Dynamics Assignment.

a) $U_{orb} = A\, U_{ref}$.

from figure 4, we rotate firstly around the x axis. by the angle I, such that,

$$\underset{\sim}{r}' = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos I & \sin I \\ 0 & -\sin I & \cos I \end{pmatrix}}_{= A_I} \underset{\sim}{r},$$

Then it is rotated around the new z-axis with an angle ω.
Such that.

$$\underset{\sim}{r}'' = \overbrace{\begin{pmatrix} \cos\omega & \sin\omega & 0 \\ -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix}}^{= A_\omega} \underset{\sim}{r}',$$

$$\text{So } \underset{\sim}{r}'' = \begin{pmatrix} \cos\omega & \sin\omega & 0 \\ -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos I & \sin I \\ 0 & -\sin I & \cos I \end{pmatrix} \underset{\sim}{r}.$$

$$= \underbrace{\begin{pmatrix} \cos\omega & \sin\omega\cos I & \sin\omega\sin I \\ -\sin\omega & \cos\omega\cos I & \cos\omega\sin I \\ 0 & -\sin I & \cos I \end{pmatrix}}_{A = A_\omega A_I} \underset{\sim}{r}.$$

6). Let $U_{ref} = \{x_1, y_1, z_1\}$,

So $|U_{ref}|^2 = x_1^2 + y_2^2 + z_1^2$.

now $U_{orb} = A\, U_{ref}$,

$= \{x_1 \cos\omega + y_1 \cos i \sin\omega + z_1 \sin i \sin\omega,$

$y_1 \cos i \cos\omega + z_1 \cos\omega \sin i - x_1 \sin\omega,$

$z_1 \cos i - y_1 \sin i\}$

now $|U_{orb}|^2$.

$= x_1^2 (\cos^2(\omega)) + y_1^2 \cos^2 i \sin^2\omega + z_1^2 \sin^2 i \sin^2\omega.$

$+ 2x_1 y_1 \cos\omega \cos i \sin\omega + 2x_1 z_1 \cos\omega \sin i \sin\omega$
$+ 2y_1 z_1 \cos i \sin i \sin^2\omega.$

$+ y_1^2 \cos^2 i \cos^2\omega + z_1^2 \cos^2\omega \sin^2 i + x_1^2 \sin^2\omega$
$+ 2y_1 z_1 \cos i \sin i \cos^2\omega - 2x_1 z_1 \cos\omega \sin i \sin\omega$
$- 2x_1 y_1 \cos i \cos\omega \sin\omega.$

$+ z_1^2 \cos^2 i + y_1^2 \sin^2 i - 2y_1 z_1 \cos i \sin i$

now group w.r.t $x_1^2, y_1^2, z_1^2, z_1 x_1, z_1 y_1, x_1 y_1$

$= x_1^2 (\cos^2\omega + \sin^2\omega) + y_1^2 (\cos^2 i \sin^2\omega + \cos^2 i \cos^2\omega + \sin^2 i)$
$+ z_1^2 (\sin^2 i \sin^2\omega + \cos^2\omega \sin^2 i + \cos^2 i)$
$+ 2x_1 y_1 (0) + 2x_1 z_1 (0) + 2y_1 z_1 (\cos i \sin i \sin^2\omega$
$+ \cos i \sin i \cos^2\omega$
$- \cos i \sin i)$

$$= x_1^2 + y_1^2 (\cos^2 i \sin^2 \omega + \cos^2 i \cos^2 \omega + \sin^2 i)$$

$$+ z_1^2 (\sin^2 i \sin^2 \omega + \cos^2 \omega \sin^2 i + \cos^2 i)$$

$$+ 2 y_1 z_1 \underbrace{(\cos i \sin i \sin^2 \omega + \cos i \sin i \cos^2 \omega - \cos i \sin i)}$$

$$= \cos i \sin i (\sin^2 \omega + \cos^2 \omega) - \cos i \sin i = 0$$

$$\cos^2 i \sin^2 \omega + \cos^2 i \cos^2 \omega + \sin^2 i.$$

$$= \cos^2 i \left(\sin^2 \omega + \cos^2 \omega + \frac{\sin^2 i}{\cos^2 i}\right) = \cos^2 i \left(1 + \frac{\sin^2 i}{\cos^2 i}\right)$$

$$= \cos^2 i + \sin^2 i = 1$$

$$(\sin^2 i \sin^2 \omega + \cos^2 \omega \sin^2 i + \cos^2 i)$$

$$= \sin^2 i + \cos^2 i = 1$$

$$\therefore \ |v_{orb}|^2 = x_1^2 + y_2^2 + z_2^2.$$

6) ii). $U_{ref} = x_1, y_1, \cancel{z_2}$ $\quad W_{ref} = (x_2, y_2, z_2)$

So. $U_{ref} \cdot W_{ref} = x_1 x_2 + y_1 y_2 + z_1 z_2$

$$W_{ref} = \{ x_2 \cos \omega + y_2 \cos i \sin \omega + z_2 \sin i \sin \omega,$$

