# 2D Viscous internal waves and streaming

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# Introduction

# Internal waves in labs

Viscosity can play an in important role for internal waves generated in labs.The Reynolds number $\mathrm{Re} = \dfrac{UL}{\nu}$ is several order of magnitude lower in labs experiments than in the ocean context.

Consequently, viscosity associated features might matter :

- Decay of internal wave beams
- Boundary layers
- Streaming

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# 2D Boussinesq model : Equations

- **Momentum equation:**

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \boldsymbol{\nabla})\, \mathbf{u} = -\boldsymbol{\nabla} P + S\mathbf{e}_z + \nu \Delta \mathbf{u}$$

- **Buoyancy advection equation**

$$\partial_t S + \mathbf{u} \cdot \boldsymbol{\nabla} S = 0$$

- **Incompressible flow**

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0$$

# 2D Boussinesq model : Equations

**Velocity field** : $\mathbf{u} = (u, w)$

- **Momentum equation:**

$$\partial_t \mathbf{u} + \left(\mathbf{u} \cdot \boldsymbol{\nabla}\right) \mathbf{u} = -\boldsymbol{\nabla} P + S\mathbf{e}_z + \nu\Delta \mathbf{u}$$

- **Buoyancy advection equation**

$$\partial_t S + \mathbf{u} \cdot \boldsymbol{\nabla} S = 0$$

- **Incompressible flow**

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# 2D Boussinesq model : Equations

**Nabla operator** $\nabla = (\partial_x, \partial_z)$ and **Laplacian operator** $\Delta = \partial_x^2 + \partial_z^2$

- **Momentum equation:**

$$\partial_t \mathbf{u} + \left(\mathbf{u} \cdot \nabla\right) \mathbf{u} = -\nabla P + S\mathbf{e}_z + \nu \Delta \mathbf{u}$$

- **Buoyancy advection equation**

$$\partial_t S + \mathbf{u} \cdot \nabla S = 0$$

- **Incompressible flow**

$$\nabla \cdot \mathbf{u} = 0$$

# 2D Boussinesq model : Equations

**Pressure field** $P$ and the **buoyancy field** $S = -g\dfrac{\rho - \rho_0}{\rho_0}$.

- **Momentum equation:**

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \boldsymbol{\nabla})\,\mathbf{u} = -\boldsymbol{\nabla} P + S\,\mathbf{e}_z + \nu \Delta \mathbf{u}$$

- **Buoyancy advection equation**

$$\partial_t S + \mathbf{u} \cdot \boldsymbol{\nabla} S = 0$$

- **Incompressible flow**

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0$$

# Casimirs, potential energy and background stratification

- The buoyancy advection equation implies the conservation of Casimirs :

$$\mathcal{C}_f\left[S\right] = \iint S\left(x, z, t\right) \mathrm{d}x\mathrm{d}z$$

where $f$ is any function.

- $\implies$ The buyancy field evolves by rearanging itself.
- The standard potential energy is defined by :

$$E_p\left[S\right] = -\iint S\left(x, z, t\right) z \ \mathrm{d}x\mathrm{d}z$$

- The background stratification is defined by :

$$E_p\left[b_{\mathrm{bg}}\right] = \min_S \left\{E_p\left[S\right] \middle| \mathcal{C}_f\left[S\right] = C_f \ \ \forall f\right\}$$

such that $b_{\mathrm{bg}}\left(x, z, t\right) = \int^{z} N^2\left(z'\right) \mathrm{d}z'$ where $N$ is the local Brunt-Väisälä frequency.

# Buoyancy and pressure disturbances

- We define the buyancy disturbance fields by $b = S - b_{\mathrm{bg}}$ and the pressure disturbance by $p = P - \int^{z} b_{\mathrm{bg}}(z')\,\mathrm{d}z'$
- Injecting this in the equations of motion, one gets :

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\,\mathbf{u} & = -\boldsymbol{\nabla} p + b\,\mathbf{e}_z + \nu\Delta\mathbf{u} \\ \partial_t b + \mathbf{u}\cdot\boldsymbol{\nabla} b + N^2(z)\,w & = 0 \\ \boldsymbol{\nabla}\cdot\mathbf{u} & = 0 \end{cases}$$

