Project on Domain Decomposition Method

This project concerns a special class of numerical methods for solving partial differential equations (PDEs), where the methods of concern are based on a physical decomposition of a global solution domain. The global solution to a PDE is then sought by solving the smaller subdomain problems collaboratively and "patching together" the subdomain solutions. These numerical methods are therefore termed as domain decomposition (DD) methods. We concentrate on one special group of DD methods, namely iterative DD methods using overlapping subdomains. Roughly speaking, overlapping DD methods operate by an iterative procedure, where the PDE is repeatedly solved within every subdomain. For each subdomain, the artificial internal boundary condition is provided by its neighboring subdomains.

1 The Classical Alternating Schwarz Method

This method was devised to solve a Poisson equation in a specially shaped domain $\Omega = \Omega_1 \cup \Omega_2$, i.e., the union of a circle and a rectangle, as depicted in Figure 1. More specifically, the boundary-value problem reads

$$\begin{aligned} -\Delta u &= f, \quad \text{in } \Omega = \Omega_1 \cup \Omega_2, \\ u &= g \quad\quad \text{on } \partial\Omega. \end{aligned}$$

The part of the subdomain boundary $\partial\Omega_i$, which is not part of the global physical boundary

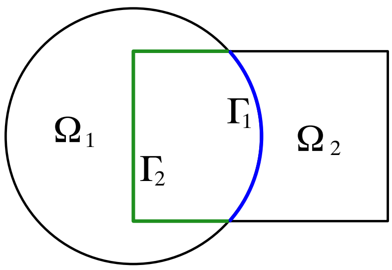

Figure 1: Solution domain for the classical alternating Schwarz method.

$\partial\Omega$, is referred to as the artificial internal boundary. In Figure 1, we see that Γ_1 is the artificial internal boundary of subdomain Ω_1, and Γ_2 is the artificial internal boundary of subdomain Ω_2.

In order to utilize analytical solution methods for solving the Poisson equation on a circle and a rectangle separately, Schwarz proposed the following iterative procedure for finding the approximate solution in the entire domain Ω. Let u_i^n denote the approximate solution in subdomain Ω_i, and f_i denote the restriction of f to Ω_i. Starting with an initial guess u^0, we iterate for $n = 1, 2, \cdots$ to find better and better approximate solutions u^1, u^2, and so on. During each iteration, we first solve the Poisson equation restricted to the circle Ω_1, using the previous iteration's solution from Ω_2 on the artificial internal boundary Γ_1:

$$\begin{aligned} -\Delta u_1^n &= f_1, \quad \text{in } \Omega_1, \\ u_1^n &= g \quad\quad \text{on } \partial\Omega_1 \backslash \Gamma_1, \\ u_1^n &= u_2^{n-1}|_{\Gamma_1} \quad \text{on } \Gamma_1. \end{aligned}$$

Then, we solve the Poisson equation within the rectangle Ω_2, using the latest solution u_1^n on the artificial internal boundary Γ_2:

$$\begin{aligned} -\Delta u_2^n &= f_2, && \text{in } \Omega_2, \\ u_2^n &= g && \text{on } \partial\Omega_2 \backslash \Gamma_2, \\ u_2^n &= u_1^n|_{\Gamma_2} && \text{on } \Gamma_2. \end{aligned}$$

The two local Poisson equations in Ω_1 and Ω_2 are coupled together in the following way: the artificial Dirichlet condition on the internal boundary Γ_1 of subdomain Ω_1 is provided by subdomain Ω_2 in form of $u_2^{n-1}|_{\Gamma_1}$, and vice versa. It is clear that $u_2^{n-1}|_{\Gamma_1}$ and $u_1^n|_{\Gamma_2}$ may change from iteration to iteration, while converging towards the true solution. Therefore, in each Schwarz iteration, the two Poisson equations need to update the artificial Dirichlet conditions on Γ_1 and Γ_2 by exchanging some data. Note also that the classical alternating Schwarz method is sequential by nature, meaning that the two Poisson solves within each iteration must be carried out in a predetermined sequence, first in Ω_1 then in Ω_2. The way to generalize this method to a parallel version is to solve

$$\begin{aligned} -\Delta u_1^n &= f_1, && \text{in } \Omega_1, \\ u_1^n &= g && \text{on } \partial\Omega_1 \backslash \Gamma_1, \\ u_1^n &= u_2^{n-1}|_{\Gamma_1} && \text{on } \Gamma_1. \end{aligned}$$

and

$$\begin{aligned} -\Delta u_2^n &= f_2, && \text{in } \Omega_2, \\ u_2^n &= g && \text{on } \partial\Omega_2 \backslash \Gamma_2, \\ u_2^n &= u_1^{n-1}|_{\Gamma_2} && \text{on } \Gamma_2. \end{aligned}$$

on to different processors and to exchange boundary values after each iteration step.

