

Viscous internal waves and streaming

Antoine Renaud

Laboratoire de Physique, ENS Lyon

Introduction

Viscosity can play an important role for internal waves generated in labs. The Reynolds number $Re = \frac{UL}{\nu}$ is several order of magnitude lower in labs experiments than in the ocean context.

Consequently, viscosity associated features might matter :

- Decay of internal wave beams
- Boundary layers
- Streaming

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The 2D Boussinesq model

2D Boussinesq model : Equations

- Momentum equation:

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla P + b \mathbf{e}_z + \nu \Delta \mathbf{u}$$

- Buoyancy advection equation

$$\partial_t b + \mathbf{u} \cdot \nabla b + N^2 w = 0$$

- Incompressible flow

$$\nabla \cdot \mathbf{u} = 0$$

2D Boussinesq model : Equations

Velocity field : $\mathbf{u} = (u, w)$

- Momentum equation:

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla P + b\mathbf{e}_z + \nu \Delta \mathbf{u}$$

- Buoyancy advection equation

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2D Boussinesq model : Equations

Nabla operator $\nabla = (\partial_x, \partial_z)$ and Laplacian operator $\Delta = \partial_x^2 + \partial_z^2$

- Momentum equation:

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla P + b \mathbf{e}_z + \nu \Delta \mathbf{u}$$

- Buoyancy advection equation

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2D Boussinesq model : Equations

Pressure field P and the **buoyancy field** $b = -g \frac{\rho - \rho_0}{\rho_0} - N^2 z$
where N is the **Brunt-Väisälä frequency** assumed constant

- **Momentum equation:**

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla P + b \mathbf{e}_z + \nu \Delta \mathbf{u}$$

- **Buoyancy advection equation**

$$\partial_t b + \mathbf{u} \cdot \nabla b + N^2 w = 0$$

- **Incompressible flow**

$$\nabla \cdot \mathbf{u} = 0$$

2D Boussinesq model : Adimensionalization

- $(\tilde{x}, \tilde{z}) = K(x, z)$ where K is a typical horizontal wave number (e.g. the wave number of the generator)
- $\tilde{t} = \Omega t$ where Ω is a typical frequency (e.g. the frequency of the generator)
- $\tilde{\mathbf{u}} = \frac{K}{\Omega} \mathbf{u}$
- $\tilde{b} = \frac{K}{N^2} b$
- $\tilde{P} = \frac{k^2}{\Omega^2} P$

2D Boussinesq model : Dimensionless parameters and adimensionalized equations

There are two independent dimensionless parameters :

- The Reynold number : $\text{Re} = \frac{\Omega}{\nu K^2}$
- The Fround number : $\text{Fr} = \frac{\Omega}{N}$

The resulting adimensionalized equations write :

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} &= -\nabla P + \frac{1}{\text{Fr}^2} b \mathbf{e}_z + \frac{1}{\text{Re}} \Delta \mathbf{u} \\ \partial_t b + \mathbf{u} \cdot \nabla b + w &= 0 \\ \nabla \cdot \mathbf{u} &= 0 \end{cases}$$

Viscous internal waves

Let us linearize the equations of motion about the rest state $\mathbf{u}, b, P = 0$:

$$\begin{cases} \partial_t u + \partial_x P - \frac{1}{\text{Re}} \Delta u & = 0 \\ \partial_t w + \partial_z P - \frac{1}{\text{Fr}^2} b - \frac{1}{\text{Re}} \Delta \mathbf{w} & = 0 \\ \partial_t b + w & = 0 \\ \nabla \cdot \mathbf{u} & = 0 \end{cases}$$

Dispersion relation

We look for **non-vanishing plane waves solutions**

$$\begin{bmatrix} u \\ w \\ b \\ p \end{bmatrix} = \begin{bmatrix} \tilde{u} \\ \tilde{w} \\ \tilde{b} \\ \tilde{p} \end{bmatrix} e^{i(\omega t - kx - mz)}.$$

