Viscous internal waves and streaming

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Introduction

Internal waves in labs

Viscosity can play an in important role for internal waves generated in labs.The Reynolds number $\mathrm{Re} = \dfrac{UL}{\nu}$ is several order of magnitude lower in labs experiments than in the ocean context.

Consequently, viscosity associated features might matter :

- Decay of internal wave beams
- Boundary layers
- Streaming

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The 2D Boussinesq model

2D Boussinesq model : Equations

- **Momentum equation:**

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \boldsymbol{\nabla}) \mathbf{u} = -\boldsymbol{\nabla} P + b\mathbf{e}_z + \nu \Delta \mathbf{u}$$

- **Buoyancy advection equation**

$$\partial_t b + \mathbf{u} \cdot \boldsymbol{\nabla} b + N^2 w = 0$$

- **Incompressible flow**

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0$$

2D Boussinesq model : Equations

Velocity field : $\mathbf{u} = \left(u, w\right)$

- **Momentum equation:**

$$\partial_t \mathbf{u} + \left(\mathbf{u} \cdot \boldsymbol{\nabla}\right) \mathbf{u} = -\boldsymbol{\nabla} P + b\mathbf{e}_z + \nu \Delta \mathbf{u}$$

- **Buoyancy advection equation**

$$\partial_t b + \mathbf{u} \cdot \boldsymbol{\nabla} b + N^2 w = 0$$

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2D Boussinesq model : Equations

Nabla operator $\nabla = (\partial_x, \partial_z)$ and **Laplacian operator** $\Delta = \partial_x^2 + \partial_z^2$

- **Momentum equation:**

$$\partial_t \mathbf{u} + \left(\mathbf{u} \cdot \nabla\right) \mathbf{u} = -\nabla P + b\mathbf{e}_z + \nu \Delta \mathbf{u}$$

- **Buoyancy advection equation**

$$\partial_t b + \mathbf{u} \cdot \nabla b + N^2 w = 0$$

- **Incompressible flow**

$$\nabla \cdot \mathbf{u} = 0$$

2D Boussinesq model : Equations

Pressure field P and the **buoyancy field** $b = -g\frac{\rho - \rho_0}{\rho_0} - N^2 z$ where N is the **Brunt-Väisälä frequency** assumed constant

- **Momentum equation:**

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \boldsymbol{\nabla})\,\mathbf{u} = -\boldsymbol{\nabla} P + b\,\mathbf{e}_z + \nu \Delta \mathbf{u}$$

- **Buoyancy advection equation**

$$\partial_t b + \mathbf{u} \cdot \boldsymbol{\nabla} b + N^2 w = 0$$

- **Incompressible flow**

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0$$

2D Boussinesq model : Adimensionalization

- $(\tilde{x}, \tilde{z}) = K\,(x, z)$ where K is a typical wave number (e.g. the wave number of the generator)
- $\tilde{t} = \Omega t$ where Ω is a typical frequency (e.g. the frequency of the generator)
- $\tilde{\mathbf{u}} = \dfrac{K}{\Omega}\mathbf{u}$
- $\tilde{b} = \dfrac{K}{N^2} b$
- $\tilde{P} = \dfrac{k^2}{\Omega^2} P$

2D Boussinesq model : Dimensionless parameters and adimensionalized equations

There are two independant dimensionless parameters :

- The Reynold number : $\mathrm{Re} = \dfrac{\Omega}{\nu K^2}$
- The Fround number : $\mathrm{Fr} = \dfrac{\Omega}{N}$

The resulting adimensionalized equations write :

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \boldsymbol{\nabla})\, \mathbf{u} & = -\boldsymbol{\nabla} P + \dfrac{1}{\mathrm{Fr}^2} b \mathbf{e}_z + \dfrac{1}{\mathrm{Re}} \Delta \mathbf{u} \\ \partial_t b + \mathbf{u} \cdot \boldsymbol{\nabla} b + w & = 0 \\ \boldsymbol{\nabla} \cdot \mathbf{u} & = 0 \end{cases}$$

Viscous internal waves

Linearization

Let us linearize the equations of motion about the rest state $\mathbf{u}, b, P = 0$:

$$\begin{cases} \partial_t u + \partial_x P - \dfrac{1}{\mathrm{Re}}\Delta u & = 0 \\ \partial_t w + \partial_z P - \dfrac{1}{\mathrm{Fr}^2} b - \dfrac{1}{\mathrm{Re}}\Delta \mathbf{w} & = 0 \\ \partial_t b + w & = 0 \\ \boldsymbol{\nabla} \cdot \mathbf{u} & = 0 \end{cases}$$

