15

Open quantum systems

Most textbooks on quantum mechanics deal exclusively, or almost exclusively, with the time evolution of closed systems, and up to now this book has been no exception, apart from a glimpse of nonunitary evolution in Section 14.4.1. The time evolution of closed systems is governed by the Schrödinger equation (4.11) or its integral form (4.14). However, a closed system is an idealization, and in practice all quantum systems (except maybe the Universe as a whole) are in contact with some kind of environment. The Hilbert space of states is then a tensor product $\mathcal{H}_A \otimes \mathcal{H}_E$, where $\mathcal{H}_A$ ($\mathcal{H}_E$) is the Hilbert space of states of the system $\mathcal{A}$ (environment $\mathcal{E}$). In Chapter 6 we learned that the state operator ρ_A of $\mathcal{A}$ is obtained by taking the trace over the degrees of freedom of $\mathcal{E}$ (see (6.30)), and the time evolution of ρ_A is not unitary: it is not governed by (6.37) with a Hermitian Hamiltonian. The von Neumann entropy $\mathrm{Tr}[\rho_A \ln \rho_A]$, which is constant for unitary evolution, is time-dependent when the system is not closed. In general, it increases because information is leaking into the environment, and irreversible behavior is observed because we are not able to control the degrees of freedom of the environment. As just mentioned, in Section 14.4.1 we gave a first example of nonunitary evolution, the system being a two-level atom and the environment the quantized electromagnetic field. In the present chapter we wish to give a general approach to the theory of quantum systems which are not closed, or *open quantum systems*.

Let us introduce the subject by looking at a specific (but very important) case, the time evolution of an open two-level system. In order that consistent notation be used throughout this chapter, we borrow the notation of quantum information (Section 6.4.2) and call $|0\rangle$ and $|1\rangle$ the basis vectors of the two-level system, with a "free" Hamiltonian H_0

$$H_0 = -\frac{1}{2}\,\hbar\omega_0\sigma_z, \tag{15.1}$$

so that the eigenstates of H_0 are $|0\rangle$ and $|1\rangle$:

$$H_0|0\rangle = -\frac{1}{2}\,\hbar\omega_0|0\rangle, \quad H_0|1\rangle = \frac{1}{2}\,\hbar\omega_0|1\rangle. \tag{15.2}$$

Then $\hbar\omega_0$ is the energy difference between the ground and excited states. The matrix elements ρ_{00} and $\rho_{11} = 1 - \rho_{00}$ of the state operator describe the populations of levels

$|0\rangle$ and $|1\rangle$, while $\rho_{01} = \rho_{10}^*$ describes the coherences. At thermal equilibrium with temperature T, the populations $\rho_{00}^{\rm eq}$ and $\rho_{11}^{\rm eq}$ are fixed by Boltzmann's law

$$\frac{\rho_{11}^{\rm eq}}{\rho_{00}^{\rm eq}} = \exp\left(-\frac{\hbar\omega_0}{k_{\rm B}T}\right). \tag{15.3}$$

For the sake of definiteness, we specialize to the NMR case (Section 5.2). If the proton spins were isolated, their time evolution would be governed by (6.37), where the Hamiltonian depends on the constant magnetic field $\vec{B}_0$ and the radiofrequency field $\vec{B}_1(t)$. As in Section 14.4.1, it is convenient to use the Bloch vector $\vec{b} = (u, v, -w)$ (6.24); $w = (\rho_{11} - \rho_{00})$ describes the population difference and (u, v) the coherences, $\rho_{01} = r = (u - {\rm i}v)/2$. If the spins were isolated from any kind of environment, the evolution equation (6.37) for ρ with the Hamiltonian (5.23) in terms of populations and coherences would read as

$$\begin{aligned} \dot{w} &= {\rm i}\omega_1 \left(r^* \, {\rm e}^{{\rm i}\omega t} - r \, {\rm e}^{-{\rm i}\omega t}\right), \\ \dot{r} &= {\rm i}\omega_0 r + \frac{{\rm i}\omega_1}{2} \, w \, {\rm e}^{{\rm i}\omega t}, \end{aligned} \tag{15.4}$$

where ω_1 is the Rabi frequency. The slight differences from (14.80)–(14.81) drop out in the rotating-wave approximation. In order to take into account the interaction with the environment in a phenomenological way, we follow Section 14.4.1 and supplement these equations by two relaxation terms

$$\begin{aligned} \dot{w} &= {\rm i}\omega_1 \left(r^* \, {\rm e}^{{\rm i}\omega t} - r \, {\rm e}^{-{\rm i}\omega t}\right) - \Gamma_1 (w - w_{\rm eq}), \\ \dot{r} &= {\rm i}\omega_0 r + \frac{{\rm i}\omega_1}{2} \, w \, {\rm e}^{{\rm i}\omega t} - \Gamma_2 \, r. \end{aligned} \tag{15.5}$$

These equations are the *Bloch equations* of NMR. The form of the relaxation term is not the most general one, but the approximations leading to (15.5) are usually justified: see the comments following (15.113). In order to give a physical interpretation of the new terms, let us assume that the radiofrequency field has been switched off at $t = 0$, so that $\omega_1 = 0$ for $t > 0$. Then the solution of (15.5) is

$$\begin{aligned} w(t) - w_{\rm eq} &= [w(t=0) - w_{\rm eq}] \, {\rm e}^{-\Gamma_1 t}, \\ r(t) &= r(t=0) \, {\rm e}^{{\rm i}\omega_0 t} \, {\rm e}^{-\Gamma_2 t}. \end{aligned} \tag{15.6}$$

The populations return to equilibrium with a relaxation time $T_1 = 1/\Gamma_1$, the longitudinal relaxation time, and the coherences with a relaxation time $T_2 = 1/\Gamma_2$, the transverse relaxation time introduced in Section 5.2. The main difference from (14.84)–(14.85) is that we now have two independent relaxation times,[1] while in (14.82)–(14.83) we had $\Gamma_2 = \Gamma_1/2 = \Gamma/2$. In the NMR case, T_1 and T_2 are of the order of a few seconds, and $T_2 \lesssim T_1$ (with $T_2 \ll T_1$ in most cases, for example $T_2 \sim 1\,{\rm ms}$ and $T_1 \sim 1\,{\rm s}$; see Levitt [2001]).

[1] Bloch equations with two independent relaxation times are also encountered in laser physics; see, e.g., Mandel and Wolf [1995], Chapter 18.

The chapter is organized as follows. In Section 15.1 we give some additional results on entanglement to supplement the more elementary approach of Chapter 6 by introducing the Schmidt decomposition of entangled states and the concept of positive operator-valued measure (POVM). Section 15.2 is devoted to establishing the general expression for the reduced state operator at time t as a function of its value at time $t = 0$, which we shall write in the Kraus form. Section 15.3 will address the particular but very important case where one is able to write the time evolution of the state operator in the form of a first-order differential equation in time, called a master equation. Finally, Section 15.4 will be devoted to the study of two models where the system of interest interacts with a thermal bath of harmonic oscillators. The first example will be that of a two-level atom and the second that of a Brownian particle. We shall derive master equations in both cases and examine their physical implications. The case of Brownian motion will be particularly important, as there we shall be able to understand the decoherence of the initially coherent superposition of two wave packets in the case of heavy particles, an example of a Schrödinger's cat.

15.1 Generalized measurements

15.1.1 Schmidt's decomposition

In this subsection, we give some further mathematical results on entangled states living in a Hilbert space of states[2] $\mathcal{H}_A \otimes \mathcal{H}_B$, in order to supplement the discussion of Chapter 6. Here $\mathcal{H}_A$ and $\mathcal{H}_B$, of dimensions N_A and N_B, are the Hilbert spaces of states of $\mathcal{A}$ and $\mathcal{B}$. The full state operator acting in $\mathcal{H}_A \otimes \mathcal{H}_B$ is denoted ρ_{AB}. We use Latin indices for $\mathcal{H}_A$ and Greek indices for $\mathcal{H}_B$, so that the matrix elements of ρ_{AB} are $\rho^{AB}_{m\mu;n\nu}$.[3] We have seen in (6.30) that the reduced state operator ρ_A of A is obtained by taking the trace over the B variables:

$$\rho_A = \mathrm{Tr}_{\mathcal{B}} \rho_{AB}, \quad \rho^A_{mn} = \sum_{\mu} \rho^{AB}_{m\mu;n\mu}. \tag{15.7}$$

Let $|\varphi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ be a pure state of the coupled $\mathcal{AB}$ system, and let $\{|m_a\rangle\}$ and $\{|\mu_B\rangle\}$ be two othonormal bases of $\mathcal{H}_A$ and $\mathcal{H}_B$. The most general decomposition of $|\varphi_{AB}\rangle$ on the basis $\{|m_A \otimes \mu_B\rangle\}$ of $\mathcal{H}_A \otimes \mathcal{H}_B$ reads

$$|\varphi_{AB}\rangle = \sum_{m,\mu} c_{m\mu} |m_A \otimes \mu_B\rangle. \tag{15.8}$$

Defining the vectors $|\tilde{m}_B\rangle \in \mathcal{H}_B$ as

$$|\tilde{m}_B\rangle = \sum_{\mu} c_{m\mu} |\mu_B\rangle,$$

[2] For the time being we do not think of system $\mathcal{B}$ as necessarily being an environment $\mathcal{E}$ for system $\mathcal{A}$.
[3] For clarity of notation, we use superscript AB when writing matrix elements.

we can rewrite (15.8) as

$$|\varphi_{AB}\rangle = \sum_m |m_A \otimes \tilde{m}_B\rangle. \tag{15.9}$$

Note that the set $\{|\tilde{m}_B\rangle\}$ need not form an orthonormal basis of $\mathcal{H}_B$. Now let us choose as a basis of $\mathcal{H}_A$ a set $\{|m_A\rangle\}$ which diagonalizes the reduced state operator ρ_A:

$$\rho_A = \mathrm{Tr}_{\mathcal{B}}|\varphi_{AB}\rangle\langle\varphi_{AB}| = \sum_{m=1}^{N_S} \mathsf{p}_m |m_A\rangle\langle m_A|. \tag{15.10}$$

If the number N_S of nonzero coefficients p_m is smaller than the dimension N_A of $\mathcal{H}_A$, we complete the set $\{|m_A\rangle\}$ by a set of $(N_A - N_S)$ orthonormal vectors, chosen to be orthogonal to the space spanned by the vectors $|m_A\rangle$ in (15.10). We use (6.34) to compute ρ_A from (15.9):

$$\rho_A = \sum_{m,n} \langle\tilde{n}_B|\tilde{m}_B\rangle |m_A\rangle\langle n_A|. \tag{15.11}$$

On comparing (15.10) and (15.11) we see that

$$\langle\tilde{n}_B|\tilde{m}_B\rangle = \mathsf{p}_m \delta_{mn},$$

and with our choice of basis $\{|m_A\rangle\}$ it turns out that the vectors $\{|\tilde{m}_B\rangle\}$ are, after all, orthogonal. To obtain an orthonormal basis, we only need to rescale the vectors $|\tilde{n}_B\rangle$

$$|n_B\rangle = \mathsf{p}_n^{-1/2} |\tilde{n}_B\rangle,$$

where we may assume that $\mathsf{p}_n > 0$ because, as explained above, it is always possible to complete the basis of $\mathcal{H}_B$ by a set of $(N_B - N_S)$ orthonormal vectors. We finally obtain Schmidt's decomposition of $|\varphi_{AB}\rangle$ on an orthonormal basis of $\mathcal{H}_A \otimes \mathcal{H}_B$:

$$\boxed{|\varphi_{AB}\rangle = \sum_n \mathsf{p}_n^{1/2} |n_A \otimes n_B\rangle.} \tag{15.12}$$

Any pure state $|\varphi_{AB}\rangle$ may be written in the form (15.12), but the bases $\{|n_A\rangle\}$ and $\{|n_B\rangle\}$ will of course depend on the state under consideration. If some of the p_n are equal, then the decomposition (15.12) is not unique, as is the case for the spectral decomposition of a Hermitian operator with degenerate eigenvalues. The reduced state operator ρ_B is readily computed from (6.35) using the orthogonality condition $\langle m_A|n_A\rangle = \delta_{mn}$:

$$\rho_B = \mathrm{Tr}_{\mathcal{A}}|\varphi_{AB}\rangle\langle\varphi_{AB}| = \sum_n \mathsf{p}_n |n_B\rangle\langle n_B|. \tag{15.13}$$

Comparing (15.10) and (15.13), we see that ρ_A and ρ_B have the same eigenvalues. The *Schmidt number* N_S is the number of nonzero eigenvalues of ρ_A (or ρ_B). A state $|\varphi_{AB}\rangle$ is a tensor product if and only if its Schmidt number is exactly equal to one. It is entangled whenever $N_S \geq 2$. If $N_A = N_B = N$, a *maximally entangled state* corresponds to $N_S = N$, $\mathsf{p}_n = 1/N$:

$$|\varphi_{AB}^{\max}\rangle = \frac{1}{\sqrt{N}} \sum_n \mathrm{e}^{\mathrm{i}\alpha(n)} |n_A \otimes n_B\rangle, \tag{15.14}$$

where $\exp[i\alpha(n)]$ is a phase factor. The Bell states

$$|\Phi_\pm\rangle = \frac{1}{\sqrt{2}}\big(|0_A \otimes 0_B\rangle \pm |1_A \otimes 1_B\rangle\big),$$
$$|\Psi_\pm\rangle = \frac{1}{\sqrt{2}}\big(|0_A \otimes 1_B\rangle \pm |1_A \otimes 0_B\rangle\big) \tag{15.15}$$

provide an example of maximally entangled states for $N_A = N_B = 2$. It can be verified directly that maximally entangled states have the property that the individual reduced state operators are proportional to the identity operators I_A and I_B. An important result is that a *local evolution* described by a unitary operator of the form $U_A \otimes U_B$ does not change the Schmidt number, because

$$(U_A \otimes U_B)|\varphi_{AB}\rangle = \sum_n \mathsf{p}_n^{1/2} |n'_A \otimes n'_B\rangle$$

with

$$|n'_A\rangle = U_A|n_A\rangle, \quad |n'_B\rangle = U_B|n'_B\rangle.$$

As a consequence, *a product state* (tensor product) *cannot be transformed into an entangled state through a local evolution* in which systems $\mathcal{A}$ and $\mathcal{B}$ evolve independently.[4] One needs nonlocal evolution, involving an interaction between the two systems, in order to entangle a state which is initially a product state. Conversely, one needs nonlocal evolution to disentangle an entangled state into a product state.

15.1.2 Positive operator-valued measures

In Chapter 4 we defined a maximal test of a quantum system whose state vector lives in a Hilbert space of dimension N as being a test with exactly N mutually exclusive outcomes, whose probabilities add up to one. Mathematically, a maximal test corresponds to defining N one-dimensional orthogonal projectors $\mathcal{P}_a$ adding up to the identity operator:

$$\mathcal{P}_a \mathcal{P}_b = \delta_{ab}, \quad \sum_{a=1}^{N} \mathcal{P}_a = I. \tag{15.16}$$

Because its eigenvalues are zero and one, $\mathcal{P}_a$ is a positive operator. If a physical property M_A of system A with nondegenerate eigenvalues λ_a is built up as

$$M_A = \sum_{a=1}^{N} \lambda_a \mathcal{P}_a,$$

then measuring M_A is equivalent to performing a maximal test. A set of projectors (15.16) is called a *von Neumann*, or *orthogonal*, *measurement.* Let ρ be the initial state operator

[4] In this chapter, "local" and "nonlocal" have the following meanings. Acting locally on a particle means that there is no interaction with the other particles, for example because the particle is far away from the others. Acting nonlocally means that there must be an interaction between this particle and other particles of the ensemble.

of a quantum system and let us perform a von Neumann measurement of M_A with result λ_a (or simply a). Recall from Section 6.2.5 that the probability $\mathsf{p}(a)$ of obtaining result a is

$$\mathsf{p}(a) = \mathrm{Tr}\,(\rho\,\mathcal{P}_a). \tag{15.17}$$

Then from the WFC postulate in the form given in Section 6.2.5 the state operator is transformed into

$$\rho \to \rho'' = \frac{\mathcal{P}_a\,\rho\,\mathcal{P}_a}{\mathrm{Tr}\,(\mathcal{P}_a\,\rho\,\mathcal{P}_a)}. \tag{15.18}$$

The denominator in (15.18) ensures that $\mathrm{Tr}\,\rho'' = 1$. If the measurement is not read (if λ_a is not observed), then the measurement destroys the coherences (see Appendix B):

$$\rho \to \rho' = \sum_{a=1}^{N} \mathcal{P}_a\,\rho\,\mathcal{P}_a. \tag{15.19}$$

The most efficient way of obtaining information on a quantum system is not always a von Neumann measurement (or a maximal test). We shall introduce generalized measurements by incorporating system $\mathcal{A}$ into a larger system $\mathcal{AB}$ and performing a joint measurement of a physical property M_{AB} acting in $\mathcal{H}_A \otimes \mathcal{H}_B$, assuming that the quantum state of $\mathcal{AB}$ is prepared as a tensor product[5] $\rho_A \otimes \rho_B$. Let us write a complete set of orthogonal projectors $\mathcal{P}_a$ (15.16) acting in $\mathcal{H}_A \otimes \mathcal{H}_B$; the probability of outcome a is

$$\mathsf{p}(a) = \mathrm{Tr}_{\mathcal{A}}\mathrm{Tr}_{\mathcal{B}}[\mathcal{P}_a(\rho_A \otimes \rho_B)] = \mathrm{Tr}_{\mathcal{A}}[\mathcal{Q}_a\rho_A], \tag{15.20}$$

with

$$\mathcal{Q}_a = \mathrm{Tr}_{\mathcal{B}}[\mathcal{P}_a\rho_B], \tag{15.21}$$

or, in terms of matrix elements (see Footnote 3),

$$\mathcal{Q}^a_{mn} = \sum_{\mu,\nu} \mathcal{P}^a_{m\mu;n\nu}\,\rho^B_{\nu\mu}. \tag{15.22}$$

The operators $\mathcal{Q}_a$ act in $\mathcal{H}_A$, and it is easy to check the following properties.

