Computational Astrophysics
Problem sheet 1 — solutions

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1a)

$$\frac{\partial \rho}{\partial t} + (\vec{v}\cdot\nabla)\rho = -\rho\nabla\cdot\vec{v} \qquad -(1)$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{\nabla P}{\rho} - \nabla\Phi \qquad -(2)$$

Assume
$$\Phi = \Phi_0 + \delta\Phi$$
$$\rho = \rho_0 + \delta\rho$$
$$\vec{v} = \vec{v}_0 + \delta\vec{v}$$; assume background state at rest $\vec{v}_0 = 0$
$$\delta P = c_s^2\,\delta\rho$$

(1) gives

$$\therefore \frac{\partial\delta\rho}{\partial t} + \underbrace{(\cancel{\vec{v}_0}^{\,0} + \delta\vec{v})\cdot\nabla\delta\rho}_{\text{2nd order}} = -(\rho_0 + \underbrace{\cancel{\delta\rho}}_{\text{2nd order}})\nabla\cdot(\delta\vec{v})$$

$$\therefore \frac{\partial\delta\rho}{\partial t} = -\rho_0\nabla\cdot(\delta\vec{v}) \qquad -(3)$$

(2) gives

$$(\rho_0 + \underbrace{\cancel{\delta\rho}}_{\text{2nd order}})\left[\frac{\partial\delta\vec{v}}{\partial t} + \underbrace{\cancel{(\delta\vec{v}\cdot\nabla)\delta\vec{v}}}_{\text{2nd order}}\right] = -\nabla(\underbrace{c_{s0}^2}_{\text{other terms are 2nd order}}\delta\rho) - \nabla(\Phi_0 + \delta\Phi)$$

$$\therefore \rho_0\frac{\partial\delta\vec{v}}{\partial t} = -c_{s0}^2\nabla(\delta\rho) - \cancel{\rho_0\nabla\Phi_0}^{\,0} - \rho_0\nabla(\delta\Phi) \qquad -(4)$$

Notice there is an inconsistency here. This is known as the 'Jeans Swindle'. Namely that the initial state cannot satisfy $\Phi_0 = \text{const}$.

Poisson's equation gives

$$\nabla^2(\Phi_0 + \delta\Phi) = 4\pi G(\rho_0 + \delta\rho)$$

$$\nabla^2\Phi_0 + \nabla^2\delta\Phi = 4\pi G\rho_0 + 4\pi G\delta\rho$$

Assume initial state satisfies $\nabla^2\Phi_0 = 4\pi G\rho_0$

then $\nabla^2\delta\Phi = 4\pi G\delta\rho$

Take $\frac{\partial}{\partial t}(3)$:

$$\frac{\partial^2\delta\rho}{\partial t^2} = -\rho_0\frac{\partial}{\partial t}\left[\nabla\cdot(\delta\vec{v})\right] \qquad -(5)$$

Take $\nabla\cdot(4)$:

$$\rho_0\frac{\partial}{\partial t}\left[\nabla\cdot\delta\vec{v}\right] = -c_{s0}^2\nabla^2\delta\rho - \cancel{\rho_0\nabla^2\Phi_0}^{\,0} - \rho_0\nabla^2\delta\Phi \qquad -(6)$$

1a) cont...

Combining (5) and (6) we have

$$\frac{\partial^2 \delta\rho}{\partial t^2} = c_{s0}^2 \nabla^2 \delta\rho + 4\pi G \rho_0 \delta\rho \qquad -(7)$$

Assume $\delta\rho = D e^{i(\vec{k}\cdot\vec{x} - \omega t)}$

$$\therefore \frac{\partial \delta\rho}{\partial t} = -i\omega D e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

$$\therefore \frac{\partial^2 \delta\rho}{\partial t^2} = i^2\omega^2 D e^{i(\vec{k}\cdot\vec{x} - \omega t)} = -\omega^2 \delta\rho$$

Similarly

$$\nabla(\delta\rho) = i\vec{k} D e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

$$\nabla^2(\delta\rho) \equiv \nabla\cdot\nabla(\delta\rho) = -k^2 \delta\rho \qquad (k^2 = \vec{k}\cdot\vec{k})$$

$\therefore$ (7) becomes

$$-\omega^2 \cancel{\delta\rho} = -c_{s0}^2 k^2 \cancel{\delta\rho} + 4\pi G \rho_0 \cancel{\delta\rho}$$

$$\therefore \omega^2 = c_s^2 k^2 - 4\pi G\rho_0$$

$$\therefore \omega^2 = c_s^2\left(k^2 - \frac{4\pi G\rho_0}{c_s^2}\right)$$

$$\boxed{\omega^2 = c_s^2\left(k^2 - k_J^2\right)} \quad ; k_J^2 \equiv \frac{4\pi G\rho_0}{c_s^2}$$

QED

1b) ω is imaginary when $k_J > k$.

$\therefore$ Critical wavenumber is k_J.

