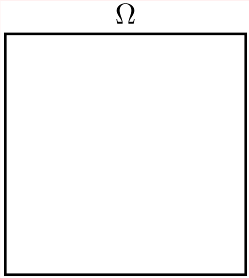
Finite volume scheme : principle



Studied problem

$$\begin{cases} \nabla \cdot \mathbf{J} = g \text{ on } \Omega, \\ \text{with} \\ \mathbf{J} = -\nabla r(u) + \mathbf{v} u \end{cases}$$

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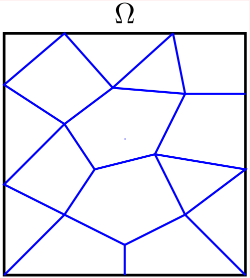

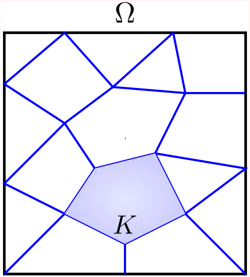

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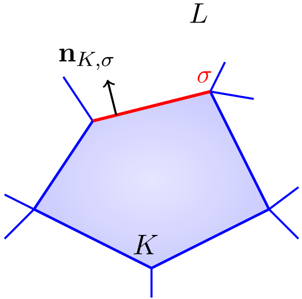
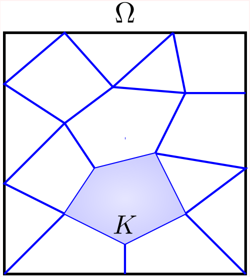
Integration of the Eq. on a volume of control





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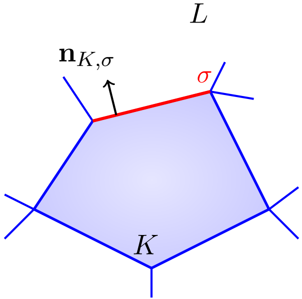
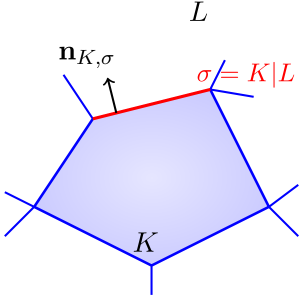


$$\sum_{\sigma \in \mathcal{E}_K} \int_\sigma \mathbf{J} \cdot \mathbf{n}_{K,\sigma} = \int_K g,$$

$\mathcal{E}_K$: set of every edges of K.

Generic form of the numerical scheme



$$\sum_{\sigma \in \mathcal{E}_K} \mathcal{F}_{K,\sigma} = m(K) g_K \quad \forall K$$

- consistency : $\mathcal{F}_{K,\sigma} \approx \int_\sigma \mathbf{J} \cdot \mathbf{n}_{K,\sigma}$
- conservativity : $\mathcal{F}_{K,\sigma} + \mathcal{F}_{L,\sigma} = 0$

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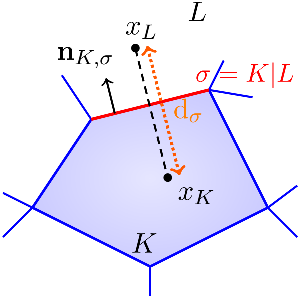

admissible mesh

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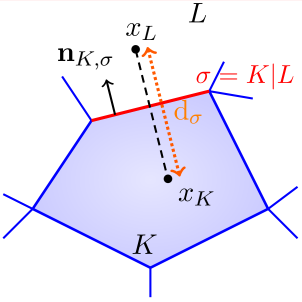
Classical numerical fluxes, $\mathbf{J} = -\nabla r(u) + \mathbf{v} u$

Centred fluxes

$$\mathcal{F}_{K,\sigma} = m(\sigma) \frac{r(u_K) - r(u_L)}{d_\sigma} + m(\sigma) v_{K,\sigma} \frac{u_K + u_L}{2}$$

$$(v_{K,\sigma} = \mathbf{v} \cdot \mathbf{n}_{K,\sigma})$$

Generic form of the numerical scheme



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Upwind fluxes

$$\mathcal{F}_{K,\sigma} = m(\sigma)\frac{r(u_K)-r(u_L)}{d_\sigma} + m(\sigma)(v^+_{K,\sigma}u_K - v^-_{K,\sigma}u_L)$$

$(v_{K,\sigma} = \mathbf{v}\cdot\mathbf{n}_{K,\sigma})$

$v^+_{K,\sigma}$: positive part of $v_{K,\sigma}$, $\quad v^-_{K,\sigma}$: negative part of $v_{K,\sigma}$

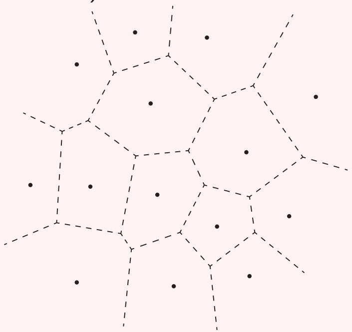
The Voronoi Diagram

Let a_i be points where the exact solution u are to be approximated

Voronoi Polygon

$$\widetilde{\Omega_i} := \{x \in \mathbb{R}^2 \,|\, |x - a_i| < |x - a_j| \text{ for all } j \neq i\}$$

The family $\{\widetilde{\Omega_i}\}_{i\in\Lambda}$ is called the Voronoi diagram of the point set $\{\widetilde{a_i}\}_{i\in\Lambda}$. ($\Lambda$: set of indices).

