

deriving the dispersion relation for waves in a dust and gas mixture,

assuming small perturbations of

$$\rho_d = \rho_d^0 + \delta \rho_d$$

$$\rho_g = \rho_g^0 + \delta \rho_g$$

$$\vec{V}_g = \delta \vec{V}_g$$

$$\vec{V}_d = \delta \vec{V}_d \quad \text{and}$$

$$\delta \rho_g = c_{s0}^2 \delta S \rho_g$$

we can linearise our two fluid equations too

$$(1) \frac{\partial \delta \rho_g}{\partial t} + \nabla \cdot (\rho_g^0 \delta \vec{V}_g) = 0,$$

$$(2) \frac{\partial \delta \rho_d}{\partial t} + \nabla \cdot (\rho_d^0 \delta \vec{V}_d) = 0,$$

$$(3) \rho_g^0 \frac{\partial \delta \vec{V}_g}{\partial t} = -c_{s0}^2 \nabla \delta \rho_g + K [\delta \vec{V}_d - \delta \vec{V}_g]$$

$$(4) \rho_d^0 \frac{\partial \delta \vec{V}_d}{\partial t} = -K [\delta \vec{V}_d - \delta \vec{V}_g]$$

for simplicity where we ignore 2nd order terms, due to small perturbations also drop the δ for \vec{V}_d, \vec{V}_g for simplicity.

Taking the divergence of (3) and (4) to get,

$$\rho_g^0 \frac{d}{dt} (\nabla \cdot \vec{V}_g) = -c_{s0}^2 \nabla^2 \delta \rho_g + K [\nabla \cdot \vec{V}_d - \nabla \cdot \vec{V}_g] \quad (5)$$

$$\rho_d^0 \frac{d}{dt} (\nabla \cdot \vec{V}_d) = -K [\nabla \cdot \vec{V}_d - \nabla \cdot \vec{V}_g] \quad (6)$$

add (5) + (6) to get,

$$\rho_g^0 \frac{d}{dt} (\nabla \cdot \vec{V}_g) + \rho_d^0 \frac{d}{dt} (\nabla \cdot \vec{V}_d) = -c_{s0}^2 \nabla^2 \delta \rho_g \quad (7)$$

...the difference between the two is ...

$$\begin{aligned} \Delta \phi &= \phi_2 - \phi_1 \\ \Delta \phi &= \phi_2 - \phi_1 \\ \Delta \phi &= \phi_2 - \phi_1 \\ \Delta \phi &= \phi_2 - \phi_1 \end{aligned}$$

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now, $\frac{d}{dt} \textcircled{1}$ and $\frac{d}{dt} \textcircled{2}$ use,

$$-\frac{1}{P_g^0} \frac{d^2 \delta P_g}{dt^2} = \frac{d}{dt} (\nabla \cdot \vec{v}_g) \quad , \quad -\frac{1}{P_d^0} \frac{d^2 \delta P_d}{dt^2} = \frac{d}{dt} (\nabla \cdot \vec{v}_d)$$

sub into get,

$$\frac{d^2 \delta P_g}{dt^2} + \frac{d^2 \delta P_d}{dt^2} = \rho C_{so}^2 \nabla^2 \delta P_g \quad \textcircled{8}$$

$\textcircled{6} - \textcircled{5}$ gives

$$\textcircled{9} \quad P_d^0 \frac{d}{dt} \nabla \cdot \vec{v}_d - P_g^0 \frac{d}{dt} \nabla \cdot \vec{v}_g = -2K [\nabla \cdot \vec{v}_d - \nabla \cdot \vec{v}_g] + C_{so}^2 \nabla^2 \delta P_g$$

and, from $\textcircled{1}$ and $\textcircled{2}$ $\nabla \cdot \vec{v}_d = -\frac{1}{P_d^0} \frac{d}{dt} \delta P_d$

$$\nabla \cdot \vec{v}_g = -\frac{1}{P_g^0} \frac{d}{dt} \delta P_g$$

sub this add in to $\textcircled{9}$ to get,

$$-\frac{d^2 \delta P_d}{dt^2} + \frac{d^2 \delta P_g}{dt^2} = -2K \left[-\frac{1}{P_d^0} \frac{d}{dt} \delta P_d + \frac{1}{P_g^0} \frac{d}{dt} \delta P_g \right] + C_{so}^2 \nabla^2 \delta P_g$$

such that,

$$\frac{d^2 \delta P_d}{dt^2} = \frac{d^2 \delta P_g}{dt^2} + 2K \left[\frac{1}{P_g^0} \frac{d}{dt} \delta P_g - \frac{1}{P_d^0} \frac{d}{dt} \delta P_d \right] - C_{so}^2 \nabla^2 \delta P_g$$

