Task_2a

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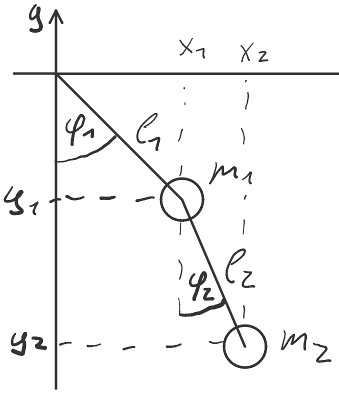

$$l_1 = l_2 = l \qquad m_1 = m_2 = m$$

$$x_1 = l \sin\varphi_1$$
$$x_2 = l\cdot(\sin\varphi_1 + \sin\varphi_2)$$
$$y_1 = -l\cos\varphi_1$$
$$y_2 = -l(\cos\varphi_1 + \cos\varphi_2)$$

$$\dot{x}_1 = l\,\dot\varphi_1 \cos\varphi_1 \qquad \dot{y}_1 = l\,\dot\varphi_1 \sin\varphi_1$$
$$\dot{x}_2 = l\cdot(\dot\varphi_1\cos\varphi_1 + \dot\varphi_2\cos\varphi_2) \qquad \dot{y}_2 = l\,(\dot\varphi_1 \sin\varphi_1 + \dot\varphi_2 \sin\varphi_2)$$

$$T = T_1 + T_2 = \frac{m}{2}\left(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2\right) \qquad V = V_1 + V_2 = mg\,(y_1 + y_2)$$

$$T = \frac{m}{2} l^2 \cdot \Big(\dot\varphi_1^2\cos^2\varphi_1 + \dot\varphi_1^2 \sin^2\varphi_1 + \dot\varphi_1^2\cos^2\varphi_1 + 2\dot\varphi_1\dot\varphi_2 \cos\varphi_1\cos\varphi_2 + \dot\varphi_2^2\cos^2\varphi_2 +$$
$$+ \dot\varphi_1^2 \sin^2\dot\varphi_1 + 2\dot\varphi_1\dot\varphi_2 \sin\varphi_1 \sin\varphi_2 + \dot\varphi_2^2 \sin^2\varphi_2\Big)$$

$$T = \frac{m}{2} l^2 \Big(\dot\varphi_1^2 + \dot\varphi_1^2 + 2\dot\varphi_1\dot\varphi_2 \frac{\cos(\varphi_1-\varphi_2)+\cos(\varphi_1+\varphi_2)}{2} + \dot\varphi_2^2 + 2\dot\varphi_1\dot\varphi_2 \frac{\cos(\varphi_1-\varphi_2)-\cos(\varphi_1+\varphi_2)}{2}$$

$$T = \frac{m}{2} l^2 \left(2\dot\varphi_1^2 + \dot\varphi_2^2 + 2\dot\varphi_1\dot\varphi_2 \cos(\varphi_1-\varphi_2)\right)$$

$$\vec\varphi = \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}, \quad \dot{\vec\varphi} = \begin{pmatrix}\dot\varphi_1\\ \dot\varphi_2\end{pmatrix}$$

$$V = mg\,(y_1 + y_2) = -mgl\,(2\cos\varphi_1 + \cos\varphi_2)$$

$$\boxed{\mathcal{L}(\vec\varphi, \dot{\vec\varphi}) = T - V} = \frac{ml^2}{2}\left(2\dot\varphi_1^2 + \dot\varphi_2^2 + 2\dot\varphi_1\dot\varphi_2\cos(\varphi_1-\varphi_2)\right) + mgl\,(2\cos\varphi_1 + \cos\varphi_2)$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial\mathcal{L}(\vec\varphi,\dot{\vec\varphi})}{\partial\dot\varphi_i}\right) - \frac{\partial\mathcal{L}(\vec\varphi,\dot{\vec\varphi})}{\partial\varphi_i} = 0, \quad i = 1, 2}$$

