Esercizio 1

- $\left\{\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right\}, \begin{pmatrix}-1\\2\\1\end{pmatrix}$

$$x\begin{pmatrix}1\\1\\1\end{pmatrix}+y\begin{pmatrix}1\\0\\1\end{pmatrix}+z\begin{pmatrix}0\\1\\1\end{pmatrix}=\begin{pmatrix}-1\\2\\1\end{pmatrix} \sim \begin{cases}x+y=-1\\x+z=2\\x+y+z=1\end{cases} \sim \left(\begin{array}{ccc|c}1&1&0&-1\\1&0&1&2\\1&1&1&1\end{array}\right) \sim \left(\begin{array}{ccc|c}1&1&0&-1\\0&-1&1&3\\0&0&1&2\end{array}\right) \sim$$

$$\sim \begin{cases}x+y=-1\\-y+z=3\\z=2\end{cases} \sim \begin{cases}x=0\\y=-1\\z=2\end{cases} \Rightarrow v_B=\begin{pmatrix}0\\-1\\2\end{pmatrix}$$

- $\{T^2-T,\ T^2+T,\ T^2+1\},\quad T+1$

$$a(T^2-T)+b(T^2+T)+c(T^2+1)=T+1 \sim (a+b+c)T^2+(-a+b)T+c=T+1$$

$$\begin{cases}a+b+c=0\\-a+b=1\\c=1\end{cases} \sim \left(\begin{array}{ccc|c}1&1&1&0\\-1&1&0&1\\0&0&1&1\end{array}\right) \sim \left(\begin{array}{ccc|c}1&1&1&0\\0&2&1&1\\0&0&1&1\end{array}\right) \sim \begin{cases}a+b+c=0\\2b+c=1\\c=1\end{cases} \sim \begin{cases}a=-1\\b=0\\c=1\end{cases} \Rightarrow (T+1)_B=\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

- $\left\{\begin{pmatrix}1&1\\1&0\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}0&-1\\1&0\end{pmatrix}\right\}, \begin{pmatrix}2&2\\-2&-1\end{pmatrix}$

$$x\begin{pmatrix}1&1\\1&0\end{pmatrix}+y\begin{pmatrix}0&1\\1&0\end{pmatrix}+z\begin{pmatrix}1&0\\0&1\end{pmatrix}+T\begin{pmatrix}0&-1\\1&0\end{pmatrix}=\begin{pmatrix}2&2\\-2&-1\end{pmatrix}$$

$$\begin{cases}x+z=2\\x+y-T=2\\x+y+T=-2\\z=-1\end{cases} \sim \left(\begin{array}{cccc|c}1&0&1&0&2\\1&1&0&-1&2\\1&1&0&1&-2\\0&0&1&0&-1\end{array}\right) \sim \left(\begin{array}{cccc|c}1&0&1&0&2\\0&1&-1&-1&0\\0&1&-1&1&-4\\0&0&1&0&-1\end{array}\right) \sim \left(\begin{array}{cccc|c}1&0&1&0&2\\0&1&-1&-1&0\\0&0&0&2&-4\\0&0&1&0&-1\end{array}\right) \sim \left(\begin{array}{cccc|c}1&0&1&0&2\\0&1&-1&-1&0\\0&0&1&0&-1\\0&0&0&2&-4\end{array}\right) \sim$$

$$\begin{cases}x+z=2\\y-z-T=0\\z=-1\\2T=-4\end{cases} \sim \begin{cases}x=3\\y=-3\\z=-1\\T=-2\end{cases} \Rightarrow \begin{pmatrix}2&2\\-2&-1\end{pmatrix}_B=\begin{pmatrix}3&-3\\-1&-2\end{pmatrix}$$

- $\left\{\begin{pmatrix}1\\1\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\1\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\\1\end{pmatrix}\right\}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$ in $C(\mathbb{F}_2)^4$

$$x\begin{pmatrix}1\\1\\1\\0\end{pmatrix}+y\begin{pmatrix}0\\1\\1\\1\end{pmatrix}+z\begin{pmatrix}1\\0\\1\\1\end{pmatrix}+T\begin{pmatrix}1\\1\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$\begin{cases}x+z+T=0\\x+y+T=0\\x+y+z=0\\y+z+T=1\end{cases} \sim \left(\begin{array}{cccc|c}1&0&1&1&0\\1&1&0&1&0\\1&1&1&0&0\\0&1&1&1&1\end{array}\right) \sim \left(\begin{array}{cccc|c}1&0&1&1&0\\0&1&1&0&0\\0&1&0&1&0\\0&1&1&1&1\end{array}\right) \sim \left(\begin{array}{cccc|c}1&0&1&1&0\\0&1&1&0&0\\0&0&1&1&0\\0&0&0&1&1\end{array}\right) \sim \begin{cases}x+z+T=0\\y+z=0\\z+T=0\\T=1\end{cases}$$

$$\begin{cases}x=0\\y=1\\z=1\\T=1\end{cases} \Rightarrow \begin{pmatrix}0\\0\\0\\1\end{pmatrix}_B=\begin{pmatrix}0\\1\\1\\1\end{pmatrix}$$