sub into $\textcircled{8}$,

$$2 \frac{d^2 \delta P_g}{dt^2} + 2K \left[\frac{1}{P_g^0} \frac{d}{dt} \delta P_g - \frac{1}{P_d^0} \frac{d}{dt} \delta P_d \right] - 2C_{so}^2 \nabla^2 \delta P_g = 0$$

expand and divide by 2, \therefore

$$\frac{d^2 \delta P_g}{dt^2} + \frac{K}{P_g^0} \frac{d}{dt} \delta P_g - \frac{K}{P_d^0} \frac{d}{dt} \delta P_d - C_{so}^2 \nabla^2 \delta P_g = 0 \quad \textcircled{10}$$

$$u = 0 \pm 2.0 \pm 0.1$$

$$(u-v) \frac{b}{x} = \frac{19.6 \cdot b}{x} \quad (5.8) \quad \frac{b}{x} = \frac{19.6 \cdot b}{x}$$

$$1.32 \pm 0.02$$

$$(2) \quad \frac{19.6 \cdot b}{x} = \frac{19.6 \cdot b}{x}$$

$$(2) - (1)$$

$$19.6 \cdot b \cdot \omega + [19.6 \cdot b - 19.6 \cdot b] \cdot 5 = 19.6 \cdot b \cdot \frac{b}{x} - 19.6 \cdot b \cdot \frac{b}{x} \quad (3)$$

$$19.6 \cdot \frac{b}{x} \cdot \frac{1}{x} = 19.6 \cdot b \cdot \frac{b}{x} - 19.6 \cdot b \cdot \frac{b}{x}$$

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$$19.6 \cdot \frac{b}{x} \cdot \frac{1}{x} = 19.6 \cdot b \cdot \frac{b}{x} \quad (4)$$

$$1.32 \pm 0.02$$

$$19.6 \cdot \frac{b}{x} \cdot \frac{1}{x} = 19.6 \cdot b \cdot \frac{b}{x} \quad (5)$$

$$(5) - (4)$$

$$0 = 19.6 \cdot \frac{b}{x} \cdot \frac{1}{x} - 19.6 \cdot b \cdot \frac{b}{x} \quad (6)$$

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$$(6) - (5) \quad 0 = 19.6 \cdot \frac{b}{x} \cdot \frac{1}{x} - 19.6 \cdot b \cdot \frac{b}{x} \quad (7)$$

$\frac{d}{dt}$ (10) gives

$$\frac{d^3 \delta P_y}{dt^3} + \frac{K}{P_y^0} \frac{d^2}{dt^2} \delta P_y - \frac{K}{P_d^0} \frac{d^2}{dt^2} \delta P_d - C_{s0}^2 \frac{d}{dt} \nabla^2 \delta P_y = 0$$

Sub in (8) to get,

$$\frac{d^3 \delta P_y}{dt^3} + \frac{K}{P_y^0} \frac{d^2}{dt^2} \delta P_y - \frac{K}{P_d^0} \left(C_{s0}^2 \nabla^2 \delta P_y - \frac{d^3 \delta P_y}{dt^3} \right) - C_{s0}^2 \frac{d}{dt} \nabla^2 \delta P_y = 0$$

$$\Rightarrow \frac{d^3 \delta P_y}{dt^3} + K \left(\frac{1}{P_y^0} + \frac{1}{P_d^0} \right) \frac{d^3 \delta P_y}{dt^3} - \frac{K}{P_d^0} C_{s0}^2 \nabla^2 \delta P_y - C_{s0}^2 \frac{d}{dt} \nabla^2 \delta P_y = 0 \quad (11)$$

where $K \left(\frac{1}{P_y^0} + \frac{1}{P_d^0} \right) = K \left(\frac{P_y^0 + P_d^0}{P_y^0 P_d^0} \right) = \frac{1}{\epsilon_s}$

assume perturbations of the form,

$$\delta P_y = P e^{i(\vec{k} \cdot \vec{x} - \omega t)} \quad \text{such that,}$$

$$\frac{d^3 \delta P_y}{dt^3} = -\omega^3 \delta P_y, \quad \text{and} \quad \frac{d^3 \delta P_y}{dt^3} = i\omega^3 \delta P_y.$$

$$\nabla^2 \delta P_y = -k^2 \delta P_y.$$

so

$$\frac{d}{dt} (\nabla^2 \delta P_y) = i\omega k^2 \delta P_y.$$

Sub into (11) to get, and divide by δP_y .