$\underline{\varphi_1:}$

$$\partial_{\dot\varphi_1}\mathcal{L}(\vec\varphi,\dot{\vec\varphi}) = \frac{ml^2}{2}\left(4\dot\varphi_1 + 2\dot\varphi_2\cos(\varphi_1-\varphi_2)\right) = ml^2\left(2\dot\varphi_1 + \dot\varphi_2\cos(\varphi_1-\varphi_2)\right)$$

$$\frac{d}{dt}\left(\partial_{\dot\varphi_1}\mathcal{L}(\vec\varphi,\dot{\vec\varphi})\right) = ml^2\left(2\ddot\varphi_1 + \ddot\varphi_2\cos(\varphi_1-\varphi_2) - \dot\varphi_2\cdot(\dot\varphi_1 - \dot\varphi_2)\sin(\varphi_1-\varphi_2)\right)$$

$$\partial_{\varphi_1}\mathcal{L}(\vec\varphi,\dot{\vec\varphi}) = \frac{ml^2}{2}\left(2\dot\varphi_1\dot\varphi_2\cdot(-\sin(\varphi_1-\varphi_2))\right) + mgl\cdot 2\cdot(-\sin\varphi_1) = -ml^2\dot\varphi_1\dot\varphi_2\sin(\varphi_1-\varphi_2) - 2mgl\sin\varphi_1$$

$$0 = ml^2\left(2\ddot\varphi_1 + \ddot\varphi_2\cos(\varphi_1-\varphi_2) + \dot\varphi_2^2\sin(\varphi_1-\varphi_2)\right) + 2mgl\sin\varphi_1 \quad /: ml^2$$

$$0 = 2\ddot\varphi_1 + \ddot\varphi_2\cos(\varphi_1-\varphi_2) + \dot\varphi_2^2\sin(\varphi_1-\varphi_2) + 2\frac{g}{l}\sin\varphi_1$$

$\underline{\varphi_2:}$

$$\mathcal{L}(\vec{\varphi},\dot{\vec{\varphi}}) = T - V = \frac{ml^2}{2}\left(2\dot{\varphi}_1^2 + \dot{\varphi}_2^2 + 2\dot{\varphi}_1\dot{\varphi}_2\cos(\varphi_1-\varphi_2)\right) + mgl\,(2\cos\varphi_1 + \cos\varphi_2)$$

$$\begin{cases} \partial_{\dot{\varphi}_2}\mathcal{L}(\vec{\varphi},\dot{\vec{\varphi}}) = \frac{ml^2}{2}\left(2\dot{\varphi}_2 + 2\dot{\varphi}_1\cos(\varphi_1-\varphi_2)\right) = ml^2\left(\dot{\varphi}_2 + \dot{\varphi}_1\cos(\varphi_1-\varphi_2)\right) \\ \frac{d}{dt}\left(\partial_{\dot{\varphi}_2}\mathcal{L}(\vec{\varphi},\dot{\vec{\varphi}})\right) = ml^2\left(\ddot{\varphi}_2 + \ddot{\varphi}_1\cos(\varphi_1-\varphi_2) - \dot{\varphi}_1^2\sin(\varphi_1-\varphi_2) + \dot{\varphi}_1\dot{\varphi}_2\sin(\varphi_1-\varphi_2)\right) \\ \partial_{\varphi_2}\mathcal{L}(\vec{\varphi},\dot{\vec{\varphi}}) = +ml^2\varphi_1\varphi_2\sin(\varphi_1-\varphi_2) - mgl\sin\varphi_2 \end{cases}$$

$$0 = ml^2\left(\ddot{\varphi}_2 + \ddot{\varphi}_1\cos(\varphi_1-\varphi_2) - \dot{\varphi}_1^2\sin(\varphi_1-\varphi_2)\right) + mgl\sin\varphi_2 \quad / : ml^2$$

$$0 = \ddot{\varphi}_2 + \ddot{\varphi}_1\cos(\varphi_1-\varphi_2) - \dot{\varphi}_1^2\sin(\varphi_1-\varphi_2) + \frac{g}{l}\sin\varphi_2$$

$$\boxed{\omega^2 = \frac{g}{l}}$$

$$\text{I}:\quad 0 = 2\ddot{\varphi}_1 + \ddot{\varphi}_2\cos(\varphi_1-\varphi_2) + \dot{\varphi}_2^2\sin(\varphi_1-\varphi_2) + 2\omega^2\sin\varphi_1$$

$$\text{II}:\quad 0 = \ddot{\varphi}_2 + \ddot{\varphi}_1\cos(\varphi_1-\varphi_2) - \dot{\varphi}_1^2\sin(\varphi_1-\varphi_2) + \omega^2\sin\varphi_2$$

$$\boxed{p_i = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}_i}}\ ,\quad \boxed{\tilde{p}_i = \frac{p_i}{ml^2}}\quad \vec{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \xrightarrow[\text{Transform.}]{\text{Legendre}} \begin{cases} \dot{p}_i = \frac{\partial H}{\partial\varphi_i} \\ \dot{\varphi}_i = \frac{\partial H}{\partial p_i} \end{cases},\quad \frac{\partial\tilde{p}_i}{\partial p_i} = \frac{1}{ml^2}$$