1c) $\lambda_J = \frac{2\pi}{k_J} = \sqrt{\frac{\cancel{4}\pi^{\cancel{2}} c_s^2}{\cancel{4\pi} G\rho_0}} = \sqrt{\frac{\pi c_s^2}{G\rho_0}}$ This is the JEANS LENGTH

1d) Maximum growth when $|\omega|$ is maximum. Occurs when $\frac{\partial \omega}{\partial k} = 0$

$$\frac{\partial \omega}{\partial k} = 2c_s^2 k = 0$$

$\therefore$ Maximum growth for $k = 0$, ie. $\lambda \to \infty$

Jeans instability grows fastest at LARGE SCALES

2. i) $$\frac{d\rho}{dt} = -\rho \nabla\cdot\vec{v} \quad - (*)$$

$$\therefore \frac{\partial \rho}{\partial t} + (\vec{v}\cdot\nabla)\rho + \rho\nabla\cdot\vec{v} = 0 \quad -(1)$$

Use identity $\nabla\cdot(\rho\vec{v}) = (\vec{v}\cdot\nabla)\rho + \rho\nabla\cdot\vec{v}$

(1) becomes

$$\therefore \boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0}$$

ii) $$\frac{d\vec{v}}{dt} = -\frac{\nabla P}{\rho}$$

Use $$\frac{d}{dt}(\rho\vec{v}) = \rho\frac{d\vec{v}}{dt} + \vec{v}\frac{d\rho}{dt}$$ ← insert (*)

$$\therefore \frac{d\vec{v}}{dt} = \frac{1}{\rho}\frac{d}{dt}(\rho\vec{v}) + \frac{\vec{v}\cancel{\rho}\nabla\cdot\vec{v}}{\cancel{\rho}}$$

$$\therefore \frac{1}{\rho}\left[\frac{\partial}{\partial t}(\rho\vec{v}) + \vec{v}\cdot\nabla(\rho\vec{v})\right] + \vec{v}\nabla\cdot\vec{v} + \frac{\nabla P}{\rho} = 0$$

$$\therefore \frac{\partial}{\partial t}(\rho\vec{v}) + (\vec{v}\cdot\nabla)(\rho\vec{v}) + \rho\vec{v}\nabla\cdot\vec{v} + \nabla P = 0$$

Now use
$$\nabla\cdot(\rho\vec{v}\vec{v}) = (\vec{v}\cdot\nabla)(\rho\vec{v}) + \rho\vec{v}\nabla\cdot\vec{v}$$

$$\therefore \frac{\partial}{\partial t}(\rho\vec{v}) + \nabla\cdot(\rho\vec{v}\vec{v} + P\bar{\bar{I}}) = 0$$

↑ this is the identity matrix

ie. $\frac{\partial}{\partial x_j}(P\delta_{ij}) = \frac{\partial P}{\partial x_i}$

or $\nabla\cdot(P\bar{\bar{I}}) = \nabla P$

2 iii) $$\frac{du}{dt} = -\frac{P}{\rho}\nabla\cdot\vec{v}$$

$$\rho e = \tfrac{1}{2}\rho v^2 + \rho u = \rho\left(\tfrac{1}{2}v^2 + u\right)$$

$$\therefore \frac{d}{dt}(\rho e) = e\frac{d\rho}{dt} + \rho\,\vec{v}\cdot\frac{d\vec{v}}{dt} + \rho\frac{du}{dt}$$

$$= -e\rho\nabla\cdot\vec{v} - \vec{v}\cdot\nabla P - P\nabla\cdot\vec{v}$$

$$\therefore \frac{\partial}{\partial t}(\rho e) + \vec{v}\cdot\nabla(\rho e) + e\rho\,\nabla\cdot\vec{v} + \vec{v}\cdot\nabla P + P\nabla\cdot\vec{v} = 0$$

use $\nabla\cdot(P\vec{v}) = \vec{v}\cdot\nabla P + P\nabla\cdot\vec{v}$

$\nabla\cdot(\rho e\vec{v}) = \vec{v}\cdot\nabla(\rho e) + \rho e\,\nabla\cdot\vec{v}$

$$\therefore \boxed{\frac{\partial}{\partial t}(\rho e) + \nabla\cdot\left[(\rho e + P)\vec{v}\right] = 0}$$

