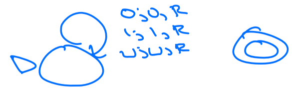
From Friday $A_{TM} \le_m HALT_{TM}$ is witnessed by computable function given by Turing machine with high level description

"On input x

1. Check if $x = \langle M, w \rangle$ for some TM M, string w.
   If not, output $\langle$ 

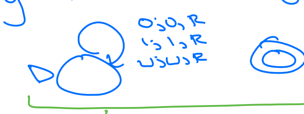
   , $\varepsilon \rangle$
2. Build M' = "On input y,
   1. Simulate M on y
   2. If accepts, accept.
      If rejects, loop."
3. Output $\langle M', w \rangle$ "

Why does this function witness mapping reduction?

- Computable b/c computed by Turing machine above (that halts and output for each input)
- Consider arbitrary x.

Case 1: $x \in A_{TM}$. So $x = \langle M, w \rangle$ for some TM M, string w. Output of function is $\langle M', w \rangle$ with M' accepting (hence halting on) w.
Thus output of function is in $HALT_{TM}$ ✓

Case 2. $x \notin A_{TM}$

Subcase 2a. $x \neq \langle M, w \rangle$ for any TM M, string w.
so output is $\langle$ 

, $\varepsilon \rangle$
which is not in $HALT_{TM}$ b/c this TM loops on ε. ✓

Subcase 2b. $x = \langle M, w \rangle$ for some TM M, string w and M loops on w. Output of function is $\langle M', w \rangle$ where M''s computation on w loops in step 2 so M' does not halt on w ✓

Subcase 2c. $x = \langle M, w \rangle$ for some TM M, string w and M rejects w. Output of function is $\langle M', w \rangle$ where M''s computation on w loops in step 3 so M' does not halt on w ✓

Monday - Memorial Day

No class today.

Recall: The class of Turing recognizable languages is not closed under complementation.

Wednesday

Recall: A is **mapping reducible to** B, written $A \leq_m B$, means there is a computable function $f : \Sigma^* \to \Sigma^*$ such that *for all* strings x in Σ^*,

$$\boxed{x \in A \qquad \text{if and only if} \qquad f(x) \in B.}$$

True or False: $\overline{A_{TM}} \leq_m \overline{HALT_{TM}}$

want a function that is ① computable ② for each $x \in \Sigma^*$, $x \in \overline{A_{TM}}$ iff $f(x) \in \overline{HALT_{TM}}$

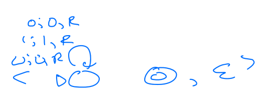
$x \notin A_{TM}$ iff $f(x) \notin HALT_{TM}$

witness function for $A_{TM} \leq_m HALT_{TM}$ also witnesses that $\overline{A_{TM}} \leq_m \overline{HALT_{TM}}$

True or False: $HALT_{TM} \leq_m A_{TM}$. BUT can't use the same function! Why?

Goal $\langle M, w \rangle$ with M halts on w $\rightsquigarrow \langle M', w \rangle$ M' accepts w

Consider function computed by TM with high level description: "On input x

1. If $x \neq \langle M, w \rangle$ M TM, w string, output $\langle$  $\rangle$
2. Otherwise, $x = \langle M, w \rangle$ for some M TM, w string and define M' = "On input y
   1. Run M on y
   2. If M accepts, accept. If M rejects, accept."
3. Output $\langle M', w \rangle$ "

To do: prove this works ...

Theorem (Sipser 5.28): If $A \leq_m B$ and B is recognizable, then A is recognizable.

Proof:

extra practice

Corollary: If $A \leq_m B$ and A is unrecognizable, then B is unrecognizable.

To witness $HALT_{TM} \leq_m A_{TM}$

a function on Σ^* needs to be computable and for each x

Case ① if $x \neq \langle M, w \rangle$ for any TM M, string w

output of function must not be in A_{TM}.

Case ② If $x = \langle M, w \rangle$ for some TM M, string w and M halts on w then output of function must be $\langle \underbrace{\sim\sim}_{TM}, \underbrace{\sim\sim}_{string} \rangle$ where this TM accepts this string.

Case ③ If $x = \langle M, w \rangle$ for some TM M, string w and M loops on w then output of function must not be in A_{TM}.

if $X \leq_m Y$ and X is undecidable then so is Y.

Strategy:

(i) To prove that a recognizable language R is undecidable, prove that $A_{TM} \leq_m R$.

(ii) To prove that a co-recognizable language U is undecidable, prove that $\overline{A_{TM}} \leq_m U$, i.e. that $A_{TM} \leq_m \overline{U}$.

$$E_{TM} = \{\langle M\rangle \mid M \text{ is a Turing machine and } L(M) = \emptyset\}$$

Example string in E_{TM} is $\langle$ 0;0,R 1;1,R ⊔;⊔,R $\rangle$. Example string not in E_{TM} is $\langle M_1, M_2 \rangle$ or $\langle \quad \rangle$.