$$i\omega^3 + \frac{1}{\epsilon_s} (-\omega^3) - \frac{K}{P_d^0} C_{s0}^2 (-k^2) - C_{s0}^2 (i\omega k^2) = 0.$$

$$= i\omega^3 - \frac{\omega^3}{\epsilon_s} + \frac{K}{P_d^0} C_{s0}^2 k^2 - i\omega C_{s0}^2 k^2 = 0$$

$$2\pi i \oint \frac{1}{z} dz$$

$$0 = \frac{1}{2\pi i} \oint \frac{1}{z} dz = \frac{1}{2\pi i} \left(\frac{1}{z} \right) \Big|_0^{2\pi i} = \frac{1}{2\pi i} \left(\frac{1}{2\pi i} - \frac{1}{0} \right)$$

Let $z = re^{i\theta}$

$$0 = \frac{1}{2\pi i} \oint \frac{1}{z} dz = \frac{1}{2\pi i} \left(\frac{1}{r} \right) \Big|_0^{2\pi i} = \frac{1}{2\pi i} \left(\frac{1}{2\pi i} - \frac{1}{0} \right)$$

$$(1) \quad 0 = \frac{1}{2\pi i} \oint \frac{1}{z} dz = \frac{1}{2\pi i} \left(\frac{1}{r} \right) \Big|_0^{2\pi i} = \frac{1}{2\pi i} \left(\frac{1}{2\pi i} - \frac{1}{0} \right)$$

$$\frac{1}{z} = \frac{1}{r} \left(\frac{1}{e^{i\theta}} \right) = \frac{1}{r} e^{-i\theta}$$

Let $z = re^{i\theta}$

$$dz = i r e^{i\theta} d\theta$$

$$\frac{1}{z} dz = \frac{1}{r} e^{-i\theta} i r e^{i\theta} d\theta = i d\theta$$

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now $\frac{k}{\rho d^0} = \frac{\rho_g^0 \rho d^0}{(\rho_g^0 + \rho d^0) \epsilon_s} \times \frac{1}{\rho d^0} = \frac{\rho_g^0}{\rho^0 \epsilon_s}$ where $\rho^0 = \rho_g^0 + \rho d^0$

so $i\omega^3 - \frac{\omega^2}{\epsilon_s} + \frac{\rho_g^0}{\rho^0 \epsilon_s} c_{s0}^2 k^2 - i\omega c_{s0}^2 k^2 = 0$

divided by $i\omega$ to get,

$$\omega^2 - \frac{\omega^2}{i\omega \epsilon_s} + \frac{\rho_g}{\rho^0 \epsilon_s i\omega} c_{s0}^2 k^2 - c_{s0}^2 k^2 = 0$$

$$\Rightarrow (\omega^2 - c_{s0}^2 k^2) - \frac{1}{i\omega \epsilon_s} (\omega^2 - \frac{\rho_g}{\rho^0} c_{s0}^2 k^2) = 0$$

Let $\frac{\rho_g}{\rho^0} c_{s0}^2 \equiv \tilde{c}_s^2$ be the modified sound speed,

so that,

$$(\omega^2 - c_{s0}^2 k^2) - \frac{1}{i\omega \epsilon_s} (\omega^2 - \tilde{c}_s^2 k^2) = 0$$

$$0 = \frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d^2 \psi}{dt^2} \right) - \frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d^2 \psi}{dt^2} \right) - \frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d^2 \psi}{dt^2} \right)$$

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the first term is zero

$$0 = \frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d^2 \psi}{dt^2} \right) - \frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d^2 \psi}{dt^2} \right) - \frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d^2 \psi}{dt^2} \right)$$

$$0 = \left(\frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d^2 \psi}{dt^2} \right) - \frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d^2 \psi}{dt^2} \right) \right) \frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d^2 \psi}{dt^2} \right) - \left(\frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d^2 \psi}{dt^2} \right) - \frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d^2 \psi}{dt^2} \right) \right) \frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d^2 \psi}{dt^2} \right)$$

$$\text{Let } \frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d^2 \psi}{dt^2} \right) \equiv \tilde{\psi} \text{ for the moment, and then}$$

so that

$$0 = \left(\frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d^2 \psi}{dt^2} \right) - \frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d^2 \psi}{dt^2} \right) \right) \frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d^2 \psi}{dt^2} \right) - \left(\frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d^2 \psi}{dt^2} \right) - \frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d^2 \psi}{dt^2} \right) \right) \frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d^2 \psi}{dt^2} \right)$$