Designing and implementing a quantum circuit, Bell state, to implement a measurement of |S| used to test the CHSH inequality.

Designing my circuit

Using IBM Quantum Platform, firstly, I started with the Hadamard gate which is defined by:

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

The Hadamard gate puts the qubit in a superposition of the states $|0\rangle$ and $|1\rangle$

Starting on q[0], the circuit with the logical state $|0\rangle$ we get:

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

Next to q[1] I add the NOT gate defined by:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Which changes the input state to the opposite output state.

On q[0] I now placed the Pauli-Z gate, defined by:

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The Pauli-X gate is equivalent to rotating the qubit state around the x-axis of the Bloch sphere by π radians. Which changes the input state to the opposite output state, for example,

$$X|0\rangle = |1\rangle$$
$$X|1\rangle = |0\rangle$$

In the logical basis, if a qubit is described by $|1\rangle$, the Pauli-Z gate provides that qubit a phase shift of $\emptyset = \pi$. Hence $Z|1\rangle = -|1\rangle$.

Next, I placed the CNOT gate with control on q[0] and target on q[1] and this is defined by:

$$\mathrm{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The CNOT gate performs a NOT operation on the target qubit only if the control qubit is in the state.

For example,

$$\mathrm{CNOT}(|00\rangle) = |00\rangle$$

$$\mathrm{CNOT}(|01\rangle) = |01\rangle$$

$$\mathrm{CNOT}(|10\rangle) = |11\rangle$$

$$\mathrm{CNOT}(|11\rangle) = |10\rangle$$

Finally I added measurement in the standard basis onto both q[0] and q[1]. This is used as you would expect, to implement any measurement when combined with gates to qubit $|00\rangle$:

Applying a Hadamard Gate to q[0]:

$$H(|00\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

Applying a NOT (Pauli-X) Gate to q[1]:

$$X(\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)) = \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle)$$

Applying a Pauli-Z Gate to q[0]:

$$Z(\frac{1}{\sqrt{2}}(|01\rangle + |11\rangle)) = \frac{1}{\sqrt{2}}(|01\rangle - |11\rangle)$$

Applying a CNOT Gate with Control on q[0] and Target on q[1]:

$$\mathrm{CNOT}(\frac{1}{\sqrt{2}}(|01\rangle - |11\rangle)) = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

After all these operations, the final state of the 2-qubit system is:

$$\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

Please see attached diagram for visual reference.

Implementing the rotations

The gates I used to implement the rotation of the qubits was the RY gates. I chose this gate as the correlation function corresponds to a rotation of the qubit around the y-axis of the Bloch sphere. I implemented this gate into my circuit just before I took measurements on q[0] and q[1]. The rotation of the state vector of the qubits around the y-axis of the Bloch sphere relates to the rotation of the measurement basis. As the rotation changes the probabilities onto the measurement basis, which in turn changes the measurement outcomes. By measuring the correlation function $D(\theta_1, \theta_2)$ in the rotated basis, you are measuring the correlation between the two qubits after they have been rotated around the y-axis of the Bloch sphere. The correlation function can be related to the predicted correlation function $C(\theta_1, \theta_2)$ for Bell's state. $|\emptyset> = \frac{1}{\sqrt{2}}$ $(|01> -|10>)$, which is given by $C(\theta_1, \theta_2) = -\cos(\theta_1 - \theta_2)$. Thus, we can test the quantum mechanical prediction for the state $|\emptyset>$ and verify its validity.

Tests made prior to running the experiment over the quantum computer

Firstly, I ran my circuit on the simulator, namely simulator_statevector, this operates very similar to an actual IBM quantum computer except it is fully simulated and predicted the outcomes rather than an official test. Once I created the original circuit tested, which was the Bell state, I was able to measure, via histogram the frequency between $|00>$, $|01>$, $|10>$ and $|00>$. Thankfully it seemed to work. I used a combination of angles of which I chose eight pares of. The angles will be stated later in a table. I ran the test first to ensure I got reasonable outcomes for the angles I selected and to ensure I had the correct gates in order to create the Bell state. For example, the circuit I explained above, a pair of angles I simulated for the RY gate was $\theta_1 = \frac{\pi}{4}$ and $\theta_2 = 0$. I measured a frequency of 533 of outcome for $|10>$ and a frequency of 491 of outcome $|01>$. Please note this was via the simulator not the quantum computer. So $D(\theta_1, \theta_2) = -\cos(\theta_1 - \theta_2)$. I expect that.

$$D\left(-\frac{\pi}{4}, 0\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

So, the value of $D\left(-\frac{\pi}{4}, 0\right)$was not exactly $\frac{1}{\sqrt{2}}$ which I expected for the correlation function $C\left(-\frac{\pi}{4}, 0\right) = \cos\left(-\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$. This could be due to some estimate of the measurement uncertainty. So, comparing the quantum computed actual results.

$$D\left(-\frac{\pi}{4}, 0\right) = \frac{417 + 390 - 92 - 125}{1024} = 0.58$$

And

$$\Delta D\left(-\frac{\pi}{4}, 0\right) = \frac{417 + 390 + 92 + 125}{1024} = \frac{1}{\sqrt{1024}} = 0.03$$

Hence, $\Delta D\left(-\frac{\pi}{4}, 0\right) = 0.58 \pm 0.03$

The results cover a range of 0.55 to 0.61, which is inconsistent with the value:

$$C\left(-\frac{\pi}{4}, 0\right) = \frac{1}{\sqrt{2}} = 0.707.$$

So, I can place this with error, perhaps the shot noise error.