1. Hermiticity: $\mathcal{Q}_a = \mathcal{Q}_a^\dagger$. Indeed,

$$(\mathcal{Q}^a_{nm})^* = \sum_{\mu,\nu}(\mathcal{P}^a_{n\mu;m\nu})^*\,(\rho^B_{\nu\mu})^* = \sum_{\mu,\nu}\mathcal{P}^a_{m,\nu;n\mu}\,\rho^B_{\mu\nu} = \mathcal{Q}^a_{mn},$$

   where we have used the Hermiticity of $\mathcal{P}_a$ and ρ_B.
2. Positivity: $\mathcal{Q}_a \geq 0$. In a basis which diagonalizes ρ_B

$$\rho_B = \sum_{\mu} \mathsf{p}_\mu |\mu_B\rangle\langle\mu_B|,$$

[5] However, the space of states of $\mathcal{AB}$ need not be a tensor product. From a mathematical point of view, the space of states may be a direct sum $\mathcal{H}_A \oplus \mathcal{H}_B$ (see Exercise 15.5.1), although, in practice, it seems difficult to implement the POVM in that case. We shall therefore limit our discussion to the case of tensor products.

we have

$$\langle\psi_A|\mathcal{Q}_a|\psi_A\rangle = \sum_\mu \mathsf{p}_\mu \langle\psi_A\otimes\mu_B|\mathcal{P}_a|\psi_A\otimes\mu_B\rangle \geq 0$$

because $\mathcal{P}_a$ is a positive operator.

3. Completeness:

$$\sum_a \mathcal{Q}_a = \mathrm{Tr}_{\mathcal{B}}\left[\sum_a (\mathcal{P}_a)\rho_B\right] = I_A,$$

because $\sum_a \mathcal{P}_a = I_{AB}$ and $\mathrm{Tr}_{\mathcal{B}}\rho_B = 1$.

In contrast to the projectors (15.16), the $\mathcal{Q}_a$ need not be orthogonal: $\mathcal{Q}_a\mathcal{Q}_b \neq \delta_{ab}$. In general, one defines a *positive operator-valued measure* (POVM) as a set of operators $\mathcal{Q}_a$ acting in $\mathcal{H}_a$ which obey

$$\boxed{\mathcal{Q}_a = \mathcal{Q}_a^\dagger, \quad \mathcal{Q}_a \geq 0, \quad \sum_a \mathcal{Q}_a = I.} \tag{15.23}$$

We can now generalize (15.17)–(15.19) to the POVM case. From (15.20) the probability of result a is

$$\mathsf{p}(a) = \mathrm{Tr}(\mathcal{Q}_a\rho). \tag{15.24}$$

If the measurement is performed but the result is not read, the state operator transforms as

$$\rho \to \rho' = \sum_a \mathcal{Q}_a\,\rho\,\mathcal{Q}_a, \tag{15.25}$$

while if the result of the measurement is read

$$\rho \to \rho'' = \frac{\mathcal{Q}_a\,\rho\,\mathcal{Q}_a}{\mathrm{Tr}(\mathcal{Q}_a\,\rho\,\mathcal{Q}_a)}. \tag{15.26}$$

We have introduced the POVM starting from an orthogonal measurement in $\mathcal{H}_A\otimes\mathcal{H}_B$. This is indeed the most general case, at least if the POVM involves rank-one operators: it follows from Neumark's theorem[6] that any POVM defined by (15.23) in $\mathcal{H}_A$ can be realized as a von Neumann measurement in some Hilbert space $\mathcal{H}_A\otimes\mathcal{H}_B$.

15.1.3 Example: a POVM with spins 1/2

Let us give as an example a POVM with spins 1/2. Let $\{\hat{n}_\alpha\}$ be a set of unit vectors in $\mathbb{R}^3$ and $\{c_\alpha\}$ a set of real coefficients such that

$$\sum_\alpha c_\alpha\hat{n}_\alpha = 0, \quad 0 \leq c_\alpha \leq 1, \quad \sum_\alpha c_\alpha = 1, \tag{15.27}$$

[6] See, e.g., Peres [1993], Chapter 9 for a proof.

and let us define the following operators for a spin 1/2:

$$\mathcal{Q}_\alpha = c_\alpha(I + \vec{\sigma}\cdot\hat{n}_\alpha) = 2c_\alpha\mathcal{P}(\hat{n}_\alpha), \tag{15.28}$$

where $\mathcal{P}(\hat{n}_\alpha)$ is the projector on the spin state $|\hat{n}_\alpha\rangle$, which is an eigenvector of $(\vec{\sigma}\cdot\hat{n}_\alpha)$ with eigenvalue $+1$:

$$(\vec{\sigma}\cdot\hat{n}_\alpha)|\hat{n}_\alpha\rangle = |\hat{n}_\alpha\rangle.$$

The state $|\hat{z}\rangle$ is identified with $|0\rangle$, and the state $|\hat{n}_\alpha\rangle$ is obtained from $|0\rangle$ by the rotation of angle θ_α around the y axis which brings the vertical unit vector $\hat{z}$ onto $\hat{n}_\alpha$:

$$|\hat{n}_\alpha\rangle = \exp\left(-\frac{\mathrm{i}}{2}\,\theta_\alpha\sigma_y\right)|0\rangle. \tag{15.29}$$

From (15.27) one sees that the $\mathcal{Q}_\alpha$ are positive operators which obey the completeness relation $\sum_\alpha \mathcal{Q}_\alpha = I$, but are not in general orthogonal. They are therefore an example of a POVM. The simplest illustration of a POVM that is not a von Neumann measurement is obtained by choosing three vectors $\{\hat{n}_\alpha\} = (\hat{n}_a, \hat{n}_b, \hat{n}_c)$ with, for example,

$$c_a = c_b = c_c = \frac{1}{3}, \quad \hat{n}_a + \hat{n}_b + \hat{n}_c = 0.$$

Then the $\mathcal{Q}_\alpha$ are

$$\mathcal{Q}_\alpha = \frac{1}{3}(I + \vec{\sigma}\cdot\hat{n}_\alpha) = \frac{2}{3}\,\mathcal{P}(\hat{n}_\alpha). \tag{15.30}$$

If we choose unit vectors $\hat{n}_\alpha$ in the xOz plane, a possible symmetric choice is as follows: $\hat{n}_a$ lying along the z axis and $\hat{n}_b$ and $\hat{n}_c$ making angles of $4\pi/3$ and $8\pi/3$ with the z axis, so that (15.29) leads to

$$\begin{aligned} |a\rangle &:= |\hat{n}_a\rangle = |0\rangle, \\ |b\rangle &:= |\hat{n}_b\rangle = -\frac{1}{2}\,|0\rangle + \frac{\sqrt{3}}{2}\,|1\rangle, \\ |c\rangle &:= |\hat{n}_c\rangle = -\frac{1}{2}\,|0\rangle - \frac{\sqrt{3}}{2}\,|1\rangle. \end{aligned} \tag{15.31}$$

Our first goal is to give an explicit verification of Neumark's theorem by constructing the POVM (15.28) from orthogonal projectors in a larger space, a space $\mathcal{H}_A \otimes \mathcal{H}_B$ of two spins 1/2. The auxiliary spin, $\mathcal{B}$, is called an *ancilla*. We build the following orthonormal basis of entangled states in $\mathcal{H}_A \otimes \mathcal{H}_B$, $\alpha = (a, b, c)$:

$$\begin{aligned} |\alpha_{AB}\rangle &= \sqrt{\frac{2}{3}}\;|\alpha_A \otimes 0_B\rangle + \sqrt{\frac{1}{3}}\;|0_A \otimes 1_B\rangle, \\ |\beta_{AB}\rangle &= |1_A \otimes 1_B\rangle. \end{aligned} \tag{15.32}$$

The orthogonality of the basis is easily checked by using the scalar products:

$$\langle a|b\rangle = \langle a|c\rangle = \langle b|c\rangle = -\frac{1}{2}.$$

Let us call $\mathcal{P}_\alpha$, $\alpha = a, b, c$, the set of orthogonal projectors on the basis vectors (15.32):

$$\mathcal{P}_\alpha = |\alpha_{AB}\rangle\langle\alpha_{AB}|, \quad \mathcal{P}_\beta = |\beta_{AB}\rangle\langle\beta_{AB}|,$$

and let us choose spin $\mathcal{B}$ in the state $|0_B\rangle$, $\rho_B = |0_B\rangle\langle 0_B|$. Then we find for the POVM $\mathcal{Q}_\alpha$ and $\mathcal{Q}_\beta$

$$\begin{aligned} \mathcal{Q}_\alpha &= \mathrm{Tr}_{\mathcal{B}}(\rho_B \mathcal{P}_\alpha) = \frac{2}{3}|\alpha_A\rangle\langle\alpha_A|, \\ \mathcal{Q}_\beta &= 0. \end{aligned} \tag{15.33}$$

These equations give an explicit verification of Neumark's theorem in this particular case: we have been able to construct the set $\{\mathcal{Q}_\alpha\}$ from a set of orthogonal projectors in $\mathcal{H}_A \otimes \mathcal{H}_B$.

Let us now describe a possible strategy to implement the POVM. Define the unit vector $\hat{u}$ in the xOz plane making an angle θ with the z axis such that

$$\cos\frac{\theta}{2} = -\sqrt{\frac{1}{3}}, \quad \sin\frac{\theta}{2} = \sqrt{\frac{2}{3}}$$

and the spin states $|\hat{u}\rangle$ and $|-\hat{u}\rangle$

$$|\hat{u}\rangle = \exp\left(-\frac{\mathrm{i}}{2}\theta\sigma_y\right)|0\rangle, \quad |-\hat{u}\rangle = \exp\left(-\frac{\mathrm{i}}{2}\theta\sigma_y\right)|1\rangle.$$

The vectors $|\alpha_{AB}\rangle$, $\alpha = a, b, c$, may be written in terms of $|\pm\hat{u}_B\rangle$:

$$\begin{aligned} |a_{AB}\rangle &= |0_A \otimes -\hat{u}_B\rangle, \\ |(b/c)_{AB}\rangle &= \frac{1}{\sqrt{2}}|0_A \otimes \hat{u}_B\rangle \pm \frac{1}{\sqrt{2}}|1_A \otimes 0_B\rangle, \end{aligned} \tag{15.34}$$

where the $+$ $(-)$ sign corresponds to b (c). To disentangle the states in the second line of (15.34), we use a basic component of quantum information, the control-U or cU gate, which has the following action in our particular case:[7]

$$\begin{aligned} \mathrm{cU}|0_A \otimes 0_B\rangle &= |0_A \otimes 0_B\rangle, \quad \mathrm{cU}|0_A \otimes 1_B\rangle = |0_A \otimes 1_B\rangle, \\ \mathrm{cU}|1_A \otimes 0_B\rangle &= |1_A \otimes \hat{u}_B\rangle, \quad \mathrm{cU}|1_A \otimes 1_B\rangle = |1_A \otimes -\hat{u}_B\rangle. \end{aligned} \tag{15.35}$$

In other words, cU leaves spin $\mathcal{B}$ unchanged if spin $\mathcal{A}$ is in state $|0_A\rangle$, and it rotates spin $\mathcal{B}$ by an angle θ if spin $\mathcal{A}$ is in state $|1\rangle$. The unitary operator cU *is a nonlocal interaction*: *it is not a tensor product* $U_A \otimes U_B$. Let us apply cU to $|\alpha_{AB}\rangle$:

$$\begin{aligned} \mathrm{cU}|a_{AB}\rangle &= |0_A \otimes -\hat{u}_B\rangle = \frac{1}{\sqrt{2}}\big(|\hat{x}_A \otimes -\hat{u}_B\rangle + |-\hat{x}_A \otimes -\hat{u}_B\rangle\big), \\ \mathrm{cU}|(b/c)_{AB}\rangle &= \frac{1}{\sqrt{2}}\big(|0_A\rangle \pm |1_A\rangle\big) \otimes |\hat{u}_B\rangle = |\pm\hat{x}_A \otimes \hat{u}_B\rangle, \end{aligned} \tag{15.36}$$

[7] A cU gate leaves spin $\mathcal{B}$ unchanged if spin $\mathcal{A}$ is in state $|0_A\rangle$, and it performs a unitary transformation $|\varphi_B\rangle \to U_B|\varphi_B\rangle$ on spin $\mathcal{B}$ if spin $\mathcal{A}$ is in state $|1_A\rangle$. The cU gate generalizes the cNOT gate defined in (6.73), which corresponds to the choice $U_B = \sigma_{Bx}$.

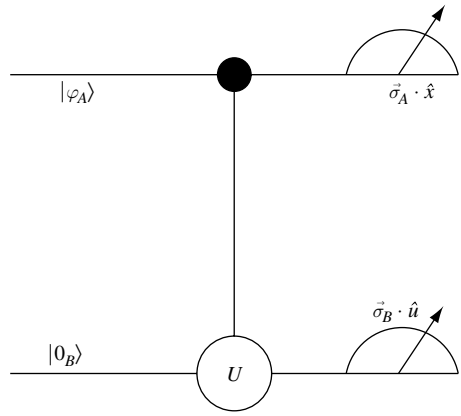

Fig. 15.1. Graphical representation of (15.36).

where the states $|\pm\hat{x}\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$ are eigenvectors of $\sigma_x = (\vec{\sigma}\cdot\hat{x})$ with eigenvalues ± 1. If we measure $(\sigma_A \cdot \hat{x})$ and $(\vec{\sigma}_B \cdot \hat{u})$ after applying the cU gate (Fig. 15.1), the results of the measurements of the pair $(\sigma_A \cdot \hat{x}\,; \vec{\sigma}_B \cdot \hat{u})$ lead to the following correspondence (0 and 1 refer to the values of the qubits measured along $\hat{x}$ or $\hat{u}$):

$$\begin{aligned} (0;1) \text{ and } (1;1) &\to a, \\ (0;0) &\to b, \\ (1;0) &\to c. \end{aligned}$$

Let us show that a POVM can in some cases give better results than a von Neumann measurement, in the sense that the former allows a better discrimination between different states of system $\mathcal{A}$ when these states are not orthogonal. Assume that Alice sends Bob a sequence of particles of spin 1/2 which are randomly distributed with equal probabilities in the states $|a_\perp\rangle$ and $|b_\perp\rangle$:[8]

$$|a_\perp\rangle = |1\rangle, \quad |b_\perp\rangle = |-\hat{n}_b\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle.$$

What is the best strategy that Bob can follow to tell *with certainty* whether a given spin was sent by Alice in state $|a_\perp\rangle$ or $|b_\perp\rangle$? Bob can perform a von Neumann measurement, by taking a Stern–Gerlach filter oriented along $\hat{z}$. If the spin is deflected upward, he can tell with certainty that spin $\mathcal{A}$ was in the state $|b_\perp\rangle$, and this occurs with probability 3/8. Thus, in 37.5% of the cases, Bob is able to decide with certainty between the states $|a_\perp\rangle$ and $|b_\perp\rangle$. He can do better by performing a POVM measurement, as we are going to

[8] We choose $|a_\perp\rangle$ and $|b_\perp\rangle$ rather than $|a\rangle$ and $|b\rangle$ in order to use the cU gate (15.35).

demonstrate. Bob entangles spin $\mathcal{A}$ with an ancilla spin $\mathcal{B}$ in the state $|0_B\rangle$ using the cU gate (15.35). An easy calculation gives

$$\begin{aligned}
\mathrm{cU}|1_A \otimes 0_B\rangle &= \frac{1}{\sqrt{2}}\,|\hat{x}_A \otimes \hat{u}_B\rangle - \frac{1}{\sqrt{2}}\,|-\hat{x}_A \otimes \hat{u}_B\rangle, \\
\mathrm{cU}|-\hat{n}_{bA} \otimes 0_B\rangle &= -\frac{1}{\sqrt{2}}\,|-\hat{x}_A \otimes \hat{u}_B\rangle + \frac{1}{2}\,|\hat{x}_A \otimes \hat{u}_B\rangle + \frac{1}{2}\,|-\hat{x}_A \otimes -\hat{u}_B\rangle, \\
\mathrm{cU}|-\hat{n}_{cA} \otimes 0_B\rangle &= -\frac{1}{\sqrt{2}}\,|\hat{x}_A \otimes \hat{u}_B\rangle + \frac{1}{2}\,|\hat{x}_A \otimes -\hat{u}_B\rangle + \frac{1}{2}\,|-\hat{x}_A \otimes -\hat{u}_B\rangle.
\end{aligned} \tag{15.37}$$

If spin $\mathcal{A}$ is in the initial state $|a_\perp\rangle = |1\rangle$, then we find the following probabilities when measuring the pair $(\sigma_A \cdot \hat{x}\,; \vec{\sigma}_B \cdot \hat{u})$:

$$\mathsf{p}(0;0) = \mathsf{p}(1;0) = \frac{1}{2}, \quad \mathsf{p}(0;1) = \mathsf{p}(1;1) = 0,$$

while if it is in the state $|b_\perp\rangle = |-\hat{n}_b\rangle$ we have

$$\mathsf{p}(0;0) = 0, \quad \mathsf{p}(1;0) = \frac{1}{2}, \quad \mathsf{p}(0;1) = \mathsf{p}(1;1) = \frac{1}{4}.$$

Then, if Bob's measurement gives $(0;0)$, he knows with certainty that spin $\mathcal{A}$ was initially in the state $|a_\perp\rangle$, while if he measures $(0;1)$ or $(1;1)$ he can be sure that it was in the state $|b_\perp\rangle$. If he measures $(1;0)$, he cannot decide. This occurs in 50% of the cases, so that he is able to distinguish between the two states with a 50% probability, instead of the 37.5% in the case of a von Neumann measurement. The same results are obtained by using the POVM $(\mathcal{Q}_a, \mathcal{Q}_b, \mathcal{Q}_c)$ (see Exercise 15.5.2). It can be shown that this is the best result Bob can achieve: a general theorem states that optimal POVMs consist of rank-one operators.[9]

15.2 Superoperators

15.2.1 Kraus decomposition

We have seen in the preceding section how an orthogonal measurement on a bipartite system whose state vector lies in a Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ is translated into a POVM on $\mathcal{A}$ alone. In the present section, which is, as we shall see later on, closely related to the preceding one, we shall attempt to answer the following question: if a state of $\mathcal{H}_A \otimes \mathcal{H}_B$ undergoes a unitary evolution U_{AB} from $t = 0$ to t, what is the general expression for the (generally nonunitary) evolution of the state operator for $\mathcal{A}$? The answer is provided by the Kraus representation, which we are going to derive. We assume that the state operator at $t = 0$ is a tensor product, with $\rho_B = |0_B\rangle\langle 0_B|$ a pure state, a kind of "reference state":

$$\rho_{AB}(t=0) = \rho_A \otimes \rho_B = \rho_A \otimes |0_B\rangle\langle 0_B|. \tag{15.38}$$

[9] E. Davies, *IEEE Trans. Inform. Theory* **IT-24**, 596 (1978).

