34th Midwest Symposium on Circuits and Systems

Matrix Mixed-Radix Conversion For RNS Arithmetic Architectures

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ABSTRACT

Processor architectures based on arithmetic cells using residue number systems are inherently parallel, modular and fault isolating. This paper describes an improved technique for transforming a residue number into a mixed-radix weighted representation Using matrix techniques. The mixed-radix digits of the proposed conversion procedure are factorized into a product of two terms each, one term is invariant and predetermined, while the other is variable and depends upon the particular residue number being converted. In comparison with existing conversion techniques, the proposed conversion method achieves a considerable reduction in the number of arithmetic multiplications needed during the conversion process.

1. INTRODUCTION

Processor architectures based on arithmetic cells using residue number systems are inherently parallel, modular and fault isolating. The fundamental characteristic of the residue number systems is that it is an unweighted numbering system [1-2]. In addition, subtraction, and multiplication any particular digit of the resultant depends solely on the corresponding digits of its suboperation. This parallel property of residue arithmetic makes it capable of performing carry-free addition and multiplication, and borrow-free subtraction. Moreover, multiplication is executed in a single step without partial products. Moreover, this special structure allows redundancy to be introduced quite easily, thus computational errors can be detected, located and corrected efficiently [3,4]. The reason why residue arithmetic does not find widespread use in general purpose signal processor architectures is that a residue number system is not suitable for executing the following arithmetic operations: firstly the sign of a residue encoded number is a function of all the residue digits of that number [5]; secondly overflows resulting from addition or multiplication are not detectable during the actual arithmetic operation [6]; thirdly magnitude comparison is not possible within the residue number system; and finally general division is slow [7]. Several special purpose signal processing architectures which do not require these arithmetic operations have been designed around residue arithmetic [8-13]. However, if these operations are required, they are best performed in a weighted (positional) numbering system like binary or decimal systems. Moreover, in many cases the final results of a residue arithmetic calculation will also need to be presented in the usual digital format. Thus techniques for converting a residue number into an equivalent decimal number is of great importance for residue number processor applications. Although the classical approach to this conversion is the Chinese Remainder Theorem [14], this requires large modulus adders and a more effective approach has been shown to be first convert the residue number to a mixed-radix representation [2,15-16] from which the decimal number may be obtained rather readily.

In an earlier work, we described a new residue number system called hierarchical residue number system which can achieve large dynamic ranges with small value moduli [17]. In this paper we describe an improved technique for transforming a residue number into a mixed-radix number representation using matrix techniques The proposed conversion algorithm is based on the cyclic patterns inherent in residue number systems and uses matrix techniques to derive mixed-radix digits. The proposed matrix approach is then compared to the mixed-radix techniques described in previous work [17].

2. RESIDUE NUMBER SYSTEMS

A *weighted number system* is defined by one or more fixed radices called bases. For example the decimal system is a weighted numbering system whose base or radix is 10. It is called a "weighted" system because each digit in a decimal number possesses a certain weight according to its place in the decimal expression. A *non-weighted number system* is defined by a set of N-radices which constitute the base. In such a system a number is represented by an N-tuple whose elements do not have any relative significance. For example the residue number system is a non-weighted numbering system whose radices are called moduli.

A *residue number system* is defined by a given a set of moduli $m_1, m_2 \ldots m_N$ which must be relatively prime. Any integer x in the range $x \varepsilon [0, D]$ can be uniquely defined by the N-tuple which is its set of residues with respect to the set of moduli.

$$x \equiv (R_1, R_2 \ldots R_N)$$

where

$$D = \left(\prod_{i=1}^{N} m_i \right) - 1, \qquad i = 0 \ldots N.$$

$$x \equiv R_1, \bmod m_1$$
$$x \equiv R_2, \bmod m_2$$

$$x \equiv R_N, \bmod m_N$$

The interval [0, D] is called the *legitimate range* and defines the total number of states which may be defined unambiguously in that particular residue number system.

3. MIXED-RADIX NUMBER SYSTEM

Given a set of radices β_i and a set of digits γ_i such that $0 \le \gamma_i < \beta_i$. The mixed-radix representation of a number x is defined as follows [2]:

$$x = \gamma_1 W_1 + \gamma_2 W_2 + \ldots + \gamma_N W_N$$

where

$$W_1 = 1$$
$$W_2 = \beta_1$$
$$W_3 = \beta_1 \cdot \beta_2$$

$$W_N = \beta_1 \cdot \beta_2 \ldots \beta_{N-1}.$$

The mixed-radix representation of a number x is denoted by $(\gamma_1, \gamma_2, \ldots \gamma_N)$ where the digits γ_i are called the mixed-radix digits and are subscripted in order of increasing significance. It is evident that the mixed-radix numbering system is a weighted (positional) system (with weights W_i). It is also clear that any number $x \varepsilon [0, D]$ is uniquely represented in the mixed-radix form by the digits γ_j, where

$$D = \left(\prod_{i=1}^{N} \beta_i \right) - 1 \qquad \text{and} \qquad i = 0 \ldots N.$$

The familiar decimal system is a special case of the mixed-radix system where $\beta_i = 10$ for all i.

