c}{a_b}\right)\left(\frac{m_b}{m_c}\right)\sum_{n=1}^{\infty}\left[\frac{\partial}{\partial e_c}\left(\mathcal{R}_{n+1\,n\,n+1}\right)\cos\phi_{n+1\,n\,n+1} + \frac{\partial}{\partial e_c}\left(\mathcal{R}_{n\,n\,n+1}\right)\cos\phi_{n\,n\,n+1}\right] \\
&= \nu_c \left(\frac{1}{e_c}\right)\left(\frac{m_b}{m_*}\right)\sum_{n=1}^{\infty}(f_n^{(c)}(\alpha)/\alpha)\cos\phi_{n\,n\,n+1}.
\end{aligned}$$

Equations (8)-(89) can be integrated directly if we take the orbital frequencies, semimajor axes, eccentricities and longitudes of periastron as constant in the expressions on the RHSs (this introduces errors of order $(m_c/m_*)^2$ and $(m_b/m_*)^2$). The mean longitudes are then

$$\lambda_b = \nu_b(t - T_0) + \lambda_b(T_0) \quad \text{and} \quad \lambda_c = \nu_c(t - T_0) + \lambda_c(T_0). \tag{90}$$

(d) Consider a planetary system for which the period ratio $P_c/P_b = \nu_b/\nu_c$ is close to 2. Show that

$$a_b(t) \simeq \langle a_b\rangle \left\{1 + 2\left(\frac{\nu_b}{\nu_c}\right)\left(\frac{m_c}{m_*}\right)\frac{e_b\, f_1^{(b)}(\alpha)\cos\phi_{212}(t) + e_c\, f_1^{(c)}(\alpha)\cos\phi_{112}(t)}{\nu_b/\nu_c - 2}\right\} \tag{91}$$

$$e_b(t) \simeq \langle e_b\rangle - \left(\frac{\nu_b}{\nu_c}\right)\left(\frac{m_c}{m_*}\right)\frac{f_1^{(b)}(\alpha)\cos\phi_{212}(t)}{\nu_b/\nu_c - 2} \tag{92}$$

and

$$\varpi_b(t) \simeq \varpi_b(T_0) + \left(\frac{\nu_b}{\nu_c}\right)\left(\frac{1}{e_b}\right)\left(\frac{m_c}{m_*}\right)\frac{f_1^{(b)}(\alpha)\left(\sin\phi_{212}(t) - \sin\phi_{212}(T_0)\right)}{\nu_b/\nu_c - 2}, \tag{93}$$

with similar expressions for the outer orbit. Here we have used the average values of $a_b(t)$ and $e_c(t)$ as reference values, but we could equally have written these in terms of the initial values as we have for $\varpi_b(t)$.

The only time-dependent quantities in expressions (8)-(89) are λ_b and λ_c via (100). Integrating thus involves terms such as

$$\begin{aligned}\nu_b \int \sin(n\lambda_b - (n+1)\lambda_c + \varpi_b)\, dt &= \nu_b \int \sin\left[(n\nu_b - (n+1)\nu_c)(t - T_0) + n\lambda_b(T_0) - (n+1)\lambda_c(T_0) + \varpi_b\right] dt \\ &= -\nu_b \cdot \frac{\cos(n\lambda_b - (n+1)\lambda_c + \varpi_b)}{n\nu_b - (n+1)\nu_c} + \text{const} \\ &= -\left(\frac{\nu_b}{\nu_c}\right) \frac{\cos(n\lambda_b - (n+1)\lambda_c + \varpi_b)}{n(\nu_b/\nu_c) - (n+1)} + \text{const}.\end{aligned}$$

For a system near the 2:1 resonance, $P_c/P_b = \nu_b/\nu_c \simeq 2$ so that $n(\nu_b/\nu_c) - (n+1) \simeq 2n - n - 1 = n - 1$. Thus when one integrates (8)-(89), the terms with $n = 1$ involve "small divisors"; the closer the period ratio is to 2, the larger the amplitudes of these terms. Thus the terms which dominate the solutions are associated with $n = 1$, and we can omit all other terms to first approximation.