3 a) $$\rho_1 v_1 = \rho_2 v_2 \quad -(1)$$

$$P_1 + \rho_1 v_1^2 = P_2 + \rho_2 v_2^2 \quad -(2)$$

$$\tfrac{1}{2}v_1^2 + \frac{\gamma P_1}{(\gamma-1)\rho_1} = \tfrac{1}{2}v_2^2 + \frac{\gamma P_2}{(\gamma-1)\rho_2} \quad -(3)$$

from $\rho e + P = \rho\tfrac{1}{2}v^2 + \rho u + P$

$= \tfrac{1}{2}\rho v^2 + \frac{\gamma P}{\gamma-1}$

since $P = (\gamma-1)\rho u$

(1) gives $$v_2^2 = \frac{\rho_1^2}{\rho_2^2}v_1^2$$

Recall $c_s^2 = \frac{\gamma P}{\rho}$

(2) gives $$\rho_1 v_1^2 + P_1 = \frac{\rho_1^2}{\rho_2}v_1^2 + P_2$$

$$\therefore P_2 = P_1 + \rho_1 v_1^2\left(1 - \frac{\rho_1}{\rho_2}\right)$$

(3) gives
$$\tfrac{1}{2}v_1^2 + \frac{c_{s1}^2}{\gamma-1} = \frac{\rho_1^2 v_1^2}{2\rho_2^2} + \frac{\gamma}{(\gamma-1)\rho_2}\left[P_1 + \rho_1 v_1^2\left(1-\frac{\rho_1}{\rho_2}\right)\right]$$

$$\therefore \tfrac{1}{2}v_1^2\left(1 - \frac{\rho_1^2}{\rho_2^2}\right) + \frac{c_{s1}^2}{\gamma-1} = \frac{\rho_1}{\rho_2(\gamma-1)}c_{s1}^2 + \frac{\gamma}{\gamma-1}\frac{\rho_1}{\rho_2}v_1^2\left(1-\frac{\rho_1}{\rho_2}\right)$$

3a), cont...

Multiply through by $2(\gamma-1)$ and define $M_1^2 = \dfrac{v_1^2}{c_{s1}^2}$

$$\therefore M_1^2(\gamma-1)\cancel{\left(1-\frac{\rho_1}{\rho_2}\right)}\left(1+\frac{\rho_1}{\rho_2}\right) + 2c_{s1}^2\cancel{\left(1-\frac{\rho_1}{\rho_2}\right)} = 2\gamma\frac{\rho_1}{\rho_2}v_1^2\cancel{\left(1-\frac{\rho_1}{\rho_2}\right)}$$

Divide by c_{s1}^2

$$\therefore M_1^2(\gamma-1)\left(1+\frac{\rho_1}{\rho_2}\right) + 2 = 2\gamma\frac{\rho_1}{\rho_2}M_1^2$$

$$\therefore M_1^2(\gamma-1) + 2 = \frac{\rho_1}{\rho_2}M_1^2\left[2\gamma - (\gamma-1)\right]$$

$$= \frac{\rho_1}{\rho_2}M_1^2(\gamma+1)$$

$$\therefore \boxed{\frac{\rho_2}{\rho_1} = \frac{M_1^2(\gamma+1)}{M_1^2(\gamma-1)+2}}$$

QED.

3b) $$P_2 = P_1 + \rho_1 v_1^2\left(1-\frac{\rho_1}{\rho_2}\right)$$

$$\therefore \frac{P_2}{P_1} = 1 + \frac{\rho_1}{P_1}v_1^2\left(1-\frac{\rho_1}{\rho_2}\right)$$

use $\dfrac{c_{s1}^2}{\gamma} = \dfrac{P_1}{\rho_1}$

$$= 1 + \gamma\underbrace{\frac{v_1^2}{c_{s1}^2}}_{M_1^2}\left[1 - \frac{M_1^2(\gamma-1)+2}{M_1^2(\gamma+1)}\right]$$

$$= 1 + \gamma M_1^2 - \frac{\gamma M_1^2(\gamma-1)+2\gamma}{\gamma+1}$$

$$= \frac{(\gamma+1) + \gamma(\gamma+1)M_1^2 - \gamma M_1^2(\gamma-1) - 2\gamma}{\gamma+1}$$

$$= \frac{-(\gamma-1) + \gamma M_1^2(\cancel{\gamma}+1-\cancel{\gamma}+1)}{\gamma+1}$$

$$\therefore \boxed{\frac{P_2}{P_1} = \frac{2\gamma M_1^2 - (\gamma-1)}{\gamma+1}}$$

QED.

3c) $\frac{T_2}{T_1} = \frac{P_2}{P_1}\frac{\rho_1}{\rho_2}$ use $P = \rho \frac{kT}{\mu m_H}$ $\therefore$ $P_2 = C\rho_2 T_2$, $P_1 = C\rho_1 T_1$

$\therefore \frac{T_2}{T_1} = \frac{P_2}{C\rho_2} \cdot \frac{C\rho_1}{P_1} = \frac{P_2}{P_1}\frac{\rho_1}{\rho_2}$

$$= \frac{2\gamma M_1^2 - (\gamma - 1)}{\gamma + 1} \cdot \frac{[2 + (\gamma - 1)M_1^2]}{M_1^2(\gamma + 1)}$$

$$\therefore \boxed{\frac{T_2}{T_1} = \frac{[2\gamma M_1^2 - (\gamma - 1)][2 + (\gamma - 1)M_1^2]}{M_1^2(\gamma + 1)^2}}$$

QED.

