2D Viscous internal waves and streaming

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Introduction

Internal waves in labs

Viscosity can play an in important role for internal waves generated in labs.The Reynolds number $\mathrm{Re} = \dfrac{UL}{\nu}$ is several order of magnitude lower in labs experiments than in the ocean context.

Consequently, viscosity associated features might matter :

- Decay of internal wave beams
- Boundary layers
- Streaming

Table of contents

1. Introduction
2. The 2D Boussinesq model
3. Wave and mean-flow decomposition
4. Inviscid internal waves
5. Viscous internal waves
6. Boundary conditions
7. Computation of Reynold stress in the large Reynold number limit

The 2D Boussinesq model

2D Boussinesq model : Equations

- **Momentum equation:**

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \boldsymbol{\nabla})\,\mathbf{u} = -\boldsymbol{\nabla} P + S\mathbf{e}_z + \nu \Delta \mathbf{u}$$

- **Buoyancy advection equation**

$$\partial_t S + \mathbf{u} \cdot \boldsymbol{\nabla} S = 0$$

- **Incompressible flow**

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0$$

2D Boussinesq model : Equations

Velocity field : $\mathbf{u} = (u, w)$

- **Momentum equation:**

$$\partial_t \mathbf{u} + \left(\mathbf{u} \cdot \boldsymbol{\nabla} \right) \mathbf{u} = -\boldsymbol{\nabla} P + S\mathbf{e}_z + \nu \Delta \mathbf{u}$$

- **Buoyancy advection equation**

$$\partial_t S + \mathbf{u} \cdot \boldsymbol{\nabla} S = 0$$

- **Incompressible flow**

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0$$

2D Boussinesq model : Equations

Nabla operator $\boxed{\nabla} = (\partial_x, \partial_z)$ and **Laplacian operator** $\boxed{\Delta} = \partial_x^2 + \partial_z^2$

- **Momentum equation:**

$$\partial_t \mathbf{u} + \left(\mathbf{u} \cdot \boxed{\nabla}\right) \mathbf{u} = -\boxed{\nabla} P + S\mathbf{e}_z + \nu \boxed{\Delta} \mathbf{u}$$

- **Buoyancy advection equation**

$$\partial_t S + \mathbf{u} \cdot \boxed{\nabla} S = 0$$

- **Incompressible flow**

$$\boxed{\nabla} \cdot \mathbf{u} = 0$$

2D Boussinesq model : Equations

Pressure field P and the **buoyancy field** $S = -g\dfrac{\rho - \rho_0}{\rho_0}$.

- **Momentum equation:**

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \boldsymbol{\nabla})\, \mathbf{u} = -\boldsymbol{\nabla} P + S\, \mathbf{e}_z + \nu \Delta \mathbf{u}$$

- **Buoyancy advection equation**

$$\partial_t S + \mathbf{u} \cdot \boldsymbol{\nabla} S = 0$$

- **Incompressible flow**

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0$$

Casimirs, potential energy and background stratification

- The buoyancy advection equation implies the conservation of Casimirs :

$$\mathcal{C}_f\left[S\right] = \iint S\left(x, z, t\right) \mathrm{d}x \mathrm{d}z$$

  where f is any function.
- $\implies$ The buyancy field evolves by rearanging itself.
- The standard potential energy is defined by :

$$E_p\left[S\right] = -\iint S\left(x, z, t\right) z \;\; \mathrm{d}x \mathrm{d}z$$

- The background stratification is defined by :

$$E_p\left[b_{\mathrm{bg}}\right] = \min_S \left\{ E_p\left[S\right] \middle| \mathcal{C}_f\left[S\right] = C_f \;\; \forall f \right\}$$

  such that $b_{\mathrm{bg}}\left(x, z, t\right) = \int^{z} N^2\left(z'\right) \mathrm{d}z'$ where N is the local Brunt-Väisälä frequency.

