HW3 scores released
Practice assignments
Project Part 2 released.

Monday May 9

	$L = L(M)$ Suppose M is a TM that recognizes L	$L = L(D)$ Suppose D is a TM that decides L	$L = L(E)$ Suppose E is an enumerator that enumerates L
If string w is in L then ...	M accepts w	D accepts w	E prints w (in finite time)
If string w is not in L then ...	M rejects w or M loops on w	D rejects w	E never prints w

Describing Turing machines (Sipser p. 185)

The Church-Turing thesis posits that each algorithm can be implemented by some Turing machine

High-level descriptions of Turing machine algorithms are written as indented text within quotation marks.

Stages of the algorithm are typically numbered consecutively.

The first line specifies the input to the machine, which must be a string. This string may be the encoding of some object or list of objects.

Notation: $\langle O \rangle$ is the string that encodes the object O. $\langle O_1, \ldots, O_n \rangle$ is the string that encodes the list of objects $O_1, \ldots, O_n$.

Assumption: There are Turing machines that can be called as subroutines to decode the string representations of common objects and interact with these objects as intended (data structures).

For example, since there are algorithms to answer each of the following questions, by Church-Turing thesis, there is a Turing machine that accepts exactly those strings for which the answer to the question is "yes"

Computational problem

- Does a string over $\{0, 1\}$ have even length?
- Does a string over $\{0, 1\}$ encode a string of ASCII characters?[1]
  7 bit
- Does a DFA have a specific number of states?
- Do two NFAs have any state names in common?
  On input $\langle N_1, N_2 \rangle$ ...
- Do two CFGs have the same start variable?

On input $\langle A, n \rangle$ the string representation of the formal definition of the DFA A and a referenc
1. Parse the string $\langle A \rangle$ to find the substring listing the set of states of A.
2. Count the num of distinct elements of this se
3. If this number equals accept. Otherwise reject.

[1]An introduction to ASCII is available on the w3 tutorial here.

The TM answering the question

- Does a string over $\{0, 1\}$ have even length?

can be defined in different ways:

* High level definition

  " On input w
  1. If $|w| \% 2 == 0$, accept
  2. otherwise, reject. "

* Implementation-level definition

  " On input w
  1. Scan across the tape considering two cells at a time.
  2. If both cells blank, halt and accept.
  3. If both cells have characters from input alphabet, cross them out and move to next-to-the-right pair of cells.
  4. If one cell blank and the other has an input character, halt and reject. "

* Formal definition (extra practice)

A **computational problem** is decidable iff language encoding its positive problem instances is decidable.

{ strings encoding objects for which answer "yes" }

The computational problem "Does a specific DFA accept a given string?" is encoded by the language

$$\{\text{representations of DFAs } M \text{ and strings } w \text{ such that } w \in L(M)\}$$
$$= \{\langle M, w\rangle \mid M \text{ is a DFA}, w \text{ is a string}, w \in L(M)\}$$

The computational problem "Is the language generated by a CFG empty?" is encoded by the language

$$\{\text{representations of CFGs } G \text{ such that } L(G) = \emptyset\}$$
$$= \{\langle G\rangle \mid G \text{ is a CFG}, L(G) = \emptyset\}$$

The computational problem "Is the given Turing machine a decider?" is encoded by the language

$$\{\text{representations of TMs } M \text{ such that } M \text{ halts on every input}\}$$
$$= \{\langle M\rangle \mid M \text{ is a TM and for each string } w, M \text{ halts on } w\}$$

Note: writing down the language encoding a computational problem is only the first step in determining if it's recognizable, decidable, or . . .

