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Finite Volume Schemes: A Tutorial

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Outline

- Hyperbolic Conservation Laws
- Finite Volume Methods
- Upwinding, Oscillations and Limiting
- Nonlinear Systems and Riemann Solvers
- Other Related Topics

Hyperbolic Conservation Laws

Finite volume schemes are most useful for modelling hyperbolic conservations laws which, in their simplest (scalar) form, are represented by the PDE

$$u_t + \vec{\nabla} \cdot \vec{f} \;=\; 0$$

on a domain Ω, with $u(\vec{x}, t)$ specified on **inflow** boundaries.

The advective form, given by

$$u_t + \vec{\lambda} \cdot \vec{\nabla} u \;=\; 0$$

where $\vec{\lambda} = \partial \vec{f} / \partial u$ is the **advection velocity**, is often referred to but it is subtly different (particularly when $\vec{\nabla} \cdot \vec{\lambda} \neq 0$).

Some Common Schemes

- **Finite Differences:**
  - problems with conservation and irregular geometry.
- **Finite Volumes:**
  - biased towards edges and one-dimensional physics.
- **Finite Elements:**
  - trouble with hyperbolic equations.
- **Discontinuous Galerkin:**
  - best (and worst) of all worlds.
- **Fluctuation Splitting:**
  - not yet widely applicable.

Sought After Properties

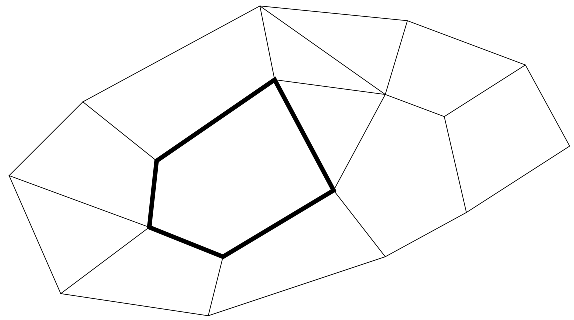
The ideal scheme would reflect the properties of the underlying physical/mathematical system and be

- **Conservative:** for correct capturing of discontinuities.
- **Accurate:** for obvious reasons.
- **Free of spurious oscillations:** for physical realism...and remarkably difficult to achieve.
- **Continuous:** for convergence to the steady state.

Upwinding aids in most of this for hyperbolic equations.

The Finite Volume Method

- Partition the computational domain into **control volumes** (which are not necessarily the cells of the mesh).



- Discretise the **integral** formulation of the conservation laws over each control volume (making use of the **Gauss divergence theorem**).
- Solve the resulting set of algebraic equations or update the values of the dependent variables.

The Integral Form

The key to the method is that the integral form of the conservation law

$$\frac{\partial}{\partial t}\int_{\Omega} U \, d\Omega + \int_{\Omega} \vec{\nabla} \cdot \vec{F} \, d\Omega \; = \; 0$$

can be rewritten, using the Gauss Divergence Theorem, as

$$\frac{\partial}{\partial t}\int_{\Omega} U \, d\Omega + \oint_{\partial\Omega} \vec{F} \cdot \vec{n} \, d\Gamma \; = \; 0$$

In other words, the rate of change of mass in the control volume is equal to the net mass flux through its boundary.

Notes

- In one dimension this becomes

$$\frac{\partial}{\partial t}\int_{\Omega} U \; d\Omega + \int_{X_L}^{X_R} F_x \; dx \; = \; \frac{\partial}{\partial t}\int_{\Omega} U \; d\Omega + (F_R - F_L) \; = \; 0$$

- There is a strong relationship to continuous Finite Element and Discontinuous Galerkin methods, which take the form

$$\int_{\Omega} W \, U_t \; d\Omega + \int_{\Omega} W \, \vec{\nabla} \cdot \vec{F} \; d\Omega$$

$$= \; \int_{\Omega} W \, U_t \; d\Omega + \oint_{\partial\Omega} W \, \vec{F} \cdot \vec{n} \; d\Gamma - \int_{\Omega} \vec{\nabla} W \cdot \vec{F} \; d\Omega \; = \; 0$$

The Discrete Form

The flux terms are discretised by

$$\oint_{\partial\Omega} \vec{F}\cdot\vec{n}\, d\Gamma \;\approx\; \sum_{\text{faces}} \vec{F}^*_k \cdot \vec{n}_k$$

where $\vec{F}^*$ is known as the **numerical flux.**

- U is assumed to be **discontinuous** across the control volume boundaries (typically constant within each volume), which is where the fluxes are evaluated.
- U_i represents an “average” value associated with the i^{th} control volume.

Updating the Solution

For simplicity, a forward Euler discretisation of the time derivative will be considered, leading to

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{V_i} \sum_{\text{faces}} \vec{F}_k^* \cdot \vec{n}_k$$

Steady state computations provide a special case.

- Inverting the system derived from $\vec{\nabla} \cdot \vec{f} = 0$ directly is usually tough.
- Usually the above is used to iterate to the steady state.
- There are many convergence acceleration techniques.

