deriving the dispersion relation for waves in a dust and gas mixture,

assuming small Perturbations of
$$\rho_d = \rho_d^0 + \delta\rho_d$$
$$\rho_g = \rho_g^0 + \delta\rho_g$$
$$\vec{V}_g = \delta\vec{V}_g$$
$$\vec{V}_d = \delta\vec{V}_d \quad \text{and}$$
$$\delta P_g = C_{s0}^2 \delta\rho_g$$

we can linearise our two Fluid equations too

$$① \quad \frac{\partial \delta\rho_g}{\partial t} + \nabla\cdot(\rho_g^0 \delta\vec{V}_g) = 0,$$

$$② \quad \frac{\partial \delta\rho_d}{\partial t} + \nabla\cdot(\rho_d^0 \delta\vec{V}_d) = 0,$$

$$③ \quad \rho_g^0 \frac{\partial \delta\vec{V}_g}{\partial t} = -C_{s0}^2 \nabla\delta\rho_g + K[\delta\vec{V}_d - \delta\vec{V}_g].$$

$$④ \quad \rho_d^0 \frac{\partial \delta\vec{V}_d}{\partial t} = -K[\delta\vec{V}_d - \delta\vec{V}_g]$$

~~[illegible]~~ where we ignore 2nd order terms, due to small Perturbations also drop the δ for $\vec{V}_d, \vec{V}_g$ for simplicity.

Taking the divergence of ③ and ④ to get,

$$\rho_g^0 \frac{\partial}{\partial t}(\nabla\cdot\vec{V}_g) = -C_{s0}^2 \nabla^2 \delta\rho_g + K[\nabla\cdot\vec{V}_d - \nabla\cdot\vec{V}_g] \quad ⑤$$

$$\rho_d^0 \frac{\partial}{\partial t}(\nabla\cdot\vec{V}_d) = -K[\nabla\cdot\vec{V}_d - \nabla\cdot\vec{V}_g] \quad ⑥,$$

add ⑤ + ⑥ to get,

$$\rho_g^0 \frac{\partial}{\partial t}(\nabla\cdot\vec{V}_g) + \rho_d^0 \frac{\partial}{\partial t}(\nabla\cdot\vec{V}_d) = -C_{s0}^2 \nabla^2 \delta\rho_g \quad ⑦.$$

now, $\frac{\partial}{\partial t}$ ① and $\frac{\partial}{\partial t}$ ② are,

$$-\frac{1}{\rho_g^0}\frac{\partial^2 \delta\rho_g}{\partial t^2} = \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{v}_g) \quad , \quad \frac{-1}{\rho_d^0}\frac{\partial^2 \delta\rho_d}{\partial t^2} = \frac{\partial}{\partial t}(\nabla\cdot\vec{v}_d)$$

sub into get,

$$\frac{\partial^2 \delta\rho_g}{\partial t^2} + \frac{\partial^2 \delta\rho_d}{\partial t^2} = + C_{s0}^2 \nabla^2 \delta\rho_g \qquad ⑧$$

⑥ - ⑤ gives

$$⑨ \quad \rho_d^0 \frac{\partial}{\partial t}\nabla\cdot\vec{v}_d - \rho_g^0\frac{\partial}{\partial t}\nabla\cdot\vec{v}_g = -2K\left[\nabla\cdot\vec{v}_d - \nabla\cdot\vec{v}_g\right] + C_{s0}^2\nabla^2\delta\rho_g.$$

and. from ① and ②

$$\nabla\cdot\vec{v}_d = -\frac{1}{\rho_d^0}\frac{\partial}{\partial t}\delta\rho_d$$

$$\nabla\cdot\vec{v}_g = -\frac{1}{\rho_g^0}\frac{\partial}{\partial t}\delta\rho_g$$

sub this add in to ⑨ to get.

$$\therefore -\frac{\partial^2\delta\rho_d}{\partial t^2} + \frac{\partial^2\delta\rho_g}{\partial t^2} = -2K\left[-\frac{1}{\rho_d^0}\frac{\partial}{\partial t}\delta\rho_d + \frac{1}{\rho_g^0}\frac{\partial}{\partial t}\delta\rho_g\right] + C_{s0}^2\nabla^2\delta\rho_g.$$

such that,

$$\frac{\partial^2\delta\rho_d}{\partial t^2} = \frac{\partial^2\delta\rho_g}{\partial t} + 2K\left[\frac{1}{\rho_g^0}\frac{\partial}{\partial t}\delta\rho_g - \frac{1}{\rho_d^0}\frac{\partial}{\partial t}\delta\rho_d\right] - C_{s0}^2\nabla^2\delta\rho_g,$$

sub into ⑧,

$$2\frac{\partial^2\rho_g}{\partial t^2} + 2K\left[\frac{1}{\rho_g^0}\frac{\partial}{\partial t}\delta\rho_g - \frac{1}{\rho_d^0}\frac{\partial}{\partial t}\delta\rho_d\right] - 2C_{s0}^2\nabla^2\delta\rho_g = 0.$$

expand and divide by 2.