Some classes of computational problems help us understand the differences between the machine models we've been studying:

Acceptance problem

. . . for DFA	A_{DFA}	$\{\langle B, w\rangle \mid B \text{ is a DFA that accepts input string } w\}$
. . . for NFA	A_{NFA}	$\{\langle B, w\rangle \mid B \text{ is a NFA that accepts input string } w\}$
. . . for regular expressions	A_{REX}	$\{\langle R, w\rangle \mid R \text{ is a regular expression that generates input string } w\}$
. . . for CFG	A_{CFG}	$\{\langle G, w\rangle \mid G \text{ is a context-free grammar that generates input string } w\}$
. . . for PDA	A_{PDA}	$\{\langle B, w\rangle \mid B \text{ is a PDA that accepts input string } w\}$

Language emptiness testing

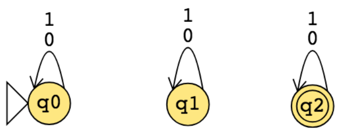
. . . for DFA	E_{DFA}	$\{\langle A\rangle \mid A \text{ is a DFA and } L(A) = \emptyset\}$
. . . for NFA	E_{NFA}	$\{\langle A\rangle \mid A \text{ is a NFA and } L(A) = \emptyset\}$
. . . for regular expressions	E_{REX}	$\{\langle R\rangle \mid R \text{ is a regular expression and } L(R) = \emptyset\}$
. . . for CFG	E_{CFG}	$\{\langle G\rangle \mid G \text{ is a context-free grammar and } L(G) = \emptyset\}$
. . . for PDA	E_{PDA}	$\{\langle A\rangle \mid A \text{ is a PDA and } L(A) = \emptyset\}$

Language equality testing

. . . for DFA	EQ_{DFA}	$\{\langle A, B\rangle \mid A \text{ and } B \text{ are DFAs and } L(A) = L(B)\}$
. . . for NFA	EQ_{NFA}	$\{\langle A, B\rangle \mid A \text{ and } B \text{ are NFAs and } L(A) = L(B)\}$
. . . for regular expressions	EQ_{REX}	$\{\langle R, R'\rangle \mid R \text{ and } R' \text{ are regular expressions and } L(R) = L(R')\}$
. . . for CFG	EQ_{CFG}	$\{\langle G, G'\rangle \mid G \text{ and } G' \text{ are CFGs and } L(G) = L(G')\}$
. . . for PDA	EQ_{PDA}	$\{\langle A, B\rangle \mid A \text{ and } B \text{ are PDAs and } L(A) = L(B)\}$

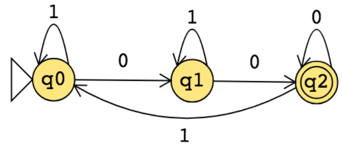
Sipser Section 4.1

DFAs



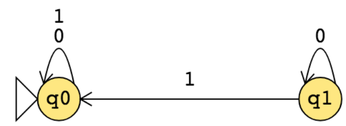
M_1

$0 \notin L(M_1)$



M_2

$0 \notin L(M_2)$



M_3

$0 \notin L(M_3)$

Example strings in A_{DFA} $= \{ \langle M, w \rangle \mid M \text{ is DFA}, w \text{ is string}, w \in L(M) \}$

specific strings in A_{DFA} depends on encodings.

$\langle M_1, 0 \rangle \notin A_{DFA}$.

$\langle M_2, 00 \rangle \in A_{DFA}$

Example strings in E_{DFA} $= \{ \langle M \rangle \mid M \text{ is DFA}, L(M) = \emptyset \}$

$\langle M_1 \rangle \in E_{DFA}$

$\langle M_3 \rangle \in E_{DFA}$

$\langle M_2 \rangle \notin E_{DFA}$

Example strings in EQ_{DFA} $= \{ \langle M_1, M_2 \rangle \mid M_1, M_2 \text{ DFA}, L(M_1) = L(M_2) \}$

$\langle M_1, M_3 \rangle \in EQ_{DFA}$

$\langle M_1, M_1 \rangle \in EQ_{DFA}$

$\langle M_1, M_2 \rangle \notin EQ_{DFA}$

Food for thought: which of the following computational problems are decidable: A_{DFA}?, E_{DFA}?, EQ_{DFA}?

Review: Week 7 Monday

Recall: Review quizzes based on class material are assigned each day. These quizzes will help you track and confirm your understanding of the concepts and examples we work in class. Quizzes can be submitted on Gradescope as many times (with no penalty) as you like until the quiz deadline: the three quizzes each week are all due on Friday (with no penalty late submission open until Sunday).