Uniform Structured Meshes

The scheme can be simplified considerably on uniform Cartesian meshes,

$$
\begin{aligned}
U_{ijk}^{n+1} &= U_{ijk}^{n} - \frac{\Delta t}{\Delta x}\left(\vec{F}^{*}_{i+1/2jk} - \vec{F}^{*}_{i-1/2jk}\right) \\
&\quad - \frac{\Delta t}{\Delta y}\left(\vec{G}^{*}_{ij+1/2k} - \vec{G}^{*}_{ij-1/2k}\right) \\
&\quad - \frac{\Delta t}{\Delta z}\left(\vec{H}^{*}_{ijk+1/2} - \vec{H}^{*}_{ijk-1/2}\right)
\end{aligned}
$$

Dimensional splitting is often used here to improve speed and stability, though accuracy may diminish (Strang, 1968).

Conservation and Convergence

Conservation is clearly satisfied because

$$\sum_{\text{volumes}} \left(\sum_{\text{faces}} \vec{F}_k^* \cdot \vec{n}_k \right)_i = \sum_{\text{boundary}} \vec{F}_k^* \cdot \vec{n}_k$$

so the net flux equal to the contribution from the boundary.

Convergence of the scheme

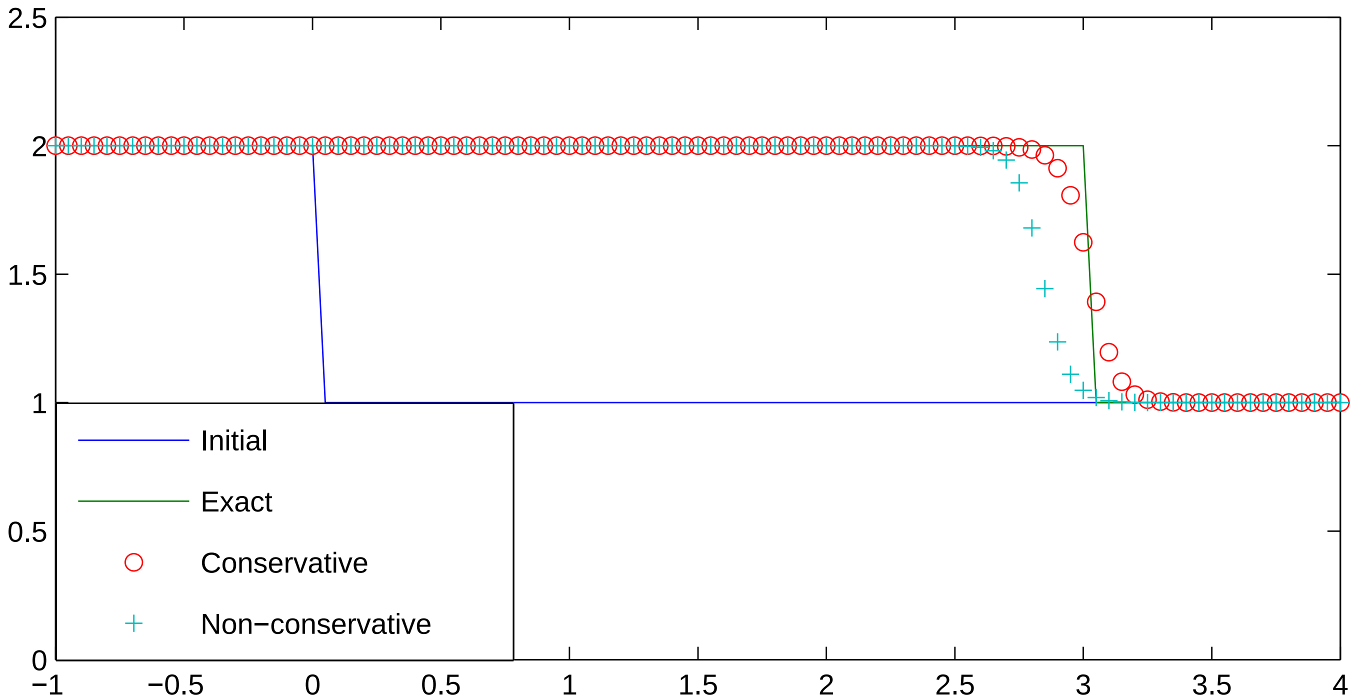
$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}(F^*_{i+1/2} - F^*_{i-1/2})$$

to a **weak solution** of $U_t + F_x = 0$ was proved by Lax and Wendroff in 1960. This has been extended to **some** more complex situations.

Why Does Conservation Matter?