We shall comment on this apparently very restrictive assumption later on. The bipartite system $\mathcal{AB}$ evolves during a time interval t according to

$$\rho_{AB}(t=0) = \rho_{AB} \to \rho_{AB}(t) = \rho'_{AB} = U_{AB}\,\rho_{AB}(t=0)\,U_{AB}^{\dagger}, \tag{15.39}$$

where U_{AB} is obtained by solving (4.17) in $\mathcal{H}_A \otimes \mathcal{H}_B$. In order to find the state operator $\rho_A(t) = \rho'_A$ of system A, we perform a partial trace (see Footnote 3):

$$\rho'_{mn}{}^{A} = \sum_{\mu} U^{AB}_{m\mu;k0}\,\rho^{A}_{kl}\,(U^{AB})^{\dagger}_{l0;n\mu}, \tag{15.40}$$

where we have made explicit use of the peculiar form (15.38) of the initial state operator ρ_{AB}. The matrix elements of U_{AB} are

$$U^{AB}_{m\mu;n\nu} = \langle m_A \otimes \mu_B | U_{AB} | n_A \otimes \nu_B \rangle.$$

Equation (15.40) can be written in operator form by introducing the *superoperator* M_μ acting in $\mathcal{H}_A$ through

$$M_\mu = \langle \mu_B | U_{AB} | 0_B \rangle. \tag{15.41}$$

Writing $\rho'_A = \mathcal{K}(\rho_A)$, (15.40) becomes (see Fig. 15.2)

$$\mathcal{K}(\rho_A) = \sum_{\mu} M_\mu\,\rho_A\,M_\mu^{\dagger}. \tag{15.42}$$

The unitarity of U_{AB} implies that the set of superoperators M_μ obeys the completeness relation (note the order of the operators; $\sum_\mu M_\mu M_\mu^{\dagger}$ has no simple expression in the general case):

$$\sum_{\mu} M_\mu^{\dagger} M_\mu = \sum_{\mu} \langle 0_B | U_{AB}^{\dagger} | \mu_B \rangle \langle \mu_B | U_{AB} | 0_B \rangle = I_A. \tag{15.43}$$

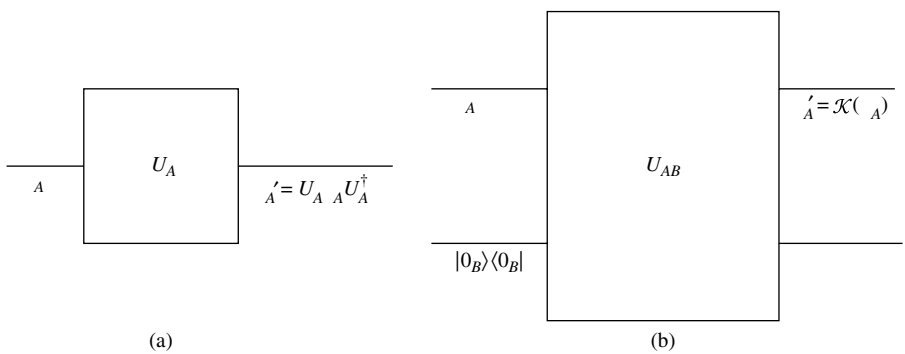

Fig. 15.2. Graphical representation of unitary evolution (a) and the evolution (15.42) (b).

Equation (15.42) is the *Kraus representation* of ρ'_A. This Kraus representation, together with the completeness relation (15.43), defines a linear map $\rho_A \to \rho'_A = \mathcal{K}(\rho_A)$. The operator ρ'_A obeys the three necessary conditions for being a bona fide state operator:

(i) ρ'_A is obviously Hermitian;
(ii) $\mathrm{Tr}\,\rho'_A = 1$ owing to (15.43);
(iii) ρ'_A is positive: indeed, with $|\psi^\mu_A\rangle = M^\dagger_\mu|\varphi_A\rangle$,

$$\langle\varphi_A|\rho'_A|\varphi_A\rangle = \sum_\mu(\langle\varphi_A|M_\mu)\rho_A(M^\dagger_\mu|\varphi_A\rangle) = \sum_\mu\langle\psi^\mu_A|\rho_A|\psi^\mu_A\rangle \geq 0.$$

Conversely, any Kraus representation (15.42) can always be derived from a unitary representation in some Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, as we now show. Let us choose as $\mathcal{H}_B$ a Hilbert space whose dimension is at least the number of terms in (15.42), and let $\{|\mu_B\rangle\}$ be an orthonormal basis in $\mathcal{H}_B$ and $|0_B\rangle$ one particular vector of this basis. *Define* the action of U_{AB} on the vector $|\varphi_A \otimes 0_B\rangle$, where $|\varphi_A\rangle$ is an arbitrary vector of $\mathcal{H}_A$, as

$$U_{AB}|\varphi_A \otimes 0_B\rangle = \sum_\mu(M_\mu \otimes I_B)|\varphi_A \otimes \mu_B\rangle. \tag{15.44}$$

Equation (15.44) describes a *quantum jump*: in the time interval $[0, t]$, the $\mathcal{AB}$ system "jumps" from $|\varphi_A \otimes 0_B\rangle$ to a superposition of states $M_\mu|\varphi_A\rangle \otimes |\mu_B\rangle$. The operator U_{AB} preserves the scalar product

$$\Big(\sum_\mu\langle\psi_A \otimes \mu_B|[M^\dagger_\mu \otimes I_B]\Big)\Big(\sum_\nu[M_\nu \otimes I_B]|\varphi_A \otimes \nu_B\rangle\Big) = \langle\psi_A|\Big(\sum_\mu M^\dagger_\mu M_\mu\Big)|\varphi_A\rangle = \langle\psi_A|\varphi_A\rangle,$$

and therefore U_{AB}, which is a priori defined only on a subset of $\mathcal{H}_A \otimes \mathcal{H}_B$, can be extended as a unitary operator on the full $\mathcal{H}_A \otimes \mathcal{H}_B$. Taking a partial trace, we find

$$\mathrm{Tr}_\mathcal{B}\Big(U_{AB}|\varphi_A \otimes 0_B\rangle\,\langle\varphi_A \otimes 0_B|U^\dagger_{AB}\Big) = \sum_\mu M_\mu|\varphi_A\rangle\langle\varphi_A|M^\dagger_\mu,$$

so that any state operator ρ_A, which can be written as $\sum_i \mathsf{p}_i|\varphi^i_A\rangle\langle\varphi^i_A|$, transforms according to (15.42). We shall see later on that any "reasonable" evolution law for ρ_A is of the form (15.42). Then the fact that one can always find a unitary representation (15.44) is somewhat surprising at first sight: in principle, at $t = 0$, systems $\mathcal{A}$ and $\mathcal{B}$ are entangled, so that assuming an initial state of the form (15.38) looks like a very restrictive assumption. But it seems that, *for the purpose of describing the evolution of* ρ_A, it is always possible to find a model environment such that there is no initial entanglement between the system and its (fictitious) environment.

Let us conclude this subsection by stating some general results on the Kraus representation. As some of the proofs are rather technical, we shall omit them and refer the reader to the bibliography. One interesting question is the following: under what "reasonable"

conditions can the Kraus representation (15.42) be proved? A priori, it would seem that one should require that

(i) $\mathcal{K}$ is a linear operation:[10]

$$\mathcal{K}(\lambda\rho_A + \mu\rho_B) = \lambda\mathcal{K}(\rho_A) + \mu\mathcal{K}(\rho_B).$$

(ii) $\mathcal{K}(\rho_A)$ is Hermitian:

$$\mathcal{K}(\rho_A) = [\mathcal{K}(\rho_A)]^\dagger.$$

(iii) $\mathcal{K}$ is trace-preserving: $\mathrm{Tr}[\mathcal{K}(\rho_A)] = 1$.

(iv) $\mathcal{K}(\rho_A)$ is a positive operator: $\mathcal{K}(\rho_A) \geq 0$.

But condition (iv) is actually too weak. Suppose $\mathcal{A}$ is coupled to a system $\mathcal{B}$ and there is a third system $\mathcal{C}$, totally uncoupled to $\mathcal{A}$, of which we are unaware. If $\mathcal{A}$ evolves and $\mathcal{C}$ does not, then $\mathcal{K}(\rho_A) \otimes I_C$ must be a positive operator. Thus $\mathcal{K}(\rho_A)$ should obey not (iv) but the stronger condition (iv′):

(iv′) $\mathcal{K}(\rho_A)$ is *completely positive*: $\mathcal{K}(\rho_A) \otimes I_C \geq 0$, for any system $\mathcal{C}$.

An example of an operator that obeys (i) to (iii), but not (iv′), is the transposition (Exercise 15.5.4)

$$\mathcal{K}(\rho_A) = \rho_A^T.$$

We can now state without proof the *Kraus representation theorem*: any operator $\rho \to \mathcal{K}(\rho)$ in a space of dimension N which obeys the conditions (i) to (iii) and (iv′) can be written in the form

$$\boxed{\mathcal{K}(\rho) = \sum_{\mu=1}^{K} M_\mu \rho M_\mu^\dagger, \quad \sum_{\mu=1}^{K} M_\mu^\dagger M_\mu = I,} \tag{15.45}$$

where the number of terms in the sum is bounded by $K \leq N_A^2$, with N_A the dimension of $\mathcal{H}_A$; K is the *Kraus number*. There always exists an expression for $\mathcal{K}(\rho)$ with a number of terms $\leq N_A^2$, independently of the dimension of the Hilbert space $\mathcal{H}_B$ of the environment, even if this dimension is infinite.

The Kraus representation is not unique, but any two representations may be related through a unitary transformation: if

$$\mathcal{K}(\rho) = \sum_{\mu=1}^{K} M_\mu \rho M_\mu^\dagger = \sum_{\mu=1}^{L} N_\mu \rho N_\mu^\dagger,$$

then N_ν is related to M_μ by a unitary transformation:

$$N_\nu = \sum_\mu U_{\nu\mu} M_\mu.$$

[10] However, see, e.g., J. Preskill, *Quantum Computation*, http://www.theory.caltech.edu/~ preskill/(1999), Section 3.2 for a discussion; the arguments that nonlinear evolution should be excluded are not entirely compelling.

As some of the matrix elements $U_{\nu\mu}$ may vanish, the number of nonzero terms need not be the same in both decompositions: it may happen that $K \neq L$.

Let us finally make the link with POVM by showing that a unitary transformation that entangles $\mathcal{A}$ with $\mathcal{B}$ followed by an orthogonal measurement on $\mathcal{B}$ can be described as a POVM. If a state $|\varphi_A \otimes 0_B\rangle$ evolves according to (15.44), then an orthogonal measurement on $\mathcal{B}$ which projects onto the $\{|\mu_B\rangle\}$ basis has a probability $\mathsf{p}(\mu)$ of finding result μ

$$\begin{aligned}\mathsf{p}(\mu) &= \sum_{\nu,\tau} \mathrm{Tr}\Big[(I_A \otimes |\mu\rangle\langle\mu|) M_\nu |\varphi_A \otimes \nu\rangle\langle\varphi_A \otimes \tau| M_\tau^\dagger\Big] \\ &= \langle\varphi_A| M_\mu^\dagger M_\mu |\varphi_A\rangle.\end{aligned}$$

Writing the state operator of $\mathcal{A}$ as $\rho_A = \sum_i \mathsf{p}_i |i\rangle\langle i|$, we find in the general case

$$\mathsf{p}(\mu) = \mathrm{Tr}(\Omega_\mu \rho_A), \quad \Omega_\mu = M_\mu^\dagger M_\mu = \Omega_\mu^\dagger \geq 0. \tag{15.46}$$

Furthermore, $\sum_\mu \Omega_\mu = \sum_\mu M_\mu^\dagger M_\mu = I$, so that the Ω_μ form a POVM. Conversely, let $\mathcal{R}_\mu$ be a set of Hermitian and positive operators which obey $\sum_\mu \mathcal{R}_\mu = I$ and $\mathsf{p}(\mu) = \mathrm{Tr}\,(\mathcal{R}_\mu \rho)$. A POVM which modifies the state operator according to

$$\rho \to \rho' = \sum_\mu \sqrt{\mathcal{R}_\mu}\, \rho \sqrt{\mathcal{R}_\mu} \tag{15.47}$$

gives $\sqrt{\mathcal{R}_\mu}$ as a special case of superoperator. Then, from the unitary representation (15.44), one can find a unitary operator U_{AB} such that

$$U_{AB} |\varphi_A \otimes 0_B\rangle = \sum_\mu \sqrt{\mathcal{R}_\mu}\, |\varphi_A \otimes \mu_B\rangle. \tag{15.48}$$

By performing an orthogonal measurement on $\mathcal{B}$ which projects onto the basis $\{|\mu_B\rangle\}$, we obtain an implementation of the POVM. However, this is not in general the most economic way to proceed, because the dimension of $\mathcal{H}_B$ is at least K, the number of different POVMs. For example, in Section 15.1.3, we had $K = 3$, so that $N_B = 3$, while we were able to use an ancilla living in a two-dimensional Hilbert space, $N_B = 2$. Thus we have (at least) two ways of implementing a POVM: (i) associate with $\mathcal{A}$ an ancilla $\mathcal{B}$ and perform a *nonlocal* measurement on $\mathcal{AB}$; (ii) entangle $\mathcal{A}$ with $\mathcal{B}$ and perform a *local* measurement on $\mathcal{B}$.

Let us apply the notion of superoperators to three important examples of physical mechanisms leading to nonunitary evolution of a two-level system. In all three examples, conventionally called "channels," a two-level system is coupled to an environment $\mathcal{E}$, so that the unitary evolution takes place in a Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_E$. In what follows we wish to think of $\mathcal{B}$ as an environment $\mathcal{E}$: $\mathcal{B} \to \mathcal{E}$. In the three examples we shall start from a unitary evolution in $\mathcal{H}_A \otimes \mathcal{H}_E$ in the form of quantum jumps, from which we shall derive the explicit form of the Kraus operators. Our three examples will be (i) the depolarizing channel; (ii) the phase-damping channel; (iii) the amplitude-damping channel.

The terminology will be justified in each of the corresponding subsections.

15.2.2 The depolarizing channel

In this example, $\mathcal{H}_E$ has dimension $N_E = 4$ and an orthonormal basis is formed by a reference state $|0_E\rangle$ and three states $|i_E\rangle$, $i = 1, 2, 3$. The quantum jump (15.44) is assumed to take the form

$$U_{AE}|\varphi_A \otimes 0_E\rangle = \sqrt{1-\mathsf{p}}\;|\varphi_A \otimes 0_E\rangle + \sqrt{\frac{\mathsf{p}}{3}}\left[\sum_{i=1}^{3}(\sigma_{iA}\otimes I_E)|\varphi_A \otimes i_E\rangle\right]. \tag{15.49}$$

Therefore, the initial state $|0_E\rangle$ is unchanged with probability $(1-\mathsf{p})$, and a change $|\varphi_A\rangle \to \sigma_i|\varphi_A\rangle$, which occurs with probability $\mathsf{p}/3$, is accompanied by a change $|0_E\rangle \to |i_E\rangle$. On comparing (15.49) with (15.44) we find

$$M_0 = \sqrt{1-\mathsf{p}}\; I, \quad M_i = \sqrt{\frac{\mathsf{p}}{3}}\;\sigma_i. \tag{15.50}$$

These four superoperators obey the completeness relation (15.43)

$$\sum_{\mu=0}^{3} M_\mu^\dagger M_\mu = \left[(1-\mathsf{p})I + 3\,\frac{\mathsf{p}}{3}\,I\right] = I,$$

where we have used $\sigma_i^2 = I$. The state operator of the system evolves according to (15.45):

$$\rho \to \mathcal{K}(\rho) = (1-\mathsf{p})\rho + \frac{\mathsf{p}}{3}\sum_{i=1}^{3}(\sigma_i\rho\sigma_i). \tag{15.51}$$

Let us write the Bloch form (6.24) of the state operator with a Bloch vector $\vec{b}$ (recall that $\vec{b}$ is the polarization in the case of a spin 1/2)

$$\rho = \frac{1}{2}\left(I + \vec{\sigma}\cdot\vec{b}\right) = \frac{1}{2}\left(I + \sum_{j=1}^{3}\sigma_j b_j\right). \tag{15.52}$$

The identities (3.49) for the Pauli matrices lead to the relation

$$\sigma_i\sigma_j\sigma_i = 2\sigma_j\delta_{ij} - \sigma_j,$$

so that

$$\rho' = \frac{1}{2}\left(I + \vec{\sigma}\cdot\vec{b}'\right), \quad \vec{b}' = \left(1 - \frac{4\mathsf{p}}{3}\right)\vec{b}. \tag{15.53}$$

This transformation corresponds to a simple rescaling of the polarization by a factor $(1-4\mathsf{p}/3)$: if the initial state is a pure state with polarization $|\vec{b}| = 1$, we see that the polarization is reduced from $|\vec{b}| = 1$ to $|1-4\mathsf{p}/3|$, hence the terminology *depolarizing channel*. Note that in all cases the norm of $\vec{b}$ is scaled down by a factor $|1-4\mathsf{p}/3| \le 1$, and that the orientation of $\vec{b}$ changes if $3/4 < \mathsf{p} \le 1$.

