Viscous internal waves and streaming

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Introduction

Internal waves in labs

Viscosity can play an in important role for internal waves generated in labs.The Reynolds number $\mathrm{Re} = \dfrac{UL}{\nu}$ is several order of magnitude lower in labs experiments than in the ocean context.

Consequently, viscosity associated features might matter :

- Decay of internal wave beams
- Boundary layers
- Streaming

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The 2D Boussinesq model

2D Boussinesq model : Equations

- **Momentum equation:**

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \boldsymbol{\nabla})\,\mathbf{u} = -\boldsymbol{\nabla} P + b\mathbf{e}_z + \nu \Delta \mathbf{u}$$

- **Buoyancy advection equation**

$$\partial_t b + \mathbf{u} \cdot \boldsymbol{\nabla} b + N^2 w = 0$$

- **Incompressible flow**

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0$$

2D Boussinesq model : Equations

Velocity field : $\mathbf{u} = \big(u, w\big)$

- **Momentum equation:**

$$\partial_t \mathbf{u} + \big(\mathbf{u} \cdot \boldsymbol{\nabla}\big) \mathbf{u} = -\boldsymbol{\nabla} P + b\mathbf{e}_z + \nu \Delta \mathbf{u}$$

- **Buoyancy advection equation**

$$\partial_t b + \mathbf{u} \cdot \boldsymbol{\nabla} b + N^2 w = 0$$

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2D Boussinesq model : Equations

Nabla operator $\nabla = (\partial_x, \partial_z)$ and **Laplacian operator** $\Delta = \partial_x^2 + \partial_z^2$

- **Momentum equation:**

$$\partial_t \mathbf{u} + \left(\mathbf{u} \cdot \nabla\right) \mathbf{u} = -\nabla P + b\mathbf{e}_z + \nu \Delta \mathbf{u}$$

- **Buoyancy advection equation**

$$\partial_t b + \mathbf{u} \cdot \nabla b + N^2 w = 0$$

- **Incompressible flow**

$$\nabla \cdot \mathbf{u} = 0$$

2D Boussinesq model : Equations

Pressure field P and the **buoyancy field** $b = -g\frac{\rho - \rho_0}{\rho_0} - N^2 z$
where N is the **Brunt-Väisälä frequency** assumed constant

- **Momentum equation:**

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \boldsymbol{\nabla})\,\mathbf{u} = -\boldsymbol{\nabla} P + b\,\mathbf{e}_z + \nu \Delta \mathbf{u}$$

- **Buoyancy advection equation**

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2D Boussinesq model : Adimensionalization

- $(\tilde{x}, \tilde{z}) = K\,(x, z)$ where K is a typical wave number (e.g. the wave number of the generator)
- $\tilde{t} = \Omega t$ where Ω is a typical frequency (e.g. the frequency of the generator)
- $\tilde{\mathbf{u}} = \dfrac{K}{\Omega}\mathbf{u}$
- $\tilde{b} = \dfrac{K}{N^2} b$
- $\tilde{P} = \dfrac{k^2}{\Omega^2} P$

2D Boussinesq model : Dimensionless parameters and adimensionalized equations

There are two independant dimensionless parameters :

- The Reynold number : $\mathrm{Re} = \dfrac{\Omega}{\nu K^2}$
- The Fround number : $\mathrm{Fr} = \dfrac{\Omega}{N}$

The resulting adimensionalized equations write :

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \boldsymbol{\nabla})\, \mathbf{u} & = -\boldsymbol{\nabla} P + \dfrac{1}{\mathrm{Fr}^2} b \mathbf{e}_z + \dfrac{1}{\mathrm{Re}} \Delta \mathbf{u} \\ \partial_t b + \mathbf{u} \cdot \boldsymbol{\nabla} b + w & = 0 \\ \boldsymbol{\nabla} \cdot \mathbf{u} & = 0 \end{cases}$$