The figure below shows results for Burgers' equation obtained using "sensible" discretisations of

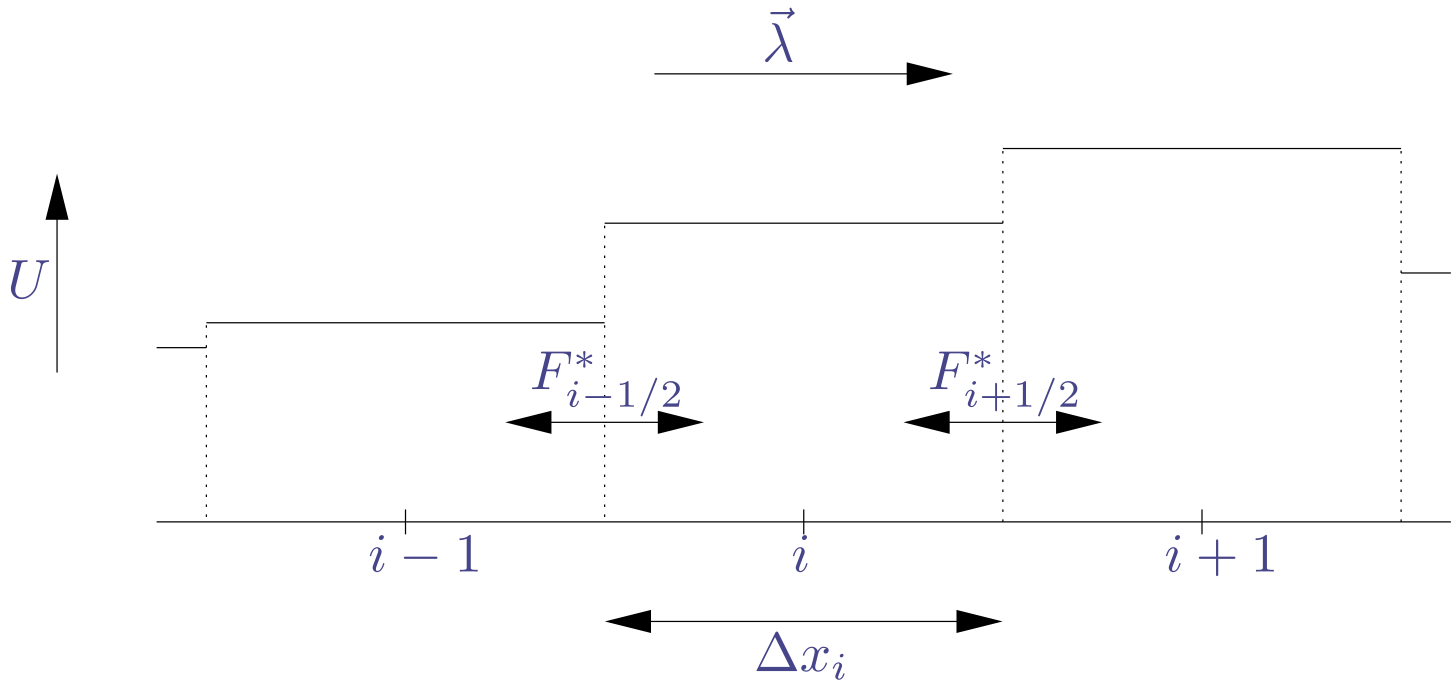
$$u_t + (u^2/2)_x = 0 \qquad \text{and} \qquad u_t + uu_x = 0$$



The One-Dimensional Case

Consider a first order cell-centre scheme in one dimension, which reduces to

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x_i}\left(F^*_{i+1/2} - F^*_{i-1/2}\right)$$



Example Numerical Fluxes

- **Central differences:**

$$F^*_{i+1/2} = \frac{1}{2}(F(U_i) + F(U_{i+1})) \quad \text{or} \quad F^*_{i+1/2} = F\left(\frac{U_i + U_{i+1}}{2}\right)$$

- **Lax-Wendroff:**

$$F^*_{i+1/2} = \frac{1}{2}(F(U_i) + F(U_{i+1})) - \frac{\lambda_{i+1/2}\Delta t}{2\Delta x_{i+1/2}}(F(U_{i+1}) - F(U_i))$$

- **Upwind:**

$$F^*_{i+1/2} = \begin{cases} F(U_i) & \text{if } \lambda_{i+1/2} \geq 0 \\ F(U_{i+1}) & \text{if } \lambda_{i+1/2} < 0 \end{cases}$$

$\lambda = \partial F/\partial U$ is the associated "wave" velocity.

Finite Difference Form

On uniform meshes, with constant advection, the equivalent finite difference schemes can easily be derived,