Buoyancy and pressure disturbances

- We define the buyancy disturbance fields by $b = S - b_{\mathrm{bg}}$ and the pressure disturbance by $p = P - \int^{z} b_{\mathrm{bg}}(z')\,\mathrm{d}z'$
- Injecting this in the equations of motion, one gets :

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\,\mathbf{u} & = -\boldsymbol{\nabla} p + b\,\mathbf{e}_z + \nu\Delta\mathbf{u} \\ \partial_t b + \mathbf{u}\cdot\boldsymbol{\nabla} b + N^2(z)\,w & = 0 \\ \boldsymbol{\nabla}\cdot\mathbf{u} & = 0 \end{cases}$$

Potential energy of the buyancy disturbance

- For the standard expression of the potential energy, the fact that $b = 0$ is a minimum is not obvious:

$$E_p\left[S\right] - E_p\left[b_{\mathrm{bg}}\right] = \Delta E_p\left[b\right] = -\iint b \,\mathrm{d}x\mathrm{z}$$

- Let us introduce $f = \left(\int^{z} b_{\mathrm{bg}}\right)^{-1}$, such that :

$$\begin{aligned}\mathcal{C}_f\left[b_{\mathrm{bg}}\right] &= \mathcal{C}_f\left[b_{\mathrm{bg}} + b\right] \\ &= \mathcal{C}_f\left[b_{\mathrm{bg}}\right] - \Delta E_p\left[b\right] + \sum_{k=2}^{\infty} \iint \frac{f^{(k)}\left(b_{\mathrm{bg}}\right)}{k!} b^k \;\; \mathrm{d}x\mathrm{d}z\end{aligned}$$

- For $|b| \ll 1$ we have

$$\Delta E_p\left[b\right] \approx \iint \frac{b^2\left(x, z, t\right)}{2N^2\left(z\right)} \mathrm{d}x\mathrm{d}z$$

This relation is exact if $N^2 = \mathrm{Cte}$.

2D Boussinesq model : Adimensionalization

- We assume $N^2 = \mathrm{Cte}$ in the following
- $(\tilde{x}, \tilde{z}) = K\,(x, z)$ where K is a typical wave number (e.g. the wave number of the generator)
- $\tilde{t} = \Omega t$ where Ω is a typical frequency (e.g. the frequency of the generator)
- $\tilde{\mathbf{u}} = \dfrac{K}{\Omega}\mathbf{u}$
- $\tilde{b} = \dfrac{K}{N^2}b$
- $\tilde{P} = \dfrac{k^2}{\Omega^2}P$

2D Boussinesq model : Dimensionless parameters and adimensionalized equations

There are two independant dimensionless parameters :

- The Reynold number : $\mathrm{Re} = \dfrac{\Omega}{\nu K^2}$
- The Fround number : $\mathrm{Fr} = \dfrac{\Omega}{N}$

The resulting adimensionalized equations write :

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \boldsymbol{\nabla}) \mathbf{u} & = -\boldsymbol{\nabla} P + \dfrac{1}{\mathrm{Fr}^2} b \mathbf{e}_z + \dfrac{1}{\mathrm{Re}} \Delta \mathbf{u} \\ \partial_t b + \mathbf{u} \cdot \boldsymbol{\nabla} b + w & = 0 \\ \boldsymbol{\nabla} \cdot \mathbf{u} & = 0 \end{cases}$$

Wave and mean-flow decomposition

Zonally symetric flows

We consider here the simplest case of zonally periodic flows :

- The averaging operator is defined by $\overline{u} = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} u \,\mathrm{d}x$.
- The wave and mean-flow decomposition is defined by $(u, w, b, P) = \left(\overline{u}, \overline{w}, \overline{b}, \overline{P}\right) + (u', w', b', P')$
- Taking the mean part of the equations of motion leads to the **mean-flow equations** :

$$\begin{cases} \partial_t \overline{u} - \dfrac{1}{\mathrm{Re}}\partial_{zz}\overline{u} & = -\partial_z \overline{u' w'} \\ \overline{w} & = 0 \\ \overline{P} & = \overline{w'^2} \\ \partial_t \overline{b} & = -\partial_z \overline{b' w'} \end{cases}$$

Waves equations

- Substracting the mean flow equations to the equations of motions leads to the **wave equations** :

$$\begin{cases} \partial_t u' + \overline{u}\partial_x u' + w'\partial_z \overline{u} + \boxed{u'\partial_x u' + w'\partial_z u' - \partial_z \overline{u'w'}} + \partial_x P' - \dfrac{1}{\mathrm{Re}}\Delta u' & = 0 \\ \partial_t w' + \overline{u}\partial_x w' + \boxed{u'\partial_x w' + w'\partial_z w' - \partial_z \overline{w'^2}} + \partial_z P' - \dfrac{1}{\mathrm{Fr}^2} b' - \dfrac{1}{\mathrm{Re}}\Delta w' & = 0 \\ \partial_t b' + \overline{u}\partial_x b' + \boxed{u'\partial_x b' + w'\partial_z b' - \partial_z \overline{b'w'}} + \underbrace{\left(1 + \partial_z \overline{b}\right)}_{N^2(z)} w' & = 0 \\ \partial_x u' + \partial_z w' & = 0 \end{cases}$$

- Non-linear terms leading to PSI.