$$p_1 = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}_1} = ml^2\left(2\dot{\varphi}_1 + \dot{\varphi}_2\cos(\varphi_1-\varphi_2)\right) \rightarrow \dot{\varphi}_1 = \frac{p_1}{2ml^2} - \frac{1}{2}\dot{\varphi}_2\cos(\varphi_1-\varphi_2) = \frac{1}{2}\left(\tilde{p}_1 - \dot{\varphi}_2\cos(\varphi_1-\varphi_2)\right)$$

$$p_2 = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}_2} = ml^2\left(\dot{\varphi}_2 + \dot{\varphi}_1\cos(\varphi_1-\varphi_2)\right) \rightarrow \dot{\varphi}_2 = \frac{p_2}{ml^2} - \dot{\varphi}_1\cos(\varphi_1-\varphi_2) = \tilde{p}_2 - \varphi_1\cos(\varphi_1-\varphi_2)$$

$$p_2 = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}_2} = ml^2\left(\dot{\varphi}_2 + \dot{\varphi}_1\cos(\varphi_1-\varphi_2)\right) = ml^2\cdot\left(\dot{\varphi}_2 + \frac{1}{2}\left(\tilde{p}_1 - \dot{\varphi}_2\cos(\varphi_1-\varphi_2)\right)\cos(\varphi_1-\varphi_2)\right) =$$

$$p_2 = \dot{\varphi}_2\cdot\left(ml^2 - \frac{1}{2}ml^2\cos^2(\varphi_1-\varphi_2)\right) + \frac{ml^2}{2}\tilde{p}_1\cos(\varphi_1-\varphi_2) \quad / - \frac{ml^2}{2}\tilde{p}_1\cos(\varphi_1-\varphi_2) / : \dot{\varphi}_2\ \text{term}$$

$$\dot{\varphi}_2 = \frac{p_2 - \frac{ml^2}{2}\tilde{p}_1\cos(\varphi_1-\varphi_2)}{ml^2\left(1 - \frac{1}{2}\cos^2(\varphi_1-\varphi_2)\right)} = \frac{2\tilde{p}_2 - \tilde{p}_1\cos(\varphi_1-\varphi_2)}{2-\cos^2(\varphi_1-\varphi_2)}$$

$[\sin^2 x + \cos^2 x = 1 \rightarrow 2 - \cos^2 x = \sin x + 1$

$$\underline{\underline{\dot{\varphi}_2 = \frac{2\tilde{p}_2 - \tilde{p}_1\cos(\varphi_1-\varphi_2)}{1+\sin^2(\varphi_1-\varphi_2)}}}$$

$$p_1 = 2ml^2\dot\varphi_1 + ml^2\dot\varphi_2\cos(\varphi_1-\varphi_2) = 2ml^2\dot\varphi_1 + ml^2\tilde p_2\cos(\varphi_1-\varphi_2) - ml^2\dot\varphi_1\cos^2(\varphi_1-\varphi_2)$$

$$= ml^2\tilde p_2\cos(\varphi_1-\varphi_2) + \dot\varphi_1\left(2ml^2 - ml^2\cos^2(\varphi_1-\varphi_2)\right)$$

$$\hookrightarrow \dot\varphi_1 = \frac{ml^2(\tilde p_1 - \tilde p_2\cos(\varphi_1-\varphi_2))}{ml^2(2-\cos^2(\varphi_1-\varphi_2))} = \underline{\underline{\frac{\tilde p_1 - \tilde p_2\cos(\varphi_1-\varphi_2)}{1+\sin^2(\varphi_1-\varphi_2)}}}$$

$$\boxed{H(\vec\varphi,\vec p) = \sum_{i=1}^{2}\dot\varphi_i p_i - \mathcal{L}(\vec\varphi,\dot{\vec\varphi})}$$

$$= \dot\varphi_1 p_1 + \dot\varphi_2 p_2 - \frac{ml^2}{2}\left(2\dot\varphi_1^2 + \dot\varphi_2^2 + 2\dot\varphi_1\dot\varphi_2\cos(\varphi_1-\varphi_2)\right) - mgl\,(2\cos\varphi_1 + \cos\varphi_2)$$

$$= \frac{ml^2}{2}\cdot\Big(2\dot\varphi_1\cdot\left[2\dot\varphi_1 + \dot\varphi_2\cos(\varphi_1-\varphi_2)\right] + 2\dot\varphi_2\cdot\left[\dot\varphi_2 + \dot\varphi_1\cos(\varphi_1-\varphi_2)\right] - 2\dot\varphi_1^2 - \dot\varphi_2^2 - 2\dot\varphi_1\dot\varphi_2\cos(\varphi_1-\varphi_2)\Big)$$
$$- mgl\,(2\cos\varphi_1 + \cos\varphi_2) =$$