e.g. for the central difference scheme

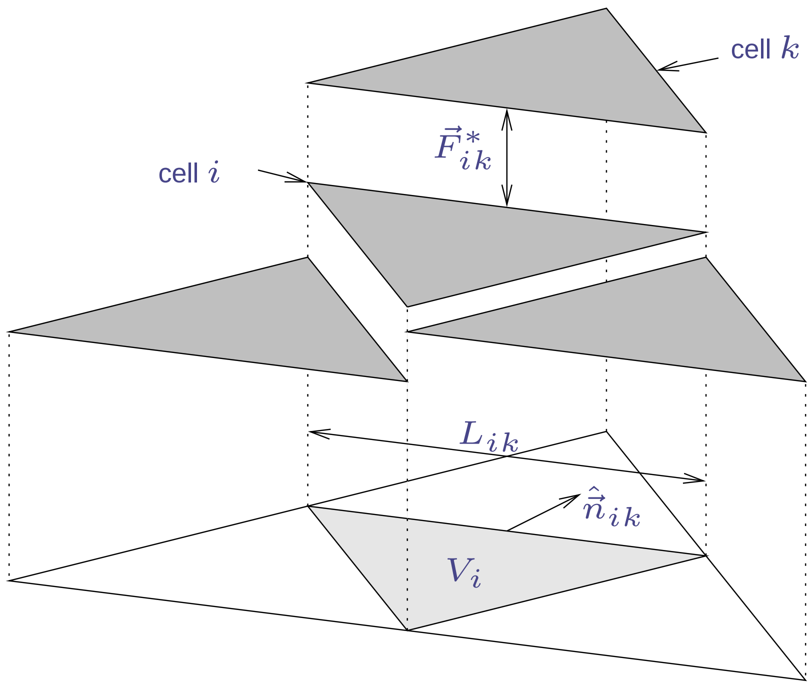
$$
\begin{aligned}
U_i^{n+1} &= U_i^n - \frac{\Delta t}{\Delta x}(F^*_{i+1/2} - F^*_{i-1/2}) \\
&= U_i^n - \frac{\Delta t}{\Delta x}\left[\frac{1}{2}(F(U_i) + F(U_{i+1})) - \frac{1}{2}(F(U_{i-1}) + F(U_i))\right] \\
&= U_i^n - \frac{\Delta t}{2\Delta x}\left(F(U_{i+1}) - F(U_{i-1})\right)
\end{aligned}
$$

Care is needed when λ changes sign in the upwind scheme.

The Two-Dimensional Case

Consider a first order cell-centre scheme in two dimensions, which reduces to

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{V_i} \sum_{\text{faces}} \vec{F}^*_{ik} \cdot \vec{n}_{ik}$$



Example Numerical Fluxes

- **Central differences:**

$$\vec{F}^*_{ik} = \frac{1}{2}(\vec{F}(U_i) + \vec{F}(U_k)) \quad \text{or} \quad \vec{F}^*_{ik} = \vec{F}\left(\frac{U_i + U_k}{2}\right)$$

- **Lax-Wendroff:**

$$\vec{F}^*_{ik} = \frac{1}{2}(\vec{F}(U_i) + \vec{F}(U_k)) - \frac{\vec{\lambda}_{ik} \cdot \hat{\vec{n}}_{ik}\,\Delta t}{2\Delta x_{ik}}(\vec{F}(U_k) - \vec{F}(U_i))$$

- **Upwind:**

$$\vec{F}^*_{ik} = \begin{cases} \vec{F}(U_i) & \text{if } \vec{\lambda}_{ik} \cdot \vec{n}_{ik} \geq 0 \\ \vec{F}(U_k) & \text{if } \vec{\lambda}_{ik} \cdot \vec{n}_{ik} < 0 \end{cases}$$

$\vec{\lambda} = \partial\vec{F}/\partial U$ is again the wave velocity.

Limiting the Flux

In 1959, Godunov showed that it is not possible for a **linear** scheme to be both higher than first order accurate **and** free of spurious oscillations.

Early attempts to address this include

- adding artificial diffusion/viscosity,
- Flux-Corrected Transport (Boris and Book, 1973),

but the most successful

- choose one low order, non-oscillatory flux and one higher order oscillatory flux,
- weight them to get the ideal combination.

