2 RG&TC-Code

1 Introduction

The mathematical description of $d\Omega^2$ is ,$d\Omega^2 = d\theta^2 + \sin^2[\theta]\, d\phi^2$

3.1 Assumptions

a) Based on the assumptions that:

1. The metric is spherically symmetric.
2. The metric is static and does not depend on time.
3. The metric is invariant under time-reversal.

The simplified expression for the most general metric would be,
$ds^2 = a[r]\, dt^2 + b[r]\, dr^2 + c[r]\, d\Omega^2$
In order to maintain spherical symmetry the general shape of $d\Omega^2$ must be conserved,
and the functions a,b,c must not depend on θ or ϕ.
For the metric it must also not have any time dependance.
This means components of the metric must not have any dependancy on time and that there are no time-space cross terms.
b) And finally for the metric to be invariant under time-reversal the off diagonal elements must be 0.

3.2 Exponential formulation

C) The requirement that we need to preserve spherical symmetry is that the form of $d\Omega^2 = d\theta^2 + \sin^2[\theta]\, d\phi^2$ must be maintained. Hence we are free to multiple all the other terms by separate coefficients as long as they are functions that depend on r alone. Hence the met-
ric
$ds^2 = -e^{2\,\alpha(r)}\, dt^2 + e^{2\,\beta(r)}\, dr^2 + e^{2\,\gamma(r)}\, r^2\, dt^2.$ (1)

D) After defining the new coordinate r_bar == $e^{\gamma(r)}\, r$. It makes sense to use this substitution since its not really clear what r really is, it is possible to interpret this after we have our solutions.

e) The form associated with $dr_{bar} = e^{\gamma(r)}\, dr + e^{\gamma(r)}\, r\, d\gamma = \left(1 + r\, \frac{d\gamma}{dr}\right) e^{\gamma}\, dr$ (2)

f) The Metric in terms of r_bar is:
$ds^2 = -e^{2\,\alpha(r)}\, dt^2 + \left(1 + r\, \frac{d\gamma}{dr}\right)^{-2} e^{2\,\beta(r) - 2\,\gamma(r)}\, dr_{bar}{}^2 + r_{bar}{}^2\, d\Omega^2,$ (3)

We can now define the new variable $e^{2\,\beta'(r)} = \left(1 + r\, \frac{d\gamma}{dr}\right)^{-2} e^{2\,\beta(r) - 2\,\gamma(r)}$ (4)

g) By setting $\gamma(r) = 0$ we do not loose any generality since we would get the same result if we began

with $\gamma(\mathrm{r}) = 0$ in the original metric.
Simplifying our metric further too
$ds^2 = -e^{2\,\alpha(r)}\,dt^2 + e^{2\,\beta'(r)}\,dr_{bar}{}^2 + r_{bar}{}^2\,d\Omega^2$ (5)
Now for simplicity we drop the Bar and prime notation to get.
$ds^2 = -e^{2\,\alpha(r)}\,dt^2 + e^{2\,\beta(r)}\,dr^2 + r^2\,d\Omega^2$ (6)
Comparing (6) to (1) they are almost Identical except $e^{2\,\gamma(r)}$is gone. It is important to note that we didn’t simply set it equal to one (or $\gamma(\mathrm{r}) = 0$). Rather we made a a specific change in radial coordinate such that the factor doesn’t exist. Thus the generality of equation (1) and (6) is preserved.

4 Solving for α and β

A) Now that we have our metric simplified as much as possible it is possible to calculate the Einstein tensor (since the metric and coordinates is all we really need to derive the properties of spacetime). Once the Einstein tensor is calculated we can relate it to actual physical problems through the field equation, specifically how matter and energy (the stress tensor) effects the space time it is embedded into.
B) The First step is to calculate the Christoffel symbols Allons-Y!

In[ ]:=
```
xCoord = {t, r, θ, ϕ};
g = {
   {-Exp[2 * a[r]], 0, 0, 0},
   {0, Exp[2 * b[r]], 0, 0},
   {0, 0, r^2, 0},
   {0, 0, 0, r^2 * Sin[θ]^2}
  };
RGtensors[g, xCoord]
```

$$gdd = \begin{pmatrix} -e^{2\,a[r]} & 0 & 0 & 0 \\ 0 & e^{2\,b[r]} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\,\mathrm{Sin}[\theta]^2 \end{pmatrix}$$

$$LineElement = e^{2\,b[r]}\,d[r]^2 - e^{2\,a[r]}\,d[t]^2 + r^2\,d[\theta]^2 + r^2\,d[\phi]^2\,\mathrm{Sin}[\theta]^2$$

$$gUU = \begin{pmatrix} -e^{-2\,a[r]} & 0 & 0 & 0 \\ 0 & e^{-2\,b[r]} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{\mathrm{Csc}[\theta]^2}{r^2} \end{pmatrix}$$