Viscous internal waves

Linearization

Let us linearize the equations of motion about the rest state $\mathbf{u}, b, P = 0$:

$$\begin{cases} \partial_t u + \partial_x P - \dfrac{1}{\mathrm{Re}} \Delta u & = 0 \\ \partial_t w + \partial_z P - \dfrac{1}{\mathrm{Fr}^2} b - \dfrac{1}{\mathrm{Re}} \Delta \mathbf{w} & = 0 \\ \partial_t b + w & = 0 \\ \boldsymbol{\nabla} \cdot \mathbf{u} & = 0 \end{cases}$$

Dispersion relation

We look for **non-vanishing plane waves solutions**

$$\begin{bmatrix} u \\ w \\ b \\ P \end{bmatrix} = \begin{bmatrix} \tilde{u} \\ \tilde{w} \\ \tilde{b} \\ \tilde{P} \end{bmatrix} e^{i\,(\omega t - kx - mz)}.$$

This leads to the following **dispersion relation** :

$$\omega \left(\omega - i \frac{k^2 + m^2}{\mathrm{Re}} \right) = \frac{1}{\mathrm{Fr}^2} \frac{k^2}{k^2 + m^2}$$

Inviscid limit

In the inviscid limit (i.e. $\mathrm{Re} = +\infty$), we recover the well known dispersion relation :

$$\omega^2 = \frac{1}{\mathrm{Fr}^2} \frac{k^2}{k^2 + m^2} = \frac{1}{\mathrm{Fr}^2} \sin^2 \theta$$

With the phase and group velocities

$$\mathbf{c}_\varphi = \pm \frac{1}{\mathrm{Fr}\left(k^2 + m^2\right)} \begin{bmatrix} k \\ m \end{bmatrix} \qquad , \qquad \mathbf{c}_g = \pm \frac{k^2}{\mathrm{Fr}\sqrt{k^2 + m2}} \begin{bmatrix} m^2 \\ -mk \end{bmatrix}$$

such that $\mathbf{c}_\varphi \cdot \mathbf{c}_g = 0$

We must have $|\omega| < \dfrac{1}{\mathrm{Fr}}$ for propagating waves.

Back to the viscous case : horizontal generator

Let us consider again the viscous case. We consider the case where $\omega = 1$ and $k = 1$. (generator set-up horizontally)

$$\mathrm{Fr}^2 \left(1 - i\frac{1+m^2}{\mathrm{Re}}\right)\left(1+m^2\right) = 1$$

We can already remark a few things

- 4^{th} order complex polynomial equation for m meaning there are 4 different complex solutions
- The symetry $m \to -m$ indicates two important branches

Two branches :

$$m^2 = \frac{\mathrm{Re}}{2i}\left(1 \pm \sqrt{1 - \frac{4i}{\mathrm{Re}\mathrm{Fr}^2}}\right) - 1$$

Large Reynold number limit

We now consider large values of the Reynold number (such that $\mathrm{Fr}^2\mathrm{Re} \gg 1$). The solution then writes :

$$m_w = \pm \left(m_0 + \frac{i}{2\mathrm{Fr}^4 m_0 \mathrm{Re}} \right)$$

$$m_{bl} = \pm (1 - i) \sqrt{\frac{\mathrm{Re}}{2}}$$

where $m_0 = \sqrt{\frac{1}{\mathrm{Fr}^2} - 1}$ is the inviscid value for m.

Few remarks :

- m_w : Propagating branche
- $L_{\mathrm{Re}} = 2\mathrm{Fr}^4\mathrm{Re}m_0$: penetration length for the wave beam
- m_{bl} : Boundary layer branche
- $\delta_{\mathrm{Re}} = \sqrt{2/\mathrm{Re}}$: Boundary layer length

Back to the viscous case : vertical generator

We consider here the case where $\omega = 1$ and $m = 1$. (generator set-up vertically)

$$\mathrm{Fr}^2\left(1+k^2\right)\left(1-i\frac{1+k^2}{\mathrm{Re}}\right)-k^2=0$$

Two branches :

$$k^2=\frac{i\mathrm{Re}}{2k_0^2}\left(1+\frac{2ik_0^2}{\mathrm{Re}}-\sqrt{1+4i\frac{\left(1+k_0^2\right)k_0^2}{\mathrm{Re}}}\right)$$

where $k_0 = \dfrac{\mathrm{Fr}}{\sqrt{1-\mathrm{Fr}^2}}$ is the inviscid value for k.