15.2.3 The phase-damping channel

In this case $\mathcal{H}_E$ is of dimension 3, and the unitary evolution (quantum jump) is assumed to be of the form

$$\begin{aligned} U_{AE}|0_A \otimes 0_E\rangle &= \sqrt{1-\mathsf{p}}\ |0_A \otimes 0_E\rangle + \sqrt{\mathsf{p}}\ |0_A \otimes 1_E\rangle, \\ U_{AE}|1_A \otimes 0_E\rangle &= \sqrt{1-\mathsf{p}}\ |1_A \otimes 0_E\rangle + \sqrt{\mathsf{p}}\ |1_A \otimes 2_E\rangle. \end{aligned} \tag{15.54}$$

Unlike the preceding case, system $\mathcal{A}$ does not make any transition. The Kraus decomposition is readily written from (15.44):

$$M_0 = \sqrt{1-\mathsf{p}}\ I, \quad M_1 = \sqrt{\mathsf{p}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \sqrt{\mathsf{p}} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{15.55}$$

and the transformed state matrix is

$$\mathcal{K}(\rho) = (1-\mathsf{p})\rho + \mathsf{p} \begin{pmatrix} \rho_{00} & 0 \\ 0 & \rho_{11} \end{pmatrix} = \begin{pmatrix} \rho_{00} & (1-\mathsf{p})\rho_{01} \\ (1-\mathsf{p})\rho_{10} & \rho_{11} \end{pmatrix}. \tag{15.56}$$

We note that the operations affect only the coherences (the off-diagonal matrix elements of ρ), hence the terminology *phase-damping channel*. Furthermore, if we apply $\mathcal{K}$ twice we get

$$\mathcal{K}^2(\rho) = \mathcal{K}[\mathcal{K}(\rho)] = \begin{pmatrix} \rho_{00} & (1-\mathsf{p})^2\rho_{01} \\ (1-\mathsf{p})^2\rho_{10} & \rho_{11} \end{pmatrix}.$$

Assume now that the quantum jump takes place in a short time interval Δt, with a probability proportional to Δt: $\mathsf{p} = \Gamma \Delta t \ll 1$.[11] Let us write $t = n\Delta t$, $n \gg 1$, and make n iterations of $\mathcal{K}$:

$$\mathcal{K}^n(\rho) = \begin{pmatrix} \rho_{00} & (1-\mathsf{p})^n\rho_{01} \\ (1-\mathsf{p})^n\rho_{10} & \rho_{11} \end{pmatrix} \to \begin{pmatrix} \rho_{00} & \rho_{01}\,\mathrm{e}^{-\Gamma t} \\ \rho_{10}\,\mathrm{e}^{-\Gamma t} & \rho_{11} \end{pmatrix}. \tag{15.57}$$

The relaxation time of the coherences (the transverse relaxation time T_2 of NMR) is $T_2 = 1/\Gamma$. If the two-level system is prepared at $t = 0$ in a pure state which is a *coherent* superposition of $|0\rangle$ and $|1\rangle$

$$|\varphi\rangle = a|0\rangle + b|1\rangle, \quad \rho_{00} = 1 - \rho_{11} = |a|^2, \quad \rho_{01} = \rho_{10}^* = ab^*,$$

then, after a time $t \gg 1/\Gamma$, the quantum state is transformed from (15.57) into an *incoherent* superposition of $|0\rangle$ and $|1\rangle$:

$$t \gg 1/\Gamma : \rho(t) \to |a|^2|0\rangle\langle 0| + |b|^2|1\rangle\langle 1|.$$

As an application, let us give a heuristic discussion of the decoherence of a quantum superposition involving macroscopic systems. Let us identify $|0_A\rangle$ and $|1_A\rangle$ with the position eigenstates $|x\rangle$ and $|-x\rangle$ (or, more realistically, with narrow nonoverlapping

[11] Note that this is a rather bold assumption. In general, we expect *amplitudes* to be proportional to Δt if transitions take place toward a single state; see (5.62). One needs transitions to a continuous set of states, as in the Fermi Golden Rule (9.170), in order to obtain *probabilities* proportional to Δt.

wave packets centered at x and $-x$) of a "dust particle,"[12] that elastically scatters photons initially in state $|0_E\rangle$. Scattering by the dust particle in state $|x\rangle$ ($|-x\rangle$) sends the photons into states $|1_E\rangle$ ($|2_E\rangle$), while the dust particle remains in its initial state. If the distance $2|x|$ between the centers of the wave packets is large compared with the photon wavelength, the states $|1_E\rangle$ and $|2_E\rangle$ will be approximately orthogonal, $\langle 1_E|2_E\rangle \simeq 0$, because photon scattering is localized in space. We are therefore in the situation described by (15.54). If the dust particle is initially in a coherent superposition of the two wave packets

$$|\varphi\rangle = \frac{1}{\sqrt{2}}(|x\rangle + |-x\rangle),$$

the coherence between the wave packets will be destroyed after a time $\sim 1/\Gamma$. The relaxation rate Γ is proportional to the elastic cross section σ. A rough estimate for σ is $\sigma \propto R^2 \propto M^{2/3}$, where R is the radius of the dust particle and M is its mass. The decoherence time $\tau_{\rm dec}$ is proportional to $M^{-2/3}$

$$\tau_{\rm dec} \simeq \frac{1}{\Gamma} \propto \frac{1}{M^{2/3}}.$$

The decoherence time is much shorter than the damping time τ characteristic of the motion of the dust particle, the time taken by the dust particle to change its momentum under photon scattering: $\tau_{\rm dec} \ll \tau$; as in Section 6.4.1, $\tau_{\rm dec}$ is controlled by the scattering of one photon and τ by the scattering of a large number of photons. This result has important consequences for the Young's slit experiment: if the time taken by the particle to travel from the slits to the screen is larger than $\tau_{\rm dec}$, no interference is possible, because the coherence of the two wave packets leaving the slits is destroyed before the particle arrives at the screen. "Which path" information is encoded in the environment.

As we have seen in Section 6.4.1, a quantum superposition of two macroscopic states is called a Schrödinger's cat. We have just seen that this Schrödinger's cat is destroyed over a time $\sim \tau_{\rm dec}$, and we are left with an incoherent mixture. The mechanism responsible for decoherence selects a preferred basis: photon scattering selects a basis of position states, because the photons scattered by different position eigenstates are sent into orthogonal states. Actually, it is not necessary that the final photon states be orthogonal. Assume, for example, that the states $|1_E\rangle$ and $|2_E\rangle$ satisfy $\langle 1_E|2_E\rangle = 1 - \varepsilon$; then the probability p would be replaced by $\mathsf{p} \to \varepsilon\mathsf{p}$ and the decoherence time by $\tau_{\rm dec} \to \varepsilon^{-1}\tau_{\rm dec}$.

15.2.4 The amplitude-damping channel

This is a schematic model for describing the spontaneous decay of a two-level atom with the emission of one photon. By detecting the emitted photon, we perform a POVM which gives us information about the state of the atom. The system is the two-level atom, and the environment is the quantized electromagnetic field. If the atom and the field are in their respective ground states $|0_A\rangle$ and $|0_E\rangle$, nothing can happen. If the atom is in its

[12] A "dust particle" is large by microscopic standards and small by macroscopic standards.

excited state $|1_A\rangle$ and the field is in $|0_E\rangle$, there is a probability p that the atom emits a photon and is left in $|0_A\rangle$. The unitary representation of the quantum jump then is

$$\begin{aligned} U_{AE}|0_A \otimes 0_E\rangle &= |0_A \otimes 0_E\rangle, \\ U_{AE}|1_A \otimes 0_E\rangle &= \sqrt{1-\mathsf{p}}\ |1_A \otimes 0_E\rangle + \sqrt{\mathsf{p}}\ |0_A \otimes 1_E\rangle. \end{aligned} \tag{15.58}$$

where $|1_E\rangle$ is a one photon state.

The Kraus operators from (15.44) are

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\mathsf{p}} \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & \sqrt{\mathsf{p}} \\ 0 & 0 \end{pmatrix}, \tag{15.59}$$

and $\rho' = \mathcal{K}(\rho)$ is given by

$$\mathcal{K}(\rho) = \begin{pmatrix} 1+(1-\mathsf{p})\rho_{11} & \sqrt{1-\mathsf{p}}\ \rho_{01} \\ \sqrt{1-\mathsf{p}}\ \rho_{10} & (1-\mathsf{p})\rho_{11} \end{pmatrix}. \tag{15.60}$$

As in the preceding example, we take $\mathsf{p} = \Gamma\Delta t \ll 1$, $\Delta t = t/n$, and make n iterations of $\mathcal{K}$:

$$\mathcal{K}^n(\rho) = \begin{pmatrix} 1+(1-\mathsf{p})^n\rho_{11} & (1-\mathsf{p})^{n/2}\rho_{01} \\ (1-\mathsf{p})^{n/2}\rho_{10} & (1-\mathsf{p})^n\rho_{11} \end{pmatrix} \to \begin{pmatrix} 1-\mathrm{e}^{-\Gamma t}\rho_{11} & \mathrm{e}^{-\Gamma t/2}\rho_{01} \\ \mathrm{e}^{-\Gamma t/2}\rho_{10} & \mathrm{e}^{-\Gamma t}\rho_{11} \end{pmatrix}. \tag{15.61}$$

In this model, $T_2 = 2T_1 = 2/\Gamma$, which explains why we chose Γ and $\Gamma/2$ as the relaxation rates of populations and coherences, respectively, in the optical Bloch equations (14.82)–(14.83).

In contrast to the preceding example, where we could not envisage detecting the photons scattered by the dust particles, it may be possible in the present case to detect the emitted photon. A coherent superposition of the two atomic states evolves as

$$\big(a|0_A\rangle + b|1_A\rangle\big) \otimes |0_E\rangle \to \big(a|0_A\rangle + b\sqrt{1-\mathsf{p}}\ |1_A\rangle\big) \otimes |0_E\rangle + b\sqrt{\mathsf{p}}\ |0_A \otimes 1_E\rangle.$$

If we detect the photon, we know with certainty that the initial state of the atom was $|1_A\rangle$. If we detect no photon, then we have prepared the (unnormalized) atomic state

$$a|0_A\rangle + b\sqrt{1-\mathsf{p}}\ |1_A\rangle.$$

The atomic state has evolved owing to our failure to detect a photon! As we have seen, a unitary transformation which entangles $\mathcal{A}$ and $\mathcal{E}$, followed by an orthogonal measurement on $\mathcal{E}$, can be described as a POVM on $\mathcal{A}$. From (15.46), the POVM are $\mathcal{R}_\mu = M_\mu^\dagger M_\mu$, so that

$$\mathcal{R}_0 = M_0^\dagger M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1-\mathsf{p} \end{pmatrix}, \quad \mathcal{R}_1 = M_1^\dagger M_1 = \begin{pmatrix} 0 & 0 \\ 0 & \mathsf{p} \end{pmatrix}, \tag{15.62}$$

and $\mathsf{p}(\mu) = \mathrm{Tr}(\mathcal{R}_\mu \rho_A)$ if the atom is initially in the mixed state ρ_A.

15.3 Master equations: the Lindblad form

15.3.1 The Markovian approximation

The system of Bloch equations (15.5), which we wrote down from heuristic arguments, is typical of what is called a master equation: the time evolution of the state operator is given by a first-order differential equation in time, or, in other words, the evolution is local in time. For example, the optical Bloch equations (14.86)–(14.87) are equivalent to (15.61) if we ignore the unitary part $(-\mathrm{i}/\hbar)[H,\rho]$ of the evolution, since (15.61) may be written as

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{\Gamma}{2}\begin{pmatrix} -2\rho_{11}(t) & \rho_{01}(t) \\ \rho_{10}(t) & 2\rho_{11}(t) \end{pmatrix}. \tag{15.63}$$

It is readily checked that (15.63) can be cast in the form

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\Gamma}{2}\big[2\sigma_+\rho\,\sigma_- - \{\sigma_-\sigma_+,\rho\}\big], \tag{15.64}$$

where $\sigma_\pm = (\sigma_x \pm \mathrm{i}\sigma_y)/2$,[13] and $\{A,B\}$ denotes the *anticommutator* of two operators A and B:

$$\{A,B\} = AB + BA. \tag{15.65}$$

Actually, one can readily supplement (15.63) by a unitary evolution as in (15.5):

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}[H,\rho] + \frac{\Gamma}{2}\big[2\sigma_+\rho\,\sigma_- - \{\sigma_-\sigma_+,\rho\}\big]. \tag{15.66}$$

For later purposes, it will be useful to introduce the interaction picture:

$$\begin{aligned} \tilde{\rho}(t) &= \mathrm{e}^{\mathrm{i}H_0 t/\hbar}\,\rho(t)\,\mathrm{e}^{-\mathrm{i}H_0 t/\hbar}, \\ \tilde{\sigma}_\pm(t) &= \mathrm{e}^{\mathrm{i}H_0 t/\hbar}\,\sigma_\pm\,\mathrm{e}^{-\mathrm{i}H_0 t/\hbar} = \mathrm{e}^{\mp\mathrm{i}\omega_0 t}\sigma_\pm, \end{aligned} \tag{15.67}$$

where H_0 is the free Hamiltonian (15.1). Equation (15.66) is an example of a master equation in the Lindblad form, of which we now give a general derivation.

In the preceding example, we have been able to go from the Kraus form (15.45) to the master equation in the Lindblad form (15.66). However, our derivation depends on the crucial (and strong) assumption that the probability p is proportional to Δt. In the general case, it is far from obvious that it is possible to obtain a differential equation for the nonunitary evolution of $\tilde{\rho}(t)$, because we expect *memory effects* to be present (a priori, a local time evolution can be valid only for $\tilde{\rho}$, not ρ). Information flows from the system to the environment, but, conversely, information also flows from the environment to the system. Schematically, an equation with memory effects has the form of an integro-differential equation which is *nonlocal in time*:

$$\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}t} = -\int_{-\infty}^{t} \gamma(t-t')\tilde{\rho}(t')\mathrm{d}t', \tag{15.68}$$

[13] Beware of the fact that Nielsen and Chuang [2000] use the opposite convention for $\sigma_\pm$. We have chosen a convention consistent with the definition (10.4) of the angular momentum operators $J_\pm$, and which is moreover consistent with that of field theory, since σ_+ (σ_-) is a positive (negative) frequency operator like a ($a^\dagger$); see (11.67).

where $\gamma(t-t')$ is the memory function, or memory kernel.[14] If the characteristic relaxation time τ_* of $\gamma(t-t')$ is much smaller that the typical evolution time τ of $\tilde{\rho}$, $\tau_* \ll \tau$, we may write (15.68) in the approximate form

$$\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}t} \simeq -\rho(t)\int_{-\infty}^{t}\gamma(t')\,\mathrm{d}t' = -\Gamma\,\tilde{\rho}(t), \tag{15.69}$$

and we obtain a master equation. The short-memory approximation in (15.69) is also called the *Markovian approximation*: $\mathrm{d}\tilde{\rho}/\mathrm{d}t$ depends only on $\tilde{\rho}$ at time t, and not on its value at earlier times $t' < t$. The Markovian approximation will hold if there are two widely separated time scales: τ, the typical evolution time of $\tilde{\rho}$, and τ_*, the typical relaxation time of the memory function, with $\tau_* \ll \tau$. The assumption of two widely separated time scales is a very common one in nonequilibrium statistical mechanics.

Let us examine the conditions under which we may hope to derive a master equation from the Kraus representation. The first step is to use a coarse-graining approximation with a typical time Δt which obeys

$$\tau_* \ll \Delta t \ll \tau. \tag{15.70}$$

Assuming this condition to be valid, we write the Kraus representation for the evolution between t and $t+\Delta t$ as

$$\frac{\mathrm{d}\rho_A}{\mathrm{d}t} \simeq \frac{\Delta\rho_A}{\Delta t} = \frac{1}{\Delta t}\big[\rho_A(t+\Delta t)-\rho_A(t)\big] = \frac{1}{\Delta t}\big[\mathcal{K}_{t,t+\Delta t}(\rho_A(t))-\rho_A(t)\big].$$

In order to derive a master equation, we need to satisfy two conditions.

(i) The state operator of the bipartite $\mathcal{AE}$ system must factorize: $\rho_{AE}(t) \simeq \rho_A(t)\otimes\rho_E(t)$. This condition is needed to write the Kraus representation at time $t+\Delta t$.
(ii) The superoperator $\mathcal{K}_{t,t+\Delta t}$ must depend only on Δt, and not on t.