gUU computed in 0.001269 sec

Gamma computed in 0.002689 sec

Riemann(dddd) computed in 0.003816 sec

Riemann(Uddd) computed in 0.003612 sec

Ricci computed in 0.004122 sec

Weyl computed in 0.009574 sec

Einstein computed in 0.002694 sec

Out[ ]= All tasks completed in 0.031724

C)

```
In[ ]:= GUdd // MatrixForm
GUdd[[1, 1, 2]];
GUdd[[1, 2, 1]];
GUdd[[2, 1, 1]];
GUdd[[2, 2, 2]];
GUdd[[2, 3, 3]];
GUdd[[2, 4, 4]];
GUdd[[3, 2, 3]];
GUdd[[3, 3, 2]];
GUdd[[3, 4, 4]];
GUdd[[4, 2, 4]];
GUdd[[4, 4, 2]];
GUdd[[4, 3, 4]];
GUdd[[4, 4, 3]];
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \begin{pmatrix} 0 \\ a'[r] \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} a'[r] \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} e^{2\,a[r]-2\,b[r]}\,a'[r] \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ b'[r] \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ -e^{-2\,b[r]}\,r \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ -e^{-2\,b[r]}\,r\,\mathrm{Sin}[\theta]^2 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ \frac{1}{r} \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ \frac{1}{r} \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta] \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{r} \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mathrm{Cot}[\theta] \end{pmatrix} & \begin{pmatrix} 0 \\ \frac{1}{r} \\ \mathrm{Cot}[\theta] \\ 0 \end{pmatrix} \end{pmatrix}$$

D)

To calculate the Christoffel symbols use the equation:

$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2} g^{\mu\lambda}(d_{\beta}\, g_{\lambda\alpha} + d_{\alpha}\, g_{\lambda\beta} - d_{\lambda}\, g_{\alpha\beta})$

Lets calculate the Christoffel symbol for $\Gamma^{0}{}_{01}$ since it seems pretty straight forward (as the only derivative is on a[r])

Therefore;

$\Gamma^{0}{}_{01} = \frac{1}{2} g^{00}(d_1\, g_{00} + d_0\, g_{01} - d_0\, g_{01})$

And since the metric is diagonal reduces too

$\Gamma^{0}{}_{01} = \frac{1}{2} g^{00}(d_1\, g_{00}) = \frac{1}{2}\left(-e^{-2\,a[r]}\, d_r\left(-e^{2\,a[r]}\right)\right.$

Applying the chain rule.

$= \frac{1}{2}\left(-e^{-2\,a[r]}\right)\left(-2\,a'[r]\,e^{2\,a[r]}\right) = a'[r]$

I now understand why the choice of exponential functions was made (and the factor of 2) since it simplifies the solutions the most. Very Nice I like.

E) Once again the whole point of calculating the Christoffel symbols is because they make up the mathematical framework of general relativity and are used to calculate the Riemann tensor and finally the Einstein tensor (which we then relate to physical phenomenon through the field equation). I would like to stress that up until that point general relativity is a rigorous mathematical tool that describes manifolds. How those manifolds are create/warped by content (energy and mass) is the physical application.

F) The Riemann tensor and the Ricci tensor. are calculated below.

In[ ]:=
```
RUddd // MatrixForm;
Rdd // MatrixForm // FullSimplify
```

Out[ ]//MatrixForm=

$$\left(\begin{array}{cccc} \frac{e^{2\,a[r]-2\,b[r]}\,(a'[r]\,(2+r\,a'[r]-r\,b'[r])+r\,a''[r])}{r} & 0 & 0 & \\ 0 & \frac{2\,b'[r]}{r}+a'[r]\,(-a'[r]+b'[r])-a''[r] & 0 & \\ 0 & 0 & 1+e^{-2\,b[r]}\,(-1-r\,a'[r]+r\,b'[r]) & \\ 0 & 0 & 0 & e^{-2} \end{array}\right.$$

G) The Riemann tensor is calculated via:

$R^{\rho}{}_{\sigma\mu\nu} = d_{\mu}\,\Gamma^{\rho}{}_{\nu\sigma} - d_{\nu}\,\Gamma^{\rho}{}_{\mu\sigma} + \Gamma^{\rho}{}_{\mu\lambda}\,\Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\lambda}\,\Gamma^{\lambda}{}_{\mu\sigma}$

And the Ricci tensor is calculated from the Riemann tensor by contraction:

$R_{\sigma\nu} = R^{\rho}{}_{\sigma\rho\nu}$

The computer calculated the components of the Riemann tensor from the partial derivatives of the Christoffel symbols. And it further calculates the Ricci tensor by contracting the Riemann tensor. Thank god for computers.

$R_{tt} = \frac{e^{2\,a[r]-2\,b[r]}\,(2\,a'[r]+r\,a'[r]^2-r\,a'[r]\,b'[r]+r\,a''[r])}{r} = e^{2\,(a-b)}\left[a'' + (a')^2 - a'\,b' + \frac{2}{r}\,a'\right]$

$R_{rr} = -\frac{r\,a'[r]^2-2\,b'[r]-r\,a'[r]\,b'[r]+r\,a''[r]}{r} = -a'' - (a')^2 + a'\,b' + \frac{2}{r}\,b'$

$R_{\theta\theta} = e^{-2\,b[r]}\left(-1+e^{2\,b[r]}-r\,a'[r]+r\,b'[r]\right) = e^{-2\,b}[r(b'-a')-1]+1$

and

$R_{\phi\phi} = \text{Sin}^2[\theta]\,R_{\theta\theta}$