Parameters used

The choice of parameters used are as follows:

I chose to make eight data points ranging from 0 to π. I felt eight was a good amount as I needed enough data for a table and to plot graphically but also enough data to be able to make a conclusion for if my circuit is sound. If my $D(\theta_1, \theta_2)$ compares well, or not, with $C(\theta_1, \theta_2)$ and to see if the CHSH sum fits well with Bells inequality. However, for example, 1000 data points would be too much and take too long to measure. I need to be able to read and interpret the data correctly, without clutter and human/computing error. Also, that amount would backlog IBM considerably, in which I have only five attempts on each platform at one time. This amount would be inconsiderate for other users of IBM. The angles I picked were picked for simplicity. I would have tried more complex angles though IBM has restricted angles.

The appropriate angles I chose were:

$$0, \frac{\pi}{8}, \frac{\pi}{4}, \frac{\pi}{2} \text{ and } \pi$$

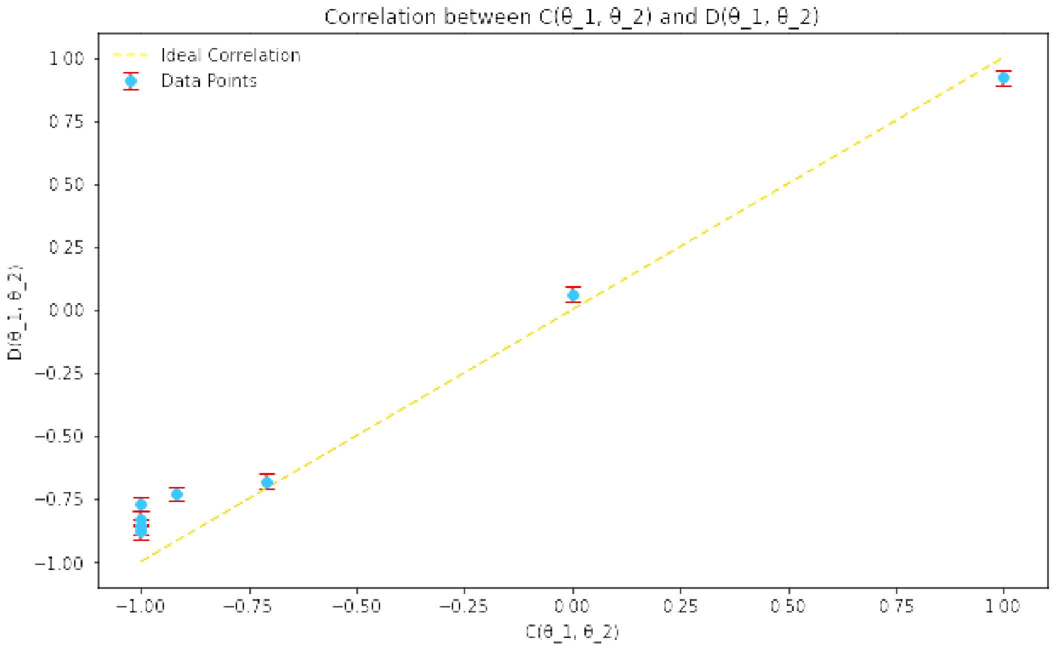
I kept my shots controlled at 1024 shots on each data point as again I felt that it was sufficient without capping the shots but also receiving enough data to make sound judgement.

The angles chosen to calculate the inequality were 0, $\frac{\pi}{4}$, $-\frac{\pi}{4}$ and $\frac{\pi}{2}$, as you will see later also.