Further comments on these conditions will be made in the next section, in the context of a specific model for $\mathcal{E}$ and its interaction with $\mathcal{A}$. A general statement is that both (i) and (ii) are valid provided $|\mathcal{V}|\tau_*/\hbar \ll 1$, where $|\mathcal{V}|$ is a typical matrix element of the $\mathcal{AE}$ interaction.

15.3.2 The Lindblad equation

Let us *assume* that conditions (i) and (ii) hold. Then we can write, from (15.45),

$$\mathcal{K}_{\Delta t}[\rho_A(t)] = \sum_{\mu} M_\mu(\Delta t)\rho_A(t)M_\mu^\dagger(\Delta t), \tag{15.71}$$

and $[\mathcal{K}_{\Delta t}[\rho_A(t)]-\rho_A(t)]$ is first-order in Δt:

$$\mathcal{K}_{\Delta t}(\rho_A(t)) = \rho_A(t)+\mathcal{O}(\Delta t).$$

[14] In general, (15.68) takes a matrix form, and some additional terms are present; see the references in "Further reading."

It follows that one of the M_μ, which we call by convention M_0, must have an expansion of the form

$$M_0(t) = I_A + \left(-\frac{\mathrm{i}}{\hbar}H - K\right)\Delta t + O(\Delta t)^2, \tag{15.72}$$

where H and K are Hermitian operators. Then the first term in (15.71) reads

$$M_0(\Delta t)\rho_A(t)M_0^\dagger(\Delta t) = \rho_A(t) - \frac{\mathrm{i}\Delta t}{\hbar}[H, \rho_A] - \Delta t\{K, \rho_A\} + O(\Delta t)^2, \tag{15.73}$$

(see (15.65)). The other terms in (15.71) must be of order $\sqrt{\Delta t}$

$$M_\mu(\Delta t) = L_\mu\sqrt{\Delta t}, \tag{15.74}$$

and the completeness relation in (15.45) leads to

$$I_A = M_0M_0^\dagger + \sum_{\mu>0} M_\mu^\dagger M_\mu = I_A - 2K\Delta t + \left(\sum_{\mu>0} L_\mu^\dagger L_\mu\right)\Delta t,$$

which implies

$$K = \frac{1}{2}\sum_{\mu>0} L_\mu^\dagger L_\mu. \tag{15.75}$$

Combining (15.71), (15.73), and (15.75) and from now on suppressing the subscript A, we find the *Lindblad equation* for the state operator ρ of $\mathcal{A}$:

$$\boxed{\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}[H, \rho] + \sum_{\mu>0}\left(L_\mu\rho L_\mu^\dagger - \frac{1}{2}\{K, \rho\}\right).} \tag{15.76}$$

The operators L_μ are the *quantum jump operators*. They describe how the state of $\mathcal{A}$ is modified by an orthogonal measurement on the environment. Provided the $L\mu S$ are bounded operators, the Lindblad equation is the most general (Markovian) master equation which preserves the positivity of the state operator.

It is instructive to rederive the Bloch equations (15.64) from the Lindblad form (15.76). Using the expressions (15.59) of M_0 and M_1 for the amplitude-damping channel, we can write M_0 and M_1 in the form

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 - \frac{\Gamma}{2}\Delta t \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & \sqrt{\Gamma\Delta t} \\ 0 & 0 \end{pmatrix},$$

or, in terms of Pauli matrices,

$$M_0 = I - \frac{\Gamma\Delta t}{2}\sigma_-\sigma_+, \quad M_1 = \sqrt{\Gamma\Delta t}\ \sigma_+.$$

The operators K and L_1 then are

$$K = \frac{\Gamma}{2}\sigma_-\sigma_+, \quad L_1 = \sqrt{\Gamma}\ \sigma_+, \tag{15.77}$$

and we recover (15.64).

15.3.3 Example: the damped harmonic oscillator

Consider a harmonic oscillator coupled to the quantized electromagnetic field, assuming for simplicity that the system is at zero temperature; the case of nonzero temperature will be dealt with in Section 15.4.3. If the oscillator is initially in an excited state, it can only cascade down due to spontaneous photon emission; it cannot absorb photons, as no photons are available at zero temperature. Hence there is only one quantum jump operator L_1, which must be proportional to a (recall the analogy between the annihilation operator a (11.6) and σ_+; see Footnote 13):

$$L_1 = \sqrt{\Gamma}\, a. \tag{15.78}$$

Then by inspection we can write down the Lindblad equation, by comparison with (15.66):

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}[H_0, \rho] + \frac{1}{2}\Gamma\Big[2a\rho a^\dagger - \{a^\dagger a, \rho\}\Big], \tag{15.79}$$

where $H_0 = \hbar\omega_0 a^\dagger a$ is the free Hamiltonian. In this derivation, we missed the radiative renormalization of the energy levels of the harmonic oscillator due to the interaction between the oscillator and the quantized electromagnetic field, which is an example of a Lamb shift, computed explicitly in Section 15.4.3. Moreover, a full derivation of (15.79) shows that it is only valid under the condition $\Gamma \ll \omega_0$. This condition allows us to ignore the coupling between matrix elements of the state operator that evolve with different eigenfrequencies, for example the coupling between populations and coherences. We get rid of the commutator by going to the interaction picture (compare with (15.67)):

$$\begin{aligned} \tilde{a}(t) &= \mathrm{e}^{\mathrm{i}H_0 t/\hbar}\, a\, \mathrm{e}^{-\mathrm{i}H_0 t/\hbar} = a\mathrm{e}^{-\mathrm{i}\omega_0 t}, \\ \tilde{a}^\dagger(t) &= \mathrm{e}^{\mathrm{i}H_0 t/\hbar}\, a^\dagger\, \mathrm{e}^{-\mathrm{i}H_0 t/\hbar} = a^\dagger \mathrm{e}^{\mathrm{i}\omega_0 t}, \end{aligned} \tag{15.80}$$

whence

$$\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}t} = \Gamma\left[\tilde{a}\,\tilde{\rho}\,\tilde{a}^\dagger - \frac{1}{2}\{\tilde{a}^\dagger\tilde{a}, \tilde{\rho}\}\right] = \Gamma\left[a\,\tilde{\rho}\,a^\dagger - \frac{1}{2}\{a^\dagger a, \tilde{\rho}\}\right]. \tag{15.81}$$

Here we have used (15.80) to obtain the second expression. In the absence of damping ($\Gamma = 0$), the average value of the operator

$$\overline{a} = \mathrm{e}^{-\mathrm{i}H_0 t/\hbar}\, a\, \mathrm{e}^{\mathrm{i}H_0 t/\hbar}$$

is time-independent. If $\Gamma \neq 0$, from (15.81) we derive the evolution equation for its average value:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\overline{a}\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Tr}\,(\overline{a}\,\rho) = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Tr}\,\left(a\,\mathrm{e}^{\mathrm{i}H_0 t}\,\rho\,\mathrm{e}^{-\mathrm{i}H_0 t}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Tr}\,(a\,\tilde{\rho}) = \mathrm{Tr}\left(a\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}t}\right),$$

while from (15.81)

$$\begin{aligned} \mathrm{Tr}\left(a\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}t}\right) &= \frac{\Gamma}{2}\mathrm{Tr}\left[2a^2\tilde{\rho}a^\dagger - aa^\dagger a\tilde{\rho} - a\tilde{\rho}a^\dagger a\right] \\ &= \frac{\Gamma}{2}\mathrm{Tr}\,\left[[a^\dagger, a]a\tilde{\rho}\right] = -\frac{\Gamma}{2}\langle\overline{a}\rangle, \end{aligned}$$

so that we find the decay law

$$\langle \overline{a}(t) \rangle = \mathrm{e}^{-\Gamma t/2} \langle \overline{a}(t=0) \rangle. \tag{15.82}$$

An analogous computation shows that the average occupation number $n(t) = \langle a^\dagger a \rangle$ decays with a relaxation time $1/\Gamma$:

$$n(t) = \mathrm{e}^{-\Gamma t}\, n(t=0). \tag{15.83}$$

As shown in Exercise 15.5.7, if the initial state of the oscillator is a coherent state (Section 11.2) $|z\rangle$ at $t = 0$, time evolution leads to

$$|z\rangle \to |z\mathrm{e}^{-\mathrm{i}\omega_0 t}\, \mathrm{e}^{-\Gamma t/2}\rangle,$$

so that the coherent state does not become entangled with its environment, although it decays slowly ($\Gamma \ll \omega_0$) toward the vacuum state. However, if one starts from a coherent superposition of coherent states $|z_1\rangle$ and $|z_2\rangle$

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \big(|z_1\rangle + |z_2\rangle \big),$$

as shown in Exercise 15.5.7, the off-diagonal terms of the state matrix decay as

$$\exp\left[-\frac{1}{2}\, \Gamma |z_1 - z_2|^2 t \right].$$

The decoherence rate Γ_{dec} is much larger than the damping rate Γ if $|z_1 - z_2|^2 \gg 1$:

$$\Gamma_{\mathrm{dec}} = \frac{1}{2}\, \Gamma |z_1 - z_2|^2. \tag{15.84}$$

It is proportional to the square of the "distance" $|z_1 - z_2|$ between the two coherent states.

15.4 Coupling to a thermal bath of oscillators

15.4.1 Exact evolution equations

In order to derive more detailed properties of the master equation, in this section we choose specific models for the system and the reservoir: in Section 15.4.3 system $\mathcal{A}$ will be a two-level system, in Section 15.4.4 it will be a Brownian particle, and in both cases the environment will be modeled by a large number of uncoupled harmonic oscillators in thermal equilibrium at temperature T. In deference to the standard terminology of thermodynamics, the environment will be called the "reservoir": $\mathcal{E} \to \mathcal{R}$. Our reservoir is thus a thermal bath of harmonic oscillators, whose Hamiltonian H_R is

$$H_R = \sum_\lambda \hbar \omega_\lambda a_\lambda^\dagger a_\lambda. \tag{15.85}$$

It is important that the frequencies ω_λ form a quasi-continuum in a large frequency interval $\sim 1/\tau_*$. The state operator of the uncoupled reservoir is given by the Boltzmann law (1.12):

$$\rho_R(t=0) = \frac{\mathrm{e}^{-H_R/k_\mathrm{B}T}}{\mathrm{Tr}\,(\mathrm{e}^{-H_R/k_\mathrm{B}T})}. \tag{15.86}$$

We shall need the following equilibrium average values, which are immediately derived from (15.86):

$$\langle a_\lambda \rangle = \langle a_\lambda^\dagger \rangle = 0, \quad \langle a_\lambda^\dagger a_\mu \rangle = n_\lambda \delta_{\lambda\mu}, \quad \langle a_\lambda a_\mu^\dagger \rangle = (n_\lambda + 1)\delta_{\lambda\mu}, \tag{15.87}$$

where the average occupation number n_λ of oscillator λ is (see (1.20))

$$n_\lambda = \frac{1}{\mathrm{e}^{\hbar\omega_\lambda/k_\mathrm{B}T} - 1}. \tag{15.88}$$

The system–reservoir coupling V is assumed to be of the form

$$V = AR, \quad R = R^\dagger = \sum_\lambda (g_\lambda a_\lambda + g_\lambda^* a_\lambda^\dagger), \tag{15.89}$$

where $A = A^\dagger$ is an operator acting in $\mathcal{H}_A$ and the total Hamiltonian H_{AR} is

$$H_{AR} = H_A + H_R + V = H_T + V, \quad H_T = H_A + H_R. \tag{15.90}$$

The evolution equation for the state operator, first written in the Schrödinger picture

$$\frac{\mathrm{d}\rho_{AR}}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}[H_{AR}, \rho_{AR}],$$

is transformed into the interaction picture, defined as previously by

$$\tilde{\rho}_{AR}(t) = \mathrm{e}^{\mathrm{i}H_T t/\hbar}\, \rho_{AR}\, \mathrm{e}^{-\mathrm{i}H_T t/\hbar}.$$

In this picture the evolution equation reads

$$\frac{\mathrm{d}\tilde{\rho}_{AR}}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}[V(t), \tilde{\rho}_{AR}(t)] = -\frac{\mathrm{i}}{\hbar}[A(t)R(t), \tilde{\rho}_{AR}(t)], \tag{15.91}$$

where $A(t)$ and $R(t)$ are given by[15]

$$\begin{aligned} A(t) &= \mathrm{e}^{\mathrm{i}H_T t/\hbar}\, A\, \mathrm{e}^{-\mathrm{i}H_T t/\hbar} = \mathrm{e}^{\mathrm{i}H_A t/\hbar}\, A\, \mathrm{e}^{-\mathrm{i}H_A t/\hbar}, \\ R(t) &= \mathrm{e}^{\mathrm{i}H_T t/\hbar}\, R\, \mathrm{e}^{-\mathrm{i}H_T t/\hbar} = \mathrm{e}^{\mathrm{i}H_R t/\hbar}\, R\, \mathrm{e}^{-\mathrm{i}H_R t/\hbar} = \sum_\lambda \left(g_\lambda a_\lambda\, \mathrm{e}^{-\mathrm{i}\omega_\lambda t} + g_\lambda a_\lambda^\dagger\, \mathrm{e}^{\mathrm{i}\omega_\lambda t} \right). \end{aligned} \tag{15.92}$$

The last expression in both lines of (15.92) is valid because H_R (H_A) does not act on the degrees of freedom of $\mathcal{A}$ ($\mathcal{R}$). The quantity that will play a central role in what follows is the *equilibrium autocorrelation function* $g(t')$ of $R(t)$:

$$g(t') = \langle R(t)R(t-t') \rangle = \langle R(t')R(0) \rangle, \tag{15.93}$$

[15] We have suppressed the tilde to simplify the notation.

where the average $\langle \bullet \rangle$ is taken with respect to the *equilibrium* state operator (15.86) of the reservoir. From time-translation invariance at equilibrium, g depends only on t' and not on t and t' separately (hence the second expression in (15.93)), while from the Hermiticity of R we have $g(t') = g^*(-t')$. The autocorrelation function $g(t')$ plays a fundamental role in *linear response theory*,[16] where it is customary to write its real and imaginary parts $C(t')$ and $-\chi(t')/2$ separately:

$$C(t') = \frac{1}{2}\langle\{R(t'), R(0)\}\rangle, \tag{15.94}$$

$$\chi(t') = \frac{\mathrm{i}}{\hbar}\langle[R(t'), R(0)]\rangle\theta(t'), \tag{15.95}$$

where the second line contains a step function $\theta(t')$, because we are interested only in the case $t' \geq 0$. The function $\chi(t')$ is called the *dynamical susceptibility* of the reservoir. In linear response theory, one shows that if the reservoir is submitted to a perturbation $-f(t)R$ (in the Schrödinger picture), where $f(t)$ is a *classical* function, then, to first order in f, the *nonequilibrium average* $\overline{R}(t)$ is

$$\overline{R}(t) = \int \mathrm{d}t'\, \chi(t') f(t - t'). \tag{15.96}$$

As a consequence, if $f(t') = f\theta(-t')$, that is, we have a constant perturbation $-fR$ for $t' < 0$, the return to equilibrium $[f(t') = 0]$ is governed by the *equilibrium* time fluctuations, a result known as the Onsager principle.

Using (15.87) and (15.89), it is easy to derive explicit expressions for $g(t')$, $C(t')$, and $\chi(t')$:

$$g(t') = \sum_\lambda |g_\lambda|^2 \left[n_\lambda \mathrm{e}^{\mathrm{i}\omega_\lambda t'} + (n_\lambda + 1)\mathrm{e}^{-\mathrm{i}\omega_\lambda t'}\right], \tag{15.97}$$

$$C(t') = \sum_\lambda |g_\lambda|^2 (2n_\lambda + 1)\cos\omega_\lambda t', \tag{15.98}$$

$$\chi(t') = \frac{2\theta(t')}{\hbar}\sum_\lambda |g_\lambda|^2 \sin\omega_\lambda t'. \tag{15.99}$$

We observe that the dynamical susceptibility does not depend on the state of the reservoir: it is independent of n_λ. Because the reservoir is large and because the frequencies ω_λ are closely spaced in a frequency interval $\sim 1/\tau_*$, we expect the correlation function to decay with a characteristic time τ_*:

$$|g(t')| \sim \mathrm{e}^{-|t'|/\tau_*}. \tag{15.100}$$

Indeed, $g(t')$ is a superposition of a large number of complex exponentials oscillating at different frequencies, and these exponentials interfere destructively once $|t'| \gtrsim \tau_*$.

[16] See "Further reading" for references on linear response theory. In these references, the "interaction picture" is called "Heisenberg picture," because coupling to another quantum system is not of interest.