Flux Limiters

First note that the upwind flux is equivalent to

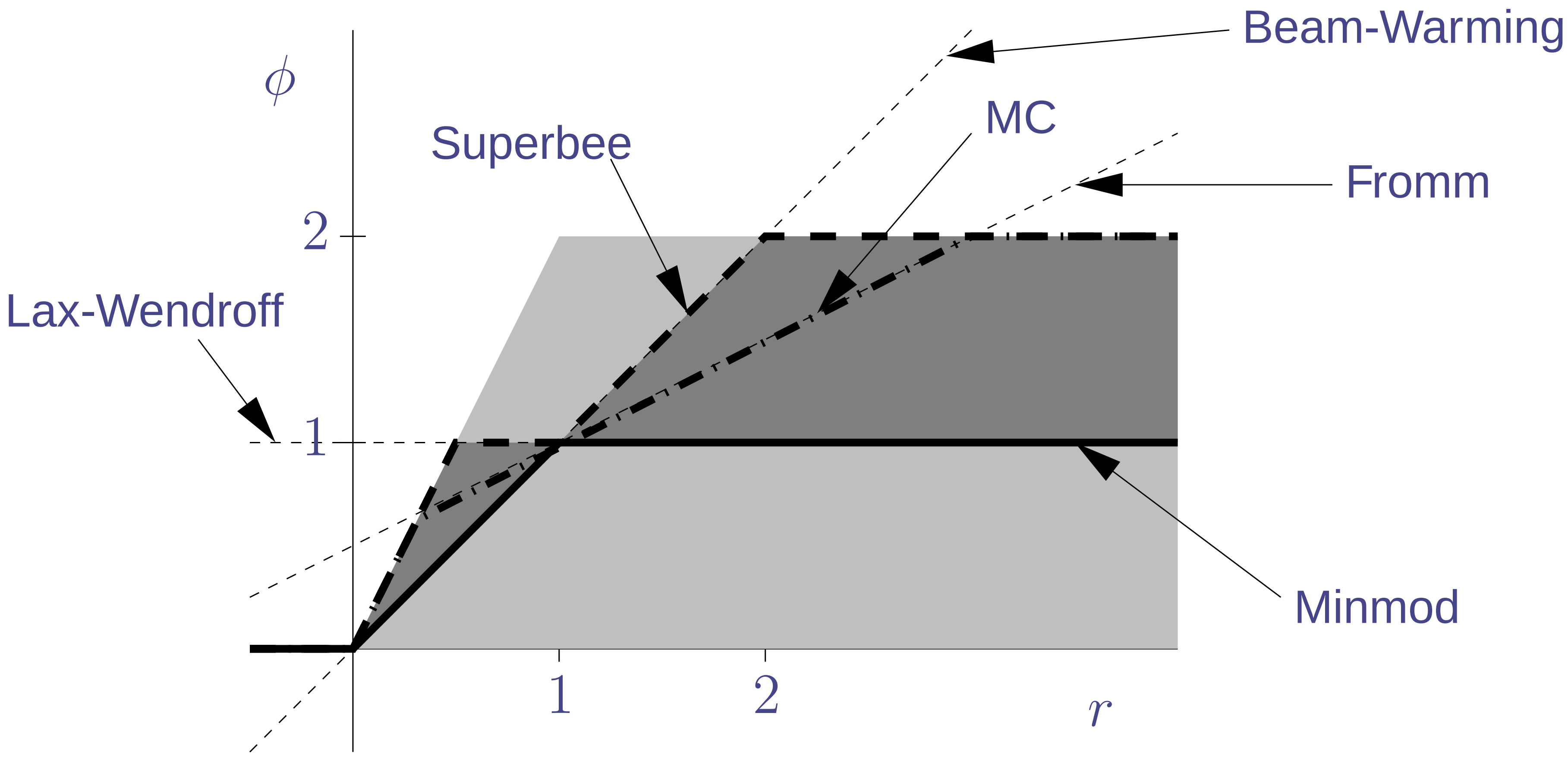
$$F^*_{i+1/2} = \frac{1}{2}(F(U_i) + F(U_{i+1})) - \frac{1}{2}\text{sign}(\lambda_{i+1/2})\,(F(U_{i+1}) - F(U_i))$$

i.e. a central difference plus a diffusion term.

The upwind and Lax-Wendroff schemes are weighted by a **limiter function** $\phi(r)$ where r is the ratio of successive solution differences, giving (with $\nu = \lambda\,\Delta t/\Delta x$)

$$\begin{aligned} F^*_{i+1/2} &= \frac{1}{2}(F(U_i) + F(U_{i+1})) \\ &\quad - \frac{1}{2}\text{sign}(\lambda_{i+1/2})\,[1 - \phi(r_{i+1/2})(1 - |\nu_{i+1/2}|)]\,(F(U_{i+1}) - F(U_i)) \end{aligned}$$

Limiter Functions



The **second order TVD region** of Sweby (1984) is dark grey.

Notes on Limiting

- It's really just parameter-free artificial diffusion.
- It can also be thought of as upwinding with additional "antidiffusion".
- For full second order accuracy the limiter should be smooth near $r = 1$.
- Ideally the limiter would also be symmetric,