Please complete the review quiz questions on Gradescope about computational problems.

Pre class reading for next time: Decidable problems concerning regular languages, Sipser pages 194-196.

Solving a computational problem means defining a TM that is a decider and whose language is the language encoding this computational problem.

Wednesday May 11

Acceptance problem		
for ...	$A_{\dots}$	$\{\langle B, w\rangle \mid B$ is a ... that accepts input string $w\}$
Language emptiness testing		
for ...	$E_{\dots}$	$\{\langle A\rangle \mid A$ is a ... and $L(A) = \emptyset\}$
Language equality testing		
for ...	$EQ_{\dots}$	$\{\langle A, B\rangle \mid A$ and B are ... and $L(A) = L(B)\}$
Sipser Section 4.1		

M_1 = "On input $\langle M, w\rangle$, where M is a DFA and w is a string:

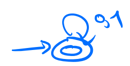
finitely many steps

0. Type check encoding to check input is correct type. ✓ if correct type, continue to step 1; if not, reject.
1. Simulate M on input w (by keeping track of states in M, transition function of M, etc.)
2. If the simulations ends in an accept state of M, accept. If it ends in a non-accept state of M, reject. "

computation of DFA always finite

finitely many steps

What is $L(M_1)$?

<  , 0100 > ∈ L(M₁)

<  , ε > ∉ L(M₁)

<  > ∉ L(M₁)

$L(M_1) = \{\langle A, w\rangle \mid A \text{ DFA}, w \text{ string}, w \in L(A)\} = A_{DFA}$

Is $L(M_1)$ a decider?

Check whether M_1 is guaranteed to halt for all input. Checking each step of high-level description, see computation of M_1 halts in finite time for all inputs.

M_2 ="On input $\langle M, w\rangle$ where M is a DFA and w is a string,

IMPLICIT TYPE CHECK

1. Run M on input w.
2. If M accepts, accept; if M rejects, reject."

What is $L(M_2)$? $L(M_2) = L(M_1) = A_{DFA}$

Is M_2 a decider? Yes.

Typo in originally posted notes had L(...)

machines are/aren't deciders

languages are/aren't decidable

$A_{REX} = \{ <R, w> \mid R$ is regular expression, w a string, $w \in L(R) \}$

$A_{NFA} = \{ <M, w> \mid M$ is NFA, w a string, $w \in L(M) \}$.

~~True~~ / False: $A_{REX} = A_{NFA} = A_{DFA}$ $\quad <(0 \cup 1)^*, 0> \in A_{REX}$

$<(0 \cup 1)^*, 0> \notin A_{NFA}$

True / ~~False~~: $A_{REX} \cap A_{NFA} = \emptyset$, $A_{REX} \cap A_{DFA} = \emptyset$, $A_{DFA} \cap A_{NFA} = \emptyset$

A Turing machine that decides A_{NFA} is:

M_{NFA} = " On input $<M, w>$ M NFA, w string

1. Use subset construction from Chapter 1 to transform M to a DFA M_D with $L(M_D) = L(M)$.
2. Run M_D on w.
   i.e. Run M_1 on input $<M_D, w>$
3. If accepts, accept ; if rejects, reject ".

To confirm $L(M_{NFA}) = A_{NFA}$ and M_{NFA} is decider.

A Turing machine that decides A_{REX} is:

(extra practice)

$E_{DFA} = \{ \langle M \rangle \mid M \text{ DFA}, L(M) = \emptyset \}$.

$\langle \text{D} \rangle \in E_{DFA}$

M_3 = "On input $\langle M \rangle$ where M is a DFA,

1. For integer $i = 1, 2, \ldots$ infinite loop.
2. Let s_i be the ith string over the alphabet of M (ordered in string order).
3. Run M on input s_i.
4. If M accepts, reject. If M rejects, increment i and keep going."

Keep coming!

$L(M_3) = \emptyset$

Choose the correct option to help fill in the blank so that M_3 recognizes E_{DFA}

Not possible - this approach won't work for recognizing E_{DFA}.

A. accepts

B. rejects

C. loop for ever

D. ~~We can't fill in the blank in any way to make this work~~

E. None of the above

$E_{DFA} \neq \emptyset$.