This leads to the following **dispersion relation** :

$$\omega \left(\omega - i \frac{k^2 + m^2}{\text{Re}} \right) = \frac{1}{\text{Fr}^2} \frac{k^2}{k^2 + m^2}$$

Inviscid limit

In the inviscid limit (i.e. $\text{Re} = +\infty$), we recover the well known dispersion relation :

$$\omega^2 = \frac{1}{\text{Fr}^2} \frac{k^2}{k^2 + m^2} = \frac{1}{\text{Fr}^2} \sin^2 \theta$$

With the phase and group velocities

$$\mathbf{c}_\varphi = \pm \frac{1}{\text{Fr} (k^2 + m^2)} \begin{bmatrix} k \\ m \end{bmatrix}, \quad \mathbf{c}_g = \pm \frac{k^2}{\text{Fr} \sqrt{k^2 + m^2}} \begin{bmatrix} m^2 \\ -mk \end{bmatrix}$$

such that $\mathbf{c}_\varphi \cdot \mathbf{c}_g = 0$

We must have $|\omega| < \frac{1}{\text{Fr}}$ for propagating waves.

Back to the viscous case

Let us consider again the viscous case. We consider the case where $\omega = 1$ and $k = 1$. (generator set-up horizontally)

$$\text{Fr}^2 \left(1 - i \frac{1 + m^2}{\text{Re}} \right) (1 + m^2) = 1$$

We can already remark a few things

- 4th order complex polynomial equation for m meaning there are 4 different complex solutions
- The symmetry $m \rightarrow -m$ indicates two important branches

Two branches :

$$m^2 = \frac{\text{Re}}{2i} \left(1 \pm \sqrt{1 - \frac{4i}{\text{ReFr}^2}} \right) - 1$$

Large Reynold number limit

We now consider large values of the Reynold number (such that $\text{Fr}^2 \text{Re} \gg 1$). The solution then writes :

$$m_w = \pm \left(m_0 + \frac{i}{2\text{Fr}^4 m_0 \text{Re}} \right)$$
$$m_{bl} = \pm (1 - i) \sqrt{\frac{\text{Re}}{2}}$$

where $m_0 = \sqrt{\frac{1}{\text{Fr}^2} - 1}$ is the inviscid value for m .

Few remarks :

- m_w : Propagating branche
- $L_{\text{Re}} = 2\text{Fr}^4 \text{Re} m_0$: Dumping scale for the wave beam
- m_{bl} : Boundary layer branche
- $\delta_{\text{Re}} = \sqrt{2/\text{Re}}$: Boundary layer length

metropolis supports 4 different titleformats:

- Regular
- SMALLCAPS
- ALLSMALLCAPS
- ALLCAPS

They can either be set at once for every title type or individually.

This frame uses the `smallcaps` titleformat.

Potential Problems

Be aware, that not every font supports small caps. If for example you typeset your presentation with pdfTeX and the Computer Modern Sans Serif font, every text in smallcaps will be typeset with the Computer Modern Serif font instead.

This frame uses the `allsmallcaps` titleformat.

Potential problems

As this titleformat also uses smallcaps you face the same problems as with the `smallcaps` titleformat. Additionally this format can cause some other problems. Please refer to the documentation if you consider using it.

As a rule of thumb: Just use it for plaintext-only titles.

This frame uses the `allcaps` titleformat.

Potential Problems

This titleformat is not as problematic as the `allsmallcaps` format, but basically suffers from the same deficiencies. So please have a look at the documentation if you want to use it.

Elements

The theme provides sensible defaults to
`\emph{emphasize}` text, `\alert{accent}` parts
or show `\textbf{bold}` results.

becomes

The theme provides sensible defaults to *emphasize* text, **accent** parts or
show **bold** results.