4) a) $$\sum_i \sum_j \sigma_{ij}(x_i - x_j) = \frac{1}{2}\sum_i \sum_j \sigma_{ij}(x_i - x_j) + \frac{1}{2}\sum_j \sum_i \sigma_{ji}(x_j - x_i)$$

$$= \frac{1}{2}\sum_i \sum_j \sigma_{ij}(x_i - x_j) - \frac{1}{2}\sum_j \sum_i \sigma_{ij}(x_i - x_j)$$

$\because \sigma_{ji} = \sigma_{ij}$

$$= 0$$

For vector quantities the same applies in y and z direction.

e.g. $$\sum_i \sum_j \sigma_{ij}(y_i - y_j) = \frac{1}{2}\sum_i \sum_j \sigma_{ij}(y_i - y_j) - \frac{1}{2}\sum_j \sum_i \sigma_{ji}(y_i - y_j)$$

$$= 0$$

b) Can write kernel gradient $\nabla_i W_{ij} = (\vec{r}_i - \vec{r}_j)F_{ij}$

$\therefore$ Linear momentum:

i) $$\frac{d}{dt}\sum_i m_i \vec{v}_i = 0$$

$$\therefore \sum_i m_i \frac{d\vec{v}_i}{dt} = 0$$

$$\therefore -\sum_i \sum_j m_i m_j \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2}\right)(\vec{r}_i - \vec{r}_j)F_{ij}$$

Define $\sigma_{ij} = m_i m_j \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2}\right)F_{ij}$; then identity above applies

4b) cont...

and we have

$$\sum_i \sum_j \sigma_{ij} (\vec{r}_i - \vec{r}_j) = 0 \qquad \text{since } \sigma_{ij} = \sigma_{ji}.$$

$\therefore$ Linear momentum is conserved.

ii) Angular momentum

$$\vec{L} = \sum_i m_i \vec{r}_i \times \vec{v}_i \qquad \text{use } \frac{d\vec{r}_i}{dt} = \vec{v}_i$$

$$\therefore \frac{d\vec{L}}{dt} = \sum_i m_i \left[\cancel{\vec{v}_i \times \vec{v}_i}^{\,0} + \vec{r}_i \times \frac{d\vec{v}_i}{dt} \right] = 0$$

$\therefore$ we need $\sum_i m_i \vec{r}_i \times \frac{d\vec{v}_i}{dt} = 0$ for AM conservation)

we find

$$\frac{d\vec{L}}{dt} = -\sum_i \sum_j m_i m_j \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2} \right) \vec{r}_i \times (\vec{r}_i - \vec{r}_j) F_{ij}$$

$$= + \sum_i \sum_j \sigma_{ij} (\vec{r}_i \times \vec{r}_j) \qquad \sigma_{ij} \text{ defined as previously.}$$

This is antisymmetric because $(\vec{r}_i \times \vec{r}_j) = -(\vec{r}_j \times \vec{r}_i)$

$\therefore$ Identity holds and AM is conserved.

5) $e = \frac{1}{2}v^2 + u$

$$\therefore \frac{de_i}{dt} = \vec{v}_i \cdot \frac{d\vec{v}_i}{dt} + \frac{du_i}{dt}$$

$$= -\sum_j m_j \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2}\right) \vec{v}_i \cdot \nabla_i W_{ij} + \sum_j m_j \frac{P_i}{\rho_i^2}(\vec{v}_i - \vec{v}_j) \cdot \nabla_i W_{ij}$$

$$\therefore \frac{de_i}{dt} = -\sum_j m_j \left[\frac{P_i}{\rho_i^2}(\cancel{\vec{v}_i} - \cancel{\vec{v}_i} + \vec{v}_j) + \frac{P_j}{\rho_j^2}\vec{v}_i\right] \cdot \nabla_i W_{ij}$$

$$\therefore \boxed{\frac{de_i}{dt} = -\sum_j m_j \left[\frac{P_i \vec{v}_j}{\rho_i^2} + \frac{P_j \vec{v}_i}{\rho_j^2}\right] \cdot \nabla_i W_{ij}} \quad - (*)$$

QED.

Total energy conservation:

$$\frac{dE}{dt} = \frac{d}{dt}\left(\sum_i m_i e_i\right) = 0$$

$$\therefore \sum_i m_i \frac{de_i}{dt} = 0$$

Using (*) we have.

$$-\sum_i \sum_j \underbrace{m_i m_j \left[\frac{P_i \vec{v}_j}{\rho_i^2} + \frac{P_j \vec{v}_i}{\rho_j^2}\right]}_{\text{symmetric}} \cdot \underbrace{\nabla_i W_{ij}}_{\text{antisymmetric}} = 0$$

$\therefore$ zero because double sum is antisymmetric

$\therefore$ total energy is conserved.

$\Rightarrow$ Since we can relate $\frac{du}{dt}$ and $\frac{de}{dt}$ analytically, it does not matter which one we use in the code $\Rightarrow$ energy will be conserved.