# Potential energy of the buyancy disturbance

- For the standard expression of the potential energy, the fact that $b = 0$ is a minimum is not obvious:

$$E_p\left[S\right] - E_p\left[b_{\mathrm{bg}}\right] = \Delta E_p\left[b\right] = -\iint b \,\mathrm{d}x\mathrm{z}$$

- Let us introduce $f = \left(\int^{z} b_{\mathrm{bg}}\right)^{-1}$, such that :

$$\begin{aligned}\mathcal{C}_f\left[b_{\mathrm{bg}}\right] &= \mathcal{C}_f\left[b_{\mathrm{bg}} + b\right] \\ &= \mathcal{C}_f\left[b_{\mathrm{bg}}\right] - \Delta E_p\left[b\right] + \sum_{k=2}^{\infty} \iint \frac{f^{(k)}\left(b_{\mathrm{bg}}\right)}{k!} b^k \;\; \mathrm{d}x\mathrm{d}z\end{aligned}$$

- For $|b| \ll 1$ we have

$$\Delta E_p\left[b\right] \approx \iint \frac{b^2\left(x, z, t\right)}{2N^2\left(z\right)} \mathrm{d}x\mathrm{d}z$$

This relation is exact if $N^2 = \mathrm{Cte}$.

## 2D Boussinesq model : Adimensionalization

- We assume $N^2 = \mathrm{Cte}$ in the following
- $(\tilde{x}, \tilde{z}) = K\,(x, z)$ where $K$ is a typical wave number (e.g. the wave number of the generator)
- $\tilde{t} = \Omega t$ where $\Omega$ is a typical frequency (e.g. the frequency of the generator)
- $\tilde{\mathbf{u}} = \dfrac{K}{\Omega}\mathbf{u}$
- $\tilde{b} = \dfrac{K}{N^2} b$
- $\tilde{P} = \dfrac{k^2}{\Omega^2} P$

# 2D Boussinesq model : Dimensionless parameters and adimensionalized equations

There are two independant dimensionless parameters :

- The Reynold number : $\mathrm{Re} = \dfrac{\Omega}{\nu K^2}$
- The Fround number : $\mathrm{Fr} = \dfrac{\Omega}{N}$

The resulting adimensionalized equations write :

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \boldsymbol{\nabla})\, \mathbf{u} & = -\boldsymbol{\nabla} P + \dfrac{1}{\mathrm{Fr}^2} b \mathbf{e}_z + \dfrac{1}{\mathrm{Re}} \Delta \mathbf{u} \\ \partial_t b + \mathbf{u} \cdot \boldsymbol{\nabla} b + w & = 0 \\ \boldsymbol{\nabla} \cdot \mathbf{u} & = 0 \end{cases}$$

# Wave and mean-flow decomposition

# Zonally symetric flows

We consider here the simplest case of zonally periodic flows :

- The averaging operator is defined by $\overline{u} = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} u \,\mathrm{d}x$.
- The wave and mean-flow decomposition is defined by $(u, w, b, P) = \left(\overline{u}, \overline{w}, \overline{b}, \overline{P}\right) + (u', w', b', P')$
- Taking the mean part of the equations of motion leads to the **mean-flow equations** :

$$\begin{cases} \partial_t \overline{u} - \dfrac{1}{\mathrm{Re}} \partial_{zz} \overline{u} & = -\partial_z \overline{u' w'} \\ \overline{w} & = 0 \\ \overline{P} & = \overline{w'^2} \\ \partial_t \overline{b} & = -\partial_z \overline{b' w'} \end{cases}$$

# Waves equations

- Substracting the mean flow equations to the equations of motions leads to the **wave equations** :

$$\begin{cases} \partial_t u' + \overline{u}\partial_x u' + w'\partial_z \overline{u} + \boxed{u'\partial_x u' + w'\partial_z u' - \partial_z \overline{u'w'}} + \partial_x P' - \frac{1}{\mathrm{Re}}\Delta u' & = 0 \\ \partial_t w' + \overline{u}\partial_x w' + \boxed{u'\partial_x w' + w'\partial_z w' - \partial_z \overline{w'^2}} + \partial_z P' - \frac{1}{\mathrm{Fr}^2} b' - \frac{1}{\mathrm{Re}}\Delta w' & = 0 \\ \partial_t b' + \overline{u}\partial_x b' + \boxed{u'\partial_x b' + w'\partial_z b' - \partial_z \overline{b'w'}} + \underbrace{\left(1 + \partial_z \overline{b}\right)}_{N^2(z)} w' & = 0 \\ \partial_x u' + \partial_z w' & = 0 \end{cases}$$

- Non-linear terms leading to PSI.

