Esercizio 1

i) In base $\mathcal{E} = \{1, T, T^2\}$

$$(W+W')_{\mathcal{E}} = \text{Span}\left\{ \overset{W}{\begin{pmatrix}1\\1\\0\end{pmatrix}}, \overset{W'}{\begin{pmatrix}-2\\-1\\1\end{pmatrix}}, \overset{W'}{\begin{pmatrix}0\\-2\\1\end{pmatrix}} \right\}$$

$$\begin{pmatrix}1 & -2 & 0\\ 1 & -1 & -2\\ 0 & 1 & 1\end{pmatrix} \sim \begin{pmatrix}1 & -2 & 0\\ 0 & 1 & -2\\ 0 & 1 & 1\end{pmatrix} \sim \begin{pmatrix}1 & -2 & 0\\ 0 & 1 & -2\\ 0 & 0 & 3\end{pmatrix} \Rightarrow W + W' = \mathbb{R}_{\leq 2}[T]$$

Per Grassman $W \cap W' = \{0\}$

ii) In base $\mathcal{E}$ risolviamo:

$$\left(\begin{array}{ccc|ccc}1 & -2 & 0 & 1 & 0 & 0\\ 1 & -1 & -2 & 0 & 1 & 0\\ 0 & 1 & 1 & 0 & 0 & 1\end{array}\right) \sim \left(\begin{array}{ccc|ccc}1 & -2 & 0 & 1 & 0 & 0\\ 0 & 1 & -2 & -1 & 1 & 0\\ 0 & 1 & 1 & 0 & 0 & 1\end{array}\right) \sim \left(\begin{array}{ccc|ccc}1 & -2 & 0 & 1 & 0 & 0\\ 0 & 1 & -2 & -1 & 1 & 0\\ 0 & 0 & 3 & 1 & -1 & 1\end{array}\right) \sim$$

$$\sim \begin{cases} x = \frac{1}{3} \\ y = -\frac{1}{3} \\ z = \frac{1}{3} \end{cases} \qquad \begin{cases} x = \frac{2}{3} \\ y = \frac{1}{3} \\ z = -\frac{1}{3} \end{cases} \qquad \begin{cases} x = \frac{4}{3} \\ y = \frac{2}{3} \\ z = \frac{1}{3} \end{cases}$$

$$\pi_W^{W'}(1) = \frac{1}{3}w - \frac{1}{3}w' + \frac{1}{3}w' = \frac{1}{3}(1+T) = \frac{1}{3} + \frac{T}{3}$$

$$\pi_W^{W'}(T) = \frac{2}{3}w + \frac{1}{3}w' - \frac{1}{3}w' = \frac{2}{3}(1+T) = \frac{2}{3} + \frac{2T}{3}$$

$$\pi_W^{W'}(T^2) = \frac{4}{3}w + \frac{2}{3}w' + \frac{1}{3}w' = \frac{4}{3}(1+T) = \frac{4}{3} + \frac{4T}{3}$$

Esercizio 2

i) é invertibile $\Leftrightarrow rk(A) = 3$:

$$\begin{pmatrix}1 & 1 & -1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix} \sim \begin{pmatrix}1 & 1 & -1\\ 0 & -1 & 2\\ 0 & 0 & 1\end{pmatrix} \Rightarrow \text{invertibile}$$

ii) $\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \sim \begin{cases} x = 1 \\ y = -1 \\ z = -1 \end{cases} \begin{cases} x = 1 \\ y = -1 \\ z = 0 \end{cases} \begin{cases} x = -1 \\ y = 2 \\ z = 1 \end{cases}$

Quindi $L_A^{-1}(e_1) = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$, $L_A^{-1}(e_2) = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ e $L_A^{-1}(e_3) = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$

Esercizio 3

Ricordiamo che $T_{B'}^{B} = (T(v_1)_{B'} \dots T(v_n)_{B'})$.

i) $T(1) = 0 - 1 + 1 = 0$
$T(T) = 1 - T + 0 = 1 - T$
$T(T^2) = 2T - T^2 + 0 = 2T - T^2$
$\Rightarrow T_{B'}^{B} = (0_{B'} \ (1-T)_{B'} \ (2T-T^2)_{B'}) =$

$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$

ii) $T(1) = \begin{pmatrix} 0+1 \\ 1-0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad T(T) = \begin{pmatrix} 1+1 \\ 1-0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$T(T^2) = \begin{pmatrix} -2+1 \\ 1-2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$

$T_{\mathcal{E}}^{B} = \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}_{\mathcal{E}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}_{\mathcal{E}} \begin{pmatrix} -1 \\ -1 \end{pmatrix}_{\mathcal{E}} \right) = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 1 & -1 \end{pmatrix}$

iii) $T_{B'}^{B} = (T(1+T)_{B'} \ T(T^2-T-2)_{B'} \ T(T^2-2T)_{B'}) = ((1+T)_{B'} \ 0_{B'} \ 0_{B'}) =$

$= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

iv) $T_{B'}^{B} = (T(1)_{B'} \ T(T)_{B'} \ T(T^2)_{B'}) = \left(\left(\frac{1}{3} + \frac{T}{3}\right)_{B'} \left(\frac{2}{3} + \frac{2}{3}T\right)_{B'} \left(\frac{4}{3} + \frac{4}{3}T\right)_{B'} \right) =$

$\begin{pmatrix} 1/3 & 2/3 & 4/3 \\ 1/3 & 2/3 & 4/3 \\ 0 & 0 & 0 \end{pmatrix}$