

Chapter 1

Vectors

1.1 Introduction

Definition

A vector is an ordered collection of n numbers

Definition

Let us consider vector $\underline{u} \in \mathbb{R}^n$. The i -th component of vector

$$\underline{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

is u_i

Definition

Let us consider vectors $\underline{u} \in \mathbb{R}^n$ and $\underline{v} \in \mathbb{R}^n$. Vector $\underline{w} \in \mathbb{R}^n$ is a sum of \underline{u} and \underline{v} , $\underline{w} = \underline{u} + \underline{v}$, if $w_i = u_i + v_i$ for all $i = 1, \dots, n$

Definition

1. Vectors $\underline{u} \in \mathbb{R}^n$ and $\underline{v} \in \mathbb{R}^n$ are equal, if $u_i = v_i$ for all $i = 1, \dots, n$
2. A scalar is just another name for real number
3. Let us consider a scalar $\alpha \in \mathbb{R}$ and vector $\underline{u} \in \mathbb{R}^n$. A product of α and \underline{u} is defined as:

$$\alpha \underline{u} = \alpha \cdot \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} \alpha \cdot u_1 \\ \vdots \\ \alpha \cdot u_n \end{pmatrix}$$

Definition

Let us consider scalars α and β , and vectors $\underline{u} \in \mathbb{R}^n$ and $\underline{v} \in \mathbb{R}^n$. A sum of $\alpha \cdot \underline{u} + \beta \cdot \underline{v}$ is called a linear combination of vectors \underline{u} and \underline{v} .

Definition

Vector $\underline{u} \in \mathbb{R}^n$ is called a zero vector if all $u_i = 0$, $i = 1, \dots, n$. The zero vector is often written as $\underline{0} \in \mathbb{R}^n$.

1.2 Vector Representations and Operations**1.3 Dot Product (Scalar product)****Definition**

Let us consider two vectors $\underline{u} \in \mathbb{R}^n$ and $\underline{v} \in \mathbb{R}^n$. The dot (or scalar) product of vectors \underline{u} and \underline{v} is defined as

$$\langle \underline{u}, \underline{v} \rangle = u_1 v_1 + u_2 v_2 + \dots + u_n v_n = \sum_{i=1}^n u_i v_i$$

1.4 Length of a Vector**Definition**

The length of vector $\underline{u} \in \mathbb{R}^n$, $\|\underline{u}\|$, is defined as $\|\underline{u}\| = \sqrt{\langle \underline{u}, \underline{u} \rangle}$. Sometimes it is also called the Euclidian norm of \underline{u} .

1.5 Unit Vectors**Definition**

A vector with length equal to 1 is called a unit vector

1.6 Angle between Vectors**1.7 Cauchy Schwarz inequality**

Chapter 2

Matrices

2.1 Matrix Operations

Definition

Let us consider matrices $A \in \mathbb{R}^{n,m}$ and $B \in \mathbb{R}^{n,m}$ where n = rows, m = columns. Matrix $C \in \mathbb{R}^{n,m}$ is a sum of A and B , $C = A + B$, if $C_{ij} = A_{ij} + B_{ij}$ for all $i = 1, \dots, n, j = 1, \dots, m$

Definition

A product of a scalar α and a matrix $A \in \mathbb{R}^{n,m}$ is defined as $(\alpha A)_{ij} = \alpha \cdot A_{ij}$, $\forall i = 1, \dots, n; j = 1, \dots, m$.