M_4 = " On input $\langle M \rangle$ where M is a DFA,

1. Mark the start state of M.
2. Repeat until no new states get marked:
3. Loop over the states of M.
4. Mark any unmarked state that has an incoming edge from a marked state.
5. If no accept state of A is marked, accept; otherwise, reject.".

halt because M is DFA so has finitely many states

Search underlying graph of state diagram of M for path from the start state to an accept state.

transition

$L(M) = \emptyset$ so $\langle M \rangle \in E_{DFA}$

$L(M) \neq \emptyset$ so $\langle M \rangle \notin E_{DFA}$

To build a Turing machine that decides EQ_{DFA}, notice that

$$L_1 = L_2 \qquad \text{iff} \qquad (\ (L_1 \cap \overline{L_2}) \cup (L_2 \cap \overline{L_1})\ ) = \emptyset$$

There are no elements that are in one set and not the other

M_{EQDFA} = "On input $\langle M_1, M_2 \rangle$ where M_1, M_2 are DFA

1. Construct (using flip states approach) DFAs $\tilde{M}_1$ and $\tilde{M}_2$ such that $L(\tilde{M}_1) = \overline{L(M_1)}$, $L(\tilde{M}_2) = \overline{L(M_2)}$
2. Construct (using Cartesian product construction) DFAs D_1 and D_2 such that $L(D_1) = L(M_1) \cap L(\tilde{M}_2)$ and $L(D_2) = L(\tilde{M}_1) \cap L(M_2)$
3. Construct (using Cartesian product construction) DFA D such that $L(D) = L(D_1) \cup L(D_2)$
4. Run M_4 on input $\langle D \rangle$
5. If accepts, accept; if rejects, reject"

to confirm: $L(M_{EQDFA}) = EQ_{DFA}$ and M_{EQDFA} is decider.

Summary: We can use the decision procedures (Turing machines) of decidable problems as subroutines in other algorithms. For example, we have subroutines for deciding each of A_{DFA}, E_{DFA}, EQ_{DFA}. We can also use algorithms for known constructions as subroutines in other algorithms. For example, we have subroutines for: counting the number of states in a state diagram, counting the number of characters in an alphabet, converting DFA to a DFA recognizing the complement of the original language or a DFA recognizing the Kleene star of the original language, constructing a DFA or NFA from two DFA or NFA so that we have a machine recognizing the language of the union (or intersection, concatenation) of the languages of the original machines; converting regular expressions to equivalent DFA; converting DFA to equivalent regular expressions, etc.

Review: Week 7 Wednesday

Please complete the review quiz questions on Gradescope about decidable computational problems.

Pre class reading for next time: An undecidable language, Sipser pages 207-209.

Friday May 13

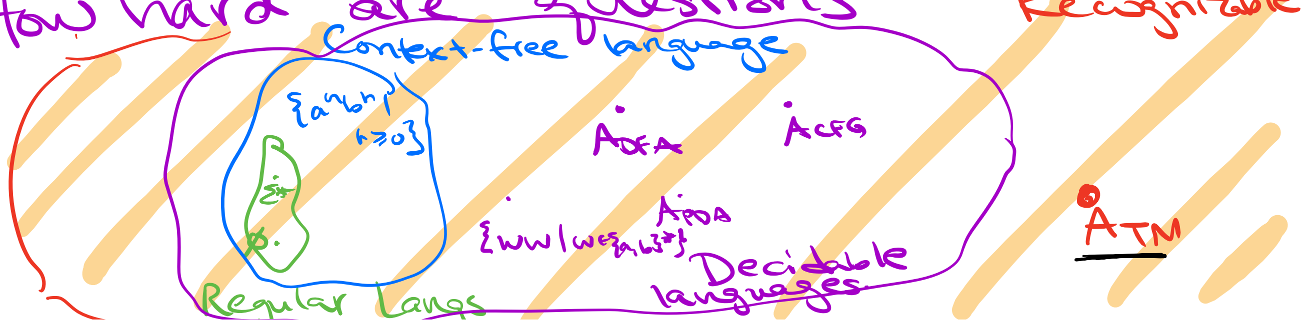
Some classes of computational problems help us understand the differences between the machine models we've been studying:

Acceptance problem			
...for DFA	A_{DFA}	$\{\langle B, w\rangle \mid B$ is a DFA that accepts input string $w\}$	Decidable
...for NFA	A_{NFA}	$\{\langle B, w\rangle \mid B$ is a NFA that accepts input string $w\}$	Decidable
...for regular expressions	A_{REX}	$\{\langle R, w\rangle \mid R$ is a regular expression that generates input string $w\}$	Decidable
...for CFG	A_{CFG}	$\{\langle G, w\rangle \mid G$ is a context-free grammar that generates input string $w\}$	Decidable Ch 4
...for PDA	A_{PDA}	$\{\langle B, w\rangle \mid B$ is a PDA that accepts input string $w\}$	Decidable using A_{CFG}.
Language emptiness testing			
...for DFA	E_{DFA}	$\{\langle A\rangle \mid A$ is a DFA and $L(A) = \emptyset\}$	
...for NFA	E_{NFA}	$\{\langle A\rangle \mid A$ is a NFA and $L(A) = \emptyset\}$	
...for regular expressions	E_{REX}	$\{\langle R\rangle \mid R$ is a regular expression and $L(R) = \emptyset\}$	
...for CFG	E_{CFG}	$\{\langle G\rangle \mid G$ is a context-free grammar and $L(G) = \emptyset\}$	
...for PDA	E_{PDA}	$\{\langle A\rangle \mid A$ is a PDA and $L(A) = \emptyset\}$	
Language equality testing			
...for DFA	EQ_{DFA}	$\{\langle A, B\rangle \mid A$ and B are DFAs and $L(A) = L(B)\}$	
...for NFA	EQ_{NFA}	$\{\langle A, B\rangle \mid A$ and B are NFAs and $L(A) = L(B)\}$	
...for regular expressions	EQ_{REX}	$\{\langle R, R'\rangle \mid R$ and R' are regular expressions and $L(R) = L(R')\}$	
...for CFG	EQ_{CFG}	$\{\langle G, G'\rangle \mid G$ and G' are CFGs and $L(G) = L(G')\}$	
...for PDA	EQ_{PDA}	$\{\langle A, B\rangle \mid A$ and B are PDAs and $L(A) = L(B)\}$	
Sipser Section 4.1			

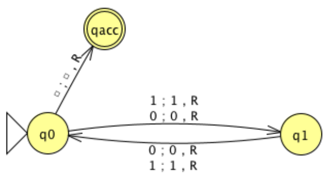
using M_1 from Wednesday

We could use an algorithm deciding A_{PDA} as a subroutine for an algorithm deciding A_{CFG}, and vice versa (relying on Chapter 2 result that there's a transformation between CFGs and PDAs). *

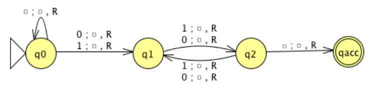
How hard are questions



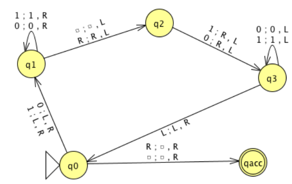
Acceptance problem for Turing machines	A_{TM}	$\{\langle M, w\rangle \mid M$ is a Turing machine that accepts input string $w\}$
Language emptiness testing for Turing machines	E_{TM}	$\{\langle M\rangle \mid M$ is a Turing machine and $L(M) = \emptyset\}$
Language equality testing for Turing machines	EQ_{TM}	$\{\langle M_1, M_2\rangle \mid M_1$ and M_2 are Turing machines and $L(M_1) = L(M_2)\}$
Sipser Section 4.1		



M_1



M_2



M_3

$L(M_1) = \{w \in \{0,1\}^* \mid |w| \text{ is even}\}$

$\varepsilon \in L(M_1)$

Example strings in A_{TM}

$\langle M_1, \varepsilon\rangle \in A_{TM}$

$\langle M_1, 1111111\rangle \notin A_{TM}$

$\langle M_1\rangle \notin A_{TM}$

Example strings in E_{TM}

Example strings in EQ_{TM}

Theorem: A_{TM} is Turing-recognizable.