6 a) $\rho_i = \sum_j m_j W_{ij}(h_i)$

$$\frac{d\rho_i}{dt} = \sum_j m_j \frac{d}{dt} W_{ij}\Big|_h + \sum_j m_j \frac{\partial W_{ij}(h_i)}{\partial h_i} \cdot \frac{\partial h_i}{\partial \rho_i}\frac{d\rho_i}{dt}$$

at constant h

$$\therefore \frac{d\rho_i}{dt}\Bigg[\underbrace{1 - \frac{\partial h_i}{\partial \rho_i}\sum_j m_j \frac{\partial W_{ij}(h_i)}{\partial h_i}}_{= \Omega_i}\Bigg] = \sum_j m_j \frac{d}{dt} W_{ij}\Big|_h$$

$$\therefore \frac{d\rho_i}{dt} = \frac{1}{\Omega_i}\sum_j m_j \frac{d}{dt} W_{ij}\Big|_h$$

Consider kernel in the form

$$W_{ab} = \frac{\sigma}{h^\nu} f(q) \qquad q = \frac{|\vec{r}_a - \vec{r}_b|}{h}$$

$$\therefore \frac{d}{dt} W_{ab}\Big|_h = \frac{\sigma}{h^\nu}\frac{\partial f}{\partial q}\frac{dq}{dt}$$

same as $\frac{\partial |x|}{\partial x} = \frac{x}{|x|}$

$$\text{where } \frac{dq}{dt} = \frac{1}{h}\frac{d}{dt}|\vec{r}_a - \vec{r}_b| = \frac{1}{h}\frac{\vec{r}_a - \vec{r}_b}{|\vec{r}_a - \vec{r}_b|} \cdot \frac{d}{dt}(\vec{r}_a - \vec{r}_b)$$

$$= \frac{1}{h}\hat{r}_{ab} \cdot [\vec{v}_a - \vec{v}_b]$$

$$\therefore \frac{d}{dt} W_{ab}\Big|_h = \frac{\sigma}{h^{\nu+1}} f'(q)\, \hat{r}_{ab} \cdot (\vec{v}_a - \vec{v}_b)$$

$$= (\vec{v}_a - \vec{v}_b) \cdot \nabla W_{ab}$$

since $\nabla W_{ab} = \frac{\sigma}{h^{\nu+1}} f'(q)\hat{r}_{ab}$

$$\therefore \boxed{\frac{d\rho_i}{dt} = \frac{1}{\Omega_i}\sum_j m_j (\vec{v}_a - \vec{v}_j) \cdot \nabla_i W_{ij}(h_i)}$$

QED.

6b) $\rho_j = \sum_k m_k \underline{W_{jk}}(h_j)$

$$\frac{\partial \rho_j}{\partial \vec{r}_i} = \sum_k m_k \left.\frac{\partial W_{jk}}{\partial \vec{r}_i}\right|_h (\delta_{ji} - \delta_{ki}) + \sum_k m_k \frac{\partial W_{jk}}{\partial h_j} \frac{\partial h_j}{\partial \rho_j} \frac{\partial \rho_j}{\partial \vec{r}_i}$$

$$\therefore \frac{\partial \rho_j}{\partial \vec{r}_i} \Bigg[\underbrace{1 - \frac{\partial h_j}{\partial \rho_j} \sum_k m_k \frac{\partial W_{jk}(h_j)}{\partial h_j}}_{\Omega_j} \Bigg] = \sum_k m_k \frac{\partial W_{jk}}{\partial \vec{r}_i} (\delta_{ji} - \delta_{ki})$$

$$\therefore \boxed{\frac{\partial \rho_j}{\partial \vec{r}_i} = \frac{1}{\Omega_j} \sum_k m_k \frac{\partial W_{jk}(h_j)}{\partial \vec{r}_i} (\delta_{ji} - \delta_{ki})} \quad \text{QED}$$

c) $L = \sum_j m_j \left(\frac{1}{2} v_j^2 - u_j \right)$

$$\frac{\partial L}{\partial \vec{v}_i} = m_i \vec{v}_i$$

$$\frac{\partial L}{\partial \vec{r}_i} = -\sum_j m_j \frac{\partial u_j}{\partial \vec{r}_i}$$

$$= -\sum_j m_j \frac{\partial u_j}{\partial \rho_j} \frac{\partial \rho_j}{\partial \vec{r}_i}$$

$$= -\sum_j m_j \frac{P_j}{\rho_j^2} \frac{1}{\Omega_j} \sum_k m_k \frac{\partial W_{jk}(h_j)}{\partial \vec{r}_i} (\delta_{ji} - \delta_{ki})$$

$$= -\cancel{\sum_i} m_i \frac{P_i}{\Omega_i \rho_i^2} \sum_k m_k \frac{\partial W_{ik}}{\partial \vec{r}_i} (h_i)$$

change this index to j

$$+ \sum_j m_j \frac{P_j}{\rho_j^2 \Omega_j} m_i \frac{\partial W_{ji}}{\partial \vec{r}_i} (h_j)$$

swap indices

6c) cont...

