FEM of higher order: Lagrange FEM

$$V_h = \{f \in C^0(\Omega),\ f|_{T_l} \in \mathbb{P}^k\} \subset H^1(\Omega), \quad T_l: \text{ triangle of the mesh.}$$

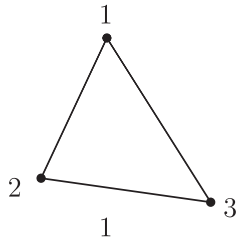
$\mathbb{P}^k$: polynomials of degree k. One needs $\frac{(k+1)(k+2)}{2}$ coefficients if 2D to determine uniquely such a polynomial. We need $\frac{(k+1)(k+2)}{2}$ constraints for each element (triangle) to be able to compute the restriction of every function on T_l.

We also need to insure continuity. Let A be an edge if a triangle, $p \in \mathbb{P}^k(\overline{\Omega})$. Then

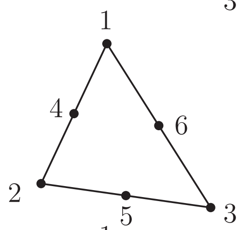
$$p(x,y)|_A = \text{polynomial of one variable, of degree } k$$

We need $k+1$ conditions to fixed it uniquely. We therefore have $\frac{(k+1)(k+2)}{2}$ constraints on each triangle and $k+1$ for each edge.

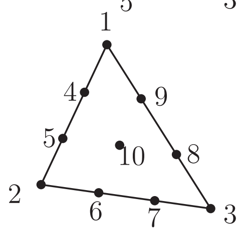
$\mathbb{P}^1$ 3 degrees of freedom $\rightarrow$ 3 constraints per triangles and 2 per edges. The only possibility is to put constraints on the three vertices



$\mathbb{P}^2$ 6 degrees of freedom $\rightarrow$ 6 constraints per triangles and 3 per edges. We choose to put constraints on the three vertices and on the middles of edges



$\mathbb{P}^3$ 10 degrees of freedom $\rightarrow$ 10 constraints per triangles and 4 per edges. We choose to put constraints on the three vertices, on the third and two third of edges, and on the center of gravity of the triangle.



Basis functions and shape functions

We have to make the distinction between physical nodes and nodes/degrees of freedom to build the basis function, *i.e* the physical nodes and nodes where unknowns live. We denote X_I the nodes of this new mesh. A basis function should satisfy $\varphi_J(X_I) = \delta_{IJ}$. A local shape function is the restriction of each φ_I to every triangle. We denote them as N_i, $i = 1, cdots, n_k$, $n_k = \frac{(k+1)(k+2)}{2}$. If A_j denotes a local node of triangle T_l, therefore $N_i(A_j) = \delta_{ij}$.

$\mathbb{P}^2$

$$\begin{aligned} &N_i = \lambda_i(2\lambda_i - 1),\ i = 1,2,3 \\ &\nabla N_i = (4\lambda_i - 1)\nabla\lambda_i \\ &N_4 = 4\lambda_1\lambda_2,\ N_5 = 4\lambda_2\lambda_3,\ N_6 = 4\lambda_3\lambda_1, \\ &\nabla N_4 = 4(\lambda_i\nabla\lambda_2 + \lambda_2\nabla\lambda_1),\ \cdots \end{aligned}$$

$\mathbb{P}^3$

$$\begin{aligned} &N_i = \lambda_i(3\lambda_i - 1)(3\lambda_i - 2)/2,\ i = 1,2,3 \\ &N_4 = 9\lambda_1(3\lambda_1 - 1)\lambda_2/2,\ \cdots \\ &N_{10} = 27\lambda_1\lambda_2\lambda_3 \end{aligned}$$

We can use the technique of the triangle of reference to compute the solution on every triangle.