Large Reynold number limit

We now consider large values of the Reynold number. The solution then writes :

$$k_w = \pm \left(k_0 - \frac{ik_0\left(1+k_0^2\right)^2}{2\mathrm{Re}} \right)$$

$$m_{bl} = \pm (1+i) \sqrt{\frac{\mathrm{Re}}{2k_0^2}}$$

where $k_0 = \dfrac{\mathrm{Fr}}{\sqrt{1-\mathrm{Fr}^2}}$ is the inviscid value for k.

Few remarks :

- k_w : Propagating branche
- $L_{\mathrm{Re}} = \dfrac{2\mathrm{Re}}{k_0\left(1+k_0^2\right)^2}$: Penetration length for the wave beam
- k_{bl} : Boundary layer branche
- $\delta_{\mathrm{Re}} = \sqrt{2k_0^2/\mathrm{Re}}$: Boundary layer length

Streaming

Wave-mean flow decomposition

- The averaging operator is defined by $\overline{u} = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} u \, \mathrm{d}x \mathrm{d}t$.
- The wave-mean decomposition is defined by $(u, w, b, P) = \left(\overline{u}, \overline{w}, \overline{b}, \overline{P}\right) + (u', w', b', P')$
- Taking the mean part of the equations of motion leads to :

$$\partial_t \overline{u} - \frac{1}{\mathrm{Re}} \Delta \overline{u} = -\partial_z \overline{u' w'}$$

  and $\overline{w}, \overline{b} = 0$.
- Streaming is induced by the waves from the Reynold stress $\partial_z \overline{u' w'}$

Waves equations

$$\begin{cases}
\partial_t u' + \overline{u}\partial x u' + w'\partial_z \overline{u} + \boxed{u'\partial_x u' + w'\partial_z u' - \partial_z \overline{u'w'}} & = -\partial_x P' + \frac{1}{\mathrm{Re}}\Delta u' \\
\partial_t w' + \overline{u}\partial x w' + \boxed{u'\partial_x w' + w'\partial_z w' - \partial_z \overline{w'^2}} & = -\partial_z P' + \frac{1}{\mathrm{Fr}^2} b' + \frac{1}{\mathrm{Re}}\Delta w' \\
\partial_t b' + \overline{u}\partial_x b' + \boxed{u'\partial_x b' + w'\partial_z b'} + w' & = 0 \\
\partial_x u' + \partial_z w' & = 0
\end{cases}$$

Non-linear terms responsible of the PSI.

Waves in shear-flows : WKB solutions

WKB ansatz and linearization

We introduce a small dimensionles parameter $a \ll 1$ and assume that the mean-flow writes $U = U\left(Z, T\right)$ where $\left(Z, T\right) = a\left(z, t\right)$.
WKB ansatz :

$$\begin{bmatrix} u \\ w \\ b \\ P \end{bmatrix} = \sum_{j=0}^{\infty} a^{j+1} \begin{bmatrix} u_j\left(Z, T\right) \\ w_j\left(Z, T\right) \\ b_j\left(Z, T\right) \\ P_j\left(Z, T\right) \end{bmatrix} \exp\left(i\frac{\Phi\left(Z, T\right)}{a} - ix \right)$$

Injecting this ansatz into the wave equation and collecting the leading order terms in a leads to :

$$\mathbf{M} \begin{bmatrix} u_0 \\ w_0 \\ b_0 \\ P_0 \end{bmatrix} + a \left(M \begin{bmatrix} u_1 \\ w_1 \\ b_1 \\ P_1 \end{bmatrix} + \begin{bmatrix} \partial_T u_0 + w_0 \partial_Z U + \frac{i}{\mathrm{Re}}\left(u_0 \partial_Z m + 2m\partial_Z u_0\right) \\ \partial_T w_0 + \partial_Z P_0 + \frac{i}{\mathrm{Re}}\left(w_0 \partial_Z m + 2m\partial_Z w_0\right) \\ \partial_T b_0 \\ \partial_Z w_0 \end{bmatrix} \right) = 0$$