Note that since the long-term average values of the trig functions are zero, taking our integrations constants as their average values gives consistent solutions. On the other hand, using the inital values as we have in (93) is also consistent since the sine terms cancel for $t = T_0$.

Some points to note are:

- Due to the "small divisor" $\nu_b/\nu_c - 2$, the closer the period ratio is to exact commensurability, the larger the amplitudes of the variations. This causes the two harmonics retained in (8)-(89) to dominate the dynamics: they are the "tall poppies" of the Fourier expansion!
- The resonant period of variation, sometimes called the "super period", is given by

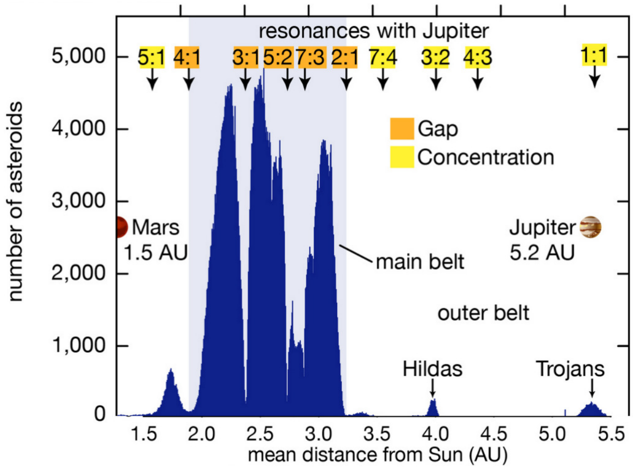
$$P_{\text{res}} = \frac{P_c}{P_c/P_b - 2} \tag{94}$$

- Unlike in the secular case, the semimajor axes variation is not zero. This means that near a commensurability, significant energy can be exchanged between the orbits. This is fundamental to the ability of one of the bodies in an unstable system to escape.
- When a system is close to a low-order commensurability, the associated Fourier angles are called *resonance angles*.
- The assumption that we can hold the elements constant while integrating is not valid for eccentricities very close to zero because of the appearance of e_b in the denominator of (93) (similarly for the outer orbit).

8. Pendulum behaviour of the resonance angles and resonance overlap

In order to demonstrate the pendulum-like behaviour of the resonance angles, we now consider the *circular-restricted three-body problem*. This involves taking one of the planets to be massless, and the eccentricity of the orbit of the other planet to be circular (and fixed, since the massless body is unable to affect the orbit of the massive planet). The problem then becomes the study of how the massless body ("test particle") responds to the presence of the massive body.

Consider the case for which the inner body (body b) is massless. An example of this configuration which has been studied extensively is that of an asteroid on an orbit interior to that of Jupiter's. The following figures shows the distribution of asteroids in the asteroid belt.



The two features of interest to us here involving first-order mean-motion resonances are

(1) The concentration of asteroids near the 3:2 mean-motion resonance (the "Hildas");

(2) The absence of asteroids at the location of the 2:1 mean-motion resonance.

We will start by considering the case of the Hildas located near the 3:2 resonance, taking the eccentricity of Jupiter to be zero (its actual eccentricity is 0.048), and noting that a typical Hilda-to-Jupiter mass ratio is around 10^{-8}. Our aim is to understand why the orbits of the Hildas are stable.

(a) Using Kepler's third law, show that

$$\frac{\dot{\nu}_b}{\nu_b} = -\frac{3}{2}\frac{\dot{a}_b}{a_b}. \tag{95}$$

From Kepler's third law

$$\nu_b^2 = \frac{Gm_*}{a_b^3}$$

we have

$$2\nu_b\dot{\nu}_b = -3\frac{Gm_*}{a_b^4}\dot{a}_b = -3\nu_b^2\frac{\dot{a}_b}{a_b}$$

and the result follows.