Font feature test

- Regular
- *Italic*
- SMALLCAPS
- **Bold**
- **Bold Italic**
- **Bold SmallCaps**
- Monospace
- *Monospace Italic*
- Monospace Bold
- *Monospace Bold Italic*

Items

- Milk
- Eggs
- Potatos

Enumerations

1. First,
2. Second and
3. Last.

Descriptions

PowerPoint Meeh.
Beamer Yeeeha.

- This is important

- This is important
- Now this

- This is important
- Now this
- And now this

- This is really important
- Now this
- And now this

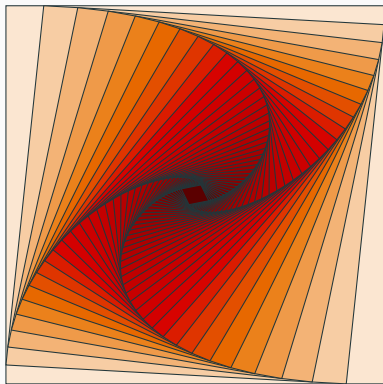


Figure 1: Rotated square from texample.net.

Table 1: Largest cities in the world (source: Wikipedia)

City	Population
Mexico City	20,116,842
Shanghai	19,210,000
Peking	15,796,450
Istanbul	14,160,467

Three different block environments are pre-defined and may be styled with an optional background color.

Default

Block content.

Alert

Block content.

Example

Block content.

Default

Block content.

Alert

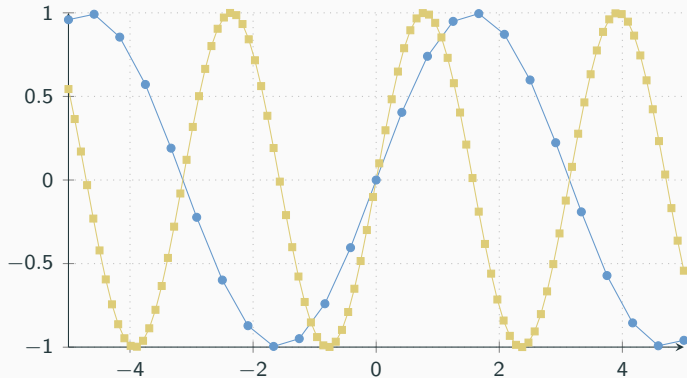
Block content.

Example

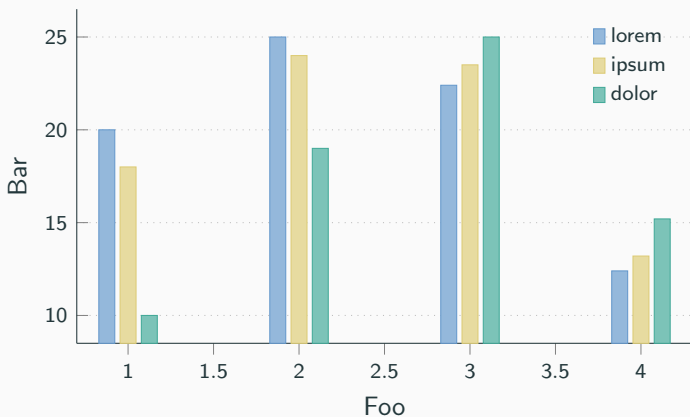
Block content.

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$$

Line plots



Bar charts



Veni, Vidi, Vici

metropolis defines a custom beamer template to add a text to the footer. It can be set via

```
\setbeamertemplate{frame footer}{My custom footer}
```

Some references to showcase `[allowframebreaks]` [?, ?, ?, ?, ?]

Conclusion

Get the source of this theme and the demo presentation from

`github.com/matze/mtheme`

The theme *itself* is licensed under a Creative Commons Attribution-ShareAlike 4.0 International License.



Questions?

Backup slides

Sometimes, it is useful to add slides at the end of your presentation to refer to during audience questions.

The best way to do this is to include the `appendixnumberbeamer` package in your preamble and call `\appendix` before your backup slides.

metropolis will automatically turn off slide numbering and progress bars for slides in the appendix.

References I