Finite differences, finite elements et finite volume methods

EXERCISE 1. *Heated and cooled plate.*

We want to determine numerically the heat field on a square plate, with boundaries kept à 20˚C, heated on a point at 100˚C, and cooled on an other one at -50˚C (see Fig. 1).

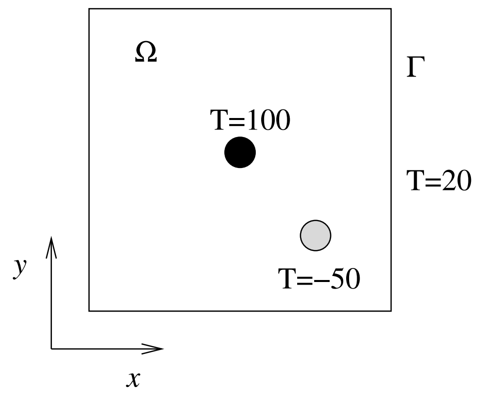


FIGURE 1 – Plaque

It can be shown that after some period of time that the temperature $T(x,y)$ stays constant, and it is solution to the stationary heat equation

$$\begin{cases} -\Delta T = S(x,y) & (x,y) \in \Omega, \\ T_{|\Gamma} = 20, \end{cases}$$

where S is a surfacic density of temperature which models the hot and cold points.

Solve numerically this problem with finite differences method.

EXERCISE 2. *Drilled plate*

We consider for this new exercice a square plate on which a square part is removed. The external boundary is maintained at 20˚C and the internal one at 100˚C (see Figure 2).

Once again, the temperature is solution to the following PDE, with boudnary conditions but without any source term

$$\begin{cases} -\Delta T = 0 & (x,y) \in \Omega, \\ T_{|\Gamma_1} = 20, & T_{|\Gamma_2} = 100. \end{cases}$$

Solve numerically this problem with finite differences methods.

EXERCISE 3. *Varying diffusion coefficient*

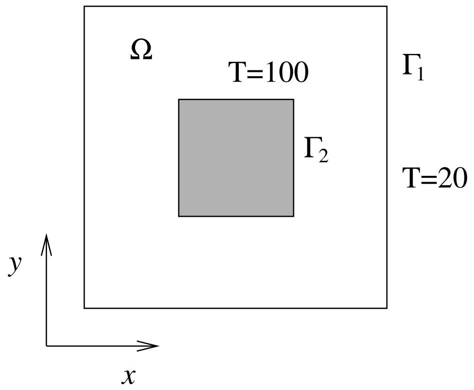


FIGURE 2 – Drilled plaque

We now assume that the plate is made of different components. In this case, the diffusion coefficients are different and the heat equation is replaced by the diffusion equation

$$-\nabla \cdot (C(x,y)\nabla T) = S(x,y) \quad (x,y) \in \Omega.$$

Reproduce the two previous exercices for various materials.

EXERCISE 4. *Gauss Quadrature*

We want to be able to compute $I_1 = \int_{-1}^{1} f(\xi)d\xi$ or $I_2 = \int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)d\xi\,d\eta$. To do so, we use a quadrature formula of Gaussian type $I_1 = \sum_{i=1}^{n} H_i f(\xi_i)$. In 2D, the idea is to first evaluate the inner integral keeping η constant,i.e. $\int_{-1}^{1} f(\xi,\eta)d\xi = \sum_{i=1}^{n} H_i f(\xi_i,\eta) = \psi(\eta)$. Evaluating the outer integral in a similar manner, we have

$$I = \int_{-1}^{1} \psi(\eta)d\eta = \sum_{i=1}^{n} H_i \psi(\eta_i) = \sum_{i=1}^{n}\sum_{j=1}^{n} H_i H_j f(\xi_j,\eta_i).$$

The values of the weights H_i are avalaible in Fig. 3.

Build function which can evaluate an integral over a finite interval or finite quadrilateral.

EXERCISE 5. *Numerical integration on triangular region.*

For a triangle, the integration is not easy and we have to select numerical quadrature points following Fig. 4. Build function which can evaluate an integral over a finite triangle.

EXERCISE 6. *Interaction with Gmsh*

Build a function that can handle two dimensional meshes generated by Gmsh (or Freefem).

EXERCISE 7. *Finite element*

Reproduce Exercice 1 and 2 for general domain.