Inviscid internal waves

Linearized

Let us first ignore the non-linear and viscous terms in the wave equations
:

$$
\begin{cases}
\partial_t u' + \overline{u}\partial_x u' + w'\partial_z \overline{u} + \partial_x P' & = 0 \\
\partial_t w' + \overline{u}\partial_x w' + \partial_z P' - \dfrac{1}{\mathrm{Fr}^2} b' & = 0 \\
\partial_t b' + \overline{u}\partial_x b' + N^2 w' & = 0 \\
\partial_x u' + \partial_z w' & = 0
\end{cases}
$$

As a direct consequence :

$$
\begin{cases}
\partial_z \overline{u'w'} & = \partial_t \left(\dfrac{1}{N^2} \overline{b'\left(\partial_x w' - \partial_z u'\right)} + \dfrac{1}{N^4} \dfrac{\overline{b'^2}}{2} \partial_{zz} \overline{u} \right) \\
\overline{b'w'} & = -\partial_t \left(\dfrac{\overline{b'^2}}{2N^2} \right)
\end{cases}
$$

WKB ansatz

- We introduce $a \ll 1$ and associated slow variables $(Z, T, T_{\mathrm{mf}}) = \left(az, at, a^3 t\right)$ and assume $\overline{u} = \overline{u}\,(Z, T_{\mathrm{mf}}) = O\,(1)$.
- WKB ansatz :

$$\begin{bmatrix} u \\ w \\ b\mathrm{Fr}/N \\ P \end{bmatrix} = a \sum_{j=0}^{\infty} a^j \begin{bmatrix} u_j\,(Z, T) \\ w_j\,(Z, T) \\ b_j\,(Z, T)\,\mathrm{Fr}/N \\ P_j\,(Z, T) \end{bmatrix} \exp\left(i \frac{\Phi\,(Z, T)}{a} - ikx \right)$$

- We define $m = -\partial_Z \Phi$, $\omega = \partial_T \Phi$ and $\hat{\omega} = \omega - k\overline{u}$

WKB ansatz

Injecting the WKB ansatz and collecting the first order terms leads to :

$$\mathbf{M}\begin{bmatrix} u_0 \\ w_0 \\ b_0\mathrm{Fr}/N \\ P_0 \end{bmatrix} + a\left(\mathbf{M}\begin{bmatrix} u_1 \\ w_1 \\ b_1\mathrm{Fr}/N \\ P_1 \end{bmatrix} + \underbrace{\begin{bmatrix} \partial_T u_0 + w_0\partial_Z \overline{u} \\ \partial_T w_0 + \partial_Z P_0 \\ \partial_T b_0\mathrm{Fr}/N \\ \partial_Z w_0 \end{bmatrix}}_{\boldsymbol{T}[\overline{u},\phi_0]}\right) = 0$$

With :

$$\mathbf{M} = \begin{bmatrix} i\hat{\omega} & 0 & 0 & -ik \\ 0 & i\hat{\omega} & -\dfrac{N}{\mathrm{Fr}} & -im \\ 0 & \dfrac{N}{\mathrm{Fr}} & i\hat{\omega} & 0 \\ -ik & -im & 0 & 0 \end{bmatrix}$$

Order zero

$$
\mathbf{M}\begin{bmatrix} u_0 \\ w_0 \\ b_0 \mathrm{Fr}/N \\ P_0 \end{bmatrix} = 0 \implies
\begin{cases}
\det \mathbf{M} & = 0 \iff \hat{\omega}^2 = \dfrac{N^2 k^2}{\mathrm{Fr}^2 \left(k^2 + m^2\right)} \\
\begin{bmatrix} u_0 \\ w_0 \\ b_0 \mathrm{Fr}/N \\ P_0 \end{bmatrix} & = \phi_0 \boldsymbol{\mathcal{P}} \\
\boldsymbol{\mathcal{P}} & = \begin{bmatrix} \hat{\omega}/k \\ -\hat{\omega}/m \\ -iN/m\mathrm{Fr} \\ \hat{\omega}^2/k^2 \end{bmatrix}
\end{cases}
$$

