Esercizio 1

- $W: x+y-z=0 \qquad W' = \text{Span}\left\{\begin{pmatrix}2\\4\\-2\end{pmatrix}\right\} \subset \mathbb{R}^3$

<u>$W+W'$</u>: portiamo W in parametrica. $W: x=-y+z \Rightarrow \begin{cases} x = -s+t \\ y = s \\ z = t \end{cases} \Rightarrow \text{Span}\left\{\begin{pmatrix}-1\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\right\}$

$$W+W' = \left\{\begin{pmatrix}-1\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}2\\4\\-2\end{pmatrix}\right\}.$$

$$\begin{pmatrix}-1&1&2\\1&0&4\\0&1&-2\end{pmatrix} \sim \begin{pmatrix}-1&1&2\\0&1&6\\0&1&-2\end{pmatrix} \sim \begin{pmatrix}-1&1&2\\0&1&6\\0&0&-8\end{pmatrix} \Rightarrow B_{W+W'} = \left\{\begin{pmatrix}-1\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}2\\4\\-2\end{pmatrix}\right\}$$

Per Grassmann abbiamo: $\dim(W\cap W') = \dim(W) + \dim(W') - \dim(W+W') = 2+1-3 = 0 \Rightarrow$
$\Rightarrow W\cap W' = \{0\}$ e $B_{W\cap W'} = \emptyset$

- $W: p(1) = 0,\ W': p(0) = 0 \subset \mathbb{R}_{\leq 2}[T]$

Per $W\cap W'$, portiamo entrambe in cartesiana:

$W: a+b+c=0$ quindi $W\cap W': \begin{cases} a+b+c=0 \\ a=0 \end{cases} \sim \begin{cases} a=0 \\ b=-c \end{cases} \Rightarrow B_{W\cap W'} = \{-T+T^2\}$
$W': a=0$

Per $W+W'$, calcoliamo $B_W = \{-1+T, -1+T^2\}$ e $B_{W'} = \{T, T^2\}$. In base $\mathcal{E}$:

$$\begin{pmatrix}-1&-1&0&0\\1&0&1&0\\0&1&0&1\end{pmatrix} \sim \begin{pmatrix}-1&-1&0&0\\0&-1&1&0\\0&1&0&1\end{pmatrix} \sim \begin{pmatrix}-1&-1&0&0\\0&-1&1&0\\0&0&1&1\end{pmatrix} \Rightarrow B_{W+W'} = \{-1+T, -1+T^2, T\}$$

- $W: \begin{cases} x+y+z=0 \\ y+z=0 \end{cases} \qquad W': \text{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\} \subset (\mathbb{F}_2)^3$

Per $W+W'$, portiamo W in parametrica:

$$W: \begin{cases} x+y+z=0 \\ y+z=0 \end{cases} \sim \begin{cases} x=0 \\ y=z \end{cases} \Rightarrow W = \text{Span}\left\{\begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$$

$$B_{W+W'} = \text{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$$

Per Grassman $\Rightarrow \dim(W\cap W') = \dim(W) + \dim(W') - \dim(W+W') = 1+1-2 = 0 \Rightarrow W\cap W' = \{0\}$

Esercizio 2

$W: \text{Span}\left\{\begin{pmatrix}1\\k\\-1\end{pmatrix},\begin{pmatrix}2\\k+1\\-2\end{pmatrix}\right\} \qquad W': \text{Span}\left\{\begin{pmatrix}2\\0\\k-2\end{pmatrix}\right\} \subset \mathbb{R}^3$

Per $W+W'$:

$$\begin{pmatrix}1&2&2\\k&k+1&0\\-1&-2&k-2\end{pmatrix} \sim \begin{pmatrix}1&2&2\\0&-k+1&-2k\\0&0&k\end{pmatrix}$$

$\dim(W+W') = 3$ per $k \neq 1, 0$
$\dim(W+W') = 2$ per $k = 1$
$\dim(W+W') = 2$ per $k = 0$

Per Grassmann : $\dim(W \cap W') = \dim(W) + \dim(W') - \dim(W + W')$

quindi:

$\dim(W \cap W') = 0$ per $K \neq 1, 0 \Rightarrow W \oplus W'$

$\dim(W \cap W') = 1$ per $K = 1$

$\dim(W \cap W') = 1$ per $K = 0$

$W : \operatorname{Ker}\begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & -K+1 & K-1 & 0 \end{pmatrix}$ $\quad W' : \operatorname{Im}\begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 1 & -1 \\ -1 & 2 \end{pmatrix} \subset \mathbb{R}^4$

Per $W \cap W'$ mettiamo W' in cartesiana. Equazione generica $ax + by + cz + dT = 0$

$W' : \begin{cases} a + c - d = 0 \\ -a + b - c + 2d = 0 \end{cases} \sim \begin{cases} a + c - d = 0 \\ b + d = 0 \end{cases} \sim \begin{cases} a = -c + d \\ b = -d \end{cases} \Rightarrow W' : \begin{cases} -X + Z = 0 \\ X - Y + T = 0 \end{cases}$

$W \cap W' : \left(\begin{array}{cccc|c} -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 2 & -K+1 & K-1 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & -K+1 & K+1 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 2 & -K+1 & 0 \end{array}\right) \sim$

$\sim \left(\begin{array}{cccc|c} -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & K-1 & 0 \end{array}\right)$

$\dim(W \cap W') = 4$ per $K \neq 1$

$\dim(W \cap W') = 3$ per $K = 1$

Per Grassmann : $\dim(W + W') = \dim(W) + \dim(W') - \dim(W \cap W')$

quindi:

$\dim(W + W') = 2 + 2 - 4 = 0$