Dispersion relation

We look for **non-vanishing plane waves solutions**

$$\begin{bmatrix} u \\ w \\ b \\ P \end{bmatrix} = \begin{bmatrix} \tilde{u} \\ \tilde{w} \\ \tilde{b} \\ \tilde{P} \end{bmatrix} e^{i\,(\omega t - kx - mz)}.$$

This leads to the following **dispersion relation** :

$$\omega \left(\omega - i \frac{k^2 + m^2}{\mathrm{Re}} \right) = \frac{1}{\mathrm{Fr}^2} \frac{k^2}{k^2 + m^2}$$

Inviscid limit

In the inviscid limit (i.e. $\mathrm{Re} = +\infty$), we recover the well known dispersion relation :

$$\omega^2 = \frac{1}{\mathrm{Fr}^2} \frac{k^2}{k^2 + m^2} = \frac{1}{\mathrm{Fr}^2} \sin^2 \theta$$

With the phase and group velocities

$$\mathbf{c}_\varphi = \pm \frac{1}{\mathrm{Fr}\left(k^2 + m^2\right)} \begin{bmatrix} k \\ m \end{bmatrix} \quad , \quad \mathbf{c}_g = \pm \frac{k^2}{\mathrm{Fr}\sqrt{k^2 + m2}} \begin{bmatrix} m^2 \\ -mk \end{bmatrix}$$

such that $\mathbf{c}_\varphi \cdot \mathbf{c}_g = 0$

We must have $|\omega| < \dfrac{1}{\mathrm{Fr}}$ for propagating waves.

Back to the viscous case : horizontal generator

Let us consider again the viscous case. We consider the case where $\omega = 1$ and $k = 1$. (generator set-up horizontally)

$$\mathrm{Fr}^2 \left(1 - i\frac{1+m^2}{\mathrm{Re}}\right)\left(1+m^2\right) = 1$$

We can already remark a few things

- 4^{th} order complex polynomial equation for m meaning there are 4 different complex solutions
- The symetry $m \to -m$ indicates two important branches

Two branches :

$$m^2 = \frac{\mathrm{Re}}{2i}\left(1 \pm \sqrt{1 - \frac{4i}{\mathrm{Re}\mathrm{Fr}^2}}\right) - 1$$

Large Reynold number limit

We now consider large values of the Reynold number (such that $\mathrm{Fr}^2\mathrm{Re} \gg 1$). The solution then writes :

$$m_w = \pm\left(m_0 + \frac{i}{2\mathrm{Fr}^4 m_0 \mathrm{Re}}\right)$$

$$m_{bl} = \pm(1-i)\sqrt{\frac{\mathrm{Re}}{2}}$$

where $m_0 = \sqrt{\frac{1}{\mathrm{Fr}^2} - 1}$ is the inviscid value for m.

Few remarks :

- m_w : Propagating branche
- $L_{\mathrm{Re}} = 2\mathrm{Fr}^4\mathrm{Re}m_0$: penetration length for the wave beam
- m_{bl} : Boundary layer branche
- $\delta_{\mathrm{Re}} = \sqrt{2/\mathrm{Re}}$: Boundary layer length

Back to the viscous case : vertical generator

We consider here the case where $\omega = 1$ and $m = 1$. (generator set-up vertically)

$$\mathrm{Fr}^2\left(1+k^2\right)\left(1-i\frac{1+k^2}{\mathrm{Re}}\right)-k^2=0$$

Two branches :

$$k^2=\frac{i\mathrm{Re}}{2k_0^2}\left(1+\frac{2ik_0^2}{\mathrm{Re}}-\sqrt{1+4i\frac{\left(1+k_0^2\right)k_0^2}{\mathrm{Re}}}\right)$$

where $k_0 = \dfrac{\mathrm{Fr}}{\sqrt{1-\mathrm{Fr}^2}}$ is the inviscid value for k.

Large Reynold number limit

We now consider large values of the Reynold number. The solution then writes :

$$k_w = \pm \left(k_0 - \frac{ik_0 \left(1 + k_0^2\right)^2}{2\mathrm{Re}} \right)$$

$$m_{bl} = \pm (1 + i) \sqrt{\frac{\mathrm{Re}}{2k_0^2}}$$

where $k_0 = \dfrac{\mathrm{Fr}}{\sqrt{1 - \mathrm{Fr}^2}}$ is the inviscid value for k.