Dispersion relation

$$\hat{\omega}^2 = \frac{N^2 k^2}{\mathrm{Fr}^2 \left(k^2 + m^2\right)}$$

- $-N \leq \mathrm{Fr}\hat{\omega} \leq N$
- Phase and Group velocities :

$$\mathbf{c}_\varphi = \frac{\hat{\omega}}{\mathbf{k}^2}\mathbf{k} \qquad , \qquad \mathbf{c}_g = -\frac{\hat{\omega}}{\mathbf{k}^2}\frac{m}{k}\mathbf{k}^\perp$$

  where $\mathbf{k}^\perp = (-m, k)$.
- $\mathbf{c}_\varphi \cdot \mathbf{c}_g = 0$

Order one

We take the scalar product of the order one terms with $\phi_0^* \mathcal{P}^\dagger$:

$$\mathcal{P}^\dagger \cdot \mathcal{T}\left[\overline{u}, \phi_0\right] = 0$$

After some algebra, we get the **wave-activity equation** :

$$\partial_t A + \partial_z \left(A w_g\right) = 0 \quad , \quad \text{with} \begin{cases} A & = \dfrac{E}{\hat{\omega}} \\ w_g & = -\dfrac{\mathrm{Fr}^2 \hat{\omega}^3 m}{N^2 k^2} \end{cases}$$

where

$$E = \frac{1}{4}\left(\left|u_0\right|^2 + \left|w_0\right|^2 + \frac{\mathrm{Fr}^2}{N^2}\left|b_0\right|^2\right) = \frac{N^2 \left|\phi_0\right|^2}{2\mathrm{Fr}^2 m^2}$$

We can check that at leading order $\overline{u' w'} = \dfrac{1}{2}\mathcal{R}\mathrm{e}\left[u_0 w_0^*\right] = k A w_g$ such that $\partial_z \overline{u' w'} \approx -\partial_t \left(kA\right)$

Viscous internal waves

Linearized

We put the viscous operator back into the linearized wave euqations :

$$\begin{cases} \partial_t u' + \overline{u}\partial_x u' + w'\partial_z \overline{u} + \partial_x P' - \frac{1}{\mathrm{Re}}\Delta u' & = 0 \\ \partial_t w' + \overline{u}\partial_x w' + \partial_z P' - \frac{1}{\mathrm{Fr}^2} b' - \frac{1}{\mathrm{Re}}\Delta w' & = 0 \\ \partial_t b' + \overline{u}\partial_x b' + N^2 w' & = 0 \\ \partial_x u' + \partial_z w' & = 0 \end{cases}$$

We still have:

$$\overline{b'w'} = -\partial_t \left(\frac{\overline{b'^2}}{2N^2} \right)$$

We use the same assumptions and consider the same WKB ansatz in the following.

WKB ansatz

Injecting the WKB ansatz and collecting the first order terms leads to :

$$\mathbf{M}\begin{bmatrix} u_0 \\ w_0 \\ b_0\mathrm{Fr}/N \\ P_0 \end{bmatrix} + a\left(\mathbf{M}\begin{bmatrix} u_1 \\ w_1 \\ b_1\mathrm{Fr}/N \\ P_1 \end{bmatrix} + \underbrace{\begin{bmatrix} \partial_T u_0 + w_0\partial_Z\overline{u} + \dfrac{i}{\mathrm{Re}}\left(u_0\partial_Z m + 2m\partial_Z u_0\right) \\ \partial_T w_0 + \partial_Z P_0 + \dfrac{i}{\mathrm{Re}}\left(w_0\partial_Z m + 2m\partial_Z w_0\right) \\ \partial_T b_0\mathrm{Fr}/N \\ \partial_Z w_0 \end{bmatrix}}_{\boldsymbol{T}[\overline{u},\phi_0]}\right) = 0$$

With :

$$\mathbf{M} = \begin{bmatrix} i\hat{\omega} + \dfrac{k^2+m^2}{\mathrm{Re}} & 0 & 0 & -ik \\ 0 & i\hat{\omega} + \dfrac{k^2+m^2}{\mathrm{Re}} & -\dfrac{N}{\mathrm{Fr}} & -im \\ 0 & \dfrac{N}{\mathrm{Fr}} & i\hat{\omega} & 0 \\ -ik & -im & 0 & 0 \end{bmatrix}$$

