

# Generation of viscous internal gravity waves by a moving boundary

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## 1 Introduction

Internal gravity waves are a field of increasing importance in fluid mechanics. The underlying reason is the major role they play in many crucial phenomena appearing in the ocean and the atmosphere where the stratification cannot be ignored. They have been extensively studied theoretically, numerically and experimentally. The experimental studies of internal gravity waves have made great progress recently. It started from the pioneer work of Long and Baines in 1955 and 1977 [9, 2] where the waves are generated by dragging a smooth bump at the bottom of a tank or Plumb and McEwan in 1978 [12] where a standing wave is excited by pistons making a rubber membrane oscillating at the bottom of a annular tank. In 2006, Gostiaux et al [7] proposed a new generator of internal gravity waves allowing for excitation of propagating monochromatic waves. This last technique led to a new class of internal gravity waves experiments (see [10, 4]). The generation by drag is still used in recent studies (see e.g. [13]).

The inviscid problem has been extensively studied (see [6]). If the inviscid theory is relevant in many oceanic and atmospheric problems, the large difference in spatial and time scaling between the geophysical internal gravity waves and their laboratory analogues makes it dodgy to ignore the dissipative processes. The theory of the quasi-biennial oscillation occurring in the equatorial stratosphere needs a dissipative process to attenuate the waves beams coming from tropical troposphere (see [8, 11]). In the atmospheric context, this dissipation comes mainly from the Newtonian cooling parametrised by a drag force in the buoyancy equation (see e.g. [5]). However, in the laboratory analogue performed by [12] the dominant dissipation process comes from the viscosity. The main difference between these two features is that the latter one involves viscous boundary layers to match the propagating waves with the boundary condition at level of the waves generator. At quasi-linear level those boundary layers can involve an important streaming close to the generator.

In high-resolution numerical simulations of stratified fluids motion over topography, the sub-grid scale dynamics is often parametrized by an effective viscosity several order of magnitude higher than the actual viscosities involved in the simulated phenomena. Therefore, viscous effects might have to be taken in account, especially if the resolution is high enough compared to the viscous boundary layer length scale, even if a free-slip boundary condition is chosen for the velocity field.

Plan du papier :

- Section 1 : Modele, equation primitive, approximation de boussinesq
- Section 2 : Zonally symmetric waves and mean-flow interaction
- Section 3 : WKB asymptotics and wave properties
- Section 4 : Boundary condition and reynold stress
- Section 5 : Applications : Lee waves, QBO Plumb
- Section 6 : Simulations
- Section 7 : Discussion/Conclusion

## 2 The Boussinesq model

We consider a stratified fluid within a 2-dimensional domain periodic in the horizontal  $x$ -direction (this direction will be referred to as “zonal” in the remaining of the paper) and semi-infinite in the vertical  $z$ -direction. We consider the incompressible Boussinesq model. Two types of dissipative mechanism will be considered: viscosity and Newtonian

Studies	Re	Fr	$\epsilon$	Pr	Generation	Type
[3]	100 – 1000	0.1	0.1	1000	ridge drag	Experimental
[13]	400 – 87000	0.1 – 34	0.1 – 1.	1000	ridge drag	Experimental
[10]	73	0.82	0.4	1000	oscillating cams	Experimental
[12]	1500 – 3000	0.02 – 0.04	0.05	1000	oscillating membranes	Experimental
[1]	650	0.1	0.06	1000	oscillating membranes	Experimental

Table 1: Referencing of some characteristic dimensionless number in internal gravity wave experiments.

cooling. The equation of state is linear. The bottom boundary is an undulating line located on average at  $z = 0$ . We are interested in the dynamics of the buoyancy anomaly defined by  $b = g \frac{(\rho_c - \rho)}{\rho_c} - \int^z N^2$  where  $\rho_c$  is a reference value for the density field  $\rho$  and  $N$  is the Brünt-Väisälä frequency (height-dependent in general and always positive such that the rest state,  $b = 0$ , is stable). The parameters of the problem are the characteristic amplitude, wave number and angular momentum of the bottom boundary undulation  $(h_b, k, \omega)$ , the kinematic viscosity  $\nu$  and the bottom value of the Brünt-Väisälä frequency  $N_0 = N(z = 0)$ . We denote  $c = \omega/k$  the associated phase-speed. We can define three independent dimensionless parameters:

$$\begin{cases} \text{Reynold number :} & \text{Re} = \frac{c}{\nu k} \\ \text{Froude number :} & \text{Fr} = \frac{\omega}{N_0} \\ \text{Bottom boundary aspect ratio :} & \epsilon = h_b k \end{cases} \quad (1)$$