Few remarks :

- k_w : Propagating branche
- $L_{\mathrm{Re}} = \dfrac{2\mathrm{Re}}{k_0 \left(1 + k_0^2\right)^2}$: Penetration length for the wave beam
- k_{bl} : Boundary layer branche
- $\delta_{\mathrm{Re}} = \sqrt{2k_0^2/\mathrm{Re}}$: Boundary layer length

Streaming

Wave-mean flow decomposition

- The averaging operator is defined by $\overline{u} = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} u \, \mathrm{d}x\mathrm{d}t$.
- The wave-mean decomposition is defined by $(u, w, b, P) = \left(\overline{u}, \overline{w}, \overline{b}, \overline{P}\right) + (u', w', b', P')$
- Taking the mean part of the equations of motion leads to :

$$\partial_t \overline{u} - \frac{1}{\mathrm{Re}} \Delta \overline{u} = -\partial_z \overline{u' w'}$$

and $\overline{w}, \overline{b} = 0$.

- Streaming is induced by the waves from the Reynold stress $\partial_z \overline{u' w'}$

Waves equations

$$
\begin{cases}
\partial_t u' + \overline{u}\partial x u' + w'\partial_z \overline{u} + \boxed{u'\partial_x u' + w'\partial_z u' - \partial_z \overline{u'w'}} & = -\partial_x P' + \frac{1}{\mathrm{Re}}\Delta u' \\
\partial_t w' + \overline{u}\partial x w' + \boxed{u'\partial_x w' + w'\partial_z w' - \partial_z \overline{w'^2}} & = -\partial_z P' + \frac{1}{\mathrm{Fr}^2} b' + \frac{1}{\mathrm{Re}}\Delta w' \\
\partial_t b' + \overline{u}\partial_x b' + \boxed{u'\partial_x b' + w'\partial_z b'} + w' & = 0 \\
\partial_x u' + \partial_z w' & = 0
\end{cases}
$$

Non-linear terms responsible of the PSI.

Waves in shear-flows : WKB solutions

WKB ansatz and linearization

We introduce a small dimensionles parameter $a \ll 1$ and assume that the mean-flow writes $U = U(Z, T)$ where $(Z, T) = a(z, t)$.
WKB ansatz :

$$\begin{bmatrix} u \\ w \\ b \\ P \end{bmatrix} = \sum_{j=0}^{\infty} a^{j+1} \begin{bmatrix} u_j(Z, T) \\ w_j(Z, T) \\ b_j(Z, T) \\ P_j(Z, T) \end{bmatrix} \exp\left(i \frac{\Phi(Z, T)}{a} - ix \right)$$

Injecting this ansatz into the wave equation and collecting the leading order terms in a leads to :

$$\mathbf{M} \begin{bmatrix} u_0 \\ w_0 \\ b_0 \\ P_0 \end{bmatrix} + a \left(M \begin{bmatrix} u_1 \\ w_1 \\ b_1 \\ P_1 \end{bmatrix} + \begin{bmatrix} \partial_T u_0 + w_0 \partial_Z U + \frac{i}{\mathrm{Re}} (u_0 \partial_Z m + 2m \partial_Z u_0) \\ \partial_T w_0 + \partial_Z P_0 + \frac{i}{\mathrm{Re}} (w_0 \partial_Z m + 2m \partial_Z w_0) \\ \partial_T b_0 \\ \partial_Z w_0 \end{bmatrix} \right) = 0$$

WKB ansatz and linearization

$$\mathbf{M}\begin{bmatrix} u_0 \\ w_0 \\ b_0 \\ P_0 \end{bmatrix} + a\left(M\begin{bmatrix} u_1 \\ w_1 \\ b_1 \\ P_1 \end{bmatrix} + \begin{bmatrix} \partial_T u_0 + w_0 \partial_Z U + \frac{i}{\mathrm{Re}}\left(u_0 \partial_Z m + 2m\partial_Z u_0\right) \\ \partial_T w_0 + \partial_Z P_0 + \frac{i}{\mathrm{Re}}\left(w_0 \partial_Z m + 2m\partial_Z w_0\right) \\ \partial_T b_0 \\ \partial_Z w_0 \end{bmatrix}\right) = 0$$