H) Yes it is ok to set the stress energy tensor to 0 because we are interested in solutions outside of the earth (Not in it!!). Hence in those regions space would be a vacuum and the stress tensor will be 0.

I) Einsteins field equations are:

$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu} = 8\,\pi\,G\,T_{\mu\nu}$

Where $G_{\mu\nu}$ is the Einstein tensor and G is the gravitational constant.

The Ricci scalar R would be 0 Because there is no curvature of spacetime in a vacuum.

J) Based on the field equation in a vacuum the stress tensor is 0 and hence there is no curvature (since the field equation relates matter and energy to curved spacetime) and therefore the Riemann tensor (which describes the curvature) will be 0.

K) Before Combining R_{tt} and R_{rr} together we can notice that the main difference is the $e^{2(a-b)}$ term which we can cancel out by multiplying by $e^{2(b-a)}$ therefore:
$e^{2(b-a)} R_{tt} + R_{rr} = a'' + (a')^2 - a' b' + \frac{2}{r} a' - a'' - (a')^2 + a' b' + \frac{2}{r} b'$
Some terms cancel out to give,
$e^{2(b-a)} R_{tt} + R_{rr} = \frac{2}{r}(a' + b') = 0$
$a' = -b'$
Since we would like to Ricci tensor to vanish we can set this to equal 0 since ($R_{tt} == R_{rr} == 0$)
Therefore after integrating w.r.t r to get.
a = -b + c
Where c is a constant.
L) We can set this constant to equal 0 by rescaling our time coordinator by t -> $e^{-c}\, t$.
This is ok to do since we are just scaling our time coordinate and the generality of the metric is not affected.
so we get a = -b
M) Now let us consider $R_{\theta\theta} = 0 = e^{-2b}[r(b' - a') - 1] + 1$
=> $e^{2a}[r(-a' - a') - 1] = -1$
=> $e^{2a}[2 ra' + 1] = 1$
=> $e^{2a}\, 2 ra' + e^{2a} = 1$
Which we can realise that the L.H.S is the result of the product rule being applied to $d_r(r * e^{2a})$ thus:
$d_r(r * e^{2a}) = 1$
Integrating to get
$r * e^{2a} = r + c \Rightarrow e^{2a} = 1 + \frac{c}{r}$
Letting the integration constant equal $-R_s$ (the Schwarzschild metric) we get
$e^{2a} = 1 - \frac{R_s}{r}$

5 Exploring the Schwarzschild metric

a) Now that we have solved for a and b we can substitute the expression back into our simplified metric to get.
$ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + r^2 d\Omega^2$
$ds^2 = -\left(1 - \frac{R_s}{r}\right) dt^2 + e^{-2a} dr^2 + r^2 d\Omega^2$
$ds^2 = -\left(1 - \frac{R_s}{r}\right) dt^2 + \left(1 - \frac{R_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2.$

Solving for the geodesic equations for a slow moving particle $\left(U^i << U^0\right)$

In[ ]:=
```
xCoord = {t, r, θ, ϕ};
g = {
  {-(1 - Rs/r), 0, 0, 0},
  {0, (1 - Rs/r)^-1, 0, 0},
  {0, 0, r^2, 0},
  {0, 0, 0, r^2 * Sin[θ]^2}
 };
RGtensors[g, xCoord]
```

$$\text{gdd} = \begin{pmatrix} -1+\frac{Rs}{r} & 0 & 0 & 0 \\ 0 & \frac{1}{1-\frac{Rs}{r}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\,\text{Sin}[\theta]^2 \end{pmatrix}$$

$$\text{LineElement} = \frac{r\,d[r]^2}{r-Rs} - \frac{(r-Rs)\,d[t]^2}{r} + r^2\,d[\theta]^2 + r^2\,d[\phi]^2\,\text{Sin}[\theta]^2$$

$$\text{gUU} = \begin{pmatrix} -\frac{r}{r-Rs} & 0 & 0 & 0 \\ 0 & \frac{r-Rs}{r} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{\text{Csc}[\theta]^2}{r^2} \end{pmatrix}$$

gUU computed in 0.015841 sec

Gamma computed in 0.003215 sec

Riemann(dddd) computed in 0.004142 sec

Riemann(Uddd) computed in 0.002236 sec

Ricci computed in 0.000237 sec

Weyl computed in 0.000019 sec

Ricci Flat

Out[ ]= All tasks completed in 0.033056

In[ ]:=
```
GUdd // MatrixForm
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GUdd[[4, 4, 2]];
GUdd[[4, 3, 4]];
GUdd[[4, 4, 3]];
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \begin{pmatrix} 0 \\ \frac{Rs}{2\,r\,(r-Rs)} \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} \frac{Rs}{2\,r\,(r-Rs)} \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} \frac{(r-Rs)\,Rs}{2\,r^3} \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ -\frac{Rs}{2\,r\,(r-Rs)} \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ -r+Rs \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ -(r-Rs)\,\mathrm{Sin}[\theta]^2 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ \frac{1}{r} \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ \frac{1}{r} \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta] \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{r} \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mathrm{Cot}[\theta] \end{pmatrix} & \begin{pmatrix} 0 \\ \frac{1}{r} \\ \mathrm{Cot}[\theta] \\ 0 \end{pmatrix} \end{pmatrix}$$