## 2.1 Equations

The flow obeys the primitive equations in the Boussinesq approximation. In the adimensionalized form, they writes:

$$\begin{cases} \partial_t u + u \partial_x u + w \partial_z u & = -\partial_x P + \frac{1}{\text{Re}} \Delta u \\ \partial_t w + u \partial_x w + w \partial_z w & = -\partial_z P + \frac{1}{\text{Fr}^2} b + \frac{1}{\text{Re}} \Delta w \\ \partial_t b + u \partial_x b + w \partial_z b + n^2(z) w & = -\alpha b \\ \partial_x u + \partial_z w & = 0 \end{cases} \quad (2)$$

where  $\Delta = \partial_{xx} + \partial_{zz}$  is the Laplacian differential operator,  $(u, w)$  are the components of the 2-dimensional velocity vector field,  $P$  is the pressure anomaly field,  $b$  is the buoyancy anomaly field,  $n$  is the rescaled Brünt-Väisälä frequency field ( $n(0) = 1$  and  $n(z > 0) = 1$  if the stratification is linear) and  $\alpha$  is a parameter parameterizing the relaxation of the buoyancy anomaly toward the rest state. The first equation will be referred to as the zonal momentum equation, the second one as the vertical momentum equation, the third one as the buoyancy equation and the last one as the incompressibility condition.

### Boundary conditions

The domain is  $2\pi$ -periodic in the zonal direction. The equation of the bottom boundary is given by  $z(x, t) = \epsilon h_b(x, t)$  with  $\int_0^{2\pi} h_b(x, t) dx = 0$ . All the following equation for the boundary conditions have to be evaluated in  $z = \epsilon h_b(x, t)$ .

If  $\text{Re} = \infty$ , we consider an impermeability condition at the bottom boundary. It amounts to fix the normal velocity at the boundary to be equal to its normal velocity:

$$w - w_{h_b} - \epsilon(u - u_{h_b}) \partial_x h_b = 0 \quad (3)$$

where  $(u_{h_b}, w_{h_b})$  are the component of the velocity of the bottom boundary.

If  $\text{Re} < \infty$ , we will either consider a no-slip or a free-slip boundary condition :

- No-slip

$$(u, w) = (u_{h_b}, w_{h_b}) \quad (4)$$

- Free-slip

$$\begin{cases} w - w_{h_b} - \epsilon(u - u_{h_b}) \partial_x h_b & = 0 \\ (\partial_z u + \partial_x w) \left(1 - (\epsilon \partial_x h_b)^2\right) - 2\epsilon \partial_x h_b (\partial_x u - \partial_z w) & = 0 \end{cases} \quad (5)$$

the first equation is the impermeability condition, the second one is the condition of vanishing tangential constraint.

### 3 Zonally symmetric wave-mean flow interaction

Let us introduce the zonal average operator:

$$\bar{\phi}(z, t) = \frac{1}{2\pi} \int_0^{2\pi} dx \phi(x, z, t), \quad (6)$$

where  $\phi$  is any field. We define the waves-mean flow decomposition by

$$(u, w, b, P) = (\bar{u}, \bar{w}, \bar{b}, \bar{P}) + (u', w', b', P'), \quad (7)$$

the fields  $(\bar{u}, \bar{w}, \bar{b}, \bar{P})$  will be referred to as the mean-flow and the fields  $(u', w', b', P')$  as the waves.

We can notice that if there is no Newtonian cooling (i.e.  $\alpha = 0$  in Eq. (2)), any integral over the domain of the total buoyancy  $b + \int^z n^2$  is conserved by the dynamics. It implies that the total buoyancy field has to be a rearrangement of its initial condition. This means that the mean buoyancy field is of order two in terms of the wave amplitude. In the following we will consider small amplitude waves. We, hereby, ignore the equation of evolution of the mean buoyancy field  $\bar{b}$  and we make the approximation  $\bar{b} = 0$  in the following. The averaging of the continuity equation in (2) leads to  $\bar{w} = 0$ . Consequently, the averaging of the vertical momentum equation simply leads to a diagnostic equation for the mean pressure field  $\bar{P}$ . The equation of interest is the averaged zonal momentum equation :

$$\partial_t \bar{u} - \frac{1}{\text{Re}} \partial_{zz} \bar{u} = -\partial_z \overline{u'w'}. \quad (8)$$