With :

$$\mathbf{M} = \begin{bmatrix} i\left(\omega - U\right) + \frac{1+m^2}{\mathrm{Re}} & 0 & 0 & -i \\ 0 & i\left(\omega - U\right) + \frac{1+m^2}{\mathrm{Re}} & -\frac{1}{\mathrm{Fr}^2} & -im \\ 0 & 1 & i\left(\omega - U\right) & 0 \\ -i & -im & 0 & 0 \end{bmatrix}$$

$$\omega = \partial_T \Phi$$

$$m = -\partial_Z \Phi$$

Order zero

$$
\mathbf{M}\begin{bmatrix} u_0 \\ w_0 \\ b_0 \\ P_0 \end{bmatrix} = 0 \implies \begin{cases} \det \mathbf{M} & = 0 \\ \begin{bmatrix} u_0 \\ w_0 \\ b_0 \\ P_0 \end{bmatrix} & = \phi_0 \mathcal{P} \\ \mathcal{P} & = \begin{bmatrix} U - \omega \\ \dfrac{\omega - U}{m} \\ \dfrac{i}{m} \\ -(\omega - U)^2 \left(1 - i \dfrac{1 + m^2}{\mathrm{Re}\,(\omega - U)}\right) \end{bmatrix} \end{cases}
$$

Order one

$$
\begin{bmatrix} U-\omega \\ \dfrac{\omega-U}{m} \\ -\dfrac{i}{m\mathrm{Fr}^2} \\ -(\omega-U)^2\left(1-i\dfrac{1+m^2}{\mathrm{Re}\,(\omega-U)}\right) \end{bmatrix} \cdot \begin{bmatrix} \partial_T u_0 + w_0\partial_Z U + \dfrac{i}{\mathrm{Re}}\left(u_0\partial_Z m + 2m\partial_Z u_0\right) \\ \partial_T w_0 + \partial_Z P_0 + \dfrac{i}{\mathrm{Re}}\left(w_0\partial_Z m + 2m\partial_Z w_0\right) \\ \partial_T b_0 \\ \partial_Z w_0 \end{bmatrix} = 0
$$

$$
\implies \mathcal{F}\left[U, \phi_0\right] = 0
$$

where $\mathcal{F}$ is differential operator (linear in ϕ_0).

Inviscid limit

For $\mathrm{Re} = \infty$, the last equation can be simplified into the **wave activity equation** :

$$\partial_T A + \partial_Z \left(A w_g \right) = 0$$

with $A = E / \left(\omega - U \right)$ and $E = \frac{1}{4} \left(\left| u_0 \right|^2 + \left| w_0 \right|^2 + \mathrm{Fr}^2 \left| b_0 \right|^2 \right)$.
Also $\overline{u_0 w_0} = A k w_g$ such that at leading order :

$$\partial_z \overline{u_0 w_0} = -\partial_t \left(k A \right)$$

Injecting this result into the mean-flow evolution equation leads to

$$\partial_t \left(U - k A \right) = 0$$

This result is known as the **non-acceleration theorem**.

Boundary conditions and full computation of the wave field

Boundary condition : transverse oscillation

Let us consider a horizontally set-up generator. The fluid is viscous wih a **no-slip** boundary condition :

$$\mathbf{u}\left(x, z = h_b\left(x, t\right), t\right) = \partial_t h_b\left(x, t\right)\mathbf{e}_z$$

If we now suppose that $||h_b|| \ll 1$, we perform the wave-decomposition and linearize this boundary condition to get :

$$\begin{cases} \overline{u}\left(z = 0, t\right) & = 0 \\ u'\left(x, z = 0, t\right) & = 0 \\ w'\left(x, z = 0, t\right) & = \partial_t h_b\left(x, t\right) \end{cases}$$

Boundary condition : Progressive wave

Here we consider $h_b(x,t) = \epsilon \mathcal{R}\mathrm{e}\left[e^{i(t-x)}\right]$ corresponding to $(\omega, k) = (1,1)$. Considering waves propagating upwardly, the we retain the solution for m with a negative imaginary part only. We first ignore the mean-flow.

$$\begin{cases} \tilde{w}'(z) & = a_w e^{-im_w z} + a_{bl} e^{-im_{bl} z} \\ \partial_z \tilde{w}'(z=0) & = 0 \\ \tilde{w}'(z=0) & = i\epsilon \end{cases} \implies \begin{cases} a_w & = i\epsilon \dfrac{m_{bl}}{m_{bl} - m_w} \\ a_{bl} & = i\epsilon \dfrac{m_w}{m_w - m_{bl}} \end{cases}$$

Small caps

This frame uses the `smallcaps` titleformat.

Potential Problems

Be aware, that not every font supports small caps. If for example you typeset your presentation with pdfTeX and the Computer Modern Sans Serif font, every text in smallcaps will be typeset with the Computer Modern Serif font instead.

all small caps

This frame uses the `allsmallcaps` titleformat.

Potential problems

As this titleformat also uses smallcaps you face the same problems as with the `smallcaps` titleformat. Additionally this format can cause some other problems. Please refer to the documentation if you consider using it.

As a rule of thumb: Just use it for plaintext-only titles.