2 The additive Schwarz method

We now extend the classical alternating Schwarz method to more than two subdomains. To this end, assume that we want to solve a linear elliptic PDE of the form:

$$\begin{aligned} Lu &= f && \text{in } \Omega, \\ u &= g && \text{on } \partial\Omega, \end{aligned}$$

where L is some linear operator.

In order to use a "divide-and-conquer" strategy, we decompose the global solution domain Ω into a set of P subdomains $\{\Omega_i\}_{i=1}^P$, such that $\Omega = \cup_{i=1}^P \Omega_i$. As before, we denote by Γ_i the internal boundary of subdomain number i, i.e., the part of $\partial\Omega_i$ not belonging to the physical global boundary $\partial\Omega$. In addition, we denote by $\mathcal{N}_i$ the index set of neighboring subdomains for subdomain number i, such that $j \in \mathcal{N}_i \rightarrow \Omega_i \cap \Omega_j \neq 0$. We require that there is explicit overlap between each pair of neighboring subdomains. (The case of non-overlapping DD methods is beyond the scope of this project.) In other words, every point on Γ_i must also lie in the interior of at least one neighboring subdomain Ω_j, $j \in \mathcal{N}_i$. When the set of overlapping subdomains is ready, we run an iterative solution procedure that starts with an initial guess $\{u_i^0\}_{i=1}^P$. The work of iteration number n consists of P sub-steps that must be carried out in sequence $i = 1, 2, \cdots, P$. For sub-step number i, the work consists in solving the PDE restricted to subdomain Ω_i:

$$\begin{array}{ll} L_i\, u_i^n = f_i & \text{in } \Omega_i, \\ u_i^n = g & \text{on } \partial\Omega_i \backslash \Gamma_i, \\ u_i^n = \tilde{g}^{n-1} & \text{on } \Gamma_i. \end{array}$$

Here, L_i means the restriction of L onto Ω_i. The right-hand side term f_i arises from restricting f onto Ω_i.

The notation $\tilde{g}^{n-1}$ means that the artificial Dirichlet condition on Γ_i is updated using solutions from all the relevant neighboring subdomains from iteration number $n-1$. In more precise terms, for every point x that lies on Γ_i, we suppose it is also an interior point of the neighboring subdomains $\Omega_{j_1}, \Omega_{j_2}, \cdots, \Omega_{j_m}$, where $j_1 < j_2 < \cdots < j_m$ with $j_k \in \mathcal{N}_i$ for $k = 1, \cdots, m$. Then, the average value

$$\frac{1}{m} \sum_{k=1}^{m} u_{j_k}^{n-1}(x)$$

will be used as the artificial Dirichlet condition on $x \in \Gamma_i$.

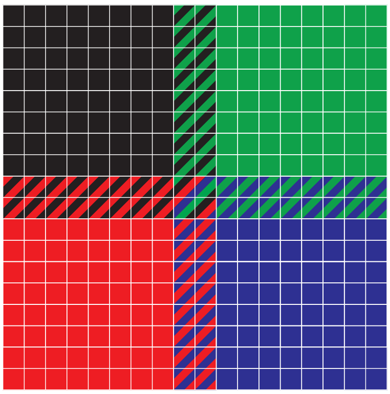

Figure 2: An example of partitioning the unit square into four overlapping subdomains.

3 Project

1. Implement the additive Schwarz method for a one-dimensional problem on a single processor, each step followed the previous computed one.

2. Implement the previous method with communications on many processors

3. Perform the two previous question for two-dimensional problems with Finite Differences Method and Finite Elements Method for the Poisson equation

4. Write a report explaining the method and your implementation technique.