# Inviscid internal waves

# Linearized

Let us first ignore the non-linear and viscous terms in the wave equations
:

$$\begin{cases} \partial_t u' + \overline{u}\partial_x u' + w'\partial_z \overline{u} + \partial_x P' & = 0 \\ \partial_t w' + \overline{u}\partial_x w' + \partial_z P' - \dfrac{1}{\mathrm{Fr}^2} b' & = 0 \\ \partial_t b' + \overline{u}\partial_x b' + N^2 w' & = 0 \\ \partial_x u' + \partial_z w' & = 0 \end{cases}$$

As a direct consequence :

$$\begin{cases} \partial_z \overline{u' w'} & = \partial_t \left( \dfrac{1}{N^2} \overline{b' \left(\partial_x w' - \partial_z u'\right)} + \dfrac{1}{N^4} \dfrac{\overline{b'^2}}{2} \partial_{zz} \overline{u} \right) \\ \overline{b' w'} & = -\partial_t \left( \dfrac{\overline{b'^2}}{2N^2} \right) \end{cases}$$

# WKB ansatz

- We introduce $a \ll 1$ and associated slow variables $(Z, T, T_{\mathrm{mf}}) = \left(az, at, a^3 t\right)$ and assume $\overline{u} = \overline{u}\,(Z, T_{\mathrm{mf}}) = O\,(1)$.
- WKB ansatz :

$$\begin{bmatrix} u \\ w \\ b\mathrm{Fr}/N \\ P \end{bmatrix} = a \sum_{j=0}^{\infty} a^j \begin{bmatrix} u_j\,(Z, T) \\ w_j\,(Z, T) \\ b_j\,(Z, T)\,\mathrm{Fr}/N \\ P_j\,(Z, T) \end{bmatrix} \exp\left( i\frac{\Phi\,(Z, T)}{a} - ikx \right)$$

- We define $m = -\partial_Z \Phi$, $\omega = \partial_T \Phi$ and $\hat{\omega} = \omega - k\overline{u}$

## WKB ansatz

Injecting the WKB ansatz and collecting the first order terms leads to :

$$\mathbf{M}\begin{bmatrix} u_0 \\ w_0 \\ b_0\mathrm{Fr}/N \\ P_0 \end{bmatrix} + a\left(\mathbf{M}\begin{bmatrix} u_1 \\ w_1 \\ b_1\mathrm{Fr}/N \\ P_1 \end{bmatrix} + \underbrace{\begin{bmatrix} \partial_T u_0 + w_0\partial_Z \overline{u} \\ \partial_T w_0 + \partial_Z P_0 \\ \partial_T b_0\mathrm{Fr}/N \\ \partial_Z w_0 \end{bmatrix}}_{\boldsymbol{T}[\overline{u},\phi_0]}\right) = 0$$

With :

$$\mathbf{M} = \begin{bmatrix} i\hat{\omega} & 0 & 0 & -ik \\ 0 & i\hat{\omega} & -\dfrac{N}{\mathrm{Fr}} & -im \\ 0 & \dfrac{N}{\mathrm{Fr}} & i\hat{\omega} & 0 \\ -ik & -im & 0 & 0 \end{bmatrix}$$

# Order zero

$$\mathbf{M}\begin{bmatrix} u_0 \\ w_0 \\ b_0\mathrm{Fr}/N \\ P_0 \end{bmatrix} = 0 \implies \begin{cases} \det \mathbf{M} & = 0 \iff \hat{\omega}^2 = \dfrac{N^2 k^2}{\mathrm{Fr}^2\left(k^2+m^2\right)} \\ \begin{bmatrix} u_0 \\ w_0 \\ b_0\mathrm{Fr}/N \\ P_0 \end{bmatrix} & = \phi_0 \mathcal{P} \\ \mathcal{P} & = \begin{bmatrix} \hat{\omega}/k \\ -\hat{\omega}/m \\ -iN/m\mathrm{Fr} \\ \hat{\omega}^2/k^2 \end{bmatrix} \end{cases}$$