The important term is the Reynold stress  $\partial_z \overline{u'w'}$ . The wave equation need to be solved to compute it. Subtracting the mean-flow equations to (2), we obtain the fully non-linear wave equation:

$$\begin{cases} \partial_t u' + \bar{u} \partial_x u' + w' \partial_z \bar{u} + \mathcal{B}(\mathbf{u}', u') &= -\partial_x P' + \frac{1}{\text{Re}} \Delta u' \\ \partial_t w' + \bar{u} \partial_x w' + \mathcal{B}(\mathbf{u}', w') &= -\partial_z P' + \frac{1}{\text{Fr}^2} b' + \frac{1}{\text{Re}} \Delta w' \\ \partial_x u' + \partial_z w' &= 0 \\ \partial_t b' + \bar{u} \partial_x b' + n^2(z) w' + \mathcal{B}(\mathbf{u}', b') &= -\alpha b' \end{cases} \quad (9)$$

where  $\mathcal{B}(\mathbf{u}', \phi') = u' \partial_x \phi' + w' \partial_z \phi' - \overline{u' \partial_x \phi' + w' \partial_z \phi'}$  is a bilinear operator with zero zonal average.

In principle, we perform the same flow decomposition at the boundaries in order to get the boundary condition for the wave and the mean part. The resulting equations are messy in general but there expressions are greatly simplified in a far-field approximation when the domain geometry is asymptotically flat in the limit  $\epsilon \ll 1$ .

### 4 Internal gravity waves within a weakly sheared mean-flow

We consider here a single topographic mode  $k = \pm 1$  and a single phase speed  $c = \pm 1$ . We consider the limit a small topographic aspect ratio (i.e.  $\epsilon \ll 1$ ). In this limit we can consider an asymptotically square domain with a bottom boundary condition expressed at  $z = 0$  in terms of power of  $\epsilon$ :

- Inviscid impermeability condition

$$w - w_{h_b} + \epsilon (h_b \partial_z w - (u - u_{h_b}) \partial_x h_b) = o(\epsilon) \quad (10)$$

- Viscous boundary conditions :

**Free-slip**

$$\partial_z u + \partial_x w + \epsilon (h_b (\partial_{zz} u + \partial_{xz} w) - 2\partial_x h_b (\partial_x u - \partial_z w)) = o(\epsilon) \quad (11)$$

**No-slip**

$$(u, w) - (u_{h_b}, w_{h_b}) + \epsilon h_b (\partial_z u, \partial_z w) = o(\epsilon) \quad (12)$$

Let us introduce a small parameter  $\epsilon^2 \ll a \ll 1$ . We will see later how it scales with  $\epsilon$ . Let us introduce a slow variable  $Z = az$ . We now assume that the mean flow  $\bar{u}$  depends only on  $Z$  in Eq. (9), we assume the same for  $n$ . For the wave field, this means that the mean flow appears weakly sheared and frozen in time. We will see later under which condition this hypothesis holds.

Let us introduce the WKB expansion of a stationary solution of the wave equation (9) :

$$\begin{bmatrix} u' \\ w' \\ b' \\ p' \end{bmatrix} = \epsilon \text{Re} \left\{ \sum_{j=0}^{+\infty} a^j \begin{bmatrix} \tilde{u}_j(Z) \\ \tilde{w}_j(Z) \\ \tilde{b}_j(Z) \\ \tilde{p}_j(Z) \end{bmatrix} \exp \left( ik(ct - x) - \frac{i\Phi(Z)}{a} \right) \right\}. \quad (13)$$

The wave amplitude is scaling with the bottom boundary aspect ratio  $\epsilon$ .  $\Phi(Z)$  is the phase of the wave. Injecting this expansion into the previous equation and collecting the leading order in  $a$  and  $\epsilon$  leads :

$$\epsilon \mathbf{M} \begin{bmatrix} \tilde{u}_0 \\ \tilde{w}_0 \\ \tilde{b}_0 \\ \tilde{p}_0 \end{bmatrix} + \epsilon a \left( \mathbf{M} \begin{bmatrix} \tilde{u}_1 \\ \tilde{w}_1 \\ \tilde{b}_1 \\ \tilde{p}_1 \end{bmatrix} + \begin{bmatrix} \tilde{w}_0 \partial_Z \bar{u} + \frac{i}{\text{Re}} (\tilde{u}_0 \partial_Z m + 2m \partial_Z \tilde{u}_0) \\ \partial_Z \tilde{P}_0 + \frac{i}{\text{Re}} (\tilde{w}_0 \partial_Z m + 2m \partial_Z \tilde{w}_0) \\ 0 \\ \partial_Z \tilde{w}_0 \end{bmatrix} \right) + o(\epsilon a) = 0 \quad (14)$$