Order zero

$$
\mathbf{M}\begin{bmatrix} u_0 \\ w_0 \\ b_0\mathrm{Fr}/N \\ P_0 \end{bmatrix} = 0 \implies \begin{cases} \det\mathbf{M} = 0 & \iff \hat{\omega}\left(\hat{\omega} - i\dfrac{k^2+m^2}{\mathrm{Re}}\right) = \dfrac{N^2}{\mathrm{Fr}^2}\dfrac{k^2}{k^2+m^2} \\ \begin{bmatrix} u_0 \\ w_0 \\ b_0\mathrm{Fr}/N \\ P_0 \end{bmatrix} & = \phi_0\mathcal{P} \\ \mathcal{P} & = \begin{bmatrix} \hat{\omega}/k \\ -\hat{\omega}/m \\ -iN/m\mathrm{Fr} \\ \dfrac{\hat{\omega}^2}{k^2}\left(1 - i\dfrac{(k^2+m^2)}{\mathrm{Re}\hat{\omega}}\right) \end{bmatrix} \end{cases}
$$

Dispersion relation

k and ω are fixed by generation process (it has to be zonally symetric for our calculation)

$$\hat{\omega}\left(\hat{\omega} - i\frac{k^2+m^2}{\mathrm{Re}}\right) = \frac{N^2}{\mathrm{Fr}^2}\frac{k^2}{k^2+m^2}$$

We can already remark a few things

- 4^{th} order complex polynomial equation for m meaning there are 4 different complex solutions
- The symetry $m \to -m$ indicates two important branches

Two branches :

$$k^2+m^2 = \frac{\mathrm{Re}\hat{\omega}}{2i}\left(1 \pm \sqrt{1 - \frac{4ik^2N^2}{\mathrm{Re}\mathrm{Fr}^2\hat{\omega}^3}}\right) - 1$$

Large Reynold number limit

We now consider large values of the Reynold number (such that $\mathrm{Fr}^2\mathrm{Re}\,|\hat{\omega}|^3/4k^2N^2 \gg 1$). The solution then writes for upwardly propagating waves :

$$m_w = -\epsilon m_0 - \frac{ik^4N^4}{2\mathrm{Fr}^4 m_0 \mathrm{Re}\,|\hat{\omega}|^5}$$

$$m_{bl} = (\epsilon - i)\sqrt{\frac{\mathrm{Re}\,|\hat{\omega}|}{2}}$$

where $m_0 = \sqrt{\frac{N^2k^2}{\mathrm{Fr}^2\hat{\omega}^2} - k^2}$ is the inviscid value for m and $\epsilon = \mathrm{sign}\,(\hat{\omega})$

Few remarks :

- m_w : Propagating branche
- $L_{\mathrm{Re}} = 2\mathrm{Fr}^4\mathrm{Re}m_0\,|\hat{\omega}|^5/k^4N^4$: penetration length for the wave beam
- m_{bl} : Boundary layer branche
- $\delta_{\mathrm{Re}} = \sqrt{2/\mathrm{Re}\,|\hat{\omega}|}$: Boundary layer length

Boundary conditions

Boundary condition : transverse oscillation

Let us consider a horizontally set-up generator. The fluid is viscous wih a **no-slip** boundary condition :

$$\mathbf{u}\left(x, z = h_b\left(x, t\right), t\right) = \partial_t h_b\left(x, t\right)\mathbf{e}_z$$

If we now su hppose that $||_b|| \ll 1$, we perform the wave and mean-flow decomposition and linearize this boundary condition to get :

$$\begin{cases} \overline{u}\left(z = 0, t\right) & = 0 \\ u'\left(x, z = 0, t\right) & = 0 \\ w'\left(x, z = 0, t\right) & = \partial_t h_b\left(x, t\right) \end{cases}$$

Important consequence : $\displaystyle\int_0^\infty \partial_z \overline{u'w'}\,\mathrm{d}z = 0.$