ALL CAPS

This frame uses the `allcaps` titleformat.

Potential Problems

This titleformat is not as problematic as the `allsmallcaps` format, but basically suffers from the same deficiencies. So please have a look at the documentation if you want to use it.

Elements

Typography

```
The theme provides sensible defaults to
\emph{emphasize} text, \alert{accent} parts
or show \textbf{bold} results.
```

becomes

The theme provides sensible defaults to *emphasize* text, accent parts or show **bold** results.

Font feature test

- Regular
- *Italic*
- SmallCaps
- **Bold**
- ***Bold Italic***
- **Bold SmallCaps**
- `Monospace`
- *`Monospace Italic`*
- **`Monospace Bold`**
- ***`Monospace Bold Italic`***

Lists

Items

- Milk
- Eggs
- Potatos

Enumerations

1. First,
2. Second and
3. Last.

Descriptions

PowerPoint Meeh.

Beamer Yeeeha.

Animation

- This is important

Animation

- This is important
- Now this

Animation

- This is important
- Now this
- And now this

Animation

- This is really important
- Now this
- And now this

Figures

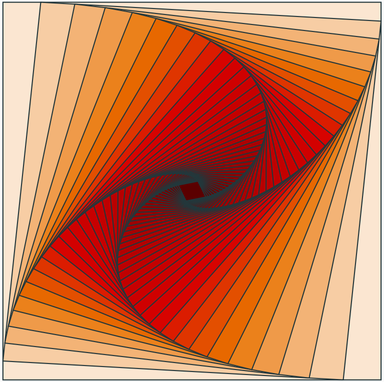


Figure 1: Rotated square from texample.net.

Tables

Table 1: Largest cities in the world (source: Wikipedia)

City	Population
Mexico City	20,116,842
Shanghai	19,210,000
Peking	15,796,450
Istanbul	14,160,467

Blocks

Three different block environments are pre-defined and may be styled with an optional background color.

Default

Block content.

Alert

Block content.

Example

Block content.

Default

Block content.

Alert

Block content.

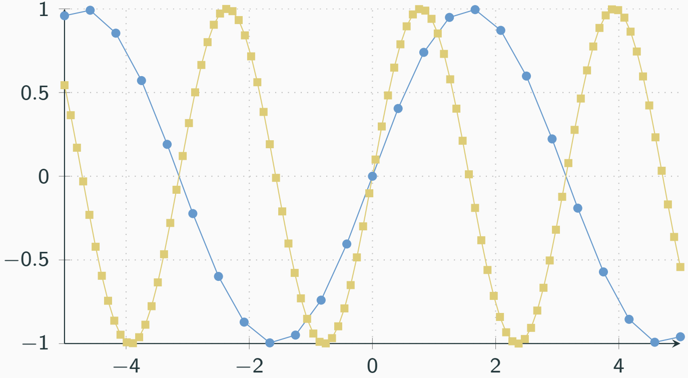
Example

Block content.

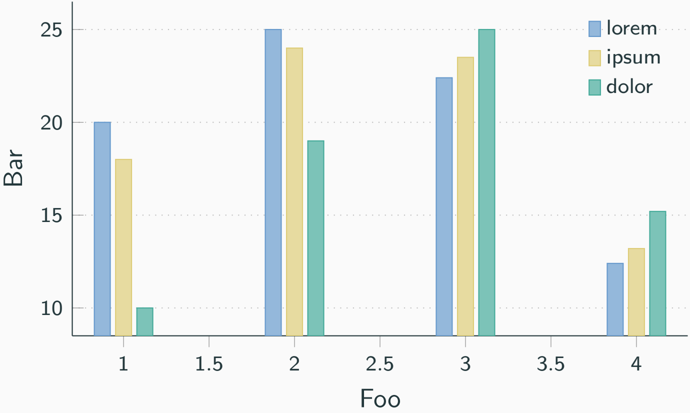
Math

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

Line plots



Bar charts



Quotes

Veni, Vidi, Vici

Frame footer

metropolis defines a custom beamer template to add a text to the footer. It can be set via

```
\setbeamertemplate{frame footer}{My custom footer}
```

References

Some references to showcase [allowframebreaks] [**?**, **?**, **?**, **?**, **?**]

Conclusion

Summary

Get the source of this theme and the demo presentation from

github.com/matze/mtheme

Questions?

Backup slides

Sometimes, it is useful to add slides at the end of your presentation to refer to during audience questions.

The best way to do this is to include the `appendixnumberbeamer` package in your preamble and call `\appendix` before your backup slides.

metropolis will automatically turn off slide numbering and progress bars for slides in the appendix.

References I