# Dispersion relation

$$\hat{\omega}^2 = \frac{N^2 k^2}{\mathrm{Fr}^2\left(k^2 + m^2\right)}$$

- $-N \leq \mathrm{Fr}\hat{\omega} \leq N$
- Phase and Group velocities :

$$\mathbf{c}_\varphi = \frac{\hat{\omega}}{\mathbf{k}^2}\mathbf{k} \qquad , \qquad \mathbf{c}_g = -\frac{\hat{\omega}}{\mathbf{k}^2}\frac{m}{k}\mathbf{k}^\perp$$

  where $\mathbf{k}^\perp = (-m, k)$.
- $\mathbf{c}_\varphi \cdot \mathbf{c}_g = 0$

# Order one

We take the scalar product of the order one terms with $\phi_0^* \mathcal{P}^\dagger$ :

$$\mathcal{P}^\dagger \cdot \mathcal{T}\left[\overline{u}, \phi_0\right] = 0$$

After some algebra, we get the **wave-activity equation** :

$$\partial_t A + \partial_z \left(A w_g\right) = 0 \quad , \quad \text{with} \begin{cases} A &= \dfrac{E}{\hat{\omega}} \\ w_g &= -\dfrac{\mathrm{Fr}^2 \hat{\omega}^3 m}{N^2 k^2} \end{cases}$$

where

$$E = \frac{1}{4}\left(\left|u_0\right|^2 + \left|w_0\right|^2 + \frac{\mathrm{Fr}^2}{N^2}\left|b_0\right|^2\right) = \frac{N^2 \left|\phi_0\right|^2}{2\mathrm{Fr}^2 m^2}$$

We can check that at leading order $\overline{u' w'} = \dfrac{1}{2}\mathcal{R}\mathrm{e}\left[u_0 w_0^*\right] = k A w_g$ such that $\partial_z \overline{u' w'} \approx -\partial_t \left(kA\right)$

# Viscous internal waves

# Linearized

We put the viscous operator back into the linearized wave euqations :

$$\begin{cases} \partial_t u' + \overline{u}\partial_x u' + w'\partial_z \overline{u} + \partial_x P' - \dfrac{1}{\mathrm{Re}}\Delta u' & = 0 \\ \partial_t w' + \overline{u}\partial_x w' + \partial_z P' - \dfrac{1}{\mathrm{Fr}^2} b' - \dfrac{1}{\mathrm{Re}}\Delta w' & = 0 \\ \partial_t b' + \overline{u}\partial_x b' + N^2 w' & = 0 \\ \partial_x u' + \partial_z w' & = 0 \end{cases}$$

We still have:

$$\overline{b' w'} = -\partial_t \left( \frac{\overline{b'^2}}{2N^2} \right)$$

We use the same assumptions and consider the same WKB ansatz in the following.

# WKB ansatz

Injecting the WKB ansatz and collecting the first order terms leads to :

$$\mathbf{M}\begin{bmatrix} u_0 \\ w_0 \\ b_0\mathrm{Fr}/N \\ P_0 \end{bmatrix} + a\left(\mathbf{M}\begin{bmatrix} u_1 \\ w_1 \\ b_1\mathrm{Fr}/N \\ P_1 \end{bmatrix} + \underbrace{\begin{bmatrix} \partial_T u_0 + w_0\partial_Z\overline{u} + \dfrac{i}{\mathrm{Re}}\left(u_0\partial_Z m + 2m\partial_Z u_0\right) \\ \partial_T w_0 + \partial_Z P_0 + \dfrac{i}{\mathrm{Re}}\left(w_0\partial_Z m + 2m\partial_Z w_0\right) \\ \partial_T b_0\mathrm{Fr}/N \\ \partial_Z w_0 \end{bmatrix}}_{\boldsymbol{T}[\overline{u},\phi_0]}\right) = 0$$

With :

$$\mathbf{M} = \begin{bmatrix} i\hat{\omega} + \dfrac{k^2+m^2}{\mathrm{Re}} & 0 & 0 & -ik \\ 0 & i\hat{\omega} + \dfrac{k^2+m^2}{\mathrm{Re}} & -\dfrac{N}{\mathrm{Fr}} & -im \\ 0 & \dfrac{N}{\mathrm{Fr}} & i\hat{\omega} & 0 \\ -ik & -im & 0 & 0 \end{bmatrix}$$