Having examined the properties of the autocorrelation function, we may now revert to the evolution equation (15.91), which can be written in integral form as

$$\tilde{\rho}_{AR}(t) = \rho_{AR}(0) - \frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}t' \big[V(t'), \tilde{\rho}_{AR}(t')\big].$$

We iterate this expression once

$$\tilde{\rho}_{AR}(t) = \rho_{AR}(0) - \frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}t' \big[V(t'), \rho_{AR}(0)\big] - \frac{1}{\hbar^2}\int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \big[V(t'), \big[V(t''), \tilde{\rho}_{AR}(t'')\big]\big],$$

and differentiate with respect to t to obtain

$$\frac{\mathrm{d}\tilde{\rho}_{AR}}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}\big[V(t), \rho_{AR}(0)\big] - \frac{1}{\hbar^2}\int_0^t \mathrm{d}t' \big[V(t), [V(t'), \tilde{\rho}_{AR}(t')]\big]. \tag{15.101}$$

As usual, we assume a factorized form for $\rho_{AR}(t=0)$

$$\rho_{AR}(t=0) = \rho(t=0) \otimes \rho_R(t=0), \tag{15.102}$$

and take the partial trace over the reservoir degrees of freedom. Then the first term in (15.101) gives (Exercise 15.5.6)

$$\mathrm{Tr}_{\mathcal{R}}\big[V(t), \rho_{AR}(0)\big] = \big[A(t), \rho_A(0)\big]\,\mathrm{Tr}_{\mathcal{R}}\big(R(t)\rho_R\big) = 0,$$

where we have made use of (15.87). Under the factorization assumption (15.102), we finally obtain an *exact* equation for the state operator $\tilde{\rho}_A(t) = \tilde{\rho}(t)$ of system $\mathcal{A}$:

$$\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}t} = -\frac{1}{\hbar^2}\int_0^t \mathrm{d}t'\,\mathrm{Tr}_{\mathcal{R}}\Big(\big[V(t), [V(t'), \tilde{\rho}_{AR}(t')]\big]\Big). \tag{15.103}$$

15.4.2 The Markovian approximation

The derivation of a master equation from the exact equation (15.103) relies on the following crucial assumption: for all times t' that are relevant for the integral in (15.103) (and not only for $t = 0$ as in (15.102)!), we can use for $\tilde{\rho}_{AR}(t)$ a factorized form similar to that in the initial state (15.102):

$$\tilde{\rho}_{AR}(t) \simeq \tilde{\rho}(t) \otimes \rho_R(t=0). \tag{15.104}$$

There are two different points to be emphasized in explaining the physical origin of (15.104).

(i) All the system–reservoir correlations which arise from third- and higher-order terms in V are neglected.
(ii) The modifications to the state of the reservoir induced by its coupling to the system are neglected.

Both items (i) and (ii) are physically reasonable if the reservoir is much "larger" than the system $\mathcal{A}$: the back action of the system on the reservoir and higher-order terms in the perturbative expansion may be neglected. One can indeed show that the true small parameter in an expansion in powers of V is $|\mathcal{V}|\tau_*/\hbar$, where $\mathcal{V}$ is a typical matrix element of V. The condition for the validity of (i) and (ii) is then $|\mathcal{V}|\tau_*/\hbar \ll 1$.[17] In particular, it can be shown that

$$|\tilde{\rho}_{AR}(t) - \tilde{\rho}_A(t) \otimes \tilde{\rho}_R(t)| = O\left(\frac{|\mathcal{V}|\tau_*}{\hbar}\right)^2.$$

Plugging (15.104) into (15.102), we obtain an equation of motion for $\tilde{\rho}$ which depends only on A and g (Exercise 15.5.6):

$$\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}t} = \frac{1}{\hbar^2}\int_0^t \mathrm{d}t'\, g(t')\Big[A(t-t')\tilde{\rho}(t-t')A(t) - A(t)A(t-t')\tilde{\rho}(t-t')\Big] + \text{H.c.}, \quad (15.105)$$

where H.c. = Hermitian conjugate and we have made the change of variable $t' \to t - t'$.

Equation (15.105) is still an integro-differential equation containing memory effects, and not a master equation. To obtain a master equation, we note from (15.100) that the times t' which contribute significantly to the integral are bounded by τ_*, $t' \lesssim \tau_*$. Hence the difference $[\tilde{\rho}(t-t') - \tilde{\rho}(t)]$ is bounded by

$$|\tilde{\rho}(t-t') - \tilde{\rho}(t)| \lesssim O\left(\frac{|\mathcal{V}|\tau_*}{\hbar}\right),$$

and we can replace $\tilde{\rho}(t'-t)$ by $\tilde{\rho}(t)$ in a manner which is consistent with the preceding approximation: the error is of higher order in the small parameter. In this way, we have justified a Markovian approximation, and $\tilde{\rho}$ is given by a first-order differential equation. Taking $t \gg \tau_*$, we can send the upper limit in the integral to infinity and write

$$\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}t} = \frac{1}{\hbar^2}\int_0^\infty \mathrm{d}t'\, g(t')\Big[A(t-t')\tilde{\rho}(t)A(t) - A(t)A(t-t')\tilde{\rho}(t)\Big] + \text{H.c.}$$

Actually, this equation can be derived from perturbation theory limited to second order. It is convenient (but by no means necessary) to revert to the Schrödinger picture and to write the master equation in its final form:

$$\boxed{\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}[H_A, \rho] + \frac{1}{\hbar^2}\big(W\rho A + A\rho W^\dagger - AW\rho - \rho W^\dagger A\big),} \quad (15.106)$$

where the operator W is given by

$$\boxed{W = \int_0^\infty g(t')A(-t')\mathrm{d}t'.} \quad (15.107)$$

[17] See C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Atom–Photon Interactions*, New York: Wiley (1992), Chapter IV for a detailed discussion.

We see that the characteristic evolution time of $\tilde{\rho}$ is

$$\tau \sim \frac{\hbar^2}{|\mathcal{V}|^2 \tau_*} = \left(\frac{\hbar}{|\mathcal{V}|\tau_*}\right)^2 \tau_* \gg \tau_*.$$

The characteristic time τ is even much larger than the "natural time" $\hbar/|\mathcal{V}|$, which one would expect if $\mathcal{A}$ were coupled to a *single* mode of the reservoir:

$$\tau_* \ll \frac{\hbar}{|\mathcal{V}|} \ll \tau \sim \frac{\hbar^2}{|\mathcal{V}|^2 \tau_*}. \tag{15.108}$$

The effective coupling is reduced owing to the fact that $\mathcal{A}$ is coupled to a large number of independent modes, a phenomenon called *motion narrowing*.

15.4.3 Relaxation of a two-level system

Let us now apply the preceding results to the case where system $\mathcal{A}$ is a two-level system coupled to a thermal bath of independent harmonic oscillators: photons, phonons... The free Hamiltonian H_A of the two-level system is now H_0 (15.2)

$$H_A \equiv H_0 = -\frac{\hbar\omega_0}{2}\sigma_z, \tag{15.109}$$

and our goal is to understand its relaxation properties. The system–reservoir interaction V must be able to induce transitions between the two levels, and a possible choice is

$$V = \sigma_x R = \sigma_x \sum_\lambda \left(g_\lambda a_\lambda + g_\lambda^* a_\lambda^\dagger\right). \tag{15.110}$$

Thus $A = \sigma_x = (\sigma_+ + \sigma_-)$. The operator W (15.107) acting on the two-level system is

$$W = \int_0^\infty g(t')A(-t')\mathrm{d}t' = G_+(\omega_0)\sigma_+ + G_-(\omega_0)\sigma_-, \tag{15.111}$$

with

$$G_\pm(\omega_0) = G_\mp(-\omega_0) = \int_0^\infty g(t')\mathrm{e}^{\pm\mathrm{i}\omega_0 t'}\,\mathrm{d}t', \tag{15.112}$$

where we have used (15.67). Plugging (15.112) into (15.106) and using $\sigma_+^2 = \sigma_-^2 = 0$, we obtain

$$\begin{aligned}\frac{\mathrm{d}\rho}{\mathrm{d}t} = {} & \frac{\mathrm{i}}{2}\omega_0[\sigma_z, \rho] \\ & + (G_+ + G_+^*)\sigma_+\rho\,\sigma_- - G_+\sigma_-\sigma_+\rho - G_+^*\rho\,\sigma_-\sigma_+ \\ & + (G_- + G_-^*)\sigma_-\rho\,\sigma_+ - G_-\sigma_+\sigma_-\rho - G_-^*\rho\,\sigma_+\sigma_- \\ & + (G_+ + G_-^*)\sigma_+\rho\,\sigma_+ + (G_- + G_+^*)\sigma_-\rho\,\sigma_-.\end{aligned} \tag{15.113}$$

Using the invariance of the trace under cyclic permutations, we can check that the equation has been written in such a way that each of the first three lines in the right-hand side has zero trace. The fourth line does not contribute to the evolution of the

populations (Exercise 15.5.8), only to that of the coherences. But even this contribution to the coherences may be neglected in the rotating-wave approximation, using the same argument as in Section 14.4.1: in the interaction picture, $\tilde{\sigma}_\pm(t) \sim \exp(\mp \mathrm{i}\omega_0 t)$, and the last term in (15.113) varies as $\exp(\mp 2\mathrm{i}\omega_0 t)$. It is rapidly oscillating and assumed to average to zero. This is a general result: if the relaxation terms are written as

$$\frac{\mathrm{d}\tilde{\rho}_{ij}}{\mathrm{d}t} = \sum_{k,l=0}^{1} \gamma_{ijkl}\tilde{\rho}_{kl},$$

it can be shown that the coefficients γ_{ijkl} can be neglected if $|\omega_{ij} - \omega_{kl}| \gg \Gamma$, $\hbar\omega_{ij} = E_i - E_j$.[18] This is called the *secular approximation*, and it allows us to justify the form of the Bloch equations (15.6).

Let us compute $G_\pm(\omega_0)$ explicitly:

$$\begin{aligned} G_+(\omega_0) &= \sum_\lambda |g_\lambda|^2 \left((n_\lambda + 1)\frac{\mathrm{i}}{\omega_0 - \omega_\lambda + \mathrm{i}\eta} + n_\lambda \frac{\mathrm{i}}{\omega_0 + \omega_\lambda + \mathrm{i}\eta} \right), \\ G_-(\omega_0) &= \sum_\lambda |g_\lambda|^2 \left(n_\lambda \frac{-\mathrm{i}}{\omega_0 - \omega_\lambda - \mathrm{i}\eta} + (n_\lambda + 1)\frac{-\mathrm{i}}{\omega_0 + \omega_\lambda - \mathrm{i}\eta} \right), \end{aligned} \tag{15.114}$$

where $\eta \to 0^+$. Using the standard formula

$$\frac{\mathrm{i}}{x \pm \mathrm{i}\eta} = \mathrm{i}\frac{\mathbb{P}}{x} \pm \pi\delta(x), \tag{15.115}$$

where $\mathbb{P}$ denotes a Cauchy principal value, for $G_+(\omega_0)$ we find

$$\begin{aligned} G_+(\omega_0) &= \frac{1}{2}\Gamma_+ - \mathrm{i}\Delta_+, \\ \Gamma_+ &= 2\pi \sum_\lambda |g_\lambda|^2 (n_\lambda + 1)\delta(\omega_0 - \omega_\lambda), \\ \Delta_+ &= -\sum_\lambda |g_\lambda|^2 \left((n_\lambda + 1)\frac{\mathbb{P}}{\omega_0 - \omega_\lambda} + n_\lambda \frac{\mathbb{P}}{\omega_0 + \omega_\lambda} \right), \end{aligned} \tag{15.116}$$

while $G_-(\omega_0)$ is given by

$$\begin{aligned} G_-(\omega_0) &= \frac{1}{2}\Gamma_- - \mathrm{i}\Delta_-, \\ \Gamma_- &= 2\pi \sum_\lambda |g_\lambda|^2 n_\lambda \delta(\omega_0 - \omega_\lambda), \\ \Delta_- &= -\sum_\lambda |g_\lambda|^2 \left(n_\lambda \frac{\mathbb{P}}{\omega_0 - \omega_\lambda} + (n_\lambda + 1)\frac{\mathbb{P}}{\omega_0 + \omega_\lambda} \right). \end{aligned} \tag{15.117}$$

[18] See C. Cohen-Tannoudji *et al.*, *Atom–Photon Interactions*, New York: Wiley (1992), Chapter IV for a detailed discussion.

Substituting the two preceding equations into (15.113) and making use of

$$\sigma_+\sigma_- = \frac{1}{2}(1+\sigma_z), \quad \sigma_-\sigma_+ = \frac{1}{2}(1-\sigma_z),$$

we obtain the final expression for the master equation in the Lindblad form:

$$\begin{aligned}\frac{\mathrm{d}\rho}{\mathrm{d}t} &= \frac{\mathrm{i}}{2}(\omega_0+\Delta)[\sigma_z,\rho] \\ &\quad +\frac{1}{2}\Gamma_+\big(2\sigma_+\rho\,\sigma_- - \{\sigma_-\sigma_+,\rho\}\big) \\ &\quad +\frac{1}{2}\Gamma_-\big(2\sigma_-\rho\,\sigma_+ - \{\sigma_+\sigma_-,\rho\}\big).\end{aligned} \tag{15.118}$$

The two quantum jump operators of the Lindblad equation are

$$L_+ = \sqrt{\frac{\Gamma_+}{2}}\,\sigma_+, \quad L_- = \sqrt{\frac{\Gamma_-}{2}}\,\sigma_-. \tag{15.119}$$

The energy shift Δ (or Lamb shift)

$$\Delta = \Delta_- - \Delta_+ = \sum_\lambda |g_\lambda|^2(2n_\lambda+1)\left(\frac{\mathbb{P}}{\omega_0-\omega_\lambda} + \frac{\mathbb{P}}{\omega_0+\omega_\lambda}\right) \tag{15.120}$$

represents the radiative correction to the energy-level difference ω_0 due to the interaction of the two-level system with the thermal bath of oscillators. Equation (15.118) generalizes (15.64) obtained at $T=0$, where only spontaneous emission was taken into account and the Lamb shift could not be computed. At nonzero temperature photon absorption must also be taken into account: the relaxation rate Γ_+ describes the transitions $|1\rangle \to |0\rangle$ and Γ_- the transitions $|0\rangle \to |1\rangle$ (Fig. 15.3). It is easy to check (Exercise 15.5.8) that $\Gamma = \Gamma_+ + \Gamma_-$ is the relaxation rate for the populations, while that for the coherences is $\Gamma/2$: the relation $T_2 = 2T_1$ also holds at nonzero temperature. In the same exercise, it is

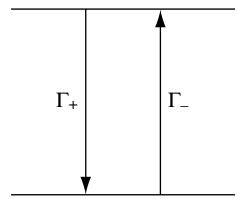

Fig. 15.3. Transition rates Γ_+ and Γ_-.

shown that in the long-time limit the populations of the levels $|0\rangle$ and $|1\rangle$ are given by Boltzmann's law (15.3), with temperature T equal to that of the thermal bath. This is a quite satisfactory result, as it shows that the system is in equilibrium with the bath in the long-time limit. The total width Γ is given explicitly by

$$\Gamma = \Gamma_+ + \Gamma_- = \frac{2\pi}{\hbar} \sum_\lambda |g_\lambda|^2 (2n_\lambda + 1) \delta(\hbar\omega_0 - \hbar\omega_\lambda). \tag{15.121}$$

This provides a nice check of the calculation, as (15.121) can be written in the form of the Fermi Golden Rule (9.170):

$$\Gamma = \frac{2\pi}{\hbar} |g(\omega_0)|^2 (2n_0 + 1) \mathcal{D}(\hbar\omega_0),$$

where $\mathcal{D}(\hbar\omega_0)$ is the density of states of the reservoir. The ratio Γ_+/Γ_- is given by a Boltzmann law

$$\frac{\Gamma_+}{\Gamma_-} = \mathrm{e}^{\hbar\omega/k_\mathrm{B}T}.$$

The master equation (15.118) allows us to write by inspection (recall the correspondence $a \to \sigma_+$, $a^\dagger \to \sigma_-$; see Footnote 12) the $T \neq 0$ generalization of (15.79), which gives the master equation for a harmonic oscillator coupled to the quantized electromagnetic field at nonzero temperature:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}[H_0, \rho] + \frac{1}{2}\Gamma_+\big[2a\rho a^\dagger - \{a^\dagger a, \rho\}\big] + \frac{1}{2}\Gamma_-\big[2a^\dagger \rho a - \{aa^\dagger, \rho\}\big]. \tag{15.122}$$

Detailed derivations of the preceding equation can be found in textbooks on quantum optics (see "Further reading").