$\pm a$		H
	$n = 1$	
0		2.000 000 000 000 000
	$n = 2$	
$1/\sqrt{3}$		1.000 000 000 000 000
	$n = 3$	
$\sqrt{0.6}$		5/9
0.000 000 000 000 000		8/9
	$n = 4$	
0.861 136 311 594 953		0.347 854 845 137 454
0.339 981 043 584 856		0.652 145 154 862 546
	$n = 5$	
0.906 179 845 938 664		0.236 926 885 056 189
0.538 469 310 105 683		0.478 628 670 499 366
0.000 000 000 000 000		0.568 888 888 888 889
	$n = 6$	
0.932 469 514 203 152		0.171 324 492 379 170
0.661 209 386 466 265		0.360 761 573 048 139
0.238 619 186 083 197		0.467 913 934 572 691
	$n = 7$	
0.949 107 912 342 759		0.129 484 966 168 870
0.741 531 185 599 394		0.279 705 391 489 277
0.405 845 151 377 397		0.381 830 050 505 119
0.000 000 000 000 000		0.417 959 183 673 469
	$n = 8$	
0.960 289 856 497 536		0.101 228 536 290 376
0.796 666 477 413 627		0.222 381 034 453 374
0.525 532 409 916 329		0.313 706 645 877 887
0.183 434 642 495 650		0.362 683 783 378 362
	$n = 9$	
0.968 160 239 507 626		0.081 274 388 361 574
0.836 031 107 326 636		0.180 648 160 694 857
0.613 371 432 700 590		0.260 610 696 402 935
0.324 253 423 403 809		0.312 347 077 040 003
0.000 000 000 000 000		0.330 239 355 001 260
	$n = 10$	
0.973 906 528 517 172		0.066 671 344 308 688
0.865 063 366 688 985		0.149 451 349 150 581
0.679 409 568 299 024		0.219 086 362 515 982
0.433 395 394 129 247		0.269 266 719 309 996
0.148 874 338 981 631		0.295 524 224 714 753

FIGURE 3 – Absissae and weight coefficients of the gaussian quadrature formula $\int_{-1}^{1} f(x)dx = \sum_{i=1}^{n} H_i f(a_i)$

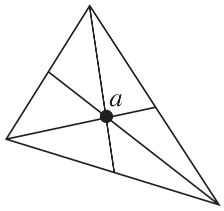
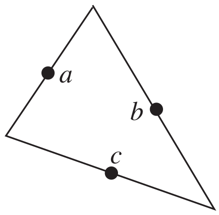
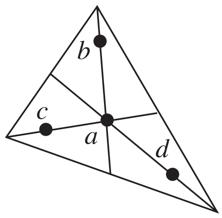
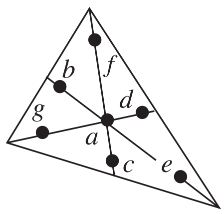
Order	Figure	Error	Points	Triangular coordinates	Weights
Linear		$R = O(h^2)$	a	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	1
Quadratic		$R = O(h^3)$	a b c	$\frac{1}{2}, \frac{1}{2}, 0$ $0, \frac{1}{2}, \frac{1}{2}$ $\frac{1}{2}, 0, \frac{1}{2}$	$\frac{1}{3}$ $\frac{1}{3}$ $\frac{1}{3}$
Cubic		$R = O(h^4)$	a b c d	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$ $\left.\begin{matrix} 0.6, 0.2, 0.2 \\ 0.2, 0.6, 0.2 \\ 0.2, 0.2, 0.6 \end{matrix}\right\}$	$-\frac{27}{48}$ $\frac{25}{48}$
Quintic		$R = O(h^6)$	a b c d e f g	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$ $\left.\begin{matrix} \alpha_1, \beta_1, \beta_1 \\ \beta_1, \alpha_1, \beta_1 \\ \beta_1, \beta_1, \alpha_1 \end{matrix}\right\}$ $\left.\begin{matrix} \alpha_2, \beta_2, \beta_2 \\ \beta_2, \alpha_2, \beta_2 \\ \beta_2, \beta_2, \alpha_2 \end{matrix}\right\}$	0.225 000 000 0 0.132 394 152 7 0.125 939 180 5
			with $\alpha_1 = 0.059\,715\,871\,7$ $\beta_1 = 0.470\,142\,064\,1$ $\alpha_2 = 0.797\,426\,985\,3$ $\beta_2 = 0.101\,286\,507\,3$		

FIGURE 4 – Numericcal integration formulae for triangles