(b) By including the 2:1, 3:2 and 4:3 terms in the expansion of $\dot{a}_b/a_b$ (ie, the 3:2 term plus the "neighbouring first-order harmonics") show that to first-order in e_b and leading order in $\alpha = a_b/a_c$,

$$\frac{\dot{\nu}_b}{\nu_b} = -3\nu_b\, e_b\left(\frac{m_c}{m_*}\right)\left[\tfrac{9}{4}\alpha^3\sin\phi_{212} + \tfrac{45}{8}\alpha^4\sin\phi_{323} + \tfrac{315}{32}\alpha^5\sin\phi_{434}\right]. \tag{96}$$

While the smallest divisor corresponds to $n = 2$, the next-smallest divisors correspond to $n = 1$ and $n = 3$. Keeping the $n = 1, 2, 3$ terms in (8) and putting $e_c = 0$, from (90) we have

$$\begin{aligned}\frac{1}{\nu_b}\frac{d\nu_b}{dt} &= +3\nu_b\left(\frac{m_c}{m_*}\right)e_b\left[f_1^{(b)}(\alpha)\sin\phi_{212} + 2f_2^{(b)}(\alpha)\sin\phi_{323} + 3f_3^{(b)}(\alpha)\sin\phi_{434}\right]\\ &\simeq 3\nu_b\left(\frac{m_c}{m_*}\right)e_b\left[(-9/4)\alpha^3\sin\phi_{212} + 2(-45/16)\alpha^4\sin\phi_{323} + 3(-105/32)\alpha^5\sin\phi_{434}\right],\end{aligned}$$

where we have used (80) and (82) for $f_1^{(b)}$ and $f_2^{(b)}$, and from (76), (74) and notes below (36), $f_3^{(b)} = c_{44}^2 x_3^{44}\alpha^5 = -105/32$.

(c) Recalling that $\dot{\lambda}_b = \nu_b$ and ignoring the variation of ϖ_b, show that

$$\ddot{\phi}_{323} + \omega_0^2 \sin\phi_{323} = -6\nu_b^2\, e_b \left(\frac{m_c}{m_*}\right) \left[\tfrac{9}{4}\alpha^3 \sin\phi_{212} + \tfrac{315}{32}\alpha^5 \sin\phi_{434}\right], \tag{97}$$

where the small-angle libration frequency is such that

$$\omega_0^2 = \tfrac{135}{4} e_b (m_c/m_*)\alpha^4 \nu_b^2. \tag{98}$$

Starting with

$$\phi_{323} = 2\lambda_b - 3\lambda_c + \varpi_b,$$

we have

$$\ddot{\phi}_{323} = 2\dot{\nu}_b - 3\dot{\nu}_c + \ddot{\varpi}_b \simeq 2\dot{\nu}_b - 3\dot{\nu}_c$$

which involves $\dot{\nu}_c$ as well as $\dot{\nu}_b$. But from (87) we see that $\dot{\nu}_b$ involves the Hilda-to-Sun mass ratio, which from above is typically $10^{-8} \times 10^{-3} = 10^{-11}$ and so (not surprisingly) the variation of ν_c can be ignored. Thus

$$\ddot{\phi}_{323} = 2\dot{\nu}_b = 2\nu_b(\dot{\nu}_b/\nu_b) = -6\nu_b^2\, e_b \left(\frac{m_c}{m_*}\right) \left[\tfrac{9}{4}\alpha^3 \sin\phi_{212} + \tfrac{45}{8}\alpha^4 \sin\phi_{323} + \tfrac{315}{32}\alpha^5 \sin\phi_{434}\right]$$

and bringing the term involving ϕ_{323} to the left-hand side we have

$$\ddot{\phi}_{323} + \left[6\nu_b^2\, e_b (\tfrac{45}{8})\alpha^4\right] \sin\phi_{323} = -6\nu_b^2\, e_b \left(\frac{m_c}{m_*}\right) \left[\tfrac{9}{4}\alpha^3 \sin\phi_{212} + \tfrac{315}{32}\alpha^5 \sin\phi_{434}\right]$$

and the result follows.