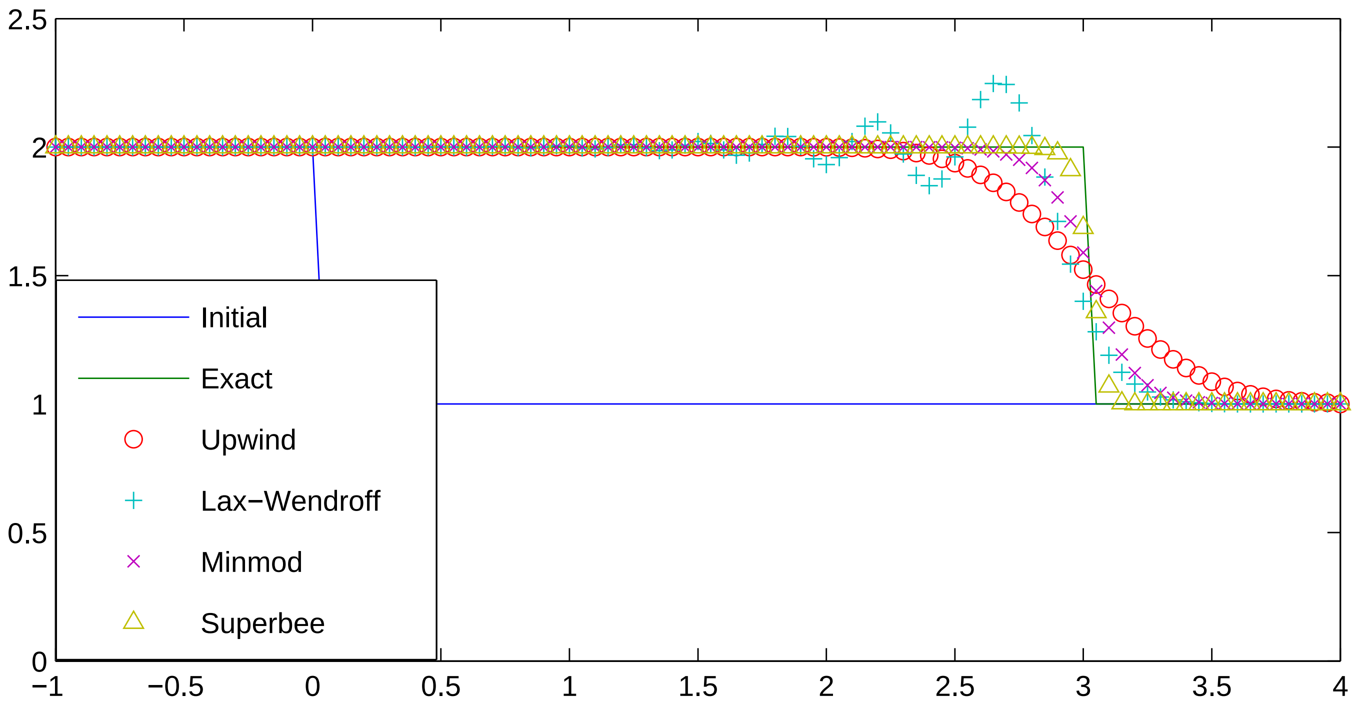
$$i.e. \quad \phi\left(\frac{1}{r}\right) = \frac{\phi(r)}{r}$$

- This is all subject to using an appropriately small time-step ($CFL \leq 1$ usually).

Do Limiters Help?

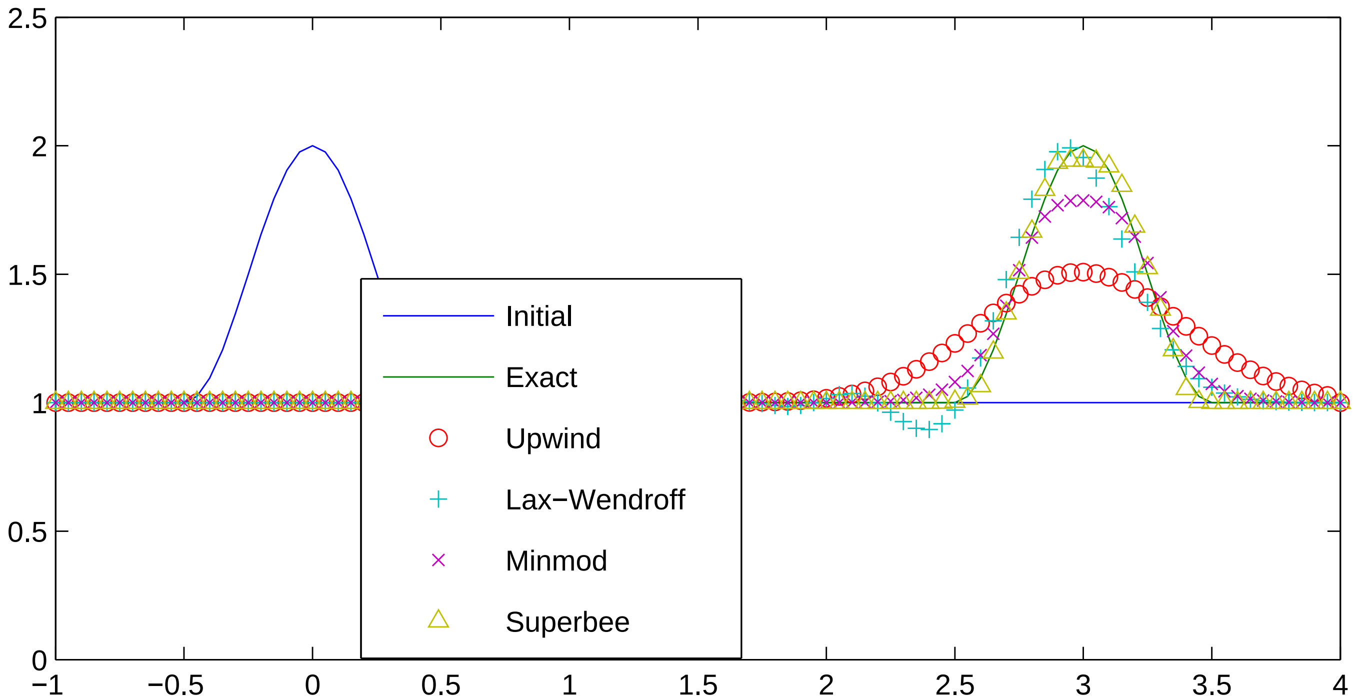
The figure below contains a series of results obtained for the advection equation

