Chapter 1

Vectors

1.1 Introduction

Definition

A vector is an ordered collection of n numbers

Definition

Let us consider vector $\underline{u} \in \mathbb{R}^n$. The $i-$th component of vector

$$\underline{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

is u_i

Definition

Let us consider vectors $\underline{u} \in \mathbb{R}^n$ and $\underline{v} \in \mathbb{R}^n$. Vector $\underline{w} \in \mathbb{R}^n$ is a sum of $\underline{u}$ and $\underline{v}$, $\underline{w} = \underline{u} + \underline{v}$, if $w_i = u_i + v_i$ for all $i = 1, \ldots, n$

Definition

1. Vectors $\underline{u} \in \mathbb{R}^n$ and $\underline{v} \in \mathbb{R}^n$ are equal, if $u_i = v_i$ for all $i = 1, \ldots, n$
2. A scalar is just another name for real number
3. Let us consider a scalar $\alpha \in \mathbb{R}$ and vector $\underline{u} \in \mathbb{R}^n$. A product of α and $\underline{u}$ is defined as:
$$\alpha\underline{u} = \alpha \cdot \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} \alpha \cdot u_1 \\ \vdots \\ \alpha \cdot u_n \end{pmatrix}$$

Definition

Let us consider scalars α and β, and vectors $\underline{u} \in \mathbb{R}^n$ and $\underline{v} \in \mathbb{R}^n$. A sum of $\alpha \cdot \underline{u} + \beta \cdot \underline{v}$ is called a linear combination of vectors $\underline{u}$ and $\underline{v}$.

Definition

Vector $\underline{u} \in \mathbb{R}^n$ is called a zero vector if all $u_i = 0$, $i = 1, \ldots, n$. The zero vector is often written as $\underline{0} \in \mathbb{R}^n$

1.2 Vector Representations and Operations

1.3 Dot Product (Scalar product)

Definition

Let us consider two vectors $\underline{u} \in \mathbb{R}^n$ and $\underline{v} \in \mathbb{R}^n$. The dot (or scalar) product of vectors $\underline{u}$ and $\underline{v}$ is defined as

$$\langle \underline{u}, \underline{v} \rangle = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = \sum_{i=1}^{n} u_i v_i$$

1.4 Length of a Vector

Definition

The length of vector $\underline{u} \in \mathbb{R}^n$, $\|\underline{u}\|$, is defined as $\|\underline{u}\| = \sqrt{\langle \underline{u}, \underline{u} \rangle}$. Sometimes it is also called the Euclidian norm of $\underline{u}$.

1.5 Unit Vectors

Definition

A vector with length equal to 1 is called a unit vector

1.6 Angle between Vectors

1.7 Cauchy Schwarz inequality

Chapter 2

Matrices

2.1 Matrix Operations

Definition

Let us consider matrices $A \in \mathbb{R}^{n,m}$ and $B \in \mathbb{R}^{n,m}$ where $n =$ rows, $m =$ columns. Matrix $C \in \mathbb{R}^{n,m}$ is a sum of A and $B, C = A + B$, if $C_{ij} = A_{ij} + B_{ij}$ for all $i = 1, \ldots, n$, $j = 1, \ldots, m$

Definition

A product of a scalar α and a matrix $A \in \mathbb{R}^{n,m}$ is defined as $(\alpha A)_{ij} = \alpha \cdot A_{ij}$, $\forall i = 1, \ldots, n; j = 1, \ldots, m$.