As discussed in lectures, the width of the libration zone of a resonance can be estimated using the formula for the separatrix of a pendulum, giving $\dot{\phi}_{\rm max,min} = \pm 2\omega_0$. The libration width can alternatively be expressed in terms of the "distance to commensurability" $\Delta\sigma$, defined for a system near the $n+1:n$ commensurability to be such that

$$\dot{\phi}_n \simeq n\nu_b - (n+1)\nu_c = n\nu_c(\nu_b/\nu_c - (n+1)/n) \equiv n\nu_c\Delta\sigma, \tag{99}$$

where we have used the shorthand $\dot{\phi}_n$ to stand for $\dot{\phi}_{n+1\,n\,n+1}$ and $\dot{\phi}_{n\,n\,n+1}$ (they are both the same if we ignore the variation of ϖ_b and ϖ_c).

(d) Show that to leading-order in α and e_b, the width of the 3:2 resonance is

$$\Delta\sigma_{3:2} = \tfrac{3}{2}\sqrt{15 e_b (m_c/m_*)\alpha}. \tag{100}$$

From (99) and $\dot{\phi}_{\rm max} = 2\omega_0$ we have

$$\Delta\sigma_{3:2} = \frac{\dot{\phi}_2^{\rm max}}{n\nu_c} = \frac{2\omega_0}{2\nu_c} = \left[\tfrac{135}{4} e_b (m_c/m_*)\alpha^4 (\nu_b/\nu_c)^2\right]^{1/2} = \left[\tfrac{135}{4} e_b (m_c/m_*)\alpha\right]^{1/2}.$$

(e) The mass ratio of Jupiter to the Sun is 0.001, while $\alpha = (3/2)^{-2/3} \simeq 0.763$. Calculate the width of the 3:2 resonance at the Hildas. Take the representative eccentricity to be 0.06 (see table below).

Substituting these values into (100) we get $\Delta\sigma_{3:2} = 0.039$.

(f) Jupiter's period is 11.86 yr. Calculate the approximate range of semimajor axes for which the Hildas can librate according to the simple theory developed here. Compare this to the values in the following table, where "`Epm`" is the eccentricity, and "`M(a)`" is the mean semimajor axis.[9]

The range of period ratios is

$$\frac{3}{2} \pm \Delta\sigma_{3:2} = 1.46 \to 1.54.$$

The range of periods that the Hildas can librate is therefore

$$11.86/1.54 \to 11.86/1.46 \text{ yr} = 7.71 \to 8.12 \text{ yr},$$

with 7.91 yr corresponding to exact commensurability. Thus the range of semimajor axes is

$$7.71^{2/3} \to 8.12^{2/3} = 3.90 \to 4.04 \text{ AU},$$

with 3.97 AU corresponding to exact commensurability.

Asteroid	Epm	M(a)	e_eq
(1256)	0.024	3.90	
(4196)	.025	3.90	
(8721)	0.022	3.89	0.026
(78867)	.024	3.89	
(29574)	.025	3.90	0.030
(7394)	.034	3.92	0.042
(17397)	.040	3.93	0.053
(51865)	.040	3.93	
(32542)	.051	3.94	0.072
(41351)	.059	3.94	
(15417)	.069	3.95	0.111
(10889)	.074	3.95	
(90704)	.074	3.95	
(83900)	.076	3.95	
(47941)	.080	3.95	
(55347)	.082	3.95	
(15626)	.083	3.95	
(11750)	.087	3.95	
(89903)	.088	3.95	
(62959)	.089	3.96	0.245

What do you notice about the true distribution compared to the theoretical distribution? *Hint*: What is the semimajor axis corresponding to exact commensurability?

All the Hildas in the list have semimajor axes *less* than exact commensurability, and hence have period ratios P_J/P_H *greater* than exact commensurability. This arrangement is expected when migration is involved at the time of planet formation (in this case, the migration of Jupiter), is also a feature of exoplanet systems near first-order resonances, as well as Jupiter's moons Io, Europa and Ganymede which participate in the so-called *Laplace resonance* such that $P_E/P_I = 2.007$ and $P_G/P_E = 2.015$.