Esercizio 2

- $\left\{ \begin{pmatrix} 1\\1\\2\\1 \end{pmatrix}, \begin{pmatrix} 0\\1\\1\\1 \end{pmatrix}, \begin{pmatrix} -1\\1\\0\\1 \end{pmatrix}, \begin{pmatrix} 1\\1\\2\\2 \end{pmatrix} \right\} \subset \mathbb{R}^4$

$$\begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 2 \\ 1 & 1 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- $\{ T^2+2T+1,\ T^3+T^2-1,\ 2T^3+T^2-2T-3,\ -T^3+2T+2 \} \subset \mathbb{R}_{\le 3}[T]$

Facciamo tutto in base $\mathcal{E} = \{1, T, T^2, T^3\}$

$$\begin{pmatrix} 1 & -1 & -3 & 2 \\ 2 & 0 & -2 & 2 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -3 & 2 \\ 0 & 2 & 4 & -2 \\ 0 & 2 & 4 & -2 \\ 0 & 1 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -3 & 2 \\ 0 & 2 & 4 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -3 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow B = \{T^2+2T+1,\ T^3+T^2-1\}$$

- $\left\{ \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \right\} \subset \mathbb{M}_{2,3}(\mathbb{F}_3)$

Tutto in base $\mathcal{E} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 0 \\ 2 & 2 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow$$

$$\Rightarrow B = \left\{ \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \right\}$$

Esercizio 3

$$\left\{ \begin{pmatrix} 0\\2\\1\\2 \end{pmatrix}, \begin{pmatrix} -2\\0\\3\\2 \end{pmatrix}, \begin{pmatrix} 3\\1\\-4\\-2 \end{pmatrix}, \begin{pmatrix} 1\\2\\1\\1 \end{pmatrix}, \begin{pmatrix} 0\\1\\2\\1 \end{pmatrix} \right\} = \{v_1, \dots, v_5\}$$

$$\begin{pmatrix} 0 & -2 & 3 & 1 & 0 \\ 2 & 0 & 1 & 2 & 1 \\ 1 & 3 & -4 & 1 & 2 \\ 2 & 2 & -2 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & -4 & 1 & 2 \\ 2 & 0 & 1 & 2 & 1 \\ 0 & -2 & 3 & 1 & 0 \\ 2 & 2 & -2 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & -4 & 1 & 2 \\ 0 & -6 & 9 & 0 & -3 \\ 0 & -2 & 3 & 1 & 0 \\ 0 & -4 & 6 & -1 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & -4 & 1 & 2 \\ 0 & -2 & 3 & 0 & -1 \\ 0 & -2 & 3 & 1 & 0 \\ 0 & -4 & 6 & -1 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & -4 & 1 & 2 \\ 0 & -2 & 3 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & 3 & -4 & 1 & 2 \\ 0 & -2 & 3 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow B_W = \left\{ \begin{pmatrix} 0\\2\\1\\2 \end{pmatrix}, \begin{pmatrix} -2\\0\\3\\2 \end{pmatrix}, \begin{pmatrix} 1\\2\\1\\1 \end{pmatrix} \right\}$$

$$(v_3)_{B_W} : \left(\begin{array}{ccc|c} 1 & 3 & 1 & -4 \\ 0 & -2 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & 1 & -4 \\ 0 & -2 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{cases} x + 3y + z = -4 \\ -2y = 3 \\ z = 0 \end{cases} \sim \begin{cases} x = \frac{1}{2} \\ y = -\frac{3}{2} \\ z = 0 \end{cases} \Rightarrow (v_3)_{B_W} = \begin{pmatrix} 1/2 \\ -3/2 \\ 0 \end{pmatrix}$$

$$(v_5)_{B_W} : \left(\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{cases} x + 3y + z = 2 \\ -2y = -1 \\ z = 1 \end{cases} \sim \begin{cases} x = -\frac{1}{2} \\ y = \frac{1}{2} \\ z = 1 \end{cases} \Rightarrow (v_5)_{B_W} = \begin{pmatrix} -1/2 \\ 1/2 \\ 1 \end{pmatrix}$$