2.2 Matrix-Matrix Multiplication

Definition

Let us consider matrix $A \in \mathbb{R}^{n,m}$ and $A \in \mathbb{R}^{m,l}$. Then $C = A \cdot B$ is an n by l matrix, $C \in \mathbb{R}^{n,l}$ such that

$$C_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

2.3 Linear System of Equations

2.4 Inverse of a Matrix

Definition

Let us consider a matrix $A \in \mathbb{R}^{n,n}$ (square matrix). Matrix $B \in \mathbb{R}^{n,n}$ is called an inverse of A, if

$$A \cdot B = I \quad \text{AND} \quad B \cdot A = I$$

(Both conditions are vital)

2.5 Special Matrices

2.6 Elementary Transition Matrices

Definition

We can define the elementary transition matrix $I_{pq} \in \mathbb{R}^{n,n}$

$$(I_{pq}) = \begin{cases} 1 & i = p, q = j \\ 0 & \text{otherwise} \end{cases}$$

If we take a matrix $A \in \mathbb{R}^{n,n}$ then when calculating I_{pq} we take row q of A, put it into row p, replace everything else with 0.

We can also define:

$$E_{pq}(l) = I + l \cdot I_{pq}, l \in \mathbb{R} \text{ - scalar}$$
$$E_{pq}(l) \cdot A = (I + lI_{pq}) \cdot A = A + l \cdot I_{pq}A$$

We take row q of A, multiply it by l, add it to row p of A

$$E_{pq}^{-1}(l) = E_{pq}(-l)$$

Chapter 3

Gaussian Elimination

Definition

Permutation matrix P is an identity matrix with rows in any order.

3.1 Matrix Transposition

Definition

Let us consider matrix $A \in \mathbb{R}^{m,n}$. Matrix $B \in \mathbb{R}^{n,m}$ is called the transpose of A if $(B)_{ij} = (A)_{ji}, i = 1 \dots n, j = 1 \dots n$

Definition

Matrix A is called symmetric if $A^t = A$. Matrix A should be a square matrix, $A \in \mathbb{R}^{n,n}$

$$\text{e.g. } A = \begin{pmatrix} 0 & 3 \\ 3 & 4 \end{pmatrix} \rightarrow A^T = \begin{pmatrix} 0 & 3 \\ 3 & 4 \end{pmatrix} \Rightarrow A^T = A$$
$$\text{e.g. } A = I \in \mathbb{R}^{n,n} \rightarrow I^T = I$$

Chapter 4

Vector Spaces

Definition

A vector space V is a set of objects, such that any two objects can be added together, any object can be multiplied by a scalar.

If two objects belong to the vector space, then their sum also belongs to the vector space.

If an object belongs to V, then the product of any scalar with this object belongs to V and the following properties are satisfied:

1. $\forall \underline{u}, \underline{v}, \underline{w} \in V$; $(\underline{u} + \underline{v}) + \underline{w} = \underline{u} + (\underline{v} + \underline{w})$
2. $\forall \underline{u}, \underline{v} \in V$; $\underline{u} + \underline{v} = \underline{v} + \underline{u}$
3. There exists unique elements $\underline{0} \in V$, such that $\forall \underline{u} \in V$; $\underline{u} + \underline{0} = \underline{0} + \underline{u} = \underline{u}$
4. For any $\underline{u} \in V, \exists! (-\underline{u}) \in V$, such that $\underline{u} + (-\underline{u}) = \underline{0}$
5. $\forall \underline{u}, \underline{v} \in V$; $\forall \alpha \in \mathbb{R}$; $\alpha(\underline{u} + \underline{v}) = \alpha\underline{u} + \alpha\underline{v}$
6. $\forall \underline{u} \in V$; $\forall \alpha, \beta \in \mathbb{R}$; $(\alpha + \beta)\underline{u} = \alpha\underline{u} + \beta\underline{u}$
7. $\forall \underline{u} \in V$; $\forall \alpha, \beta \in \mathbb{R}$; $(\alpha\beta)\underline{u} = \alpha(\beta\underline{u})$
8. $\forall \underline{u} \in V$; $1 \cdot \underline{u} = \underline{u}$ (1 is a scalar here)

4.1 Subspace of the Vector Space

Definition

A subspace W of the vector space V, is a set of vectors in V, such that:

1. If $\underline{u}, \underline{v} \in W$ then $\underline{u} + \underline{v} \in W$
2. If $\alpha \in \mathbb{R}$, $\underline{u} \in W$ then $\alpha\underline{u} \in W$