Boundary condition : Progressive wave

Here we consider $h_b(x,t) = a\mathcal{R}\mathrm{e}\left[e^{i(t-x)}\right]$ corresponding to $(\omega, k) = (1,1)$. Considering waves propagating upwardly, using the polarisation :

$$\begin{cases} -\left(\dfrac{\phi_{0,w}(0)}{m_w(0)} + \dfrac{\phi_{0,bl}(0)}{m_{bl}(0)}\right) & = ia \\ \phi_{0,w}(0) + \phi_{0,bl}(0) & = 0 \end{cases}$$

$$\implies \begin{cases} \phi_{0,w}(0) & = -ia\dfrac{m_{bl}(0)\, m_w(0)}{m_{bl}(0) - m_w(0)} \\ \phi_{0,bl}(0) & = ia\dfrac{m_{bl}(0)\, m_w(0)}{m_{bl}(0) - m_w(0)} \end{cases}$$

Computation of Reynold stress in the large Reynold number limit

General expression

$$\overline{u'w'} = -\frac{\hat{\omega}^2}{2}\left(\left|\phi_{0,w}\right|^2 \frac{m_w'}{\left|m_w\right|^2} e^{2\int m_w''} + \left|\phi_{0,bl}\right|^2 \frac{m_{bl}'}{\left|m_{bl}\right|^2} e^{2\int m_{bl}''} \right.$$
$$+ \mathcal{R}\mathrm{e}\left[\phi_{0,w}^*\phi_{0,bl}\frac{m_{bl}+m_w^*}{m_{bl}m_w}\right]\mathcal{R}\mathrm{e}\left[e^{i\int(m_w^*-m_{bl})}\right]$$
$$\left. + \mathcal{I}\mathrm{m}\left[\phi_{0,w}^*\phi_{0,bl}\frac{m_{bl}+m_w^*}{m_{bl}m_w}\right]\mathcal{I}\mathrm{m}\left[e^{i\int(m_w^*-m_{bl})}\right]\right)$$

- Bulk streaming
- Boundary streaming

Large Reynold limit without mean-flow

$$\overline{u'w'} = \epsilon\frac{\hat{\omega}^2 a^2}{2}\left\{\frac{1}{m_0}\left(e^{-\frac{z}{2\mathrm{Fr}^4 m_0 \mathrm{Re}|\hat{\omega}|^5}} - e^{-z\sqrt{\frac{\mathrm{Re}|\hat{\omega}|}{2}}}\cos z\sqrt{\frac{\mathrm{Re}\,|\hat{\omega}|}{2}}\right)\right.$$
$$\left. - \frac{1}{\sqrt{2\mathrm{Re}\,|\hat{\omega}|}}\left(e^{-z\sqrt{2\mathrm{Re}|\hat{\omega}|}} - e^{-z\sqrt{\frac{\mathrm{Re}|\hat{\omega}|}{2}}}\left(\cos z\sqrt{\frac{\mathrm{Re}\,|\hat{\omega}|}{2}} + \sin z\sqrt{\frac{\mathrm{Re}}{2}}\right)\right)\right\}$$

- Bulk streaming
- We have $\overline{u'w'}\,(z=0) = 0$

Large Reynold limit with mean-flow for the bulk term

$$\overline{u'w'} = \epsilon \frac{\hat{\omega}^2 \left|\phi_{0,w}\right|^2}{2m_0} e^{-\int \frac{\mathrm{d}z}{2\mathrm{Fr}^4 m_0 \mathrm{Re} |\hat{\omega}|^5}} = \epsilon \frac{a^2}{2m_0\left(0\right)} e^{-\int \frac{\mathrm{d}z}{2\mathrm{Fr}^4 m_0 (1-\overline{u}) \mathrm{Re} |1-\overline{u}|^5}}$$

where $\epsilon = \mathrm{sign}\left(1 - \overline{u}\right)$

Large Reynold limit with mean-flow for the bulk term : Stading wave forcing

We now consider a standing wave forcing corresponding in the superposition of two waves with equal amplitudes and opposite phase speeds :

$$\overline{u'w'} = \frac{a^2}{2m_0\left(0\right)}\left(\epsilon_- e^{-\int \frac{\mathrm{d}z}{2\mathrm{Fr}^4 m_0 (1-\overline{u})\mathrm{Re}|1-\overline{u}|^5}} - \epsilon_+ e^{-\int \frac{\mathrm{d}z}{2\mathrm{Fr}^4 m_0 (1+\overline{u})\mathrm{Re}|1+\overline{u}|^5}}\right)$$

with $\epsilon_\pm = \mathrm{sign}\left(1 \pm \overline{u} +\right)$