WKB ansatz and linearization

$$\mathbf{M}\begin{bmatrix}u_0\\w_0\\b_0\\P_0\end{bmatrix}+a\left(M\begin{bmatrix}u_1\\w_1\\b_1\\P_1\end{bmatrix}+\begin{bmatrix}\partial_T u_0+w_0\partial_Z U+\dfrac{i}{\mathrm{Re}}\left(u_0\partial_Z m+2m\partial_Z u_0\right)\\ \partial_T w_0+\partial_Z P_0+\dfrac{i}{\mathrm{Re}}\left(w_0\partial_Z m+2m\partial_Z w_0\right)\\ \partial_T b_0\\ \partial_Z w_0\end{bmatrix}\right)=0$$

With :

$$\mathbf{M}=\begin{bmatrix}i\left(\omega-U\right)+\dfrac{1+m^2}{\mathrm{Re}} & 0 & 0 & -i\\ 0 & i\left(\omega-U\right)+\dfrac{1+m^2}{\mathrm{Re}} & -\dfrac{1}{\mathrm{Fr}^2} & -im\\ 0 & 1 & i\left(\omega-U\right) & 0\\ -i & -im & 0 & 0\end{bmatrix}$$

$$\omega=\partial_T\Phi$$

$$m=-\partial_Z\Phi$$

Order zero

$$\mathbf{M}\begin{bmatrix} u_0 \\ w_0 \\ b_0 \\ P_0 \end{bmatrix} = 0 \implies \begin{cases} \det \mathbf{M} & = 0 \\ \begin{bmatrix} u_0 \\ w_0 \\ b_0 \\ P_0 \end{bmatrix} & = \phi_0 \mathcal{P} \\ \mathcal{P} & = \begin{bmatrix} U - \omega \\ \dfrac{\omega - U}{m} \\ \dfrac{i}{m} \\ -\left(\omega - U\right)^2 \left(1 - i\dfrac{1+m^2}{\mathrm{Re}\left(\omega - U\right)}\right) \end{bmatrix} \end{cases}$$

Order one

$$
\begin{bmatrix} U-\omega \\ \dfrac{\omega - U}{m} \\ -\dfrac{i}{m\mathrm{Fr}^2} \\ -(\omega-U)^2\left(1-i\dfrac{1+m^2}{\mathrm{Re}\,(\omega-U)}\right) \end{bmatrix} \cdot \begin{bmatrix} \partial_T u_0 + w_0 \partial_Z U + \dfrac{i}{\mathrm{Re}}\left(u_0\partial_Z m + 2m\partial_Z u_0\right) \\ \partial_T w_0 + \partial_Z P_0 + \dfrac{i}{\mathrm{Re}}\left(w_0\partial_Z m + 2m\partial_Z w_0\right) \\ \partial_T b_0 \\ \partial_Z w_0 \end{bmatrix} = 0
$$

$$
\implies \mathcal{F}\left[U, \phi_0\right] = 0
$$

where $\mathcal{F}$ is differential operator (linear in ϕ_0).

Inviscid limit

For $\mathrm{Re} = \infty$, the last equation can be simplified into the **wave activity equation** :

$$\partial_T A + \partial_Z \left(A w_g \right) = 0$$

with $A = E / \left(\omega - U \right)$ and $E = \dfrac{1}{4} \left(\left| u_0 \right|^2 + \left| w_0 \right|^2 + \mathrm{Fr}^2 \left| b_0 \right|^2 \right)$.
Also $\overline{u_0 w_0} = A k w_g$ such that at leading order :

$$\partial_z \overline{u_0 w_0} = -\partial_t \left(kA \right)$$

Injecting this result into the mean-flow evolution equation leads to

$$\partial_t \left(U - kA \right) = 0$$

This result is known as the **non-acceleration theorem**.