# Order zero

$$\mathbf{M}\begin{bmatrix} u_0 \\ w_0 \\ b_0\mathrm{Fr}/N \\ P_0 \end{bmatrix} = 0 \implies \begin{cases} \det\mathbf{M} = 0 & \iff \hat{\omega}\left(\hat{\omega} - i\dfrac{k^2+m^2}{\mathrm{Re}}\right) = \dfrac{N^2}{\mathrm{Fr}^2}\dfrac{k^2}{k^2+m^2} \\ \begin{bmatrix} u_0 \\ w_0 \\ b_0\mathrm{Fr}/N \\ P_0 \end{bmatrix} & = \phi_0\mathcal{P} \\ \mathcal{P} & = \begin{bmatrix} \hat{\omega}/k \\ -\hat{\omega}/m \\ -iN/m\mathrm{Fr} \\ \dfrac{\hat{\omega}^2}{k^2}\left(1 - i\dfrac{(k^2+m^2)}{\mathrm{Re}\hat{\omega}}\right) \end{bmatrix} \end{cases}$$

# Dispersion relation

$k$ and $\omega$ are fixed by generation process (it has to be zonally symetric for our calculation)

$$\hat{\omega}\left(\hat{\omega} - i\frac{k^2+m^2}{\mathrm{Re}}\right) = \frac{N^2}{\mathrm{Fr}^2}\frac{k^2}{k^2+m^2}$$

We can already remark a few things

- $4^{\mathrm{th}}$ order complex polynomial equation for $m$ meaning there are 4 different complex solutions
- The symetry $m \to -m$ indicates two important branches

Two branches :

$$k^2 + m^2 = \frac{\mathrm{Re}\hat{\omega}}{2i}\left(1 \pm \sqrt{1 - \frac{4ik^2N^2}{\mathrm{Re}\mathrm{Fr}^2\hat{\omega}^3}}\right) - 1$$

# Large Reynold number limit

We now consider large values of the Reynold number (such that $\mathrm{Fr}^2\mathrm{Re}\,|\hat{\omega}|^3/4k^2N^2 \gg 1$). The solution then writes for upwardly propagating waves :

$$m_w = -\epsilon m_0 - \frac{ik^4N^4}{2\mathrm{Fr}^4 m_0 \mathrm{Re}\,|\hat{\omega}|^5}$$

$$m_{bl} = (\epsilon - i)\sqrt{\frac{\mathrm{Re}\,|\hat{\omega}|}{2}}$$

where $m_0 = \sqrt{\frac{N^2k^2}{\mathrm{Fr}^2\hat{\omega}^2} - k^2}$ is the inviscid value for $m$ and $\epsilon = \mathrm{sign}\,(\hat{\omega})$

Few remarks :

- $m_w$ : Propagating branche
- $L_{\mathrm{Re}} = 2\mathrm{Fr}^4\mathrm{Re} m_0\,|\hat{\omega}|^5/k^4N^4$ : penetration length for the wave beam
- $m_{bl}$ : Boundary layer branche
- $\delta_{\mathrm{Re}} = \sqrt{2/\mathrm{Re}\,|\hat{\omega}|}$ : Boundary layer length

# Boundary conditions

# Boundary condition : transverse oscillation

Let us consider a horizontally set-up generator. The fluid is viscous wih a **no-slip** boundary condition :

$$\mathbf{u}\left(x, z = h_b\left(x, t\right), t\right) = \partial_t h_b\left(x, t\right) \mathbf{e}_z$$

If we now su hppose that $||_b|| \ll 1$, we perform the wave and mean-flow decomposition and linearize this boundary condition to get :

$$\begin{cases} \overline{u}\left(z = 0, t\right) & = 0 \\ u'\left(x, z = 0, t\right) & = 0 \\ w'\left(x, z = 0, t\right) & = \partial_t h_b\left(x, t\right) \end{cases}$$