E_{TM} is decidable / undecidable and recognizable / unrecognizable.

$\overline{E_{TM}}$ is decidable / undecidable and recognizable / unrecognizable. informally, need to prove.

Claim: A_{TM} $\leq_m \overline{E_{TM}}$. i.e. $\overline{A_{TM}} \leq_m E_{TM}$

Proof: Need computable function $F : \Sigma^* \to \Sigma^*$ such that $x \in A_{TM}$ iff $F(x) \notin E_{TM}$. Define

$F =$ " On input x,

1. Type-check whether $x = \langle M, w\rangle$ for some TM M and string w. If so, move to step 2; if not, output $\langle$ 0;0,R 1;1,R ⊔;⊔,R $\rangle$
2. Construct the following machine M'_x:

   M'_x = " On input y
   1. Ignore input.
   2. Run M on w
   3. If M accepts w, accept y.
   4. If M rejects w, reject y. "

3. Output $\langle M'_x\rangle$."

Verifying correctness:

Input string	Output string
$x \in A_{TM}$ $\langle M, w\rangle$ where $w \in L(M)$	$F(x) \notin E_{TM}$? $L(M'_x) = \Sigma^*$ ✓
$x \notin A_{TM}$ $\langle M, w\rangle$ where $w \notin L(M)$	$F(x) \in E_{TM}$? $L(M'_x) = \emptyset$ ✓
x not encoding any pair of TM and string	$F(x) \in E_{TM}$? Yes b/c $L(\langle$ 0;0,R 1;1,R ⊔;⊔,R $\rangle) = \emptyset$ ✓

Review: Week 9 Wednesday

Please complete the review quiz questions on Gradescope about mapping reductions.

Pre class reading for next time: Introduction to Chapter 7.

Friday

Recall: A is **mapping reducible to** B, written $A \leq_m B$, means there is a computable function $f : \Sigma^* \to \Sigma^*$ such that *for all* strings x in Σ^*,

$$x \in A \qquad \text{if and only if} \qquad f(x) \in B.$$

$$EQ_{TM} = \{\langle M, M' \rangle \mid M \text{ and } M' \text{ are both Turing machines and } L(M) = L(M')\}$$

Example string in EQ_{TM} is __________________ . Example string not in EQ_{TM} is __________________ .

EQ_{TM} is decidable / undecidable and recognizable / unrecognizable .

$\overline{EQ_{TM}}$ is decidable / undecidable and recognizable / unrecognizable .

To prove, show that __________________ $\leq_m EQ_{TM}$ and that __________________ $\leq_m \overline{EQ_{TM}}$.

Verifying correctness:

Input string	Output string
$\langle M, w \rangle$ where M halts on w	
$\langle M, w \rangle$ where M loops on w	
x not encoding any pair of TM and string	

In practice, computers (and Turing machines) don't have infinite tape, and we can't afford to wait unboundedly long for an answer. "Decidable" isn't good enough - we want "Efficiently decidable".

For a given algorithm working on a given input, how long do we need to wait for an answer? How does the running time depend on the input in the worst-case? average-case? We expect to have to spend more time on computations with larger inputs.

A language is **recognizable** if ____

A language is **decidable** if ____

A language is **efficiently decidable** if ____________________________________

A function is **computable** if ____

A function is **efficiently computable** if ___________________________________

Definition (Sipser 7.1): For M a deterministic decider, its **running time** is the function $f : \mathbb{N} \to \mathbb{N}$ given by

$$f(n) = \text{max number of steps } M \text{ takes before halting, over all inputs of length } n$$

Definition (Sipser 7.7): For each function $t(n)$, the **time complexity class** $TIME(t(n))$, is defined by

$$TIME(t(n)) = \{L \mid L \text{ is decidable by a Turing machine with running time in } O(t(n))\}$$

An example of an element of $TIME(1)$ is

An example of an element of $TIME(n)$ is

Note: $TIME(1) \subseteq TIME(n) \subseteq TIME(n^2)$

Definition (Sipser 7.12) : P is the class of languages that are decidable in polynomial time on a deterministic 1-tape Turing machine

$$P = \bigcup_k TIME(n^k)$$

Compare to exponential time: brute-force search.

Theorem (Sipser 7.8): Let $t(n)$ be a function with $t(n) \geq n$. Then every $t(n)$ time deterministic multitape Turing machine has an equivalent $O(t^2(n))$ time deterministic 1-tape Turing machine.

Review: Week 9 Friday

Please complete the review quiz questions on Gradescope about complexity.

Pre class reading for next time: Skim Chapter 7.