The quantity "`e_eq`" in the table above is the "forced" or "equilibrium" eccentricity which results from the balance between forcing from Jupiter and damping from collisions. An estimate for this is $(m_c/m_*)\sqrt{\alpha} f_2^{(b)}(\alpha)/(2\Delta\sigma)$, where $\Delta\sigma$ is calculated from `M(a)`.

(g) What do you notice about both the measured and equilibrium eccentricities of the Hildas?

The measured and equilibrium eccentricities are similar for Hildas which are furthest from exact commensurability, but diverge as $\Delta\sigma$ becomes smaller. The reason for this is beyond the scope of this course.

[9] Taken from `http://www.rzuser.uni-heidelberg.de/~s24/hilda.htm`.

The strength of the forcing from neighbouring harmonics can be characterized by calculating the ratio of their forcing amplitudes to ω_0^2.

(h) Using (97), what are the scaled forcing amplitudes of the 2:1 and 4:3 harmonics?

Here we have in mind the analysis of Section 3.3.2 "Linear Versus Non-Linear Resonance" (especially Fig. 3.5) in the lecture notes "Resonance, Chaos and Stability: The Three-Body Problem in Astrophysics".

$$\frac{A_{2:1}}{\omega_0^2} \equiv \overline{A}_{2:1} = \frac{6\nu_b^2\, e_b(m_c/m_*)\frac{9}{4}\alpha^3}{\frac{135}{4}e_b(m_c/m_*)\alpha^4\nu_b^2} = \frac{2/5}{\alpha} \simeq 0.524$$

and

$$\frac{A_{4:3}}{\omega_0^2} \equiv \overline{A}_{4:3} = \frac{6\nu_b^2\, e_b(m_c/m_*)\frac{315}{32}\alpha^5}{\frac{135}{4}e_b(m_c/m_*)\alpha^4\nu_b^2} = (7/4)\alpha \simeq 1.336,$$

where $\alpha = (3/2)^{-2/3}$.

(i) Comment on their dependence on the eccentricities of the Hildas and the mass of Jupiter.

The scaled forcing amplitudes are independent of the eccentricities of the Hildas and the mass of Jupiter.

(j) What are the corresponding forcing frequencies in units of ω_0?

The frequency associated with the 2:1 harmonic in units of ω_0 is

$$\frac{\dot\phi_{212}}{\omega_0} = \frac{\nu_b - 2\nu_c}{\nu_b\sqrt{\frac{135}{4}e_b(m_c/m_*)\alpha^4}} = \frac{1-2/\sigma}{\alpha^2\sqrt{\frac{135}{4}e_b(m_c/m_*)}} = -12.72,$$

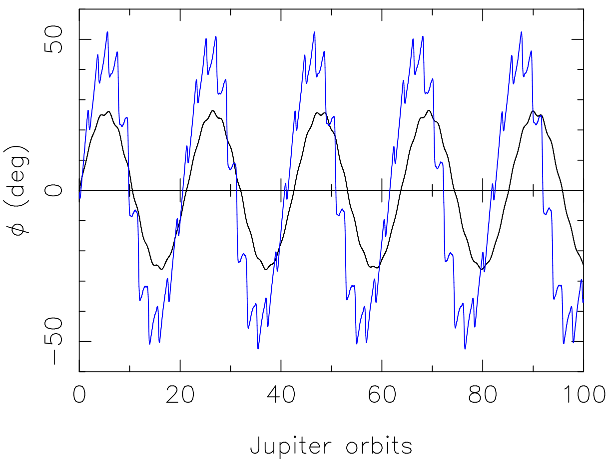
where $\sigma = 3/2$ is the period ratio at exact commensurability (departures from this make little difference to the value of the forcing frequency), while that associated with the 4:3 harmonic is

$$\frac{\dot\phi_{434}}{\omega_0} = \frac{3\nu_b - 4\nu_c}{\nu_b\sqrt{\frac{135}{4}e_b(m_c/m_*)\alpha^4}} = \frac{3-4/\sigma}{\alpha^2\sqrt{\frac{135}{4}e_b(m_c/m_*)}} = 12.72.$$

