The Finite Element Method for Linear Elliptic Boundary Value Problems of Second Order

Variational Equations and Sobolev Spaces

general assumption:

$$\begin{aligned} &V \text{ is a vector space with scalar product } \langle\cdot,\cdot\rangle \text{ and the norm } \|\cdot\| \\ &\text{ induced by } \langle\cdot,\cdot\rangle \text{ (for this, } \|v\| := \langle v,v\rangle^{1/2} \text{ for } v \in V is satisfied) ; \\ &V \text{ is complete with respect to } \|\cdot\|, \text{ i.e. a Hilbert space} ; \\ &a : V\times V \to \mathbb{R} \text{ is a (not necessarily symmetric) bilinear form} ; \\ &b : V \to \mathbb{R} \text{ is a linear form} . \end{aligned} \tag{3.1}$$

Theorem 3.1 (Lax–Milgram) *Suppose the following conditions are satisfied:*

- *a is continuous* *that is, there exists some constant $M > 0$ such that*

$$|a(u,v)| \quad \le \quad M\|u\|\,\|v\| \quad \textit{for all } u,v \in V ; \tag{3.2}$$

- *a is V-elliptic* *that is, there exists some constant $\alpha > 0$ such that*

$$a(u,u) \quad \ge \quad \alpha\|u\|^2 \quad \textit{for all } u \in V ; \tag{3.3}$$

- *b is continuous; that is, there exists some constant $C > 0$ such that*

$$|b(u)| \quad \le \quad C\|u\| \quad \textit{for all } u \in V . \tag{3.4}$$

Then the variational equation

$$\textit{find } \bar{u} \in V \textit{ such that} \quad a(\bar{u},v) = b(v) \quad \textit{for all } v \in V, \tag{3.5}$$

has one and only one solution.

Here, one cannot avoid the assumptions (3.1) *and* (3.2)–(3.4) *in general.*

Definition 3.2 Suppose $\Omega \subset \mathbb{R}^d$ is a (bounded) domain.
The *Sobolev space $H^k(\Omega)$* is defined by

$$\boxed{\begin{aligned}H^k(\Omega) := \big\{ v : \Omega \to \mathbb{R} \,\big|\, v \in L^2(\Omega)\,, \text{ the weak derivatives } \partial^\alpha v \text{ exist} \\ \text{in } L^2(\Omega) \text{ and for all multi-indices } \alpha \text{ with } |\alpha| \le k \big\}\,.\end{aligned}}$$

A scalar product $\langle \cdot, \cdot \rangle_k$ and the resulting norm $\| \cdot \|_k$ in $H^k(\Omega)$ are defined as follows:

$$\langle v, w \rangle_k := \int_\Omega \sum_{\substack{\alpha \text{ multi-index} \\ |\alpha| \le k}} \partial^\alpha v \, \partial^\alpha w \, dx\,, \tag{3.6}$$

$$\begin{aligned} \|v\|_k &:= \langle v, v \rangle_k^{1/2} = \Bigg(\int_\Omega \sum_{\substack{\alpha \text{ multi-index} \\ |\alpha| \le k}} \big|\partial^\alpha v\big|^2 \, dx \Bigg)^{1/2} \\ &= \Bigg(\sum_{\substack{\alpha \text{ multi-index} \\ |\alpha| \le k}} \int_\Omega \big|\partial^\alpha v\big|^2 \, dx \Bigg)^{1/2} = \Bigg(\sum_{\substack{\alpha \text{ multi-index} \\ |\alpha| \le k}} \big\|\partial^\alpha v\big\|_0^2 \Bigg)^{1/2}. \end{aligned} \tag{3.7}$$

Besides the norms $\| \cdot \|_k$, there are seminorms $| \cdot |_l$ for $0 \le l \le k$ in $H^k(\Omega)$, defined by

$$|v|_l = \Bigg(\sum_{\substack{\alpha \text{ multi-index} \\ |\alpha| = l}} \big\|\partial^\alpha v\big\|_0^2 \Bigg)^{1/2},$$

such that

$$\|v\|_k = \left(\sum_{l=0}^{k} |v|_l^2 \right)^{1/2},$$

In particular, these definitions are compatible with those in (2.18),

$$\langle v, w \rangle_1 := \int_\Omega vw + \nabla v \cdot \nabla w \, dx\,,$$

and with the notation $\| \cdot \|_0$ for the $L^2(\Omega)$ norm, giving a meaning to this one.