15.4.4 Quantum Brownian motion

Our last example will be that of a heavy free particle with mass M coupled to a thermal bath of harmonic oscillators with masses m_λ and frequencies ω_λ. This is a typical situation for Brownian particle motion. A heavy particle interacts with a thermal bath of light particles (molecules), and one may identify two widely separated time scales: the time scale τ_* for the bath and the time scale τ for the motion of the heavy particle, with $\tau_* \ll \tau$. The full Hamiltonian H_{AR} is assumed to have a translation-invariant form

$$H_{AR} = \frac{P^2}{2M} + \sum_\lambda \frac{P_\lambda^2}{2m_\lambda} + \frac{1}{2}\sum_\lambda m_\lambda \omega_\lambda^2 (X - X_\lambda)^2, \tag{15.123}$$

where (P, P_λ) and (X, X_λ) are momentum and position operators for the particle and the oscillators. For the sake of simplicity, we have limited ourselves to one-dimensional

motion, without losing any essential physics. The decomposition (15.90) of H_{AR} reads

$$H_A = \frac{P^2}{2M}, \tag{15.124}$$

$$H_R = \sum_\lambda \left(\frac{P_\lambda^2}{2m_\lambda} + \frac{1}{2}\,\omega_\lambda^2\, X_\lambda^2 \right) = \sum_\lambda \hbar\omega_\lambda a_\lambda^\dagger a_\lambda, \tag{15.125}$$

$$V = \frac{1}{2}\,\kappa X^2 + XR = H_{\rm CT} + X\Big[-\sum_\lambda g_\lambda \Big(a_\lambda + a_\lambda^\dagger \Big) \Big], \tag{15.126}$$

with $g_\lambda = \sqrt{\hbar m_\lambda \omega_\lambda^3/2}$, $\kappa = \sum_\lambda m_\lambda \omega_\lambda^2$, and CT standing for "counter-term" for reasons to be explained below. The operator A is therefore to be identified with the position operator of the Brownian particle, and we have neglected the zero-point energy of the oscillators. It may appear that translation invariance has been broken in (15.126), but this is of course an artefact of the decomposition: as we shall see later on, the contribution of the translation-noninvariant counter-term

$$H_{\rm CT} = \frac{1}{2}\,\kappa X^2 \tag{15.127}$$

is canceled by another contribution from the interaction. It will be convenient but by no means necessary (see the comments following (15.142)) to work in the high-temperature limit where (15.88) becomes

$$n_\lambda \simeq n_\lambda + 1 \simeq \frac{k_{\rm B} T}{\hbar\omega_\lambda} \gg 1. \tag{15.128}$$

We recall that the frequencies ω_λ are assumed to be closely spaced in an interval $\sim 1/\tau_*$, so that the sums over λ can be replaced by integrals over ω. It is convenient to define the spectral function $J(\omega)$:

$$J(\omega) = \frac{\pi}{\hbar} \sum_\lambda |g_\lambda|^2 \delta(\omega - \omega_\lambda) = \frac{\pi}{2} \sum_\lambda m_\lambda \omega_\lambda^3 \delta(\omega - \omega_\lambda). \tag{15.129}$$

From (15.98), (15.99), and (15.126) we find the expressions for the real and imaginary parts of the autocorrelation function $g(t')$:

$$\begin{aligned} C(t') &= \frac{2k_{\rm B}T}{\pi} \int_0^\infty \frac{{\rm d}\omega}{\omega}\, J(\omega) \cos\omega t', \\ \chi(t') &= \frac{2\theta(t')}{\pi} \int_0^\infty {\rm d}\omega\, J(\omega) \sin\omega t' = -\frac{\theta(t')}{k_{\rm B}T}\, \frac{{\rm d}C}{{\rm d}t'}. \end{aligned} \tag{15.130}$$

In order to proceed further, we must now choose a specific form for the spectral function $J(\omega)$. The typical frequency scale for $J(\omega)$ being $\omega_* = 1/\tau_*$, we choose $J(\omega)$ to vanish for $\omega \gg \omega_*$: ω_* plays the role of a frequency cutoff. The most convenient model for

analytic calculations is that of Caldeira and Leggett,[19] where $J(\omega)$ is linear for $\omega \leq \omega_*$ and vanishes for $\omega > \omega_*$:

$$\begin{aligned} J(\omega) &= M\gamma\omega, \quad 0 \leq \omega \leq \omega_*, \\ J(\omega) &= 0, \qquad\quad \omega > \omega_*. \end{aligned} \tag{15.131}$$

The coefficient γ has the dimension of a frequency, and will be interpreted physically as a friction coefficient, as in the equation of motion (14.104) $\dot{v} = -\gamma v$. We expect that the results do not depend qualitatively on the exact shape of $J(\omega)$, the only important feature being the existence of a high-frequency cutoff ω_*. In Exercise 15.5.10 it is shown that equivalent results are obtained using

$$J(\omega) = M\gamma\,\omega\left(\frac{\omega_*^2}{\omega^2+\omega_*^2}\right).$$

With the choice (15.131), $C(t')$ has a simple analytic form:

$$C(t') = 2k_{\mathrm{B}}TM\gamma\,\frac{\sin\omega_* t'}{\pi t'}. \tag{15.132}$$

The function $\sin\omega_* t'/\pi t'$ has a peak of height ω_*/M and width $\sim 1/\omega_* = \tau_*$ at $t' = 0$, and it becomes a delta function in the limit $\omega_* \to \infty$. We shall call $\delta_*(t')$ a smeared delta function of width τ_*. With this notation, the autocorrelation function reads

$$g(t') = 2M\gamma k_{\mathrm{B}}T\delta_*(t') + \mathrm{i}\hbar M\gamma\delta_*'(t') = 2D\delta_*(t') + \mathrm{i}\hbar M\gamma\delta_*'(t'), \tag{15.133}$$

where we have used Einstein's relation (14.113) linking the momentum diffusion coefficient D to γ and T, $D = M\gamma k_{\mathrm{B}}T$.

After these preliminaries, we are now ready to give an explicit form for the general master equation (15.106), which in the present case becomes

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}\left[\frac{P^2}{2M},\rho\right] - \frac{\mathrm{i}}{\hbar}\left[\frac{1}{2}\kappa X^2,\rho\right] - \frac{1}{\hbar^2}\left(W\rho X + X\rho W - XW\rho - \rho W^\dagger X\right), \tag{15.134}$$

with

$$W = \int_0^\infty g(t')\tilde{X}(-t')\mathrm{d}t'. \tag{15.135}$$

The operator $\tilde{X}$ in the interaction picture is given by

$$\tilde{X}(t') = \exp\left[\frac{\mathrm{i}P^2t'}{2M\hbar}\right] X \exp\left[-\frac{\mathrm{i}P^2t'}{2M\hbar}\right] = X + \frac{Pt}{M}, \tag{15.136}$$

[19] A. Caldeira and A. Leggett, Path integral approach to quantum Brownian motion, *Physica* **121**A, 587 (1983).

a result which is immediately derived from (8.67). The term proportional to D on the right-hand side of the master equation involves the integral

$$\frac{2D}{\hbar^2}\int_0^\infty \delta_*(t')\ \left[\left(X-\frac{Pt'}{M}\right)\rho(t)X+X\rho(t)\left(X-\frac{Pt'}{M}\right)\right.$$
$$\left.-X\left(X-\frac{Pt'}{M}\right)\rho(t)-\rho(t)\left(X-\frac{Pt'}{M}\right)X\right]\mathrm{d}t'.$$

Owing to the narrow width of $\delta_*(t')$, the terms proportional to Pt'/M are negligible and we are left with the double commutator

$$-\frac{D}{\hbar^2}\big[X,\big[X,\rho(t)\big]\big]. \tag{15.137}$$

The term proportional to $M\gamma$ is

$$\frac{\mathrm{i}M\gamma}{\hbar}\int_0^\infty \delta'_*(t')\left[\left(X-\frac{Pt'}{M}\right)\rho(t)X+X\rho(t)\left(X-\frac{Pt'}{M}\right)\right.$$
$$\left.-X\left(X-\frac{Pt'}{M}\right)\rho(t)-\rho(t)\left(X-\frac{Pt'}{M}\right)X\right]\mathrm{d}t'. \tag{15.138}$$

The two integrals that we need are

$$\text{(i)}\ \int_0^\infty \delta'_*(t')t'\,\mathrm{d}t'=-\frac{1}{2},$$
$$\text{(ii)}\quad \int_0^\infty \delta'_*(t')\mathrm{d}t'=\int_0^\infty \mathrm{d}t'\,\frac{\mathrm{d}}{\mathrm{d}t'}\left(\frac{\sin\omega_* t'}{\pi t'}\right)=-\frac{\omega_*}{\pi}. \tag{15.139}$$

Equation (15.138) can be written as a sum of two terms. The first one, which depends on (i), is

$$\frac{\gamma}{2\mathrm{i}\hbar}\big[X,\{P,\rho(t)\}\big], \tag{15.140}$$

and the second one depending on (ii) is

$$\frac{\mathrm{i}M\gamma\omega_*}{\pi\hbar}\big[X^2,\rho(t)\big]=\frac{\mathrm{i}}{\hbar}\left[\frac{1}{2}\kappa X^2,\rho(t)\right], \tag{15.141}$$

because in the Caldeira–Leggett model κ is given by

$$\kappa=\sum_\lambda m_\lambda\omega_\lambda^2=\frac{2}{\pi}\int_0^\infty\frac{\mathrm{d}\omega}{\omega}J(\omega)=\frac{2M\gamma\omega_*}{\pi}.$$

Then the term in (15.141) exactly cancels the contribution of H_{CT} to the evolution of the state operator. Collecting all the contributions to $\mathrm{d}\rho/\mathrm{d}t$, we finally obtain the master equation describing the quantum evolution of the Brownian particle:

$$\boxed{\frac{\mathrm{d}\rho}{\mathrm{d}t}=-\frac{\mathrm{i}}{\hbar}\left[\frac{P^2}{2M},\rho(t)\right]-\frac{\mathrm{i}\gamma}{2\hbar}\big[X,\{P,\rho(t)\}\big]-\frac{D}{\hbar^2}\big[X,\big[X,\rho(t)\big]\big].} \tag{15.142}$$

Equation (15.142) is one of the basic results in the theory of open quantum systems. It should be observed that this equation is not of the Lindblad form, although it preserves the positivity of the state operator. The first term gives the unitary evolution of the wave packet, the second one describes friction, and the last one governs decoherence, as we shall see in detail in the next subsection. A Fokker–Planck equation for the probability distribution of p can be derived from (15.142); see Exercise 15.5.11.

Since the model defined in Eq. (15.123) is linear (that is, its classical equations of motion are linear), it can be solved exactly without taking the high-temperature limit. This is done in practice using path-integral methods. One can even put the Brownian particle in a harmonic potential well with frequency Ω. The exact solution at time t is

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}\left[\frac{P^2}{2M} + \frac{1}{2}M\Omega^2(t)X^2, \rho(t)\right] - \frac{\mathrm{i}\gamma(t)}{2\hbar}\big[X, \{P, \rho(t)\}\big] - \frac{D(t)}{\hbar^2}\big[X, \big[X, \rho(t)\big]\big] - \frac{f(t)}{\hbar}\big[X, [P, \rho]\big].$$

We note the presence of a fourth term, called anomalous diffusion, which is negligible in the long-time limit $t \to \infty$. The functions $\Omega(t)$, $\gamma(t)$, $D(t)$, and $f(t)$ are given by integrals which must, in general, be computed numerically. In the long-time limit which has been taken in (15.142), analytical evaluation of the integrals is sometimes possible.

15.4.5 Decoherence and Schrödinger's cats

The preceding results are of the utmost importance, because they exhibit precise mechanisms for decoherence. A Brownian particle is a large object by macroscopic standards, and by constructing a quantum state of the particle which is a coherent superposition of two nonoverlapping wave packets we exhibit an example of a Schrödinger's cat. To be specific, let us assume that at $t = 0$ we have a coherent superposition of two Gaussian wave packets centered at $x = \pm a$ and having width $\sigma \ll a$, so that the overlap of the two wave packets is negligible. The initial wave function of the Brownian particle then is

$$\varphi(x) \simeq \frac{1}{\sqrt{2}}\left(\frac{1}{\pi\sigma^2}\right)^{1/4}\left(\exp\Big[-\frac{(x-a)^2}{2\sigma^2}\Big] + \exp\Big[-\frac{(x+a)^2}{2\sigma^2}\Big]\right). \tag{15.143}$$

The Fourier transform $\tilde{\varphi}(p)$ of (15.143) is readily computed and the momentum probability distribution $|\tilde{\varphi}(p)|^2$ is found to be

$$|\tilde{\varphi}(p)|^2 = \frac{2\sigma}{\hbar\sqrt{\pi}}\exp\left(-\frac{\sigma^2 p^2}{\hbar^2}\right)\cos^2\frac{pa}{\hbar}. \tag{15.144}$$

$|\tilde{\varphi}(p)|^2$ is a Gaussian of width $\sim\hbar/\sigma$ modulated by fast oscillations of period $\pi\hbar/a \ll \hbar/\sigma$. These oscillations originate in the coherence of the two wave packets in (15.143). Before exploiting (15.142), let us give a qualitative physical explanation for decoherence. The Brownian particle undergoes a large number of collisions with the light particles (molecules) in the thermal bath. Because of these collisions the particle follows a random

walk in momentum space[20] with a diffusion coefficient D (14.113), and the momentum dispersion Δp is

$$\Delta p^2 = 2Dt. \tag{15.145}$$

Each of the peaks in $|\tilde{\varphi}(p)|^2$ is broadened under the influence of collisions, and the peaks will be completely blurred out after a decoherence time τ_{dec} found from (15.145) as

$$\Delta p^2 \sim \left(\frac{\pi\hbar}{a}\right)^2 = 2D\tau_{\text{dec}},$$

or

$$\tau_{\text{dec}} \sim \frac{\hbar^2}{Da^2}. \tag{15.146}$$

Let us derive this result from the master equation (15.142). We limit ourselves to short times, so that the motion of the Brownian particle can be neglected.[21] This is equivalent to taking the limit $M \to \infty$ in the master equation, and in this limit only the last term on the right-hand side survives (see Exercise 15.5.11 for a study of the general case). The off-diagonal matrix elements of the state operator obey the differential equation

$$\frac{\partial}{\partial t}\langle x|\rho(t)|x'\rangle = -\frac{D}{\hbar^2}(x-x')^2\langle x|\rho(t)|x'\rangle. \tag{15.147}$$

The off-diagonal matrix elements of ρ decay with a relaxation time τ_{dec}:

$$\boxed{\tau_{\text{dec}} \simeq \frac{\hbar^2}{4Da^2},} \tag{15.148}$$

because $|x-x'| \simeq 2a$, in agreement with the preceding heuristic estimate.

Let us give a very rough estimate for a typical decoherence time. Consider a Brownian particle of radius $R \simeq 1\,\mu\text{m}$ in air with viscosity $\eta \sim 10^{-5}$. The friction coefficient γ is given by the Stokes law $\gamma = 6\pi\eta R/M$. For $a = 10\,\mu\text{m}$ we find $\tau_{\text{dec}} \sim 10^{-27}\,\text{s}$. "Large" Schrödinger's cats are really quite short-lived! In Appendix B we describe experiments in which one is able to build Schrödinger's cats small enough that decoherence can be observed and τ_{dec} measured, thus allowing an experimental verification of the decoherence mechanism.

There are other ways of writing the result (15.148). Using $D = M\gamma k_{\text{B}}T$ and introducing the thermal wavelength

$$\lambda_T = \frac{h}{\sqrt{2\pi M k_{\text{B}} T}},$$

that is, the de Broglie wavelength at temperature T, (15.148) becomes

$$\tau_{\text{dec}} \sim \frac{1}{\gamma}\left(\frac{\lambda_T}{a}\right)^2. \tag{15.149}$$

[20] Not to be confused with diffusion in position space!

[21] This is a general result. In the short-time limit, Brownian motion is dominated by diffusion; see "Further reading."

The results of the present chapter allow us to give a general picture of decoherence. The first general feature is that one finds privileged states in the Hilbert space of states: coherent states in the case of Section 15.3.3 and position states in that of Section 15.4.4. These states are called *pointer states*, and they define the preferred basis of Section 6.4.1. A generic state of the Hilbert space is not stable when the system is put in contact with an environment but decays into an incoherent superposition of pointer states, which do not become entangled with their environment and are therefore the stable states. The stability of the pointer states can be traced back to the form the system interaction with the environment. For example, the pointer states of the Brownian particle are position states, because the interaction with its environment is proportional to the position operator X. As already mentioned in Section 15.3.3, a mode of the quantized electromagnetic field in a coherent state remains in a coherent state, because of the form of its interaction with a $T = 0$ environment. Coherent states are therefore also pointer states. The second general feature is that the decoherence time is inversely proportional to the square of the "distance" between pointer states: this distance is the ordinary one in the case of the position states of Section 15.4.4, and $|z_1 - z_2|$ in the case of the coherent states $|z_1\rangle$ and $|z_2\rangle$ of Section 15.3.3. The decoherence time is nothing other than the lifetime of Schrödinger's cats, and this lifetime is extremely short for macroscopic, and even mesoscopic, objects. As explained in Appendix B, decoherence is very likely an essential ingredient in the theory of quantum measurements. It explains why the measurement apparatus cannot be found in a quantum superposition, but can only exist in one of its pointer states.

15.5 Exercises

15.5.1 POVM as projective measurement in a direct sum

Let us consider the POVM defined by (15.30) and (15.31). Define the unnormalized vectors $|\tilde{\alpha}\rangle = \sqrt{2/3}\,|\alpha\rangle$, $\alpha = a, b, c$, and use these three vectors belonging to $\mathcal{H}^{(2)}$ to construct two vectors belonging to a three-dimensional space $\mathcal{H}^{(3)}$, written as the first two *rows* of a 3×3 matrix M. Complete M by a third vector orthogonal to the two preceding ones to obtain

$$M = \begin{pmatrix} \sqrt{2/3} & -\sqrt{1/6} & -\sqrt{1/6} \\ 0 & \sqrt{1/2} & -\sqrt{1/2} \\ \sqrt{1/3} & \sqrt{1/3} & \sqrt{1/3} \end{pmatrix}.$$

Why is M an orthogonal matrix, $M^T M = I$? Consider a projective measurement in $\mathcal{H}^{(3)}$ built from the three *columns* $|u_\alpha\rangle$ of M. Show that an observer unaware of the third component of $|u_\alpha\rangle$ will conclude that she has performed a POVM in $\mathcal{H}^{(2)}$.

15.5.2 Using a POVM to distinguish between states

Assume that Alice sends Bob qubits that can be either in state $|a_\perp\rangle$ or in state $|b_\perp\rangle$, each with 50% probability (the notation is that of Section 15.1.3). Bob performs a POVM with

elements $\mathcal{Q}_a$, $\mathcal{Q}_b$, and $\mathcal{Q}_c$ (15.30). Show that if he finds the result b (a), he can be sure that the qubit was in state $|a_\perp\rangle$ ($|b_\perp\rangle$), but if he finds result c, he cannot decide. Show that in 50% of the cases Bob will be able to decide with certainty between the two states.

15.5.3 A POVM on two arbitrary qubit states

1. Consider the following two qubit states:

$$|a\rangle = \cos\alpha\,|0\rangle + \sin\alpha\,|1\rangle, \quad |b\rangle = \sin\alpha\,|0\rangle + \cos\alpha\,|1\rangle.$$

What are the projectors $\mathcal{P}_{a_\perp}$ and $\mathcal{P}_{b_\perp}$ onto the states $|a_\perp\rangle$ and $b_\perp\rangle$ respectively orthogonal to $|a\rangle$ and $|b\rangle$? Let $|c\rangle$,

$$|c\rangle = \lambda|0\rangle + \mu|1\rangle,$$

be a third qubit state vector. Build a POVM with $\mathcal{P}_{a_\perp}$, $\mathcal{P}_{b_\perp}$, and $\mathcal{P}_c$ by writing

$$A(\mathcal{P}_{a_\perp} + \mathcal{P}_{b_\perp}) + B\mathcal{P}_c = I.$$

Show that A and B can be expressed in terms of the scalar product $S = \langle a|b\rangle = \sin 2\alpha$.