The geodesic equations are therefore $\frac{du^{\mu}}{d\lambda} + \Gamma^{\mu}{}_{\rho\sigma} U^{\rho} U^{\sigma} = 0$

$$\frac{du^0}{d\lambda} + 2\,\Gamma^0{}_{01} U^0 U^1 = 0$$

$$\frac{du^1}{d\lambda} + \Gamma^1{}_{00} U^0 U^0 + \Gamma^1{}_{11} U^1 U^1 + \Gamma^1{}_{22} U^2 U^2 + \Gamma^1{}_{33} U^3 U^3 = 0$$

$$\frac{du^2}{d\lambda} + 2\,\Gamma^2{}_{21} U^2 U^1 + \Gamma^2{}_{33} U^3 U^3 = 0$$

$$\frac{du^3}{d\lambda} + 2\,\Gamma^3{}_{23} U^2 U^3 + 2\,\Gamma^3{}_{13} U^1 U^3 = 0$$

Yielding,

$$\frac{du^0}{d\lambda} + 2\,\frac{Rs}{2\,r\,(r-Rs)} U^0 U^1 = 0$$

$$\frac{du^1}{d\lambda} + \frac{(r-Rs)\,Rs}{2\,r^3} U^0 U^0 + -\frac{Rs}{2\,r\,(r-Rs)} U^1 U^1 + (-r + Rs) U^2 U^2 - (r - Rs)\,Sin[\theta]^2 U^3 U^3 = 0$$

$$\frac{du^2}{d\lambda} + \frac{2}{r} U^2 U^1 - Cos[\theta]\,Sin[\theta] U^3 U^3 = 0$$

$$\frac{du^3}{d\lambda} + 2\,Cot[\theta] U^2 U^3 + \frac{2}{r} U^1 U^3 = 0$$

The gravitational acceleration is defined as $-\frac{GM}{r^2}$ (towards the mass) and considering that the spherically symmetric mass $U^2 = U^3 = 0$ and since its slow moving $U^1 \simeq 0$ and the velocity through time (U^0) will be 1

Hence the geodesic for the radial direction reduces too:

$$\frac{du^1}{d\lambda} + \frac{(r-Rs)\,Rs}{2\,r^3} = 0 \Rightarrow a_r = -\frac{(r-Rs)\,Rs}{2\,r^3}$$

And we can equate the radial acceleration now we are in the Newtonian limit.

$$a_r = -\frac{(r-Rs)\,Rs}{2\,r^3} = -\frac{GM}{r^2}$$

Now consider if we were a away from the black hole $r >> R_s$ then $(r - R_s) \simeq r$.

$$\frac{r\,Rs}{2\,r^3} = \frac{GM}{r^2} \Rightarrow R_s = 2\,GM$$

Where M is the mass of the object.

Which gives us the metric of.

$$ds^2 = -\left(1 - \frac{2\,GM}{r}\right) dt^2 + \left(1 - \frac{2\,GM}{r}\right)^{-1} dr^2 + r^2\,d\Omega^2$$

B) In the limit where $M \to 0$ the metric reduces too.

$$-1\,dt^2 + 1\,dr^2 + r^2\,d\Omega^2$$

Which is flat space as expected.

C) As $r \to \infty$ the metric similarly reduces too

$$-1\,dt^2 + 1\,dr^2 + r^2\,d\Omega^2$$

Which makes sense as we get further from the mass the spacetime should become flat and if the mass disappears we should also expect spacetime to become flat.