$$u_t + u_x \;=\; 0$$



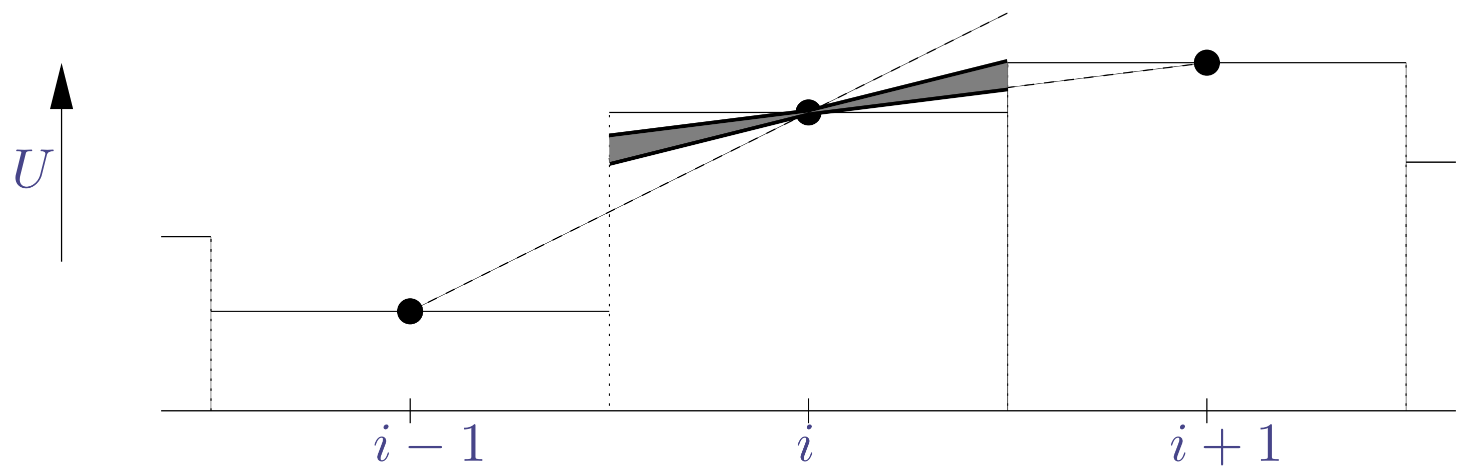
Potential Pitfalls

The figure below contains a series of results obtained for the advection equation. Careful inspection reveals that the profile is artificially steepened by the superbee limiter.



Slope Limiters

Instead of limiting the flux, create a limited higher order (piecewise linear discontinuous) reconstruction of the solution and then use the same flux functions with the new interface values.



Terminology

These schemes are often described as any of

- **Total Variation** ($\sum_{i=1}^{n} |U_i - U_{i-1}|$) **Diminishing** (TVD)
- **Monotone**
- **Monotonicity Preserving**
- **Positive**
- **Positivity Preserving**
- **Non-Oscillatory** (free of **spurious** oscillations)
- satisfying a **Maximum Principle**

though each has a slightly different meaning.

Multidimensional Limiting

For **flux limiting**, simply(!) use

$$\begin{aligned}\vec{F}^*_{ik} &= \frac{1}{2}(\vec{F}(U_i) + \vec{F}(U_k)) \\ &\quad - \frac{1}{2}\text{sign}(\lambda_{ik})\left[1 - \phi(r_{ik})(1 - |\nu_{ik}|)\right](\vec{F}(U_k) - \vec{F}(U_i))\end{aligned}$$

For **slope limiting**, many options exist for creating a piecewise linear representation of the dependent variable on an arbitrary mesh.

- The limiting is then similar to one dimension.
- Simply put the new interface values in to the numerical flux functions.