$$y_2 \cos i \cos \omega + z_2 \cos \omega \sin i - x_2 \sin \omega,$$

$$z_2 \cos i - y_2 \sin i \}$$

now

$$U_{orb} \cdot W_{orb}$$

$$= (z_1 z_2 \cos^2 i + x_1 x_2 \cos^2 \omega + y_1 y_2 \cos^2 i \cos^2 \omega$$

$$- y_2 z_1 \cos i \sin i - y_1 z_2 \cos i \sin i + y_2 z_1 \cos i \cos^2 \omega \sin i$$
$$+ y_1 z_2 \cos i \cos^2 \omega \sin i + y_1 y_2 \sin^2 i + z_1 z_2 \cos\omega^2 \sin^2 i$$
$$+ x_1 x_2 \sin^2 \omega + y_1 y_2 \cos^2 i \sin^2 \omega + y_2 z_1 \cos i \sin i \sin^2 \omega$$
$$+ y_1 z_2 \cos i \sin i \sin^2 \omega + z_1 z_2 \sin^2 i \sin^2 \omega$$

group all $x_1 x_2$ ...... to get.

$$x_1 x_2 \underbrace{(\cos^2\omega + \sin^2\omega)}_{=1} + z_1 z_2 \underbrace{(\cos^2 i + \cos\omega^2 \sin^2 i + \sin^2 i \sin^2 \omega)}_{=1}$$
$$+ y_1 y_2 \underbrace{(\cos^2 i \cos^2 \omega + \sin^2 i + \cos^2 i \sin^2 \omega)}_{=1}$$
$$+ y_2 z_1 \underbrace{(\cos i \cos^2 \omega \sin i - \cos i \sin i + \cos i \sin i \sin^2 \omega)}_{=0}$$
$$+ y_1 z_2 \underbrace{(-\cos i \sin i + \cos i \cos^2 \omega \sin i + \cos i \sin i \sin^2 \omega)}_{=0}$$

Where, $\cos^2 i + \cos\omega^2 \sin^2 i + \sin^2 i \sin^2 \omega$

$$= \cos^2 i + \sin^2 i (1) = 1,$$

simmilarly for $\cos^2 i \cos^2 \omega + \sin^2 i + \cos^2 i \sin^2 \omega$,

and. $\cos i \cos^2 \omega \sin i + \cos i \sin i \sin^2 \omega - \cos i \sin i$
$$= \cos i \sin i (\cos^2 \omega + \sin^2 \omega) - \cos i \sin i = 0.$$

So $U_{orb} \cdot W_{orb} = x_1 x_2 + y_1 y_2 + z_1 z_2 = U_{ref} \cdot W_{ref}$.

also since $U_{orb} \cdot W_{orb} = |U_{orb}||W_{orb}| \cos(\theta) = |U_{ref}||W_{ref}| \cos(\theta)$.

implies that the angle is conserverd, between vectors since $|U_{orb}| = |U_{ref}|$

c) $r = \frac{a(1-e^2)}{1+e\cos f} = a(1-e^2)(1+e\cos f)^{-1}$

so $\frac{d}{dt} r = \dot{r} = a(1-e^2)\frac{d}{dt}(1+e\cos f)^{-1}$

$= a(1-e^2)e\sin(f)(1+e\cos f)^{-2}\dot{f}$

$= \frac{a(1-e^2)e\sin(f)}{(1+e\cos f)^2}\dot{f}$

now $r^2\dot{f} = a^2 v\sqrt{1-e^2} \Rightarrow \dot{f} = \frac{a^2 v\sqrt{1-e^2}}{r^2}$

and $r^2 = \frac{a^2(1-e^2)^2}{(1+e\cos f)^2}$

So $\dot{f} = \cancel{a^2} v\sqrt{1-e^2}\left(\frac{(1+e\cos f)^2}{\cancel{a^2}(1-e^2)^2}\right) = \frac{v(1+e\cos f)^2\sqrt{1-e^2}}{(1-e^2)^2}$

now $\frac{\sqrt{1-e^2}}{(1-e^2)^2} = (1-e^2)^{1/2}(1-e^2)^{-2} = (1-e^2)^{-3/2}$

So $\dot{r} = \frac{a(1-e^2)e\sin(f)}{(1+e\cos f)^2}v(1+e\cos f)^2(1-e^2)^{-3/2}$

$= av(1-e^2)^{-1/2}e\sin(f) = \frac{ave\sin(f)}{\sqrt{1-e^2}}$

recall $\dot{f} = v\,(1+e\cos f)^2 (1-e^2)^{-3/2}$

and $r = \dfrac{a\,(1-e^2)}{1+e\cos f}$

so

$$r\dot{f} = \frac{a(1-e^2)}{\cancel{1+e\cos f}} \times \left(v\,(1+e\cos f)^2 (1-e^2)^{-3/2} \right)$$

$$= \frac{av(1+e\cos f)}{\sqrt{1-e^2}}$$

D) The velocity of the star is defined as

$$\vec{V}_* = -\frac{M_p}{M_p + M_*}\dot{\vec{r}} \quad \text{where} \quad \vec{V} = \dot{r}e_r + r\dot{f}\,e_f.$$

$$\therefore \vec{V} = (\dot{r}\cos(f) - r\dot{f}\sin(f))\,i'' + (\dot{r}\sin(f) + r\dot{f}\cos(f))\,j''$$
$$+ (0)\,k''$$