Theorem 3.3 *The bilinear form $\langle \cdot, \cdot \rangle_k$ is a scalar product on $H^k(\Omega)$; that is, $\| \cdot \|_k$ is a norm on $H^k(\Omega)$.*

$H^k(\Omega)$ is complete with respect to $\| \cdot \|_k$, and is thus a Hilbert space.

Theorem 3.5 (Trace Theorem) *Suppose Ω is a bounded Lipschitz domain. We define*

$$C^\infty(\mathbb{R}^d)|_\Omega \quad := \quad \left\{ v : \Omega \to \mathbb{R} \;\middle|\; v \text{ can be extended to } \tilde{v} : \mathbb{R}^d \to \mathbb{R} \text{ and } \tilde{v} \in C^\infty(\mathbb{R}^d) \right\}.$$

Then, $C^\infty(\mathbb{R}^d)|_\Omega$ is dense in *$H^1(\Omega)$; that is, with respect to $\|\cdot\|_1$ an arbitrary $w \in H^1(\Omega)$ can be approximated arbitrarily well by some $v \in C^\infty(\mathbb{R}^d)|_\Omega$.*

The mapping that restricts v to $\partial\Omega$,

$$\begin{aligned} \gamma_0 : \left(C^\infty(\mathbb{R}^d)|_\Omega, \| \cdot \|_1\right) &\to \left(L^2(\partial\Omega), \| \cdot \|_0\right) , \\ v &\mapsto v|_{\partial\Omega} , \end{aligned}$$

is continuous.

Thus there exists a unique, linear, and continuous extension

$$\gamma_0 : \left(H^1(\Omega), \| \cdot \|_1\right) \to \left(L^2(\partial\Omega), \| \cdot \|_0\right).$$

Therefore, in short form, $\gamma_0(v) \in L^2(\partial\Omega)$, and there exists some constant $C > 0$ such that

$$\|\gamma_0(v)\|_0 \le C\|v\|_1 \quad \text{for all } v \in H^1(\Omega).$$

Here $\gamma_0(v) \in L^2(\partial\Omega)$ is called the *trace* of $v \in H^1(\Omega)$.

Definition 3.6 $H_0^1(\Omega) := \left\{ v \in H^1(\Omega) \;\middle|\; \gamma_0(v) = 0 \text{ (as a function on } \partial\Omega) \right\}$.

Theorem 3.8 *Suppose $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain. The outer unit normal vector $\nu = (\nu_i)_{i=1,\dots,d} : \partial\Omega \to \mathbb{R}^d$ is defined almost everywhere and $\nu_i \in L^\infty(\partial\Omega)$.*

For $v, w \in H^1(\Omega)$ and $i = 1, \dots, d$,

$$\int_\Omega \partial_i v \, w \, dx = - \int_\Omega v \, \partial_i w \, dx + \int_{\partial\Omega} v \, w \, \nu_i \, d\sigma .$$

$$v \in H^2(\Omega) \overset{\text{continuous}}{\mapsto} \partial_i v \in H^1(\Omega) \quad \overset{\text{continuous}}{\mapsto} \quad \partial_i v|_{\partial\Omega} \in L^2(\partial\Omega)$$

$$\overset{\text{continuous}}{\mapsto} \quad \partial_i v|_{\partial\Omega}\, \nu_i \in L^2(\partial\Omega)$$

Corollary 3.9 *Suppose $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain.*