2. Assume that Alice sends Bob a random sequence of states $|a\rangle$ and $|b\rangle$, each occurring with 50% probability. Bob performs a POVM

$$\{\mathcal{Q}_{a_\perp}, \mathcal{Q}_{b_\perp}, \mathcal{Q}_c\}.$$

What is the probability that he can be sure that Alice sent $|a\rangle$ or $b\rangle$? Application: in the quantum cryptography setup explained in Section 3.1.3, $\alpha = \pi/8$. Show that Eve can fool Bob in 79% of the cases.

15.5.4 Transposition is not completely positive

Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be two Hilbert spaces of dimension N. Consider the maximally entangled state

$$|\varphi_{AB}\rangle = \frac{1}{\sqrt{N}} \sum_{m=1}^{N} |m_A \otimes m'_B\rangle,$$

and the corresponding state operator $\rho_{AB} = |\varphi_{AB}\rangle\langle\varphi_{AB}|$. The transposition operator $\mathcal{T}_A$ in $\mathcal{H}_A$ has the following action on ρ_{AB}:

$$(\mathcal{T}_A \otimes I_B)\rho_{AB} = \frac{1}{N} \sum_{m,n} (|n\rangle\langle m|)_A \otimes (|m'\rangle\langle n'|)_B.$$

Define $\mathcal{N} = N\rho_{AB}$ and show that applying $\mathcal{N}$ to a vector $|\varphi_A \otimes \psi_B\rangle$ has the result

$$\mathcal{N}|\varphi_A \otimes \psi_B\rangle = |\psi_A \otimes \varphi_B\rangle.$$

Show that $\mathcal{N}^2 = 1$. Write the explicit form of $\mathcal{N}$ in the case $N = 2$: this is the so-called SWAP matrix. Show, first in the case $N = 2$ and then in general, that $\mathcal{N}$ must have negative eigenvalues.

15.5.5 Phase and amplitude damping

1. Let us examine the following model with simultaneous phase and amplitude damping for a two-level system. Three Kraus operators are given by

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda-\gamma} \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix}.$$

   Check that

$$\sum_{\mu=0}^{2} M_\mu^\dagger M_\mu = I.$$

   What are the restrictions on λ and γ?
2. Show that the transformed state matrix $\mathcal{K}[\rho]$ is

$$\begin{pmatrix} \rho_{00} + \gamma\rho_{11} & \rho_{01}\sqrt{1-\lambda-\gamma} \\ \rho_{10}\sqrt{1-\lambda-\gamma} & \rho_{11}(1-\gamma) \end{pmatrix}.$$

3. What is the result after n iterations of the Kraus operator? Setting

$$\gamma = \frac{\Gamma t}{n}, \quad \lambda = \frac{\Lambda t}{n}, \quad n \gg 1,$$

   show that

$$\rho(t) = \begin{pmatrix} 1 - \rho_{11}\mathrm{e}^{-\Gamma t} & \rho_{01}\mathrm{e}^{-(\Lambda+\Gamma)t/2} \\ \rho_{10}\mathrm{e}^{-(\Lambda+\Gamma)t/2} & \rho_{11}\mathrm{e}^{-\Gamma t} \end{pmatrix}.$$

   What are the relaxation times T_1 and T_2? Check that $T_2 \leq 2T_1$.

15.5.6 Details of the proof of the master equation

1. Show that if $\rho_{AR}(0) = \rho_A(0) \otimes \rho_R(0)$, then

$$\mathrm{T}_{r\mathcal{R}}\left[V(t), \rho_{AR}(0)\right] = \left[A(t), \rho_A(0)\right]\mathrm{Tr}\left(R(t)\rho_R(0)\right) = 0.$$

2. Fill in the details of the calculations leading from (15.101) to (15.105).

15.5.7 Superposition of coherent states

We wish to study the decoherence of a superposition of two coherent states in the damped oscillator model of Section 15.3.3. The time evolution of the state operator is given by (15.79):

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}[H_0, \rho] + \frac{1}{2}\Gamma\Big[2a\rho a^\dagger - \{a^\dagger a, \rho\}\Big].$$

In this problem it is instructive to keep the H_0 part of the evolution.

1. Let us consider eigenstates $|n\rangle$ of the free Hamiltonian $H_0 = \omega_0 a^\dagger a$, and let ρ_{nm} be the matrix element $\langle n|\rho|m\rangle$ of ρ. Show that the diagonal matrix element ρ_{nn} obeys

$$\frac{\mathrm{d}\rho_{nn}}{\mathrm{d}t} = -n\Gamma\rho_{nn} + (n+1)\Gamma\rho_{n+1,n+1}.$$

Can you give a physical interpretation for the two terms of this equation? Argue that Γ is the rate for spontaneous emission of a photon (or a phonon). What is the evolution equation for the coherence $\rho_{n+1,n}$?

2. Let us introduce the function $C(\lambda, \lambda^*; t)$ by

$$C(\lambda, \lambda^*; t) = \mathrm{Tr}\left(\rho\, \mathrm{e}^{\lambda a^\dagger}\, \mathrm{e}^{-\lambda^* a}\right).$$

Show that partial derivatives with respect to λ have the following effect in the trace:

$$\frac{\partial}{\partial\lambda} \to \rho\, a^\dagger, \qquad \left(\frac{\partial}{\partial\lambda} - \lambda^*\right) \to a^\dagger \rho.$$

Hint: use the identity (2.54) to commute $a^\dagger$ and $\exp(-\lambda^* a)$. What are the corresponding identities for $\partial/\partial\lambda^*$?

3. Show that $C(\lambda, \lambda^*; t)$ obeys the partial differential equation

$$\left[\frac{\partial}{\partial t} + \left(\frac{\Gamma}{2} - \mathrm{i}\omega_0\right)\frac{\partial}{\partial \ln\lambda} + \left(\frac{\Gamma}{2} + \mathrm{i}\omega_0\right)\frac{\partial}{\partial \ln\lambda^*}\right] C(\lambda, \lambda^*; t) = 0.$$

This equation is solved by the method of characteristics. The solution is (derive it or check it!)

$$C(\lambda, \lambda^*; t) = C_0\left(\lambda \exp[-(\Gamma/2 - \mathrm{i}\omega_0)t], \lambda^* \exp[-(\Gamma/2 + \mathrm{i}\omega_0)t]\right)$$

with

$$C(\lambda, \lambda^*; t = 0) = C_0(\lambda, \lambda^*).$$

4. Assume that the initial state $t = 0$ is a coherent state $|z\rangle$:

$$|z\rangle = \mathrm{e}^{-|z|^2/2}\, \mathrm{e}^{z a^\dagger} |0\rangle.$$

Show that in this case

$$C_0 = \exp(\lambda z^* - \lambda^* z),$$

and that the state at time t is the coherent state $|z(t)\rangle$ with

$$z(t) = z\, \mathrm{e}^{-\mathrm{i}\omega_0 t}\, \mathrm{e}^{-\Gamma t/2}.$$

Therefore, a coherent state remains a coherent state when $\Gamma \neq 0$ (compare with (11.38)), but $|z(t)| \to 0$ for $t \gg 1/\Gamma$. In the complex plane, $z(t)$ spirals to the origin. As $\Gamma \ll \omega_0$, one observes many turns around the origin.

5. Let us now consider a superposition of two coherent states at $t = 0$:

$$|\Phi\rangle = c_1|z_1\rangle + c_2|z_2\rangle.$$

Show that at $t = 0$

$$C_{12}(t = 0) = \mathrm{Tr}\left(|z_1\rangle\langle z_2|\mathrm{e}^{\lambda a^\dagger}\mathrm{e}^{-\lambda^* a}\right) = \langle z_2|z_1\rangle \mathrm{e}^{\lambda z_2^*}\mathrm{e}^{-\lambda^* z_1}.$$

What is the interpretation of $C_{12}(t)$? Let us define

$$\eta(t) = \frac{\langle z_2|z_1\rangle}{\langle z_2(t)|z_1(t)\rangle}$$

and write $C_{12}(t)$ in the form

$$C_{12}(t) = \eta(t)\langle z_2(t)|z_1(t)\rangle\, \mathrm{e}^{\lambda z_2^*(t)}\, \mathrm{e}^{-\lambda^* z_1(t)}.$$

Show that

$$|\eta(t)| = \exp\left[-\frac{1}{2}|z_1 - z_2|^2\left(1 - \mathrm{e}^{-\Gamma t}\right)\right] \simeq \exp\left[-\frac{\Gamma}{2}|z_1 - z_2|^2\right],$$

where the last expression holds for $\Gamma t \ll 1$. The decoherence time is therefore

$$\tau_{\mathrm{dec}} = \frac{2}{\Gamma|z_1 - z_2|^2}.$$

6. Let us choose $z_1 = 0$ (ground state of the oscillator) and $z_2 = z$. From question 1, the average time for the emission of *one* photon is $\sim (\Gamma|z_2|^2)^{-1}$. Argue that taking the trace over the environment (here the radiation field) shows that the coherence between the components $z_1 = 0$ and z of $|\Psi\rangle$ will be lost after the spontaneous emission of a single photon.

15.5.8 Dissipation in a two-level system

1. Starting from (15.113), derive the evolution equation for the matrix elements of the state operator ρ:

$$\frac{\mathrm{d}\rho_{00}}{\mathrm{d}t} = \left(G_+ + G_+^*\right)\rho_{11} - \left(G_- + G_-^*\right)\rho_{00},$$
$$\frac{\mathrm{d}\rho_{01}}{\mathrm{d}t} = \mathrm{i}\omega_0\rho_{01} - \left(G_+^* + G_-\right)\rho_{01} + \left(G_+ + G_-^*\right)\rho_{10}.$$

   The last line in (15.113) therefore does not contribute to the evolution of populations.
2. Argue that in the rotating-wave approximation one can neglect the term $\left(G_+ + G_-^*\right)\rho_{10}$ in the evolution of ρ_{01}. Within this approximation, rewrite the evolution equations in terms of $\Gamma_\pm$ and $\Delta_\pm$ (15.116)–(15.117). Check that the relaxation rate is $\Gamma = \Gamma_+ + \Gamma_-$ for the populations and $\Gamma/2$ for the coherences.
3. From the expressions for Γ_+ and Γ_-, show that at equilibrium the relative populations of the levels $|0\rangle$ and $|1\rangle$ are

$$\mathsf{p}_0 = \frac{\Gamma_-}{\Gamma}, \quad \mathsf{p}_1 = \frac{\Gamma_+}{\Gamma},$$

   and that their ratio is given by Boltzmann's law

$$\frac{\mathsf{p}_1}{\mathsf{p}_0} = \exp\left(-\frac{\hbar\omega_0}{k_{\mathrm{B}}T}\right).$$

15.5.9 Simple models of relaxation

1. In the first model a two-level atom A is prepared in a superposition of ground ($|0_A\rangle$) and excited ($|1_A\rangle$) states at $t = 0$. The electromagnetic field is assumed to be in its ground (vacuum) state $|0_B\rangle$, so that the initial state vector is

$$|\Psi(t=0)\rangle = (\lambda|0_A\rangle + \mu|1_A\rangle) \otimes |0_B\rangle, \quad |\lambda|^2 + |\mu|^2 = 1.$$

Guided by the Wigner–Weisskopf method (Appendix C), we write the state vector at time t as

$$|\Psi(t)\rangle = \lambda|0_A \otimes 0_B\rangle + \alpha\mu|0_A \otimes 1_B\rangle + \mu\, \mathrm{e}^{-(\mathrm{i}\omega_0 + \Gamma/2)t}|1_A \otimes 0_B\rangle,$$

where $|1_A\rangle$ is a normalized one-photon state. Use the conservation of the norm $||\Psi(t)||^2 = 1$ to compute α and deduce from your computation the matrix elements of the state operator at time t. Compare with the damping models of Section 15.2 and find the Kraus operators. Show that $T_2 = 2T_1$.

2. In the second model the state $|1\rangle$ is assumed to be stable, but the resonance frequency is time-dependent. This will be the case, for example, in NMR where a spin 1/2 is submitted to a fluctuating magnetic field $\vec{B}_0(t)$. The state vector of the spin system at time t is

$$|\Psi(t)\rangle = \lambda(t)|0\rangle + \mu(t)|1\rangle,$$

with $\lambda(t)$ and $\mu(t)$ given by

$$\mathrm{i}\dot{\lambda}(t) = -\frac{1}{2}\,\omega_0(t)\lambda(t), \quad \mathrm{i}\dot{\mu}(t) = \frac{1}{2}\,\omega_0(t)\mu(t), \quad \lambda(0) = \lambda_0, \quad \mu(0) = \mu_0.$$

The solution is

$$\lambda(t) = \lambda_0 \exp\left[\frac{\mathrm{i}}{2}\int_0^t \omega_0(t')\mathrm{d}t'\right], \quad \mu(t) = \mu_0 \exp\left[-\frac{\mathrm{i}}{2}\int_0^t \omega_0(t')\mathrm{d}t'\right].$$

Assume that $\omega_0(t)$ is a Gaussian stationary random function with connected autocorrelation function

$$C(t') = \langle\omega_0(t+t')\omega_0(t)\rangle - \langle\omega_0\rangle^2,$$

where $\langle\bullet\rangle$ is an ensemble average over all realizations of the random function. Also assume that

$$C(t') \simeq C \exp\left(-\frac{|t'|}{\tau}\right).$$

Show that the populations ρ_{00} and ρ_{11} are time-independent, but that the coherences are given by

$$\rho_{01}(t) = \rho_{01}(t=0)\mathrm{e}^{\mathrm{i}\langle\omega_0\rangle t}\,\mathrm{e}^{-C\tau t}, \quad t \gg \tau.$$

Which of the models in Section 15.2 corresponds to this situation?

15.5.10 Another choice for the spectral function $J(\omega)$

Instead of (15.131), we use another choice for the spectral function $J(\omega)$, namely

$$J(\omega) = M\gamma\omega\,\frac{\omega_*^2}{\omega^2 + \omega_*^2}.$$

Show that the real part $C(t)$ of the autocorrelation function is

$$C(t) = k_B T M \gamma \omega_* e^{-\omega_* |t|}.$$

Show that all the steps leading to (15.142) remain valid with this new spectral function.

15.5.11 The Fokker–Planck–Kramers equation for a Brownian particle

1. Let $\rho(t)$ be the state operator of the Brownian particle of Section 15.4.4. Let us define the *Wigner function* $w(x, p; t)$ by

$$w(x, p; t) = \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} e^{-ipy/\hbar} \langle x + \frac{y}{2} | \rho(t) | x - \frac{y}{2} \rangle dy.$$

   Show that another expression for $w(x, p; t)$ is

$$w(x, p; t) = \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} e^{-ixz/\hbar} \langle p + \frac{z}{2} | \rho(t) | p - \frac{z}{2} \rangle dz.$$

   Show that integrating the Wigner function over $x[p]$ gives the probability density $w_x(x; t)$ $[w_p(p; t)]$.
2. Unlike $w_x(x; t)$ and $w_p(p; t)$, the Wigner function, although real, is not necessarily positive and cannot be interpreted in a straightforward way as a probability distribution in phase space. First, compute the Wigner function for a Gaussian wave packet and check that it is positive in this particular case. Then compute the Wigner function of the superposition (15.143) of two wave packets and check that it is not positive everywhere.
3. Derive from (15.142) the following partial differential equation for $w(x, p; t)$:

$$\frac{\partial w}{\partial t} + \frac{p}{M} \frac{\partial w}{\partial x} = \gamma \frac{\partial}{\partial p} [pw] + D \frac{\partial^2 w}{\partial p^2}.$$

4. Integrate over x to obtain a Fokker–Planck equation for the probability density $w_p(p; t)$:

$$\frac{\partial w_p}{\partial t} = \gamma \frac{\partial}{\partial p} [p w_p] + D \frac{\partial^2 w_p}{\partial p^2}.$$

   Show that the long-time limit of w_p is a Maxwell distribution and recover the Einstein relation between γ and $k_B T$.

15.6 Further reading

The present chapter has drawn on the following sources: Peres [1993], Chapter 9; J. Preskill, *Quantum Computation*, http://www.theory.caltech.edu/~preskill/ (1999), Chapter 3; Nielsen and Chuang [2000], Chapters 2 and 8; J. Dalibard, *Cohérence quantique et dissipation*, graduate lecture notes, Ecole Normale Supérieure, Paris (2003); S. Haroche, *Superpositions mésoscopiques d'états*, Collège de France lectures, 2003/2004. These last two references (in French) are available from the website http://www.lkb.ens.fr. Levitt [2001], Chapter 16, provides a study of the relaxation mechanisms in NMR. The concept of open quantum systems is widely used in quantum optics: see for example

H. Carmichael, *An Open System Approach to Quantum Optics*, Berlin: Springer-Verlag (1993) or M. Scully and M. Zubairy, *Quantum Optics*, Cambridge: Cambridge University Press (1997). Memory effects, linear response, and Brownian motion are studied in detail in, e.g., D. Foerster, *Hydrodynamics Fluctuations, Broken Symmetry and Correlation Functions*, New York: Benjamin (1975), Chapters 1 to 6, or Le Bellac *et al*. [2004], Chapter 9. The model in Section 15.4 was first introduced by A. Caldeira and A. Leggett, *Physica* **121** A, 587 (1983); see also C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Atom–Photon Interactions*, New York: Wiley (1992), Chapter IV. Recent references to the Caldeira–Leggett model can be traced from the review article by Zurek [2003].