4. MIXED-RADIX CONVERSION

In a residue number system with a given set of moduli m_i, the mixed-radix system defined by the set of radices $\beta_i = m_i$ is called *associated system*. The mixed-radix conversion problem is defined as follows: given a residue number system defined by the set of moduli $(m_1, m_2 \ldots m_N)$, it is required to find the associated mixed-radix digits $(\gamma_1, \gamma_2 \ldots \gamma_N)$ of a given residue number $(R_1, R_2 \ldots R_N)$. Instead of determining the γ_i directly, it is more efficient to evaluate related variables U_i given by [17]:

$$\gamma_i \equiv | U_i \cdot V_i |_{m_i}$$

where the V_i are constant predetermined factors given by

$V_1 \equiv 1$
$V_2 \equiv |m_1^*|_{m_2}$
$V_3 \equiv |(m_1 . m_2)^*|_{m_3}$

$V_N \equiv |(m_1 . m_2 ... m_{N-1})^*|_{m_N}$

where m_1^* is the multiplicative inverse of m_1 with respect to m_2. , and is given by:

$$|m_1^* \; . \; m_1|_{m_2} = 1$$

The magnitude of a number (decimal value), x, is then given in terms of the mixed-radix weights of the associated system by:

$$x = W_1 |U_1 V_1|_{m_1} + W_2 |U_2 V_2|_{m_2} ... + W_N |U_N V_N|_{m_N}$$

Using the analysis described in [17] we derive the following mixed-radix equations:

$U_1 = R_1$
$U_2 = |R_2 - R_1|_{m_2}$
$U_3 = \left| R_3 - R_1 - W_2 |U_2 . V_2|_{m2} \right|_{m_3}$
$U_4 = \left| R_4 - R_1 - W_2 |U_2 . V_2|_{m2} - W_3 |U_3 . V_3|_{m3} \right|_{m_4}$

5. MATRIX-MIXED-RADIX CONVERSION

The conversion procedure described in this section helps to understand in greater depth the internal mechanisms of the mixed-radix conversion process. Furthermore, the mixed-radix approach described in this section may have certain advantages in the hardware implementation by using matrix operations. The mixed-radix conversion is based upon the cyclic characteristics of residue numbers. By looking at an ordered table of residue numbers, e.g. Table 1 below, it can be seen that the residues cycle in fixed periods. Given a residue number, its decimal magnitude is derived by jumping backwards to the nearest residue number which has at least one residue equal to "0" and recording the value of that jump. This process is repeated until all the residues become zeros. An example is used to describe the proposed mixed conversion algorithm. Given a set of moduli

$$m_i = (11, 7, 5, 3, 2)$$

the following table expresses all the different residue numbers defined by such a set of moduli.

Table 1. Residue table

Decimal	m_1=11	m_2=7	m_3=5	m_4=3	m_5=2
0	0	0	0	0	0
1	1	1	1	1	1
2	2	2	2	2	0
3	3	3	3	0	1
4	4	4	4	1	0
5	5	5	0	2	1
5	6	6	1	0	0
7	7	0	2	1	1
8	8	1	3	2	0
9	9	2	4	0	1
10	10	3	0	1	0
11	0	4	1	2	1
12	1	5	2	0	0
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.

Note that the residue sequence of modulus m_1 has a period of 11 entries, while m_2 has a period of 7 entries etc..
The mixed-radix digits of residue numbers are derived as follows: assume we are given a residue number

$$x \equiv (4, 0, 1, 2, 1)$$

which occupies a certain location somewhere within Table 1. The aim is to make a maximum of 5 (corresponding to number of moduli) consecutive jumps in the residue table such that each location has one more zero residue than the previous location and that the fourth location is

$$0 = (0, 0, 0, 0, 0).$$

The first jump is defined by a number J_1 which corresponds to the first residue in X. Thus

$$J_1 = 4$$

The first location is $(X - J_1)$. Therefore

$$X - 4 \equiv \begin{pmatrix} 4 - 4 = 0, \text{ mod } 11 \\ 0 - 4 = 3, \text{ mod } 7 \\ 1 - 4 = 2, \text{ mod } 5 \\ 2 - 4 = 1, \text{ mod } 3 \\ 1 - 4 = 1, \text{ mod } 2 \end{pmatrix}$$

