Assume that a problem has been identified

1. Which equations model the related problem ?
2. Do the equations have a solution ?
3. Is yes, is this solution unique ?
4. Can we compute the, one or all solutions ?
5. If not, what do we know about the solution(s) ?
6. Can we compute an approximation of the solution ?
7. If yes, what is the quality of the approximation ?
8. Can we improve it ? Which cost ?
9. Do the result comply with the observations ?
10. If not, the equations must be changed et go back to the begining.

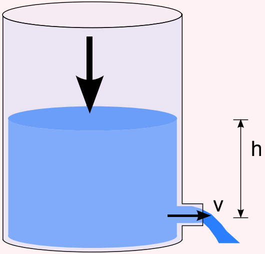
Drilled bucket and Carbon-14 dating

- Is it possible to know if and when were an empty drilled bucket filled with water ?
- Thanks to the disintegration ot the radio-active C^{14}, it is possible to date approximately the remains of organic materials.

In the second situation, it is possible to go back in the past while it is not possible in the first example.

Why ?

DRILLED BUCKET



The Torricelli law states, in ideal conditions,

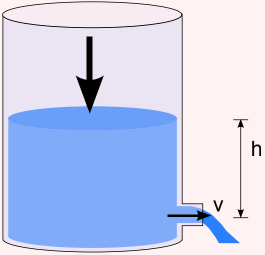
$$h'(t) = -C\sqrt{(h)}, \quad C = \frac{a}{A}\sqrt{2g}.$$

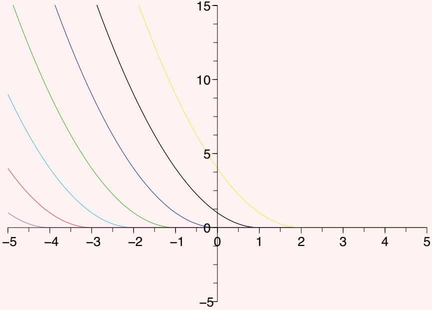
Assume initial datum is $h(4) = 0$.
For every $\alpha \leq 4$, the function

$$h(t) = \begin{cases} \frac{C^2}{4}(t-\alpha)^2, & t \leq \alpha, \\ 0, & t > \alpha. \end{cases}$$

is solution
Therefore, there exists an infinity of solutions.

DRILLED BUCKET





The empty bucket will remain empty, but it is impossible to know the past ot the bucket content.

Carbon-14 datation

The proportion of C^{14} is almost constant in a living organism.
After the death, the proportion of C^{14} loses $1/8000$ of its value each year.
Let $x(t)$ denotes the quantity of C^{14} with respect with time t (unit : year)

$$x'(t) = -\frac{1}{8000}x(t).$$

If $x(t_0)$ is known, then there exists a unique solution :

$$x(t) = x(t_0)\exp\left(-(t-t_0)/8000\right).$$

The knowledge of $x(t_0)$ allows to deduce the function x. In a same way, the knowledge of $x(t)$ and $x(t_0)$ allows to compute $(t-t_0)$.

What are the differences ?

The equations

$$(1)\ h'(t) = C\sqrt{h(t)} \qquad (2)\ x'(t) = -Cx(t)$$

are not very complicated et do not seem really different.
However, we have to avoid the problems of type (1).
Indeed, before any simulation, it is useful to know if the solution exists and is unique, because otherwise, what do we compute ?
Which criteria does allow to know, without computation, if a solution exists and is unique ?

The Cauchy Problem

Let $f : [a, b] \to \mathbb{R}$. One looks to an unkonwn function $y : t \mapsto y(t)$, $t \in [a, b]$ satisfiyng

$$\begin{aligned} &y'(t) = f(t, y(t)), \quad t \in [a, b], \\ &y(a) = y_0. \end{aligned}$$

Theorem (Cauchy-Lipschitz

If

- f in continuous on $[a, b] \times \mathbb{R}$
- it exists $L > 0$ such that $\forall (t, y_1, y_2) \in [a, b] \times \mathbb{R} \times \mathbb{R}$,

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$$

then there exists a unique solution y to the initial value problem within the range $[a, b]$.

Drilled bucket and datation

- The function $y \mapsto \sqrt{y}$ does not verify the criteria of Cauchy-Lipschitz theorem. Indeed
$$\lim_{y \to 0} \sqrt{y}/y = +\infty.$$
- Clearly, the function $y \mapsto y$ satisfy the critera. Thus, existence and uniqueness of the solution to ODE.

Higher order ODE

Any equation of order $n \geq 2$ s.t.

$$y^{(n)}(t) = f(t, y(t), y'(t), \cdots, y^{(n-1)})$$

can be recast as a n dimensions first order ODE

$$Y'(t) = F(t, Y(t))$$

where $Y(t) : [a, b] \to \mathbb{R}^n$
Idea : define $Y(t) = (y_1, \cdots y_n) = (y(t), y'(t), \cdots, y^{(n-1)})$ and $F(t, Y(t)) = f(t, y_1, \cdots, y_n)$.
Example : $y''(t) = 2y'(t) - y(t) + 3$ is transformed in

$$Y'(t) = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} Y(t) + \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

CAUCHY PROBLEM IN DIMENSION n

Let $F : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ and $Y_0 \in \mathbb{R}^n$. One looks to an unkonwn function $Y : t \mapsto Y(t)$, $t \in [a, b]$ satisfiyng

$$\begin{aligned} &Y'(t) = F(t, Y(t)), \quad t \in [a, b],\\ &Y(a) = Y_0. \end{aligned}$$

THEOREM (CAUCHY-LIPSCHITZ

If

- F in continuous on $[a, b] \times \mathbb{R}^n$
- F is Lipschitz continuous in Y : it exists $L > 0$ such that $\forall (t, y_1, y_2) \in [a, b] \times \mathbb{R}^n \times \mathbb{R}^n$,

$$\|F(t, Y_1) - F(t, Y_2)\|_2 \leq L\|Y_1 - Y_2\|_2$$

then there exists a unique solution Y to the initial value problem within the range $[a, b]$.