Change k to j in first sum, swap indices on $\nabla W_{ji} = -\nabla W_{ij}$

$$\therefore -\frac{\partial L}{\partial \vec{r}_i} = -m_i \sum_j m_j \left[\frac{P_i}{\Omega_i \rho_i^2} \frac{\partial W_{ij}}{\partial \vec{r}_i}(h_i) + \frac{P_j}{\Omega_j \rho_j^2} \frac{\partial W_{ij}}{\partial \vec{r}_i}(h_j) \right]$$

$\therefore$ Putting this in Euler Lagrange equations we have

$$\frac{d}{dt}(m_i \vec{v}_i) = \frac{\partial L}{\partial \vec{r}_i} = -m_i \sum_j m_j [\ldots]$$

$$\therefore \boxed{\frac{d\vec{v}_i}{dt} = -\sum_j m_j \left[\frac{P_i}{\Omega_i \rho_i^2} \nabla W_{ij}(h_i) + \frac{P_j}{\Omega_j \rho_j^2} \nabla W_{ij}(h_j) \right]}$$

QED.

7) a)

i) $\sum_j \frac{m_j}{\rho_j} (\vec{v}_j - \vec{v}_i) \cdot \nabla W_{ij}$

$= \nabla \cdot \vec{v} - \vec{v} \cdot \nabla 1$

$= \nabla \cdot \vec{v}$

Can also use

$\frac{1}{\rho_i} \sum_j m_j (\vec{v}_j - \vec{v}_i) \cdot \nabla W_{ij}$

$= \frac{1}{\rho} \nabla \cdot (\rho \vec{v}) - \frac{\vec{v}}{\rho} \cdot \nabla \rho$

$= \nabla \cdot \vec{v} + \frac{\vec{v}}{\rho} \cdot \nabla \rho - \left(\frac{\vec{v}}{\rho} \cdot \nabla \right) \rho$

$= \nabla \cdot \vec{v}$

ii) $\nabla \times \vec{v} = -\sum_j \frac{m_j}{\rho_j} \vec{v}_j \times \nabla W_{ij}$

Subtract $\vec{v}_i$ inside sum, get

$= -\sum_j \frac{m_j}{\rho_j} (\vec{v}_j - \vec{v}_i) \times \nabla W_{ij}$

Check $= \nabla \times \vec{v} - \vec{v} \times \nabla 1 = \nabla \times \vec{v}$ ✓ OK

7 a) ii) cont ..

can also use

$$\boxed{\nabla \times \vec{v} = -\frac{1}{\rho_i}\sum_j m_j (\vec{v}_j - \vec{v}_i) \times \nabla W_{ij}}$$

$$= \frac{1}{\rho} \nabla \times (\rho \vec{v}) + \frac{v}{\rho} \times \nabla \rho$$

Use vector identity $\nabla \times (\rho \vec{v}) = \rho \nabla \times \vec{v} - \vec{v} \times \nabla \rho$

$$= \nabla \times \vec{v} - \cancel{\frac{\vec{v}}{\rho} \times \nabla \rho} + \cancel{\frac{\vec{v}}{\rho} \times \nabla \rho}$$

$$= \nabla \times \vec{v} \quad \checkmark \text{ok.}$$

iii) Same again:

$$\boxed{\frac{\partial v^\alpha}{\partial x^\beta} = \sum_j \frac{m_j}{\rho_j} (v_j^\alpha - v_i^\alpha) \frac{\partial W_{ij}}{\partial x^\beta}}$$

also $$\boxed{\frac{\partial v^\alpha}{\partial x^\beta} = \frac{1}{\rho_i} \sum_j m_j (v_j^\alpha - v_i^\alpha) \frac{\partial W_{ij}}{\partial x^\beta}}$$

translate this:

$$\frac{1}{\rho} \frac{\partial (\rho v^\alpha)}{\partial x^\beta} - \frac{v^\alpha}{\rho} \frac{\partial \rho}{\partial x^\beta}$$

$$= \frac{\partial v^\alpha}{\partial x^\beta} + \cancel{\frac{v^\alpha}{\rho} \frac{\partial \rho}{\partial x^\beta}} - \cancel{\frac{v^\alpha}{\rho} \frac{\partial \rho}{\partial x^\beta}}$$

$$= \frac{\partial v^\alpha}{\partial x^\beta} \quad \checkmark \text{OK}$$

iv) $\nabla \vec{v}$ is same as in part iii), so $(\vec{B} \cdot \vec{\nabla}) \vec{v}$ is $B^\beta \frac{\partial v^\alpha}{\partial x^\beta}$