2.2 Matrix-Matrix Multiplication

Definition

Let us consider matrix $A \in \mathbb{R}^{n,m}$ and $B \in \mathbb{R}^{m,l}$. Then $C = A \cdot B$ is an n by l matrix, $C \in \mathbb{R}^{n,l}$ such that

$$C_{ij} = \sum_{k=1}^m a_{ik} b_{kj}$$

2.3 Linear System of Equations

2.4 Inverse of a Matrix

Definition

Let us consider a matrix $A \in \mathbb{R}^{n,n}$ (square matrix). Matrix $B \in \mathbb{R}^{n,n}$ is called an inverse of A , if

$$A \cdot B = I \quad \text{AND} \quad B \cdot A = I$$

(Both conditions are vital)

2.5 Special Matrices**2.6 Elementary Transition Matrices****Definition**

We can define the elementary transition matrix $I_{pq} \in \mathbb{R}^{n,n}$

$$(I_{pq}) = \begin{cases} 1 & i = p, q = j \\ 0 & \text{otherwise} \end{cases}$$

If we take a matrix $A \in \mathbb{R}^{n,n}$ then when calculating I_{pq} we take row q of A , put it into row p , replace everything else with 0.

We can also define:

$$\begin{aligned} E_{pq}(l) &= I + l \cdot I_{pq}, l \in \mathbb{R} - \text{scalar} \\ E_{pq}(l) \cdot A &= (I + lI_{pq}) \cdot A = A + l \cdot I_{pq}A \end{aligned}$$

We take row q of A , multiply it by l , add it to row p of A

$$E_{pq}^{-1}(l) = E_{pq}(-l)$$

Chapter 3

Gaussian Elimination

Definition

Permutation matrix P is an identity matrix with rows in any order.

3.1 Matrix Transposition

Definition

Let us consider matrix $A \in \mathbb{R}^{m,n}$. Matrix $B \in \mathbb{R}^{n,m}$ is called the transpose of A if $(B)_{ij} = (A)_{ji}, i = 1 \dots n, j = 1 \dots m$

Definition

Matrix A is called symmetric if $A^t = A$. Matrix A should be a square matrix, $A \in \mathbb{R}^{n,n}$

$$\text{e.g. } A = \begin{pmatrix} 0 & 3 \\ 3 & 4 \end{pmatrix} \rightarrow A^T = \begin{pmatrix} 0 & 3 \\ 3 & 4 \end{pmatrix} \Rightarrow A^T = A$$

$$\text{e.g. } A = I \in \mathbb{R}^{n,n} \rightarrow I^T = I$$

Chapter 4

Vector Spaces

Definition

A vector space V is a set of objects, such that any two objects can be added together, any object can be multiplied by a scalar.

If two objects belong to the vector space, then their sum also belongs to the vector space.

If an object belongs to V , then the product of any scalar with this object belongs to V and the following properties are satisfied:

1. $\forall \underline{u}, \underline{v}, \underline{w} \in V; (\underline{u} + \underline{v}) + \underline{w} = \underline{u} + (\underline{v} + \underline{w})$
2. $\forall \underline{u}, \underline{v} \in V; \underline{u} + \underline{v} = \underline{v} + \underline{u}$
3. There exists unique elements $\underline{0} \in V$, such that $\forall \underline{u} \in V; \underline{u} + \underline{0} = \underline{0} + \underline{u} = \underline{u}$
4. For any $\underline{u} \in V, \exists!(-\underline{u}) \in V$, such that $\underline{u} + (-\underline{u}) = \underline{0}$
5. $\forall \underline{u}, \underline{v} \in V; \forall \alpha \in \mathbb{R}; \alpha(\underline{u} + \underline{v}) = \alpha\underline{u} + \alpha\underline{v}$
6. $\forall \underline{u} \in V; \forall \alpha, \beta \in \mathbb{R}; (\alpha + \beta)\underline{u} = \alpha\underline{u} + \beta\underline{u}$
7. $\forall \underline{u} \in V; \forall \alpha, \beta \in \mathbb{R}; (\alpha\beta)\underline{u} = \alpha(\beta\underline{u})$
8. $\forall \underline{u} \in V; 1 \cdot \underline{u} = \underline{u}$ (1 is a scalar here)

4.1 Subspace of the Vector Space

Definition

A subspace W of the vector space V , is a set of vectors in V , such that:

1. If $\underline{u}, \underline{v} \in W$ then $\underline{u} + \underline{v} \in W$
2. If $\alpha \in \mathbb{R}, \underline{u} \in W$ then $\alpha\underline{u} \in W$

Definition

Let us consider a set of vectors $\{\underline{u}_1, \dots, \underline{u}_n\}$. The span of vectors $\{\underline{u}_1, \dots, \underline{u}_n\}$ is defined as

$$\mathcal{S} = \text{span}\{\underline{u}_1, \dots, \underline{u}_n\} = \{\alpha_1 \underline{u}_1 + \dots + \alpha_n \underline{u}_n \mid \forall \alpha_1 \dots \alpha_n \in \mathbb{R}\}$$

4.2 Linear Independence**Definition**

Let us consider vector space V and $\underline{v}_1, \dots, \underline{v}_n \in V$. $\underline{v}_1, \dots, \underline{v}_n$ are linearly dependent if there exists scalars $\alpha_1, \dots, \alpha_n$ not all equal to zero, such that $\alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n = \underline{0}$

If no such scalars exist, the vectors $\underline{v}_1, \dots, \underline{v}_n$ are linearly independent.

Definition

Vectors $\underline{v}_1, \dots, \underline{v}_n \in V$ are linearly independent if the following is true:

$$\alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n = \underline{0} \Rightarrow \text{all } \alpha_i = 0, i = 1, \dots, n$$

Definition

If vector space v is generated by $\{\underline{v}_1, \dots, \underline{v}_n\}$ (in other words, $V = \text{span}\{\underline{v}_1, \dots, \underline{v}_n\}$) and $\underline{v}_1, \dots, \underline{v}_n$ are linearly independent, then $\{\underline{v}_1, \dots, \underline{v}_n\}$ is called basis of V

Definition

Let us consider vector space V and vectors $\underline{v}_1, \dots, \underline{v}_n$ that form a basis of V . If vector $\underline{x} \in V$ can be written as $\underline{x} = x_1 \underline{v}_1 + \dots + x_n \underline{v}_n$ then (x_1, \dots, x_n) are called the coordinates of \underline{x} with respect to basis $\{\underline{v}_1, \dots, \underline{v}_n\}$

4.3 Rank of Matrix**Definition**

The row rank of matrix A is a maximum number of linearly independent rows of matrix A .

Definition

The column rank of matrix A is a maximum number of linearly independent columns of matrix A .

Definition

Two subspaces U and W of vector space V are orthogonal, if $\forall \underline{u} \in U$ and $\forall \underline{w} \in W$, we have $\langle \underline{u}, \underline{w} \rangle = 0$

Definition

Orthogonal complement of subspace M of vector space V contains every vector orthogonal to M . This subspace is usually denoted by M^\perp

Chapter 5

Orthogonality

5.1 Linear Mapping

Definition

Let us consider two vector spaces V and W . A function (or mapping) $L : V \rightarrow W$ is called a *linear mapping* if the following two conditions are satisfied:

1. For any $\underline{v} \in V$ and $\underline{v}' \in V$, $L(\underline{v} + \underline{v}') = L(\underline{v}) + L(\underline{v}')$
2. For any $\underline{v} \in V$ and any scalar α , $L(\alpha \underline{v}) = \alpha \cdot L(\underline{v})$

5.2 The 4 Fundamental Subspaces of a Matrix

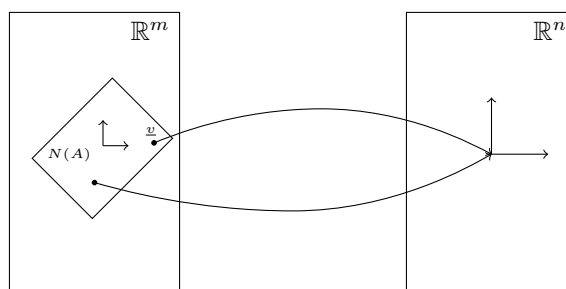
Definition

The span (or all possible linear combinations) of columns of the matrix $A \in \mathbb{R}^{n,m}$ is called a *column space* of A , denoted by $C(A)$, where $C(A) \subset \mathbb{R}^n$.