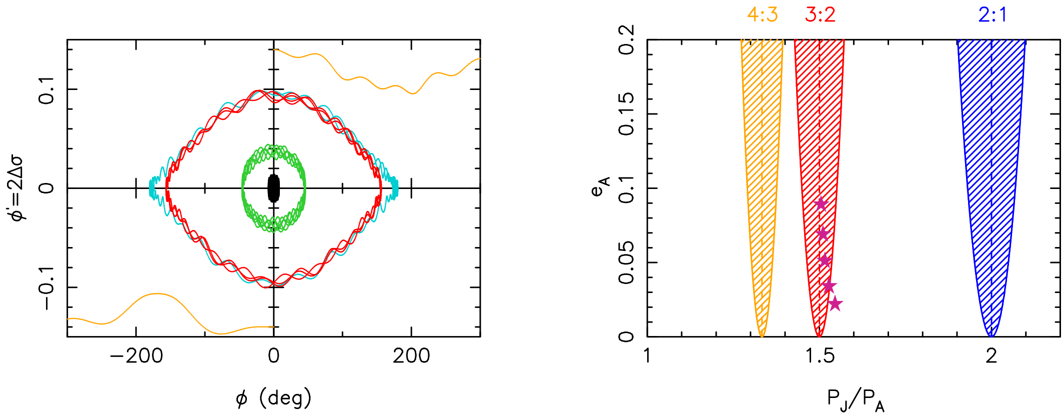
(k) What do you conclude about the orbital stability of the Hildas?

While the forcing amplitudes are quite high, the forcing frequencies are far from the "small-angle frequency" ω. Thus one would expect the Hildas to be stable. Note, however, the answer to Question (l) below - the mass of Jupiter needs to be increased by a factor of around 3 for "resonance overlap" to occur. This suggests that a scaled forcing frequency of around 7 is sufficient to produce chaos, given the rather large amplitudes of the forcing harmonics.

The following figure compares the solution to (97) with $e_b = 0.088$, $\phi(0) = 0$ and $\dot\phi(0)/\nu_c \equiv \phi'(0) = 2\Delta\sigma = 2 \times 0.012$ (black curve), with a direct 3-body integration with $m_b = 0$, $m_c/m_* = 0.001$, $\lambda_b(0) = \lambda_c(0) = \varpi_b(0) = \varpi_c(0) = 0$, $a_b(0) = 3.95$ AU, $a_c = 5.204$ AU (so that $P_c/P_b = 1.512$), $e_b(0) = 0.088$ and $e_c = 0$ (blue curve). The libration frequency is slightly underestimated and the libration amplitude and the effect of the forcing terms more so, nonetheless it is quite remarkable how well the simple forced-pendulum model captures the dynamics of this resonant system.



The left panel of the following figure shows a series of solutions to the differential equation (97).



The initial conditions are $\phi(0) = 0$ and

(i) $\phi'(0) = 2 \times 0.0047$ (black: librating)

(ii) $\phi'(0) = 0.02$ (green: librating);

(iii) $\phi'(0) = 2 \times 0.04891$ (cyan: this is very close to the actual separatrix);

(iv) $\phi'(0) = 2 \times 0.0479$ (red: this is the theoretical separatrix);

(v) $\phi'(0) = 0.14$ (orange: circulating forwards);

(vi) $\phi'(0) = -0.14$ (orange: circulating backwards).

Things to note:

- The effect of the two forcing harmonics on the solutions is evident (compare to Figure 3.2 in lecture notes);
- The time spent near $\pm 180°$ by the solution very close to the separatrix (the solution is bunched up near those points);
- The circulating solutions for $\phi'(0) = \pm 0.07$ are not symmetric (the forcing frequencies are different);
- The analytical estimate of the initial condition for the separatrix is slightly too small - the forcing changes this slightly.

The right panel of the figure above shows the resonance widths as a function of the asteroid eccentricities (with the latter on the $y-$axis). Also shown are the positions of the Hildas listed in the table above (pink stars). The resonance overlap stability criterion of Chirikov[10] states that the overlap of neighbouring resonances is a good predictor of chaos. The existence of the Hildas is consistent with no overlap of neighbouring first-order resonances.