$$= ml^2\left(\dot\varphi_1^2 + \dot\varphi_2^2 + \dot\varphi_1\dot\varphi_2\cos(\varphi_1-\varphi_2)\right) - mgl\,(2\cos\varphi_1 + \cos\varphi_2)$$

$$\dot p_1 = \frac{\partial H}{\partial\varphi_1} = -ml^2\dot\varphi_1\dot\varphi_2\cdot\sin(\varphi_1-\varphi_2) - 2mgl\sin\varphi_1 \xrightarrow{:ml^2} \dot{\tilde p}_1 = -\dot\varphi_1\dot\varphi_2\sin(\varphi_1-\varphi_2) - 2\omega^2\sin\varphi_1$$

$$\dot p_2 = \frac{\partial H}{\partial\varphi_2} = +ml^2\dot\varphi_1\dot\varphi_2\sin(\varphi_1-\varphi_2) - mgl\sin\varphi_1 \xrightarrow{:ml^2} \dot{\tilde p}_2 = \dot\varphi_1\dot\varphi_2\sin(\varphi_1-\varphi_2) - \omega^2\sin\varphi_1$$

$$\dot\varphi_1\dot\varphi_2\sin(\varphi_1-\varphi_2) = \frac{\tilde p_1 - \tilde p_2\cos(\varphi_1-\varphi_2)}{1+\sin^2(\varphi_1-\varphi_2)}\cdot\frac{2\tilde p_2 - \tilde p_1\cos(\varphi_1-\varphi_2)}{1+\sin^2(\varphi_1-\varphi_2)}\cdot 2ml^2\sin(\varphi_1-\varphi_2)$$

$$= \frac{2\tilde p_1\tilde p_2 - \tilde p_1^2\cos(\varphi_1-\varphi_2) - 2\tilde p_2^2\cos(\varphi_1-\varphi_2) + \tilde p_1\tilde p_2\cos^2(\varphi_1-\varphi_2)}{(1+\sin^2(\varphi_1-\varphi_2))^2}\cdot 2ml^2\sin(\varphi_1-\varphi_2)$$

$$= \left(\frac{\tilde p_1\tilde p_2\,(2-\cos^2(\varphi_1-\varphi_2))}{(1+\sin^2(\varphi_1-\varphi_2))^2} - \frac{(\tilde p_1^2 + 2\tilde p_2)\cos(\varphi_1-\varphi_2) - 2\tilde p_1\tilde p_2\cos^2(\varphi_1-\varphi_2)}{(1+\sin^2(\varphi_1-\varphi_2))^2}\right)\cdot 2ml^2\sin(\varphi_1-\varphi_2)$$

$$= \frac{\tilde p_1\tilde p_2\sin(\varphi_1-\varphi_2)}{1+\sin^2(\varphi_1-\varphi_2)} - \frac{\tilde p_1^2 + 2\tilde p_2^2 - 2\tilde p_1\tilde p_2\cos(\varphi_1-\varphi_2)}{(1-\sin^2(\varphi_1-\varphi_2))^2}\cdot\cos(\varphi_1-\varphi_2)\cdot\sin(\varphi_1-\varphi_2)$$

$$A = \frac{\tilde{p}_1 \tilde{p}_2 \sin(\varphi_1 - \varphi_2)}{1 + \sin^2(\varphi_1 - \varphi_2)}$$

$$B = \frac{\tilde{p}_1^2 + 2\tilde{p}_2^2 - 2\tilde{p}_1 \tilde{p}_2 \cos(\varphi_1 - \varphi_2)}{\left(1 - \sin^2(\varphi_1 - \varphi_2)\right)^2} \cdot \cos(\varphi_1 - \varphi_2) \cdot \sin(\varphi_1 - \varphi_2)$$

$$\dot{\tilde{p}}_1 = B - A - 2\omega^2 \sin\varphi_1$$

$$\dot{\tilde{p}}_2 = A - B - \omega^2 \sin\varphi_1$$

$$\dot{\varphi}_1 = \frac{\tilde{p}_1 - \tilde{p}_2 \cos(\varphi_1 - \varphi_2)}{1 + \sin^2(\varphi_1 - \varphi_2)}$$

$$\dot{\varphi}_2 = \frac{2\tilde{p}_2 - \tilde{p}_1 \cos(\varphi_1 - \varphi_2)}{1 + \sin^2(\varphi_1 - \varphi_2)}$$