The second jump is defined by a number J_2 such that

$$J_2 = k_2.11. \text{ and } 3 - J_2 = 0, \text{ mod } 7$$

Therefore

$$k_2 = 6$$

The second location is defined by $(X - J_1 - J_2)$. Thus

$$X - 4 - 66 \equiv \begin{pmatrix} 0 - 66 \equiv 0, \text{ mod } 11 \\ 3 - 66 \equiv 0, \text{ mod } 7 \\ 2 - 66 \equiv 1, \text{ mod } 5 \\ 1 - 66 \equiv 1, \text{ mod } 3 \\ 1 - 66 \equiv 1, \text{ mod } 2 \end{pmatrix}$$

The third jump is given by J_3 such that

$$J_3 = k_3(11. 7) \text{ and } 1 - J_3 = 0, \text{ mod } 5$$

Therefore

$$k_3 = 3$$

The third location is defined by $(X - J_1 - J_2 - J_3)$. Thus

$$X - 4 - 66 - 231 = \begin{pmatrix} 0 - 231 = 0, \text{ mod } 11 \\ 0 - 231 = 0, \text{ mod } 7 \\ 1 - 231 = 0, \text{ mod } 5 \\ 1 - 231 = 1, \text{ mod } 3 \\ 1 - 231 = 0, \text{ mod } 2 \end{pmatrix}$$

The procedure is repeated until all the residues of the final number are zeros, i.e.,

$$X - 4 - 66 - 231 - 385 - 1155 = \begin{pmatrix} 0, \text{ mod } 11 \\ 0, \text{ mod } 7 \\ 0, \text{ mod } 5 \\ 0, \text{ mod } 3 \\ 0, \text{ mod } 2 \end{pmatrix}$$

6. COMPARISON WITH MIXED-RADIX

A closer look at the above described procedure reveals that the process of finding the jumps J_1, J_2, J_3..., is fixed for the entire given residue number system. and corresponds to the product of the weights by the their multiplicative inverses. Thus the residue number system

$$m_i = (11, 7, 5, 3, 2)$$

is characterised by the following factors:

$$\begin{aligned} V_1W_1 &= 1.1 && = 1 \\ V_2W_2 &= (11)^*{}_7.11 && = 22 \\ V_3W_3 &= (11.7)^*{}_5.11.7 && = 231 \\ V_4W_4 &= (11.7.5)^*{}_3.11.7.5 && = 385 \\ V_5W_5 &= (11.7.5.3)^*{}_2.11.7.5.3 && = 1155 \end{aligned}$$

Using the residues of these factored weights we derive the following matrices which define the relation between the residue number system and its associated mixed-radix system

$$M_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} |22|_{11} \\ |22|_7 \\ |22|_5 \\ |22|_3 \\ |22|_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} |231|_{11} \\ |231|_7 \\ |231|_5 \\ |231|_3 \\ |231|_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$M_4 = \begin{pmatrix} |385|_{11} \\ |385|_7 \\ |385|_5 \\ |385|_3 \\ |385|_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

$$M_5 = \begin{pmatrix} |1155|_{11} \\ |1155|_7 \\ |1155|_5 \\ |1155|_3 \\ |1155|_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Thus for a residue number

$$x = \begin{pmatrix} 4 \\ 0 \\ 1 \\ 2 \\ 1 \end{pmatrix},$$

The variable factors of associated mixed-radix digits are derived by matrix operators in residue arithmetic where row one is to mod m_1, row two is to mod m_2 and so on...

$$\begin{pmatrix} 4 \\ 0 \\ 1 \\ 2 \\ 1 \end{pmatrix} - 4. \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 3 \\ 2 \\ 1 \\ 1 \end{pmatrix} - 3. \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} -1. \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} -1. \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} -1. \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The multipliers in the front of these vectors are the required variable factors of the mixed-radix digits.

$$\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Using Eq (4) the required mixed-radix digits are derived as follows:

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 1 \\ 1 \\ 1 \end{pmatrix} . \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ 6 \\ 3 \\ 1 \\ 1 \end{pmatrix}$$

The corresponding decimal value is derived using equations in section 4.

$$X = 4(1) + 6(11) + 3(77) + 1(385) + 1(1155) = 1841$$

It should be noted that these matrix operations are in residue arithmetic and each operation can start only after the scalar multiplier has been derived in the previous matrix.

7. CONCLUSION

We have a new approach for the mixed-radix conversion using matrix techniques. The conversion algorithm is based on the cyclic patterns inherent in the residue number system. The described approach simplifies mixed-radix transformations. However the implementation of the new approach still to be described. Whether for achieving speed through parallelism in computer architectures or for enhancing arithmetic reliability, residue number architectures are a promising approach.

8. ACKNOWLEGEMENT

The Author acknowleges the support of Brunel University which support this work through the BRIEF Award.

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