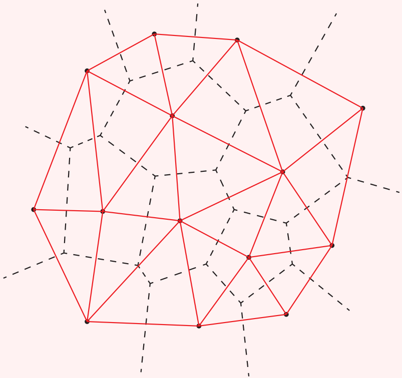


- The Voronoi polygons are convex, but not necessarily bounded, sets.
- Their boundaries are polygons.
- The vertices of these polygons are called Voronoi vertices.

Control volumes Ω_i : $\Omega_i := \widetilde{\Omega_i} \cap \Omega$, $i \in \Lambda$

- Set of indices of neighbouring nodes : $\Lambda_i := \{j \in \Lambda \ \{i\} \ : \ \partial\Omega_i \cap \partial\Omega_j \neq \emptyset\}$, $i \in \Lambda$.
- Joint piece of the boundaries of neighbouring control volumes : $\Gamma_{ij} := \{\partial\Omega_i \cap \partial\Omega_j,\ j \in \Lambda$.
- Length of Γ_{ij} : m_{ij}.

Dual graph of the Voronoi diagram : Delaunay triangulation

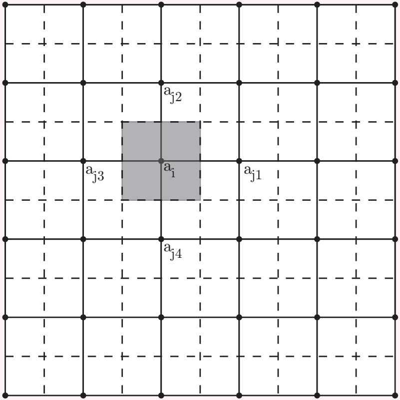


Theorem

If a triangulation consists of nonobtuse triangles exclusively, then it is a Delaunay triangulation, and the corresponding Voronoi diagram can be constructed by means of the perpendicular bisector of the triangles' edges.

Nonobtuse triangle : no interior angle is bigger than 90^o.

$$-\Delta u = f \quad \text{in } \Omega = (0,1)^2,$$
$$u = 0 \quad \text{on } \partial\Omega.$$



Control volumes

$$\Omega_i := \{x \in \Omega : |x - a_i|_\infty < h/2\}$$

$$-\sum_{k=1}^{4}\int_{\Gamma_{ij_k}} \mathbf{n}_{ij_k}\cdot\nabla u\, d\sigma = \int_{\Omega_i} f\, dx.$$

We see that

$$\begin{aligned}\mathbf{n}_{ij_1}\cdot\nabla u &= \partial_1 u, \quad & \mathbf{n}_{ij_2}\cdot\nabla u &= \partial_2 u,\\ \mathbf{n}_{ij_3}\cdot\nabla u &= -\partial_1 u, \quad & \mathbf{n}_{ij_4}\cdot\nabla u &= -\partial_2 u.\end{aligned}$$

Approximate integrals on Γ_{ij_k} by means of midpoint rule and replace the derivatives by difference quotients

$$\begin{aligned}&-\sum_{k=1}^{4}\int_{\Gamma_{ij_k}} \mathbf{n}_{ij_k}\cdot\nabla u\, d\sigma \approx -\sum_{k=1}^{4}\mathbf{n}_{ij_k}\cdot\nabla u\left(\frac{a_i+a_{j_k}}{2}\right)h\\ &\approx -\left[\frac{u(a_{j_1})-u(a_i)}{h}+\frac{u(a_{j_2})-u(a_i)}{h}+\frac{u(a_{j_3})-u(a_i)}{h}+\frac{u(a_{j_4})-u(a_i)}{h}\right]h\\ &= 4u(a_i)-\sum_{k=1}^{4}u(a_{j_k})\end{aligned}$$

$\int_{\Omega_i} f\, dx \approx f(a_i)h^2$

If $a_i \in \partial\Omega$, then parts of the boundary $\partial\Omega_i$ lie on $\partial\Omega$. At these nodes, the Dirichlet boundary conditions already prescribe values of the unknown function, and so there is no need to include the boundary control volumes into the balance equations.