While the resonance overlap stability criterion is a *heuristic* stability indicator, we can see from the present analysis why it works. Recall from the lecture notes that there are two factors which influence the stability of the forced oscillator:

(i) The forcing amplitude, and

(ii) The forcing frequency.

When a system is near the position of the separatrices of two neighbouring resonances (ie, mid-way between the two commensurabilities), the libration frequency and the forcing frequency are similar, as is the forcing amplitude in units of the square of the libration frequency: these two conditions result in chaos in the forced pendulum, forcing otherwise librating solutions to alternate randomly between circulation and libration. This is the situation for weakly interacting systems near the stability boundary. Alternatively, if the forcing frequency is not so close to the pendulum frequency but the forcing amplitude is large enough, chaotic behaviour is also possible; this tends to be the situation in strongly interacting systems (ie far from the stability boundary).

(1) By insisting that the resonance width be half way between the 4:3 and 3:2 resonances, estimate how much the perturber mass (ie, Jupiter's mass) needs to be increased to obtain an unstable solution. Take $e_b = 0.089$.
Answer: $m_c/m_* = 0.003$.
The mid-point between 4/3 and 3/2 is 4/3+1/12. From (100) we therefore need to have

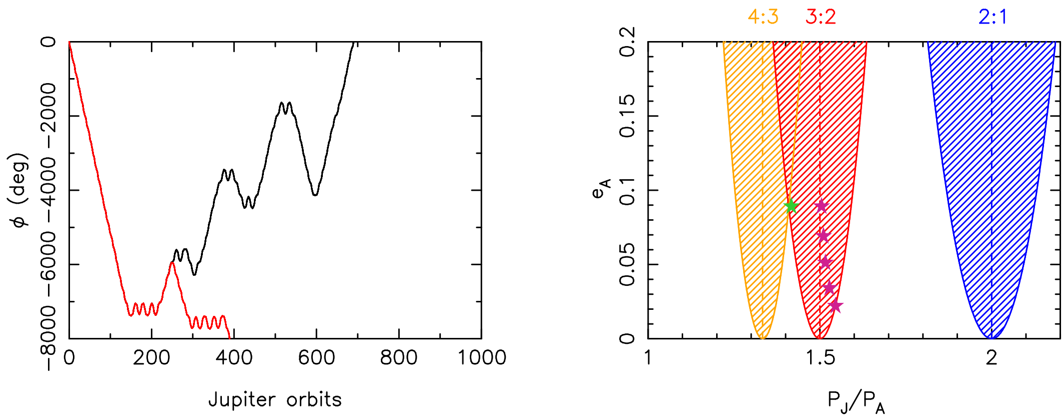
$$\tfrac{3}{2}\sqrt{15e_b(m_c/m_*)\alpha} = 1/12$$

so that for $e_b = 0.089$, overlap occurs when

$$m_c \simeq 0.003.$$

The right panel of the following figure shows the resonances widths with $m_c = 0.0037$ - slightly higher than the predicted value above. We find that below this value the angle ϕ circulates and the solutions do not diverge. Chaos is predicted to occur where the 4:3 and 3:2 resonances overlap. The left-hand panel shows the solution at the position of the green star (black curve), as well as a solution with $\phi(0)$ different by 10^{-7}. The solutions follow each other for a while, then diverge when the black solution continues to circulate backwards while the red solution starts to circulate forwards.

[10] Chirikov B. V., 1979, Physical Reviews, 52, 263.



(m) What are the corresponding values of the forcing amplitude (in units of ω_0^2) and forcing frequency (in units of ω_0) from the 4:3 harmonic?
Answer: 1.34 and 6.03 respectively.

Since the scaled forcing amplitude is independent of the perturber mass (Question (i) above), it is the same as before - 1.336.

From question (j) the forcing frequency is a factor $1/\sqrt{(0.089/0.06) \times 3}$ lower and so is $12.72/2.11 = 6.03$.