with

$$\mathbf{M} = \begin{bmatrix} ik(c - \bar{u}) \left( 1 - i \frac{k^2 + m^2}{\text{Re}k(c - \bar{u})} \right) & 0 & 0 & -ik \\ 0 & ik(c - \bar{u}) \left( 1 - i \frac{k^2 + m^2}{\text{Re}k(c - \bar{u})} \right) & -\frac{1}{\text{Fr}^2} & -im \\ 0 & n^2 & ik(c - \bar{u}) \left( 1 - i \frac{\alpha}{k(c - \bar{u})} \right) & 0 \\ -ik & -im & 0 & 0 \end{bmatrix} \quad \text{and } m = \partial_Z \phi \quad (15)$$

This implies

$$\det \mathbf{M} = 0 \implies \text{Fr}^2 (c - \bar{u})^2 \left( 1 - i \frac{(k^2 + m^2)}{\text{Re}k(c - \bar{u})} \right) \left( 1 - i \frac{\alpha}{k(c - \bar{u})} \right) (k^2 + m^2) = n^2 \quad (16)$$

and

$$\begin{bmatrix} \tilde{u}_0 \\ \tilde{w}_0 \\ \tilde{b}_0 \\ \tilde{p}_0 \end{bmatrix} = \phi_0(Z) \mathcal{P}(Z) \quad (17)$$

where  $\tilde{\phi}_0$  measures the global amplitude of the wave and  $\mathcal{P}$  is the polarization of the wave given by

$$\mathcal{P} = \begin{bmatrix} \frac{c - \bar{u}}{k} \\ -\frac{k}{m} (c - \bar{u}) \\ i \\ -\frac{m \left( 1 - i \frac{\alpha}{k(c - \bar{u})} \right)}{(c - \bar{u})^2 \left( 1 - i \frac{(k^2 + m^2)}{\text{Re}k(c - \bar{u})} \right)} \end{bmatrix}. \quad (18)$$

The equation (16) is the dispersion relation equation of the wave which as to be satisfied locally. For every value of  $Z$ , it gives  $m(Z)$ . This equation has multiple solution. Depending on the type of wave we want to study, we consider one of the value. The complete linear solution of the problem is a linear combination of all the solution satisfying the boundary conditions.

To get an evolution equation for  $\phi_0$ , we project the order one terms on

$$\begin{bmatrix} \frac{c - \bar{u}}{k} \\ -\frac{k}{m} (c - \bar{u}) \\ in^2 \\ \frac{\text{Fr}^2 m \left( 1 - i \frac{\alpha}{k(c - \bar{u})} \right)}{(c - \bar{u})^2 \left( 1 - i \frac{(k^2 + m^2)}{\text{Re}k(c - \bar{u})} \right)} \end{bmatrix}. \quad (19)$$

This cancel out the order one terms of the wave and leave us with an equation for the amplitude of the zero-th order wave  $\phi_0(Z)$  :

$$\frac{1}{\phi_0(c-\bar{u})} \partial_Z (\phi_0(c-\bar{u})) = \frac{2(k^2 - m^2) + ik \operatorname{Re}(c - \bar{u})}{4m^2(2(k^2 + m^2) + ik \operatorname{Re}(c - \bar{u}))} \partial_Z m^2. \quad (20)$$

This last equation has to be solved for every solution  $m(Z)$  of the dispersion relation (16).