Since the orbit rotates in the x''-y'' Plane,

in order to calculate the radial velocity.

we need to change our coordinates to that of the observer with the predefined Matrix A,

Since $\vec{U}_{orb} = A\,\vec{U}_{ref}$

$$A^T\vec{U}_{orb} = A^T A\,\vec{U}_{ref} = \vec{U}_{ref}$$

$$\therefore \vec{V}_{ref} = \begin{pmatrix} \cos(\omega) & -\sin(\omega) & 0 \\ \cos(i)\sin(\omega) & \cos(i)\cos(\omega) & -\sin(i) \\ \sin(i)\sin(\omega) & \cos(\omega)\sin(i) & \cos(i) \end{pmatrix} \begin{pmatrix} \dot{r}\cos f - r\dot{f}\sin(f) \\ \dot{r}\sin(f) + r\dot{f}\cos(f) \\ 0 \end{pmatrix}$$

Since the radial velocity is along the Z axis of the observer's Frame:

$$\vec{V}_r = \sin(i)\sin(\omega)(\dot{r}\cos f - r\dot{f}\sin(f)) + \cos(\omega)\sin(i)(\dot{r}\sin(f) + r\dot{f}\cos(f)) + 0$$

where $\dot{r} = a\nu\frac{e\sin f}{\sqrt{1-e^2}}$ and $r\dot{f} = \frac{a\nu(1+e\cos f)}{\sqrt{1-e^2}}$

now $V_{*r} = -\frac{M_P}{M_{*P}}\left(\dot{r}\left(\cos f \sin(i)\sin(\omega) + \cos(\omega)\sin(i)\sin(f)\right)\right.$

$\left. + r\dot{f}\left(\sin(i)\cos(f)\cos(\omega) - \sin(i)\sin(\omega)\sin(f)\right)\right)$

$$= -\frac{M_P}{M_{*P}}\left(\frac{a v e \sin(f)}{\sqrt{1-e^2}}\sin(i)\left(\cos f \sin\omega + \cos(\omega)\sin(f)\right)\right.$$

$$\left. + \frac{a v(1+e\cos f)}{\sqrt{1-e^2}}\sin(i)\left(\cos(f)\cos(\omega) - \sin(\omega)\sin(f)\right)\right)$$

$$= \underbrace{-\frac{M_P}{M_{*P}}\frac{a v \sin(i)}{\sqrt{1-e^2}}}_{=K}\left(e\sin(f)\left(\cos f\sin(\omega) + \cos(\omega)\sin(f)\right)\right.$$

$$\left. + (1+e\cos(f))\left(\cos(f)\cos(\omega) - \sin(\omega)\sin(f)\right)\right)$$

since $\frac{M_P}{M_{*P}} a = a_*$ and $K = \frac{a_* v \sin(i)}{\sqrt{1-e^2}}$

So $V_{*r} = -K\left(e\left(\cancel{\sin(f)\cos f\sin\omega} + \cos(\omega)\sin^2(f) + \cos(\omega)\cos^2(f) \cancel{- \sin(\omega)\sin(f)\cos(f)}\right)\right.$

$\left. + \cos(f)\cos(\omega) - \sin(\omega)\sin(f)\right)$

and $\cos(\omega)(\sin^2 f + \cos^2(f)) = \cos(\omega)$

$$V_{*r} = -K\left(e\cos(\omega) + \cos(f)\cos(\omega) - \sin(\omega)\sin(f)\right)$$

using the trig identity for Products of 2 angles.

$$\sin(\omega)\sin(f) = \frac{1}{2}\left[\cos(\omega-f) - \cos(\omega+f)\right]$$

$$\cos(\omega)\cos(f) = \frac{1}{2}\left[\cos(\omega-f) + \cos(\omega+f)\right]$$

So $\cos(\omega)\cos(f) - \sin(\omega)\sin(f) = \frac{1}{2}\left[2\cos(\omega+f)\right] = \cos(\omega+f)$

and,

$$V_* r = -K[e\cos(\omega) + \cos(\omega + f)].$$

now recall that $\omega = \omega_p = \omega_* - \pi$
and $f = f_p = f_*$.

We get,

$$V_* r = -K[e\cos(\omega_* - \pi) + \cos((\omega_* + f_*) - \pi)]$$

$$= -K[-e\cos(\omega_*) - \cos(\omega_* + f_*)]$$

$$= K[e\cos(\omega_*) + \cos(\omega_* + f_*)]$$