Definition

Let us consider a set of vectors $\{\underline{u}_1, \dots, \underline{u}_n\}$. The span of vectors $\{\underline{u}_1, \dots, \underline{u}_n\}$ is defined as

$$\mathcal{S} = \text{span}\{\underline{u}_1, \dots, \underline{u}_n\} = \{\alpha_1 \underline{u}_1 + \cdots + \alpha_n \underline{u}_n \mid \forall \alpha_1 \dots \alpha_n \in \mathbb{R}\}$$

4.2 Linear Independence

Definition

Let us consider vector space V and $\underline{v}_1, \dots, \underline{v}_n \in V$. $\underline{v}_1, \dots, \underline{v}_n$ are linearly dependent if there exists scalars $\alpha_1, \dots, \alpha_n$ not all equal to zero, such that $\alpha_1 \underline{v}_1 + \cdots + \alpha_n \underline{v}_n = \underline{0}$

If no such scalars exist, the vectors $\underline{v}_1, \dots, \underline{v}_n$ are linearly independent.

Definition

Vectors $\underline{v}_1, \dots, \underline{v}_n \in V$ are linearly independent if the following is true:

$$\alpha_1 \underline{v}_1 + \cdots + \alpha_n \underline{v}_n = 0 \Rightarrow \text{ all } \alpha_i = 0, i = 1, \dots, n$$

Definition

If vector space v is generated by $\{\underline{v}_1, \dots, \underline{v}_n\}$ (in other words, $V = \text{span}\{\underline{v}_1, \dots, \underline{v}_n\}$) and $\underline{v}_1, \dots, \underline{v}_n$ are linearly independent, then $\{\underline{v}_1, \dots, \underline{v}_n\}$ is called basis of V

Definition

Let us consider vector space V and vectors $\underline{v}_1, \dots, \underline{v}_n$ that form a basis of V. If vector $\underline{x} \in V$ can be written as $\underline{x} = x_1 \underline{v}_1 + \cdots + x_n \underline{v}_n$ then $(x_1, \dots, x_n)$ are called the coordinates of $\underline{x}$ with respect to basis $\{v_1, \dots, v_n\}$

4.3 Rank of Matrix

Definition

The row rank of matrix A is a maximum number of linearly independent rows of matrix A.

Definition

The column rank of matrix A is a maximum number of linearly independent columns of matrix A.

Definition

Two subspaces U and W of vector space V are orthogonal, if $\forall \underline{u} \in U$ and $\forall \underline{w} \in W$, we have $\langle \underline{u}, \underline{w} \rangle = 0$

Definition

Orthogonal complement of subspace M of vector space V contains every vector orthogonal to M. This subspace is usually denoted by $M^{\perp}$

Chapter 5

Orthogonality

5.1 Linear Mapping

Definition

Let us consider two vector spaces V and W. A function (or mapping) $L : V \to W$ is called a *linear mapping* if the following two conditions are satisfied:

1. For any $\underline{v} \in V$ and $\underline{v}' \in V$, $L(\underline{v} + \underline{v}') = L(\underline{v}) + L(\underline{v}')$
2. For any $\underline{v} \in V$ and any scalar α, $L(\alpha\underline{v}) = \alpha \cdot L(\underline{v})$

5.2 The 4 Fundamental Subspaces of a Matrix

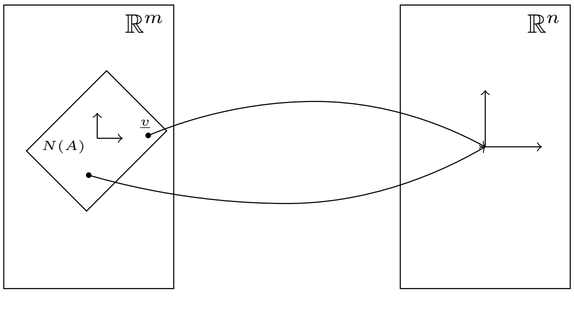
Definition

The span (or all possibile linear combinations) of columns of the matrix $A \in \mathbb{R}^{n,m}$ is called a *column space* of A, denoted by $C(A)$, where $C(A) \subset R^n$.