(1) Let $w \in H^1(\Omega)$, $q_i \in H^1(\Omega)$, $i = 1, \ldots, d$. Then

$$\int_\Omega q \cdot \nabla w \, dx = -\int_\Omega \nabla \cdot q \, w \, dx + \int_{\partial\Omega} q \cdot \nu \, w \, d\sigma \, .$$

(2) Let $v \in H^2(\Omega)$, $w \in H^1(\Omega)$. Then

$$\int_\Omega \nabla v \cdot \nabla w \, dx = -\int_\Omega \Delta v \, w \, dx + \int_{\partial\Omega} \partial_\nu v \, w \, d\sigma \, .$$

Elliptic Boundary Value Problems of Second Order

We consider the equation

$$\boxed{(Lu)(x) := -\nabla \cdot (K(x)\nabla u(x)) + c(x) \cdot \nabla u(x) + r(x)u(x) = f(x) \text{ for } x \in \Omega}$$

with the data

$$K : \Omega \to \mathbb{R}^{d,d}, \quad c : \Omega \to \mathbb{R}^d, \quad r, f : \Omega \to \mathbb{R}.$$

Boundary Conditions

suppose $\Gamma_1, \Gamma_2, \Gamma_3$ is a disjoint decomposition of the boundary $\partial\Omega$

$$\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 ,$$

where Γ_3 is a closed subset of the boundary. For given functions $g_j : \Gamma_j \to \mathbb{R}$, $j = 1, 2, 3$, and $\alpha : \Gamma_2 \to \mathbb{R}$ we assume on $\partial\Omega$

- Neumann boundary condition

$$K\nabla u \cdot \nu = \partial_{\nu_K} u = g_1 \quad \text{on } \Gamma_1 ,$$

- mixed boundary condition

$$K\nabla u \cdot \nu + \alpha u = \partial_{\nu_K} u + \alpha u = g_2 \quad \text{on } \Gamma_2 ,$$

- Dirichlet boundary condition

$$u = g_3 \quad \text{on } \Gamma_3 .$$

Assumptions

$$k_{ij}\, , c_i\, , \nabla \cdot c\, , r \in L^\infty(\Omega)\, , \; f \in L^2(\Omega)\, , \quad i, j \in \{1, \dots, d\}\, ,$$
$$\text{and if } |\Gamma_1 \cup \Gamma_2|_{d-1} > 0\, , \quad \nu \cdot c \in L^\infty(\Gamma_1 \cup \Gamma_2)\, .$$

Furthermore, the *uniform ellipticity* of L is assumed: There exists some constant $k_0 > 0$ such that for (almost) every $x \in \Omega$,

$$\sum_{i,j=1}^{d} k_{ij}(x)\xi_i\xi_j \geq k_0|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^d$$

Moreover, K should be symmetric.

$$g_j \in L^2(\Gamma_j)\, , \quad j = 1, 2, 3\, , \quad \alpha \in L^\infty(\Gamma_2)$$

Step 1: Multiplication of the differential equation by test functions that are chosen compatible with the type of boundary condition and subsequent integration over the domain Ω.

Step 2: Integration by parts under incorporation of the boundary conditions in order to derive a suitable bilinear form.

Step 3: Verification of the required properties like ellipticity and continuity.

(I) Homogeneous Dirichlet Boundary Condition
$\partial\Omega = \Gamma_3\,,\ g_3 \equiv 0\,,\ V := H_0^1(\Omega)$

$$\begin{aligned} a(u,v) &:= \int_\Omega \{K\nabla u \cdot \nabla v + c \cdot \nabla u\, v + r\, uv\}\, dx \\ &= b(v) := \int_\Omega f v\, dx \qquad \text{for all } v \in C_0^\infty(\Omega) \text{ or } v \in V \end{aligned}$$

Theorem 2.18 (Poincaré) *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Then a constant $C > 0$ exists (depending on Ω) such that*

$$\|u\|_0 \leq C \left(\int_\Omega |\nabla u(x)|^2\, dx \right)^{1/2} \quad \textit{for all } u \in H_0^1(\Omega)\,.$$