$\therefore$ use $(\vec{B}^\beta \cdot \nabla) \vec{v}^\alpha = \frac{B_i^\beta}{\rho_i} \sum_j m_j (v_j^\alpha - v_i^\alpha) \frac{\partial W_{ij}}{\partial x^\beta}$

or, in vector form:

$$\boxed{(\vec{B} \cdot \nabla) \vec{v} = \sum_j m_j (\vec{v}_j - \vec{v}_i) \frac{\vec{B}_i}{\rho_i} \cdot \nabla W_{ij}}$$

7b) i)

$$\frac{d\rho_i}{dt} = \sum_j m_j \left(\vec{v}_i - \vec{v}_j\right) \cdot \nabla_i W_{ij}$$

$$= \vec{v}_i \cdot \sum_j m_j \nabla_i W_{ij} - \sum_j m_j \vec{v}_j \cdot \nabla_i W_{ij}$$

$$= \vec{v} \cdot \nabla\rho - \nabla \cdot (\rho \vec{v})$$

$$= \cancel{\vec{v} \cdot \nabla\rho} - \rho \nabla \cdot \vec{v} - \cancel{(\vec{v} \cdot \nabla)\rho}$$

$$\therefore \boxed{\frac{d\rho}{dt} = -\rho \nabla \cdot \vec{v}}$$ QED.

ii)

$$\frac{d\rho_i}{dt} = \rho_i \sum_j \frac{m_j}{\rho_j} \left(\vec{v}_i - \vec{v}_j\right) \cdot \nabla_i W_{ij}$$

$$= \rho \vec{v} \cdot \nabla 1 - \rho \nabla \cdot \vec{v}$$

$$= -\rho \nabla \cdot \vec{v}$$ QED.

iii)

$$\frac{d\rho_i}{dt} = \phi_i \sum_j \frac{m_j}{\phi_j} \left(\vec{v}_i - \vec{v}_j\right) \cdot \nabla_i W_{ij}$$

$$= \phi \vec{v} \cdot \nabla\left(\frac{\rho}{\phi}\right) - \phi \nabla \cdot \left(\frac{\rho \vec{v}}{\phi}\right)$$

$$= \cancel{\phi \rho \vec{v} \cdot \nabla\left(\tfrac{1}{\phi}\right)} + \vec{v} \cdot \nabla\rho - \cancel{\phi \rho \vec{v} \cdot \nabla\left(\tfrac{1}{\phi}\right)} - \nabla \cdot (\rho \vec{v})$$

$$= \cancel{\vec{v} \cdot \nabla\rho} - \rho \nabla \cdot \vec{v} - \cancel{\vec{v} \cdot \nabla\rho}$$

$$= -\rho \nabla \cdot \vec{v}$$ QED.

7c)

$$\frac{dv_i}{dt} = -\sum_j m_j \left[\frac{P_i}{\rho_i^2} \frac{\phi_i}{\phi_j} + \frac{P_j}{\rho_j^2} \frac{\phi_j}{\phi_i}\right] \nabla_i W_{ij}$$

$$= -\frac{\phi P}{\rho^2} \nabla\left(\frac{\rho}{\phi}\right) - \frac{1}{\phi} \nabla\left(\frac{P\phi}{\rho}\right)$$

$$= -\frac{\phi P}{\rho} \nabla\left(\tfrac{1}{\phi}\right) - \frac{P}{\rho^2} \nabla\rho - \frac{P}{\phi\rho} \nabla\phi - \nabla\left(\frac{P}{\rho}\right)$$

$$= \cancel{\frac{P}{\rho\phi} \nabla\phi} - \cancel{\frac{P}{\rho^2} \nabla\rho} - \cancel{\frac{P}{\phi\rho} \nabla\phi} - \frac{\nabla P}{\rho} + \cancel{\frac{P}{\rho^2} \nabla\rho}$$

$$= -\frac{\nabla P}{\rho}$$ QED.

8) $\nabla^2 A = \sum_j \frac{m_j}{\rho_j} A_j \nabla^2 Y_{ij}$

$\therefore \quad -\sum_j \frac{m_j}{\rho_j}\left[\frac{1}{2}(K_i + K_j)(A_i - A_j)\right] \nabla^2 Y_{ij}$

$= -\frac{1}{2}\sum_j \frac{m_j}{\rho_j}\left(K_i A_i - K_i A_j + K_j A_i - K_j A_j\right) \nabla^2 Y_{ij}$

term $= -\frac{1}{2}\left[K A \cancel{\nabla^2 1}^{\,0} - K\nabla^2 A + A\nabla^2 K - \nabla^2(KA)\right]$

Expand:

$\nabla^2(KA) = \nabla.\left[K\nabla A + A\nabla K\right]$

$= \nabla.(K\nabla A) + \nabla A . \nabla K + A\nabla^2 K$

$\therefore$ term $= -\frac{1}{2}\left[-K\nabla^2 A + \cancel{A\nabla^2 K} - \nabla.(K\nabla A) - \nabla A.\nabla K - \cancel{A\nabla^2 K}\right]$

Now use:

$\nabla.(K\nabla A) = \nabla K.\nabla A - K\nabla^2 A$

$\therefore$ term $= -\frac{1}{2}\left[-\nabla.(K\nabla A) - \nabla.(K\nabla A)\right]$

$= -\frac{1}{2}\left[-2\nabla.(K\nabla A)\right]$

$\therefore \boxed{\text{term} = \nabla.(K\nabla A)}$

QED

9) a) $\nu\nabla^2\vec{v} = 2\nu\sum_j \frac{m_j}{\rho_j}\left(\vec{v}_i - \vec{v}_j\right)\frac{F_{ij}}{|r_{ij}|}$

b) α = const, c_s = const, h = const, $\epsilon = 0$

$\alpha c_s h \sum_j \frac{m_j}{\rho_j}\frac{(\vec{v}_{ij}.\vec{r}_{ij})}{|r_{ij}|}\frac{\nabla_i W_{ij}}{|r_{ij}|}$

$= \alpha c_s h \sum_j \frac{m_j}{\rho_j}\left(\vec{v}_{ij}.\hat{r}_{ij}\right)\frac{\hat{r}_{ij}F_{ij}}{|r_{ij}|}$ where $\nabla_i W_{ij} = \hat{r}_{ij}F_{ij}$

$= \alpha c_s h\left[\frac{1}{5}\sum_j \frac{m_j}{\rho_j}\left[5\left(\vec{v}_{ij}.\hat{r}_{ij}\right)\hat{r}_{ij} - \vec{v}_{ij}\right]\frac{F_{ij}}{|r_{ij}|} + \frac{1}{10}.2\sum_j \frac{m_j}{\rho_j}\vec{v}_{ij}\frac{F_{ij}}{|r_{ij}|}\right]$

$= \alpha c_s h\left[\frac{1}{5}\nabla(\nabla.\vec{v}) + \frac{1}{10}\nabla^2\vec{v}\right]$ QED.

9c) $\boxed{\nu = \frac{1}{10}\alpha c_s h}$, same as coefficient on $\nabla^2\vec{v}$ term.

From coefficient on $\nabla(\nabla\cdot\vec{v})$ term we have

$$\zeta + \nu/3 = \frac{1}{5}\alpha c_s h = 2\nu$$

$$\therefore \zeta = \nu\left[\frac{6}{3} - \frac{1}{3}\right] = \frac{5}{3}\nu$$

$$\therefore \zeta = \frac{5}{30}\alpha c_s h$$

$$\therefore \boxed{\zeta = \frac{1}{6}\alpha c_s h}$$

QED.

10)
$$\frac{dE}{dt} = \frac{d}{dt}\sum_i m_i e_i = \sum_i m_i \frac{de_i}{dt} = 0$$

$$e_i = \frac{1}{2}v_i^2 + u_i$$

$$\frac{de_i}{dt} = \vec{v}_i \cdot \frac{d\vec{v}_i}{dt} + \frac{du_i}{dt}$$

$$\therefore \sum_i m_i\left[\vec{v}_i \cdot \frac{d\vec{v}_i}{dt} + \frac{du_i}{dt}\right] = 0$$

$$\therefore \sum_i m_i \frac{du_i}{dt} = -\sum_i m_i \vec{v}_i \cdot \frac{d\vec{v}_i}{dt}$$

$$= -2\nu \sum_i m_i \vec{v}_i \cdot \sum_j \frac{m_j}{\bar{\rho}_{ij}}(\vec{v}_i - \vec{v}_j)\frac{F_{ij}}{|r_{ij}|}$$

split sum into two half-sums, and swap indices in second term

$$= -2\nu\left[\frac{1}{2}\sum_i\sum_j \frac{m_i m_j}{\bar{\rho}_{ij}} \vec{v}_i \cdot (\vec{v}_i - \vec{v}_j)\frac{F_{ij}}{|r_{ij}|} + \frac{1}{2}\sum_j\sum_i \frac{m_j m_i}{\bar{\rho}_{ij}} \vec{v}_j \cdot (\vec{v}_j - \vec{v}_i)\frac{F_{ji}}{|r_{ji}|}\right]$$

$$= -2\nu\left[\frac{1}{2}\sum_i\sum_j \frac{m_i m_j}{\bar{\rho}_{ij}}(\vec{v}_i - \vec{v}_j)^2\frac{F_{ij}}{|r_{ij}|}\right]$$

since $F_{ij} = F_{ji}$, $|r_{ji}| = |r_{ij}|$

$$\therefore \sum_i m_i \frac{du_i}{dt} = -\nu\sum_i m_i \sum_j \frac{m_j}{\bar{\rho}_{ij}}(\vec{v}_i - \vec{v}_j)^2\frac{F_{ij}}{|r_{ij}|}$$

$$\therefore \boxed{\frac{du_i}{dt} = -\nu\sum_j \frac{m_j}{\bar{\rho}_{ij}}(\vec{v}_i - \vec{v}_j)^2\frac{F_{ij}}{|r_{ij}|}}$$

QED.