Definition

Let us consider the matrix $A \in \mathbb{R}^{n,m}, A : \mathbb{R}^m \to \mathbb{R}^n$. The null space of A is defined as

$$N(A) = \{\underline{v} \in \mathbb{R}^m \mid A\underline{v} = \underline{0}\}, N(A) \subset \mathbb{R}^m$$



Definition

The row space of a matrix $A \in R^{n,m}$ is the span of the rows of A. Clearly, $R(A) = C(A^T)$ and $R(A) \subset \mathbb{R}^m$.

Definition

The left nullspace of A is defined as $N(A^T)$. $N(A^T) \subset \mathbb{R}^n$.

5.3 Orthogonal Basis and Gram-Schmidt process

Definition

Vectors $\underline{q}_1, \ldots, \underline{q}_m$ are *orthogonal* if:

$$\langle \underline{q}_i, \underline{q}_j \rangle = \underline{q}_i^T \underline{q}_j = 0 \quad \text{if } i \neq j$$

In other words, two vectors $\underline{q}_i$ and $\underline{q}_j$ are orthogonal if their dot product is zero, so two orthogonal vectors must be perpendicular.

Definition

Vectors $\underline{q}_1, \ldots, \underline{q}_m$ are *orthonormal* if:

$$\langle \underline{q}_i, \underline{q}_j \rangle = \underline{q}_i^T \underline{q}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

In other words, two vectors $\underline{q}_i$ and $\underline{q}_j$ are orthonormal if they are orthogonal and they both have length of 1 (they are *unit vectors*).

Definition

A square matrix is called orthogonal (if its columns are orthonormal vectors) if $Q^T Q = I$. In this case, since it is a square matrix, $QQ^T = I$

Chapter 6

Determinant

6.1 Compute the Determinant

6.2 Cramer's Rule

6.3 Inverse of a Matrix

Chapter 7

Linear Mappings

Definition

A mapping $L : \mathbb{R}^m \to \mathbb{R}^n$ is said to be a *linear mapping* (or linear transformation or linear map) if for any $\underline{u}, \underline{v} \in \mathbb{R}^m$ and any scalar α

$$L(\underline{u} + \underline{v}) = L(\underline{u}) + L(\underline{v})$$
$$L(\alpha\underline{u}) = \alpha L(\underline{u})$$

For any matrix $A \in \mathbb{R}^{m,n}$ we can associate with it a linear mapping L_A as

$$L_A(\underline{u}) = A\underline{u} \qquad \forall \underline{u} \in \mathbb{R}^n$$

where L_A is a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$.

In principle, any linear mapping is completely defined by its values on the basis vectors.

7.1 Matrix Associated with Linear Mapping in a Particular Basis

7.2 Change of Basis (Matrix)

Definition

Assume that $N \in \mathbb{R}^{n,n}$, N^{-1} exists. $A' = N^{-1}AN$ is called similarity transformation.

Definition

Matrices A' and A are called similar matrices, if $\exists N$ such that

$$A' = N^{-1}AN$$

Chapter 8

Eigenvalues and Eigenvectors

Definition

A vector $\underline{v} \in V$, $\underline{v} \neq 0$ is called an eigenvector of A, if there exists scalar λ, such that $A\underline{v} = \lambda\underline{v}$. This scalar λ is called an eigenvalue, corresponding to eigenvector $\underline{v}$.

Sometimes, eigenvectors are called characteristic vectors, and eigenvalues are called characteristic values.

8.1 Characteristic Polynomial

Definition

Consider $A \in \mathbb{R}^{n,n}$. The characteristic polynomial is defined as

$$p_a(t) = \det(tI - A)$$

Chapter 9

Change of Basis

Definition

A sot of all eigenvalues of matrix $A \in \mathbb{R}^{n,n}$ is called spectrum of A