(II) Mixed Boundary Conditions $\partial\Omega = \Gamma_2\,,\ V = H^1(\Omega)$

$$\begin{aligned} a(u,v) &:= \int_\Omega \{K\nabla u \cdot \nabla v + c \cdot \nabla u\, v + r\, uv\}\, dx + \int_{\partial\Omega} \alpha\, uv\, d\sigma \\ &= b(v) := \int_\Omega f v\, dx + \int_{\partial\Omega} g_2 v\, d\sigma \quad \text{for all } v \in V\,. \end{aligned}$$

(III) General Case

First, we consider the case of a **homogeneous Dirichlet boundary condition** on Γ_3 with $|\Gamma_3|_{d-1} > 0$. For this, we define

$$V := \left\{ v \in H^1(\Omega) : \gamma_0(v) = 0 \text{ on } \Gamma_3 \right\}$$

$$\boxed{\begin{aligned} a(u,v) \quad &:= \quad \int_\Omega \{K\nabla u \cdot \nabla v + c \cdot \nabla u\, v + r\, uv\}\, dx + \int_{\Gamma_2} \alpha\, uv\, d\sigma \\ &= \quad b(v) := \int_\Omega f v\, dx + \int_{\Gamma_1} g_1 v\, d\sigma + \int_{\Gamma_2} g_2 v\, d\sigma \quad \text{for all } v \in V \end{aligned}}$$

Now we address the case of **inhomogeneous Dirichlet boundary conditions** ($|\Gamma_3|_{d-1} > 0$).

This situation can be reduced to the case of homogeneous Dirichlet boundary conditions, if we are able to choose some (fixed) element $w \in H^1(\Omega)$ in such a way that (in the sense of trace) we have

$$\gamma_0(w) = g_3 \quad \text{on } \Gamma_3\,.$$

$$\tilde{V} := \left\{ v \in H^1(\Omega) : \gamma_0(v) = g_3 \text{ on } \Gamma_3 \right\} = \left\{ v \in H^1(\Omega) : v - w \in V \right\}$$

Find $\tilde{u} \in \tilde{V}$ such that

$$a(\tilde{u}, v) = b(v) \quad \text{for all } v \in V\,.$$

However, this formulation does not fit into the theoretical concept since the space $\tilde{V}$ is not a linear one

If we put $\boxed{\tilde{u} := u + w,}$ then this is equivalent to the following:
Find $u \in V$ such that

$$\boxed{\begin{aligned} &a(u,v) = b(v) - a(w,v) =: \tilde{b}(v) \quad \text{for all } v \in V\,. \\ &V = \left\{ v \in H^1(\Omega) : \gamma_0(v) = 0 \text{ on } \Gamma_3 \right\} \end{aligned}}$$

The Galerkin approximation of the variational equation

Find some $u \in V_h$ such that

$$a(u_h, v) = b(v) - a(w, v) = \tilde{b}(v) \quad \text{for all } v \in V_h$$

The space V_h that is to be defined has to satisfy $V_h \subset V$

For a given vector space V_h let

$$P_K := \{v|_K \mid v \in V_h\} \quad \text{for } K \in \mathcal{T}_h ,$$

that is,

$$V_h \subset \left\{v : \Omega \to \mathbb{R} \;\middle|\; v|_K \in P_K \text{ for all } K \in \mathcal{T}_h\right\} .$$

Definition 3.19 A *triangulation* $\mathcal{T}_h$ of a set $\Omega \subset \mathbb{R}^d$ consists of a finite number of subsets K of Ω with the following properties:

(T1) Every $K \in \mathcal{T}_h$ is closed.
(T2) For every $K \in \mathcal{T}_h$ its nonempty interior int (K) is a Lipschitz domain.
(T3) $\overline{\Omega} = \cup_{K \in \mathcal{T}_h} K$.
(T4) For different K_1 and K_2 of $\mathcal{T}_h$ the intersection of int (K_1) and int (K_2) is empty.

$$h = \max\left\{\operatorname{diam}(K) \;\middle|\; K \in \mathcal{T}_h\right\}$$