Definition

Let us consider the matrix $A \in \mathbb{R}^{n,m}$, $A : \mathbb{R}^m \rightarrow \mathbb{R}^n$. The null space of A is defined as

$$N(A) = \{\underline{v} \in \mathbb{R}^m \mid A\underline{v} = \underline{0}\}, N(A) \subset \mathbb{R}^m$$



Definition

The row space of a matrix $A \in \mathbb{R}^{n,m}$ is the span of the rows of A . Clearly, $R(A) = C(A^T)$ and $R(A) \subset \mathbb{R}^m$.

Definition

The left nullspace of A is defined as $N(A^T)$. $N(A^T) \subset \mathbb{R}^n$.

5.3 Orthogonal Basis and Gram-Schmidt process**Definition**

Vectors $\underline{q}_1, \dots, \underline{q}_m$ are *orthogonal* if:

$$\langle \underline{q}_i, \underline{q}_j \rangle = \underline{q}_i^T \underline{q}_j = 0 \quad \text{if } i \neq j$$

In other words, two vectors \underline{q}_i and \underline{q}_j are orthogonal if their dot product is zero, so two orthogonal vectors must be perpendicular.

Definition

Vectors $\underline{q}_1, \dots, \underline{q}_m$ are *orthonormal* if:

$$\langle \underline{q}_i, \underline{q}_j \rangle = \underline{q}_i^T \underline{q}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

In other words, two vectors \underline{q}_i and \underline{q}_j are orthonormal if they are orthogonal and they both have length of 1 (they are *unit vectors*).

Definition

A square matrix is called orthogonal (if its columns are orthonormal vectors) if $Q^T Q = I$. In this case, since it is a square matrix, $Q Q^T = I$.

Chapter 6

Determinant

6.1 Compute the Determinant

6.2 Cramer's Rule

6.3 Inverse of a Matrix

Chapter 7

Linear Mappings

Definition

A mapping $L : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is said to be a *linear mapping* (or linear transformation or linear map) if for any $\underline{u}, \underline{v} \in \mathbb{R}^m$ and any scalar α

$$\begin{aligned}L(\underline{u} + \underline{v}) &= L(\underline{u}) + L(\underline{v}) \\L(\alpha \underline{u}) &= \alpha L(\underline{u})\end{aligned}$$

For any matrix $A \in \mathbb{R}^{n,m}$ we can associate with it a linear mapping L_A as

$$L_A(\underline{u}) = A\underline{u} \quad \forall \underline{u} \in \mathbb{R}^m$$

where L_A is a linear mapping from \mathbb{R}^m to \mathbb{R}^n .

In principle, any linear mapping is completely defined by its values on the basis vectors.

7.1 Matrix Associated with Linear Mapping in a Particular Basis

7.2 Change of Basis (Matrix)

Definition

Assume that $N \in \mathbb{R}^{n,n}$, N^{-1} exists. $A' = N^{-1}AN$ is called similarity transformation.

Definition

Matrices A' and A are called similar matrices, if $\exists N$ such that

$$A' = N^{-1}AN$$

Chapter 8

Eigenvalues and Eigenvectors

Definition

A vector $\underline{v} \in V$, $\underline{v} \neq 0$ is called an eigenvector of A , if there exists scalar λ , such that $A\underline{v} = \lambda\underline{v}$. This scalar λ is called an eigenvalue, corresponding to eigenvector \underline{v} .

Sometimes, eigenvectors are called characteristic vectors, and eigenvalues are called characteristic values.

8.1 Characteristic Polynomial

Definition

Consider $A \in \mathbb{R}^{n,n}$. The characteristic polynomial is defined as

$$p_a(t) = \det(tI - A)$$

Chapter 9

Change of Basis

Definition

A set of all eigenvalues of matrix $A \in \mathbb{R}^{n,n}$ is called spectrum of A