#### 4.1 Dispersion relation for $\operatorname{Re} = \infty$ and $\alpha > 0$ :

Without viscous dissipation, the linear wave obeys the following dispersion relation locally :

$$\operatorname{Fr}^2 (c - \bar{u})^2 \left( 1 - i \frac{\alpha}{k(c - \bar{u})} \right) (k^2 + m^2) = n^2. \quad (21)$$

This is an 2nd-order equation. There is only one propagative branch for  $m$ . In the limit of  $\alpha \ll 1$ , we have :

$$m = \pm \left( m_0 + i \frac{\alpha n^2}{2k(c - \bar{u})^3 \operatorname{Fr}^2 m_0} \right) + o(\alpha) \quad (22)$$

where

$$m_0 = \sqrt{\frac{n^2}{(c - \bar{u})^2 \operatorname{Fr}^2} - k^2}. \quad (23)$$

is the positive inviscid solution for a purely propagating unattenuated wave. We assume that the wave remains propagating such that  $(c - \bar{u})^2 \operatorname{Fr}^2 k^2 < n^2$  is always verified. The sign depend on the direction of propagation of the considered wave. We can extract an attenuation length scale for the wave :

$$L_\alpha = \frac{2|k| |c - \bar{u}|^3 \operatorname{Fr}^2 m_0}{\alpha n^2}. \quad (24)$$

The definition of the vertical group velocity of an inviscid wave is naturally defined by  $w_g = -k/\partial_c m_0$ . We extend this definition to attenuated waves :

$$w_g \equiv \frac{k}{\partial_c \mathbb{R}[m]}. \quad (25)$$

In the limit  $\alpha \ll 1$ , it leads to the inviscid group velocity :

$$w_g = \mp \frac{k m_0}{k^2 + m_0^2} (c - \bar{u}) + o(\alpha). \quad (26)$$

The expression of the group velocity will be useful later to compare the time scale of evolution of the mean flow with the time scale of the propagation of the wave.

We can now consider back the equation for amplitude of the wave (20) in the limit  $\alpha \ll 1$ . It simplifies into :

$$\partial_Z \left( \frac{\phi_0^2 (c - \bar{u})^2}{m_0} \right) = o(\alpha). \quad (27)$$

#### 4.2 Dispersion relation for $\alpha = 0$ and $\operatorname{Re} < \infty$ :

Without Newtonian cooling, the linear wave obeys the following dispersion relation locally :

$$\operatorname{Fr}^2 (c - \bar{u})^2 \left( 1 - i \frac{(k^2 + m^2)}{\operatorname{Re} k (c - \bar{u})} \right) (k^2 + m^2) = n^2. \quad (28)$$

This an 4th- order equation. There is two branches for  $m$ . A propagative one and a boundary layer one. In the limit of  $\operatorname{Re} \gg 1$ , we have :

$$m_{\text{Prop}} = \pm \left( m_0 + \frac{in^4}{2k(c - \bar{u})^5 \operatorname{Fr}^4 \operatorname{Re} m_0} \right) + o(\operatorname{Re}^{-1}), \quad (29)$$

$$m_{\text{bl}} = \pm (1 - \operatorname{sign}(k(c - \bar{u}))) i \sqrt{\frac{|c - \bar{u}| |k| \operatorname{Re}}{2}} + o(\sqrt{\operatorname{Re}}). \quad (30)$$

We assume that the waves are always non-evanescent (i.e.  $(c - \bar{u})^2 \text{Fr}^2 k^2 < n^2$ ).  $m_{\text{Prop}}$  corresponds to the propagating branch. Its attenuation length scale is given by

$$L_{\text{Re}} = \frac{2k(c - \bar{u})^5 \text{Fr}^4 \text{Re} m_0}{n^2}. \quad (31)$$

$m_{\text{bl}}$  correspond to the boundary layer branch. The existence of this supplementary branch allows the matching of the propagative waves with the bottom boundary condition. The characteristic length of this boundary layer is given by :

$$\delta_{\text{Re}} = \sqrt{\frac{2}{|c - \bar{u}| |k| \text{Re}}}. \quad (32)$$

$$m_{\text{Prop}}^2 = \frac{1}{(c - \bar{u})^2 \text{Fr}^2} - k^2 + \frac{i}{(c - \bar{u})^5 \text{Fr}^4 k \text{Re}} + o(\text{Re}^{-1}) \quad (33)$$

$$m_{\text{bl}}^2 = -i(c - \bar{u}) k \text{Re} + o(\text{Re}) \quad (34)$$

Then for  $m_{\text{Prop}}$ , we have at leading order in  $\text{Re}$ :

$$\partial_Z \left( \frac{\phi_{0,\text{Prop}}^2 (c - \bar{u})^2}{\sqrt{\frac{1}{(c - \bar{u})^2 \text{Fr}^2} - k^2}} \right) = 0 \quad (35)$$

and for  $m_{\text{bl}}$ , we have at leading order in  $\text{Re}$ :

$$\partial_Z (\phi_{0,\text{bl}} m_{\text{bl}}) = 0 \quad (36)$$

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