Boundary conditions: computation of the full wave field

Boundary condition : transverse oscillation

Let us consider a horizontally set-up generator. The fluid is viscous wih a **no-slip** boundary condition :

$$\mathbf{u}\left(x, z = h_b\left(x,t\right), t\right) = \partial_t h_b\left(x,t\right)\mathbf{e}_z$$

If we now suppose that $||h_b|| \ll 1$, we perform the wave-decomposition and linearize this boundary condition to get :

$$\begin{cases} \overline{u}\left(z=0,t\right) & = 0 \\ u'\left(x,z=0,t\right) & = 0 \\ w'\left(x,z=0,t\right) & = \partial_t h_b\left(x,t\right) \end{cases}$$

Important consequence : $\displaystyle\int_0^\infty \partial_z \overline{u'w'}\,\mathrm{d}z = 0.$

Boundary condition : Progressive wave

Here we consider $h_b(x,t) = \epsilon \mathcal{R}\mathrm{e}\left[e^{i(t-x)}\right]$ corresponding to $(\omega, k) = (1, 1)$. Considering waves propagating upwardly, the we retain the solution for m with a negative imaginary part only. We first ignore the mean-flow.

$$\begin{cases} \tilde{w}'(z) & = a_w(Z)\, e^{-i\int_0^z m_w\,\mathrm{d}z} + a_{bl}(Z)\, e^{-i\int_0^z m_{bl},\mathrm{d}z} \\ \partial_z \tilde{w}'(z=0) & = 0 \\ \tilde{w}'(z=0) & = i\epsilon \end{cases}$$

$$\implies \begin{cases} a_w(0) & = i\epsilon \dfrac{m_{bl}(0)}{m_{bl}(0) - m_w(0)} \\ a_{bl}(0) & = i\epsilon \dfrac{m_w(0)}{m_w(0) - m_{bl}(0)} \end{cases}$$

Computation of Reynold stress in large Reynold number limit

General expression

For a wave field of the form $\tilde{w}(z) = a_w e^{-i\int m_w} + a_{bl} e^{-i\int m_{bl}}$, we have :

$$\overline{u'w'} = -\frac{1}{2}\Bigg(|a_w|^2 m'_w e^{2\int m''_w} + |a_{bl}|^2 m'_{bl} e^{2\int m''_{bl}} + \mathcal{R}\mathrm{e}\left[a_w^* a_{bl}\left(m_{bl} + m_w^*\right)\right]\mathcal{R}\mathrm{e}\left[e^{i\int(m_w^* - m_{bl})}\right] + \mathcal{I}\mathrm{m}\left[a_w^* a_{bl}\left(m_{bl} + m_w^*\right)\right]\mathcal{I}\mathrm{m}\left[e^{i\int(m_w^* - m_{bl})}\right]\Bigg)$$

- Bulk streaming
- Boundary streaming

Large Reynold limit without mean-flow

$$\overline{u'w'} = -\frac{\epsilon^2}{2}\left\{m_0\left(e^{-\frac{z}{\mathrm{Fr}^4 m_0 \mathrm{Re}}} - e^{-z\sqrt{\frac{\mathrm{Re}}{2}}}\cos z\sqrt{\frac{\mathrm{Re}}{2}}\right)\right.$$

$$\left. - \frac{m_0^2}{\sqrt{2\mathrm{Re}}}\left(e^{-z\sqrt{2\mathrm{Re}}} - e^{-z\sqrt{\frac{\mathrm{Re}}{2}}}\left(\cos z\sqrt{\frac{\mathrm{Re}}{2}} + \sin z\sqrt{\frac{\mathrm{Re}}{2}}\right)\right)\right\}$$

- Bulk streaming
- We have $\overline{u'w'}\,(z=0)=0$

Large Reynold limit with mean-flow for the bulk term

$$\overline{u'w'} = -\frac{\epsilon^2 m_0\left(\overline{u}\right)}{2\left(1-\overline{u}\right)^2} e^{-\int \frac{\mathrm{d}z}{\mathrm{Fr}^4 (1-\overline{u})^5 m_0(\overline{u}) \mathrm{Re}}}$$

$$\partial_z \overline{u'w'} = \frac{\epsilon^2}{2\left(1-\overline{u}\right)^7 \mathrm{Fr}^4 \mathrm{Re}} e^{-\int \frac{\mathrm{d}z}{\mathrm{Fr}^4 (1-\overline{u})^5 m_0(\overline{u}) \mathrm{Re}}}$$