Asteroids near the 2:1 resonance

Clearly the theory does not predict the lack of asteroids close to the 2:1 resonance (both figures above shows that the 2:1 and 3:2 resonances are well separated). What is missing? Many ingredients go into a thorough study of the Jupiter-asteroid belt system.[11] For example:

- The non-zero eccentricity of Jupiter;
- The Jupiter's apsidal advance due to Saturn;
- The motion of Jupiter and Saturn at the formation stage;
- Apsidal advance of the asteroid orbits and the associated variation of their eccentricities.

. It turns out that several "secular resonances" are responsible for the depletion of the asteroids short of the 2:1 resonance. For example, the near 1:1 commensurability between the rate of apsidal advance of the asteroids in this vicinity and Jupiter's rate of apsidal advance due to Saturn excites the asteroid eccentricities to such an extent that they cross the orbit of Mars and collide or escape.

Summary

Resonance was responsible for sculpting the Solar System at the time of formation, and continues to influence its structure as (for example) the moons of Jupiter move deeper into resonance, planetary rings with their resonant structure come in and out of existence, and debris (in the form of comets and asteroids) collide with the planets or are ejected from the system. Resonance protects and destroys; for example, while Pluto's orbit crosses the orbit of Neptune, the two bodies never collide thanks to the protective libration of the 3:2 resonance angle ϕ_{223} (they are never in the same place at the same time),[12] while

[11] See, for example, Clement et al 2019, *Excitation and Depletion of the Asteroid Belt in the Early Instability Scenario*, The Astronomical Journal, 157, 38, and Morbidelli & Moons 1993,

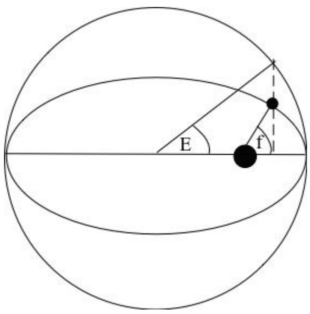
[12] Actually, Pluto's orbit is inclined to Neptunes by some 17° so that precession of Pluto's orbit relative to Neptune's plays and important role in the system's stability.

in contrast, the formation of the planets in the Solar System relied on unstable neighbouring orbits (the word "build" rather than "destroy" may therefore be more appropriate here).

Whenever some combination of angles librates, and some combination of frequencies is similar to the libration frequency, a weakly interacting system is susceptible to instability (as long as the forcing amplitude is sufficient). For example, shortly after the formation of the Earth-Moon system when the orbital period was only a few hours and the semimajor axis was only a few Earth radii, the apsidal motion period of the system was around 1 year, that is, equal to its orbital period around the Sun. This *secular-mean motion resonance* caused the eccentricity to increase to high values (giving it the special name "evection resonance"), bringing the Earth-Moon system dangerously close at periastron (because the semimajor axis was not affected), and the associated tidal heating is likely to have affected the internal structure of both bodies.[13] In the end it was a competition of tidal and apsidal timescales which prevented an ultimate collision, as the system passed through the evection resonance and settled at a small eccentricity. The latter persists today as the Earth is pushed towards the spin-synchronous state (which will occur when one Earth "day" equals one lunar "month" at 60-something current days).

It is clear from the multitude of discoveries in the last 25 years that resonance plays a fundamental role in sculpting exoplanetary systems, resulting in systems which share similarities with the Solar System as well as many big differences. An understanding of how it operates is one of the keys to understanding the structure of our Universe.

Appendix: The eccentric anomaly



The eccentric anomoly is an alternative to the true anomoly f and the mean anomoly M and is useful for

[13]See Figure 10 of Mardling & Lin 2002, The Astrophysical Journal, 573, 829.

performing orbit averaging integrals. It is related to these angles as follows (see Murray & Dermott p33):

$$
\begin{aligned}
r &= \frac{a(1-e^2)}{1+e\cos f} = a(1-e\cos E) \\
\cos f &= \frac{\cos E - e}{1-e\cos E} \\
\sin f &= \frac{\sqrt{1-e^2}\sin E}{1-e\cos E} \\
\frac{dE}{dt} &= \frac{\nu}{1-e\cos E} \\
M &= E - e\sin E
\end{aligned}
$$