**Important consequence :** $\int\limits_0^\infty \partial_z \overline{u'w'}\, \mathrm{d}z = 0.$

# Boundary condition : Progressive wave

Here we consider $h_b(x,t) = a\mathcal{R}\mathrm{e}\left[e^{i(t-x)}\right]$ corresponding to $(\omega, k) = (1,1)$. Considering waves propagating upwardly, using the polarisation :

$$\begin{cases} -\left(\dfrac{\phi_{0,w}(0)}{m_w(0)} + \dfrac{\phi_{0,bl}(0)}{m_{bl}(0)}\right) & = ia \\ \phi_{0,w}(0) + \phi_{0,bl}(0) & = 0 \end{cases}$$

$$\Longrightarrow \begin{cases} \phi_{0,w}(0) & = -ia\dfrac{m_{bl}(0)\, m_w(0)}{m_{bl}(0) - m_w(0)} \\ \phi_{0,bl}(0) & = ia\dfrac{m_{bl}(0)\, m_w(0)}{m_{bl}(0) - m_w(0)} \end{cases}$$

# Computation of Reynold stress in the large Reynold number limit

# General expression

$$\overline{u'w'} = -\frac{\hat{\omega}^2}{2}\left(|\phi_{0,w}|^2 \frac{m'_w}{|m_w|^2} e^{2\int m''_w} + |\phi_{0,bl}|^2 \frac{m'_{bl}}{|m_{bl}|^2} e^{2\int m''_{bl}} + \mathcal{R}\mathrm{e}\left[\phi^*_{0,w}\phi_{0,bl}\frac{m_{bl}+m^*_w}{m_{bl}m_w}\right]\mathcal{R}\mathrm{e}\left[e^{i\int(m^*_w - m_{bl})}\right] + \mathcal{I}\mathrm{m}\left[\phi^*_{0,w}\phi_{0,bl}\frac{m_{bl}+m^*_w}{m_{bl}m_w}\right]\mathcal{I}\mathrm{m}\left[e^{i\int(m^*_w - m_{bl})}\right]\right)$$

- Bulk streaming
- Boundary streaming

# Large Reynold limit without mean-flow

$$\overline{u'w'} = \epsilon \frac{\hat{\omega}^2 a^2}{2} \left\{ \frac{1}{m_0} \left( e^{-\frac{z}{2\mathrm{Fr}^4 m_0 \mathrm{Re}|\hat{\omega}|^5}} - e^{-z\sqrt{\frac{\mathrm{Re}|\hat{\omega}|}{2}}} \cos z \sqrt{\frac{\mathrm{Re}\,|\hat{\omega}|}{2}} \right) - \frac{1}{\sqrt{2\mathrm{Re}\,|\hat{\omega}|}} \left( e^{-z\sqrt{2\mathrm{Re}|\hat{\omega}|}} - e^{-z\sqrt{\frac{\mathrm{Re}|\hat{\omega}|}{2}}} \left( \cos z \sqrt{\frac{\mathrm{Re}\,|\hat{\omega}|}{2}} + \sin z \sqrt{\frac{\mathrm{Re}\,|\hat{\omega}|}{2}} \right) \right) \right\}$$

- Bulk streaming
- We have $\overline{u'w'}\,(z=0) = 0$

# Large Reynold limit with mean-flow for the bulk term

$$\overline{u'w'} = \epsilon \frac{\hat{\omega}^2 \left|\phi_{0,w}\right|^2}{2m_0} e^{-\int \frac{\mathrm{d}z}{2\mathrm{Fr}^4 m_0 \mathrm{Re} |\hat{\omega}|^5}} = \epsilon \frac{a^2}{2m_0\left(0\right)} e^{-\int \frac{\mathrm{d}z}{2\mathrm{Fr}^4 m_0 (1-\overline{u}) \mathrm{Re} |1-\overline{u}|^5}}$$

where $\epsilon = \mathrm{sign}\left(1 - \overline{u}\right)$

# Large Reynold limit with mean-flow for the bulk term : Stading wave forcing

We now consider a standing wave forcing corresponding in the superposition of two waves with equal amplitudes and opposite phase speeds :

$$\overline{u'w'} = \frac{a^2}{2m_0\left(0\right)}\left(\epsilon_- e^{-\int \frac{\mathrm{d}z}{2\mathrm{Fr}^4 m_0 (1-\overline{u})\mathrm{Re}|1-\overline{u}|^5}} - \epsilon_+ e^{-\int \frac{\mathrm{d}z}{2\mathrm{Fr}^4 m_0 (1+\overline{u})\mathrm{Re}|1+\overline{u}|^5}}\right)$$

with $\epsilon_\pm = \mathrm{sign}\left(1 \pm \overline{u}+\right)$