$$\therefore \frac{\partial^2\rho_g}{\partial t^2} + \frac{K}{\rho_g^0}\frac{\partial}{\partial t}\delta\rho_g - \frac{K}{\rho_d^0}\frac{\partial}{\partial t}\delta\rho_d - C_{s0}^2\nabla^2\delta\rho_g = 0. \qquad ⑩$$

$\frac{\partial}{\partial t}$ (10) gives

$$\frac{\partial^3 \delta\rho_g}{\partial t^3} + \frac{K}{\rho_g^0}\frac{\partial^2}{\partial t^2}\delta\rho_g - \frac{K}{\rho_d^0}\frac{\partial^2}{\partial t}\delta\rho_d - C_{s0}^2\frac{\partial}{\partial t}\nabla^2\rho_g = 0.$$

Sub in (8) to get,

$$\frac{\partial^3\rho_g}{\partial t^3} + \frac{K}{\rho_g^0}\frac{\partial^2}{\partial t^2}\delta\rho_g - \frac{K}{\rho_d^0}\left(C_{s0}^2\nabla^2\delta\rho_g - \frac{\partial^2\delta\rho_g}{\partial t^2}\right) - C_{s0}^2\frac{\partial}{\partial t}\nabla^2\rho_g = 0$$

$$\Rightarrow \frac{\partial^3\delta\rho_g}{\partial t^3} + K\left(\frac{1}{\rho_g^0} + \frac{1}{\rho_d^0}\right)\frac{\partial^2\delta\rho_g}{\partial t^2} - \frac{K}{\rho_d^0}C_{s0}^2\nabla^2\delta\rho_g - C_{s0}^2\frac{\partial}{\partial t}\nabla^2\delta\rho_g = 0 \quad (11)$$

Where $K\left(\frac{1}{\rho_g^0}+\frac{1}{\rho_d^0}\right) = K\left(\frac{\rho_g^0+\rho_d^0}{\rho_g^0\rho_d^0}\right) \equiv \frac{1}{t_s}$

assume purturbations of the form.

$\delta\rho_g = De^{i[\vec{k}\cdot\vec{x} - \omega t]}$ such that,

$\frac{\partial^2\delta\rho_g}{\partial t} = -\omega^2\delta\rho_g$, and $\frac{\partial^3}{\partial t^3}\delta\rho_g = i\omega^3\delta\rho_g$.

$\nabla^2\delta\rho_g = -k^2\delta\rho_g$.

so

$\frac{\partial}{\partial t}(\nabla^2\delta\rho_g) = i\omega k^2\delta\rho_g$.

Sub into (11) to get, and divide by $\delta\rho_g$.

$$i\omega^3 + \frac{1}{t_s}(-\omega^2) - \frac{K}{\rho_d^0}C_{s0}^2(-k^2) - C_{s0}^2(i\omega k^2) = 0.$$

$$= i\omega^3 - \frac{\omega^2}{t_s} + \frac{K}{\rho_d^0}C_{s0}^2k^2 - i\omega C_{s0}^2k^2 = 0$$

now $\quad \dfrac{k}{\rho_d^0} = \dfrac{\rho_g^0 \rho_d^0}{(\rho_g^0 + \rho_d^0)\epsilon_s} \times \dfrac{1}{\rho_d^0} = \dfrac{\rho_g^0}{\rho^0 \epsilon_s}$ where $\rho^0 = \rho_g^0 + \rho_d^0$

so $\quad i\omega^3 - \dfrac{\omega^2}{\epsilon_s} + \dfrac{\rho_g^0}{\rho^0 \epsilon_s} C_{s0}^2 k^2 - i\omega C_{s0}^2 k^2 = 0$

divided by $i\omega$ to get,

$$\omega^2 - \frac{\omega^2}{i\omega\epsilon_s} + \frac{\rho_g}{\rho \epsilon_s i\omega} C_{s0}^2 k^2 - C_{s0}^2 k^2 = 0.$$

$$\Rightarrow (\omega^2 - C_{s0}^2 k^2) - \frac{1}{i\omega\epsilon_s}\left(\omega^2 - \frac{\rho_g}{\rho} C_{s0}^2 k^2\right) = 0$$

Let $\dfrac{\rho_g}{\rho} C_{s0}^2 \equiv \tilde{C}_s^2$ be the modified sound speed.

so that,

$$(\omega^2 - C_{s0}^2 k^2) - \frac{1}{i\omega\epsilon_s}\left(\omega^2 - \tilde{C}_s^2 k^2\right) = 0$$