Esercizio 1

$B = B' =$ base canonica

- $T\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix}$, $T\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = \begin{pmatrix}0\\-2\\0\end{pmatrix}$, $T\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$, $T\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1\\0\\-2\end{pmatrix}$

$$T_{B'}^{B} = \left(\begin{pmatrix}1\\1\\0\end{pmatrix}_{B'} \begin{pmatrix}0\\-2\\0\end{pmatrix}_{B'} \begin{pmatrix}0\\0\\1\end{pmatrix}_{B'} \begin{pmatrix}1\\0\\-2\end{pmatrix}_{B'}\right) = \begin{pmatrix}1 & 0 & 0 & 1\\ 1 & -2 & 0 & 0\\ 0 & 0 & 1 & -2\end{pmatrix}$$

$$\begin{pmatrix}1 & 0 & 0 & 1\\ 1 & -2 & 0 & 0\\ 0 & 0 & 1 & -2\end{pmatrix} \sim \begin{pmatrix}1 & 0 & 0 & 1\\ 0 & -2 & 0 & -1\\ 0 & 0 & 1 & -2\end{pmatrix} \Rightarrow Im(T_{B'}^{B}) = Span\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\-2\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$$

Per il $Ker(T_{B'}^{B})$:

$$\begin{cases} a + d = 0\\ -2b - d = 0\\ c - 2d = 0\end{cases} \sim \begin{cases} a = -d\\ b = -\frac{1}{2}d\\ c = 2d\end{cases} \Rightarrow Ker(T) = Span\left\{\begin{pmatrix}-1 & -1/2\\ 2 & 1\end{pmatrix}\right\}$$

- $T(1) = T^2 - 2 \qquad T(T) = -T^2 - 1 \qquad T(T^2) = T^2 - 2T$

$$T_{B'}^{B} = \left((T^2-2)_{B'} \ (-T^2-1)_{B'} \ (T^2-2T)_{B'}\right) =$$

$$= \begin{pmatrix}-2 & -1 & 0\\ 0 & 0 & -2\\ 1 & -1 & 1\end{pmatrix} \sim \begin{pmatrix}1 & -1 & 1\\ 0 & 0 & -2\\ 0 & -3 & 2\end{pmatrix} \sim \begin{pmatrix}1 & -1 & 1\\ 0 & -3 & 2\\ 0 & 0 & -2\end{pmatrix} \Rightarrow Im(T) = \mathbb{R}_{\leq 2}[T]$$

Per $Ker(T)$

$$\begin{cases} a - b + c = 0\\ -3b + 2c = 0\\ -2c = 0\end{cases} \sim \begin{cases} a = 0\\ b = 0\\ c = 0\end{cases} \Rightarrow Ker(T) = 0$$

• $T(1) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ $T(T) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ $T(T^2) = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$ $T(T^3) = \begin{pmatrix} -2 & -5 \\ -5 & 0 \end{pmatrix}$

$$T_{B'}^{B} = \begin{pmatrix} 1 & 0 & -1 & -2 \\ -1 & 1 & 1 & -5 \\ -1 & 1 & 1 & -5 \\ 1 & 1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 0 & -7 \\ 0 & 1 & 0 & -7 \\ 0 & 1 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathrm{Im}(T) = \mathrm{Span}\left\{ \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

Per $\mathrm{Ker}(T)$:

$$\begin{cases} a - c - 2d = 0 \\ b - 7d = 0 \\ c + 9d = 0 \end{cases} \sim \begin{cases} a = -7d \\ b = 7d \\ c = -9d \end{cases} \Rightarrow \mathrm{Ker}(T) = \mathrm{Span}\{-7 + 7T - 9T^2 + T^3\}$$

• $T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$ $T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}$

$$T_{B'}^{B} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{B'} \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}_{B'} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{B'} \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}_{B'} \right) =$$

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

$$\mathrm{Im}(T) = \mathrm{Span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix} \right.$$

$\mathrm{Ker}(T) = 0$

Esercizio 2

Partiamo da quelli immediati :

$$Id_{\mathbb{R}^3}{}^{B}_{\mathcal{E}} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & 2 & -1 \end{pmatrix} \quad e \quad Id_{\mathbb{R}^3}{}^{B'}_{\mathcal{E}} = \begin{pmatrix} -2 & 0 & -1 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$Id^{\mathcal{E}}_{B} = \left(Id^{B}_{\mathcal{E}}\right)^{-1} = \left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ -1 & 2 & -1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 & -2 & 0 \\ 0 & 3 & 0 & 0 & 1 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 & -2 & 0 \\ 0 & 0 & -3 & 3 & -5 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1/3 & 0 \\ 0 & 0 & 1 & -1 & 5/3 & -1/3 \end{array}\right) \Rightarrow Id^{\mathcal{E}}_{B} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1/3 & 0 \\ -1 & 5/3 & -1/3 \end{pmatrix}$$

$$Id^{\mathcal{E}}_{B'} = \left(Id^{B'}_{\mathcal{E}}\right)^{-1} = \left(\begin{array}{ccc|ccc} -2 & 0 & -1 & 1 & 0 & 0 \\ 1 & -1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & -2 & -2 & 0 & 1 & -1 \\ 0 & 2 & 1 & 1 & 0 & 2 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & -2 & -2 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1/2 & 1/2 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 1 & 0 & 1 & 1/2 & 3/2 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{array}\right) \Rightarrow$$

$$Id^{\mathcal{E}}_{B'} = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 1/2 & 3/2 \\ 1 & -1 & -1 \end{pmatrix}$$

$$Id^{B}_{B'} = Id^{\mathcal{E}}_{B'}\, Id^{B}_{\mathcal{E}} = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 1/2 & 3/2 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1/2 & 0 \\ 1 & 9/2 & 0 \\ 2 & -2 & 1 \end{pmatrix}$$

$$Id^{B'}_{B} = Id^{\mathcal{E}}_{B}\, Id^{B'}_{\mathcal{E}} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1/3 & 0 \\ -1 & 5/3 & -1/3 \end{pmatrix} \begin{pmatrix} -2 & 0 & -1 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 1 & 0 \\ -1/3 & 1/3 & 1/3 \\ 10/3 & -2 & -1 \end{pmatrix}$$

Esercizio 3

- $T_B^B = \left(T(v_1)_B \; T(v_2)_B \; T(v_3)_B\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$Id_B^{\mathcal{E}} = \left(Id_{\mathcal{E}}^B\right)^{-1} = \left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ -1 & 2 & -1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 1 \end{array}\right)$$

I calcoli sono sempre gli stessi...