In range	θ_1/rad	θ_2/rad	$C(\theta_1, \theta_2)$/rad	$D(\theta_1, \theta_2)$/rad	error
No	0	0	-1	-0.88	0.03
No	$\frac{\pi}{8}$	0	-0.92	-0.73	0.03
Yes	$\frac{\pi}{4}$	0	-0.71	-0.68	0.03
No	$\frac{\pi}{2}$	0	0	0.06	0.03
No	$\frac{\pi}{8}$	$\frac{\pi}{8}$	-1	-0.86	0.03
No	$\frac{\pi}{4}$	$\frac{\pi}{4}$	-1	-0.77	0.03
No	$\frac{\pi}{2}$	$\frac{\pi}{2}$	-1	-0.83	0.03
No	π	0	1	0.92	0.03

As you can see from the table above, the majority of the results are just outside of the predicted $C(\theta_1, \theta_2)$, but lie outside regardless. I believe my circuit is valid as my results were closely predicted. I would like to note here, that I originally pared most of θ_1 and θ_2 with the same values as for simplicity. With hindsight I would have probably received more interesting and calculable values if I had mixed up my paring. I could place some inaccuracies due but not limited to decoherence: Quantum states are extremely sensitive to their environment. Interactions with the external world cause the qubit states to decohere, essentially losing their quantum nature over time. Initialization errors: errors can occur when preparing the initial state of the qubits. If the qubits aren't perfectly set to the $|0\rangle$ state, this can cause deviations in the final state. Thermal fluctuations: even though quantum computers are kept at extremely low temperatures to reduce thermal noise, residual thermal effects can still impact qubit stability. Cross-talk: when multiple qubits are used, unwanted interactions between adjacent qubits can cause errors, especially in densely packed quantum processors. Also, when repeating this experiment alongside mixing up the data sets for θ_1 and θ_2. I would repeat this experiment more frequently over the quantum computer not just as a simulation.

A graph to show the relationship between $C(\theta_1, \theta_2)$ and $D(\theta_1, \theta_2)$



plotted using Matplotlib

Again, re-evaluating my graph and looking at the results plotted I realise I need to mix the pairs of θ_1 and θ_2 to be able to retain a more interesting and data driven graph, or alternatively used a wider range of angles for a bigger range of data.

Evaluation the CHSH inequality

My value of the CHSH sum as follows:

$$S = E(\theta_1, \theta_2) - E(\theta_1, \theta_2') + E(\theta_1', \theta_2) + E(\theta_1', \theta_2')$$

The expectation value $E(\theta_1, \theta_2)$ can be calculated as $C(\theta_1, \theta_2) \times D(\theta_1, \theta_2)$.

So as seen previously, the angles used to calculate the inequality were, $\theta_1 = 0, \theta_2 = \frac{\pi}{4}, \theta'_1 = -\frac{\pi}{4}, \theta'_2 = \frac{\pi}{2}$. I chose these values because we can quantify the amount by which the CHSH sum violates Bells inequality using the concurrence. T. The three states under these angles have different entanglement when $t = 0$, the state has no entanglement and satisfies $|S| \leq 2$. While $T = \frac{1}{2}$ is a superposition of a product state and an entangled one. When $T = 1$ it allows the state to be maximally entangled. The values C and D are as also mentioned previously as $C = \frac{1}{\sqrt{2}}$, and $D = 0.58$.

Substituting these values, we get:

$$S = E(0, 45) - E(0, -45) + E(90, 45) + E(90, -45)$$

$$S = C(0, 45) \times D(0, 45) - C(0, -45) \times D(0, -45)$$

$$+ C(90, 45) \times D(90, 45) + C(90, -45) \times D(90, -45)$$

$$S = \frac{1}{\sqrt{2}} \times 0.58 - \frac{1}{\sqrt{2}} \times 0.58 + \frac{1}{\sqrt{2}} \times 0.58 + \frac{1}{\sqrt{2}} \times 0.58$$

$$S = 4 \times \left(\frac{1}{\sqrt{2}} \times 0.58\right)$$

$$S = 2 \times \left(\sqrt{2} \times 0.58\right)$$

$$S = 2 \times 0.82$$

$$S = 1.64$$

According to quantum mechanics, S can range from $-\frac{2}{\sqrt{2}}$ to $\frac{2}{\sqrt{2}}$. In classical mechanics, the Bell inequality states that S should be between -2 and 2. In this case, the value of $S = 1.64$ falls within both ranges, not showing a violation of the classical or quantum limits. Thus, the calculated S value doesn't help discriminate between classical and quantum theories but is consistent with both.

The CHSH circuit I ran over IBM is as follows:

θ_1/rad	θ_2/rad	θ'_1/rad	θ'_2/rad	$C(\theta_1,\theta_2)$/rad	$D(\theta_1,\theta_2)$/rad	error	f_{00}	f_{11}	f_{01}	f_{10}
0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$-\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	0.60	0.03	425	393	90	116

So as:

$$\begin{aligned} D(\theta\_1,\theta\_2\,) &= (f\_00 + f\_11 - f\_01 - f\_10)/1024 \\ &= \ (425 + 393 - \ 90 - 116)/1024 \\ &= 0.60 \end{aligned}$$

$$\begin{aligned} \Delta D(\theta_1,\theta_2) &= \frac{1}{\sqrt{1024}} \\ &= 0.03 \end{aligned}$$

$$S = 0.60\ \pm 0.03$$

And such local realism is included. The significance of the CHSH result provides experimental evidence that the world is quantum mechanical, and that local realism is not a valid description of reality.