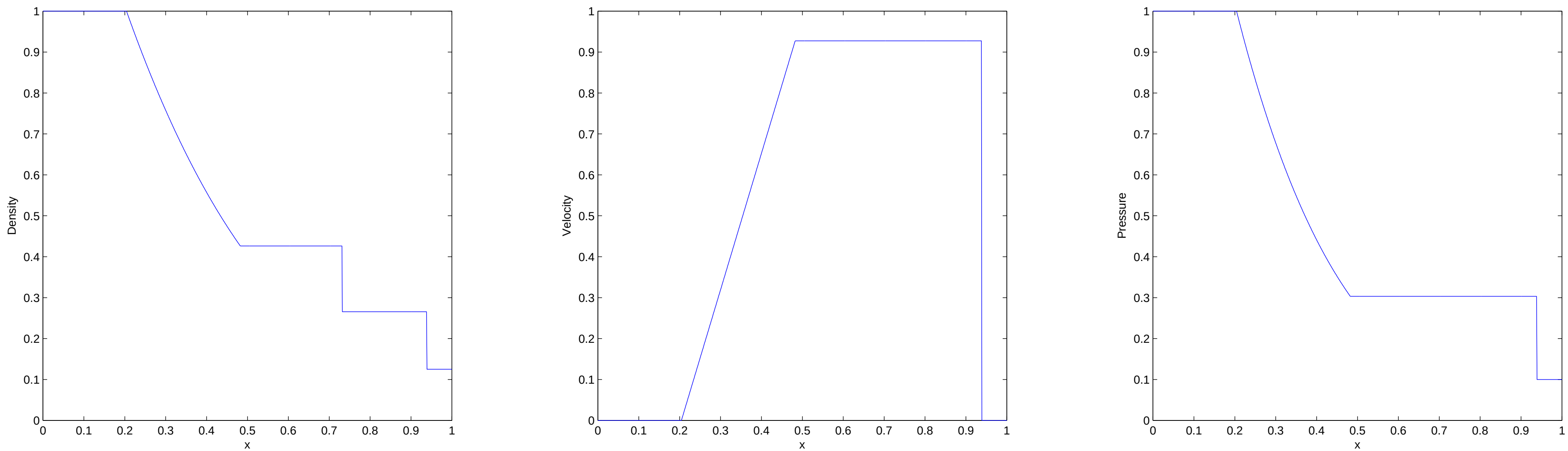
Notes

- High order, TVD time-stepping schemes also exist (Shu and Osher, 1988).
- The TVD property aids in convergence proofs, particularly when discontinuities are allowed.
- Unfortunately, Osher and Chakravarthy (1984) show that TVD schemes must degenerate to first order accuracy at extremal points.
- The loss of order propagates downstream!
- Discontinuous Galerkin is, in many ways, a more natural extension to high order accuracy.

Nonlinear Systems of Equations

The piecewise constant representation suggests the use of Riemann problems (Godunov, 1959).

Exact solutions are known, *e.g.* for the Euler equations,



Riemann Solvers

- **Godunov's scheme** (1959) solves the Riemann problem exactly to find the interface fluxes at the required time.
- **Roe's scheme** (1981) solves a linearised problem exactly, giving a piecewise constant representation.
- The **HLL scheme** (1983) and its offspring solve simplified Riemann problems with fewer intermediate states.
- The **Engquist-Osher scheme** (1980) integrates along a trajectory between the two initial states.

Beware of unphysical (**entropy violating**) solutions.

Multidimensional Flows

Typically the Riemann solvers are applied perpendicular to the control volume edges, reducing the flow to a series of quasi-one-dimensional problems,

$$U_t + (\vec{\lambda} \cdot \vec{n})\, U_\xi \;=\; 0$$

where ξ is the coordinate perpendicular to the edge.

- **Rotated Riemann solvers** (Davis, 1984) align with the flow for greater physical realism but are less robust.
- **Generalised Riemann solvers** exist for both higher order reconstructions and multidimensional problems.

Higher Derivatives

Second derivatives are simple to incorporate within the numerical fluxes on uniform meshes because

$$u_{xx} \approx \frac{U_{i+1} - 2U_i + U_{i-1}}{(\Delta x)^2} = \frac{1}{\Delta x}\left(\frac{U_{i+1} - U_i}{\Delta x} - \frac{U_i - U_{i-1}}{\Delta x}\right)$$

but this is very sensitive to mesh perturbations.

- Implicit time-stepping is often used.
- Higher derivatives are even more unpredictable.
- Often the Galerkin finite element scheme is used to approximate second derivatives (though issues remain).

Higher Order Accuracy

The TVD condition is typically relaxed when very high order accuracy is required. **Total Variation Bounded** (TVB) schemes are often used instead, such as

- The **Piecewise Parabolic Method** (PPM) of Colella and Woodward (1984).
- **Essentially Non-Oscillatory** (ENO) schemes (Harten et al., 1987) pick a number of reconstructions and choose the "smoothest".
- **Weighted ENO** (WENO) schemes (Liu et al., 1994) do something similar, but use a weighted average of the reconstructions to try to get higher accuracy.

Source terms

It is often very difficult to maintain equilibria of the form

$$\vec{\nabla} \cdot \vec{F} = S$$

using a typical finite volume approximation

$$\frac{\partial}{\partial t} \int_{\Omega} U \, d\Omega + \oint_{\partial\Omega} \vec{F} \cdot \vec{n} \, d\Gamma = \int_{\Omega} S \, d\Omega$$

because of the use of the divergence theorem, which can't be applied to the source term (Bermúdez and Vázquez, 1994).

Stiff source terms can also cause serious problems (LeVeque and Yee, 1990).

Summary

Finite volume methods are very commonly used for modelling hyperbolic systems of conservation laws.

- They are robust, efficient, flexible and reliable.
- They are typically very accurate...
- ...but there is limited theory to back this up...
- ...and there are many open questions associated with non-standard systems.
- They are **not** the answer to everything.

Further Reading

Finite Volume methods on uniform structured meshes are treated thoroughly and clearly in

- **Finite Volume Methods for Hyperbolic Problems,** R.J.LeVeque, Cambridge University Press, 2002,

(along with the CLAWPACK software), while

- **Numerical Schemes for Conservation Laws,** D.Kröner, Wiley-Teubner, 1997,

covers the more theoretical side comprehensively.

There are many other books which have chapters on the subject but I don't know of one which details fully the multidimensional case (and is less than 15 years old).