Strategy: To prove this theorem, we need to define a Turing machine R_{ATM} such that $L(R_{ATM}) = A_{TM}$.

Define R_{ATM} = "On input <M,w>   M TM, w string
implicit type check: reject strings not well-typed.
1. Run M on w
start at the initial configuration, apply transition function, check at each step if comp'n enters q_{accept} or q_{reject}
2. If computation of M on w accepts, accept.
3. If computation of M on w rejects, reject"

Proof of correctness:

WTS ① For each string in A_{TM}, R_{ATM} accepts this string.

Let $x \in A_{TM}$. WTS R_{ATM} accepts x. Trace comp'n of R_{ATM} on x: By def of A_{TM}, x = <M,w> for some M TM, w string so type check passes.
In step 1 of R_{ATM}, simulate M on w.
By def of A_{TM}, since $x \in A_{TM}$ and x = <M,w> have $w \in L(M)$, namely computation of M on w halts and accepts. So R_{ATM} will move to step 2 in finite time and accept ✓

WTS ② For each string not in A_{TM}, R_{ATM} rejects this string OR loops on this string.

Let $x \notin A_{TM}$. Either $x \neq$ <M,w> for any TM M, w string or x = <M,w> for M TM, w string and $w \notin L(M)$.
Trace R_{ATM} in either case to show R_{ATM} does not accept x.

(keep going for extra practice...)

We will show that A_{TM} is undecidable. (Monday)

A **Turing-recognizable** language is a set of strings that is the language recognized by some Turing machine. We also say that such languages are recognizable.

A **Turing-decidable** language is a set of strings that is the language recognized by some decider. We also say that such languages are decidable.

An **unrecognizable** language is a language that is not Turing-recognizable.

An **undecidable** language is a language that is not Turing-decidable.

~~**True**~~ or **False**: Any undecidable language is also unrecognizable.

counterexample: A_{TM}.

True or ~~**False**~~: Any unrecognizable language is also undecidable.

To prove that a computational problem is **decidable**, we find/ build a Turing machine that recognizes the language encoding the computational problem, and that is a decider.

How do we prove a specific problem is **not decidable**?

How would we even find such a computational problem?

Counting arguments for the existence of an undecidable language:

- The set of all Turing machines is countably infinite.
- Each Turing-recognizable language is associated with a Turing machine in a one-to-one relationship, so there can be no more Turing-recognizable languages than there are Turing machines.
- Since there are infinitely many Turing-recognizable languages (think of the singleton sets), there are countably infinitely many Turing-recognizable languages.
- Such the set of Turing-decidable languages is an infinite subset of the set of Turing-recognizable languages, the set of Turing-decidable languages is also countably infinite.

Since there are uncountably many languages (because $\mathcal{P}(\Sigma^*)$ is uncountable), there are uncountably many unrecognizable languages and there are uncountably many undecidable languages.

What's a specific example of a language that is unrecognizable or undecidable?

Key idea: self-referential disagreement.

Review: Week 7 Friday

Please complete the review quiz questions on Gradescope about undecidability and unrecognizability.

Some classes of computational problems help us understand the differences between the machine models we've been studying:

Acceptance problem		
...for DFA	A_{DFA}	$\{\langle B, w\rangle \mid B$ is a DFA that accepts input string $w\}$
...for NFA	A_{NFA}	$\{\langle B, w\rangle \mid B$ is a NFA that accepts input string $w\}$
...for regular expressions	A_{REX}	$\{\langle R, w\rangle \mid R$ is a regular expression that generates input string $w\}$
...for CFG	A_{CFG}	$\{\langle G, w\rangle \mid G$ is a context-free grammar that generates input string $w\}$
...for PDA	A_{PDA}	$\{\langle B, w\rangle \mid B$ is a PDA that accepts input string $w\}$
Language emptiness testing		
...for DFA	E_{DFA}	$\{\langle A\rangle \mid A$ is a DFA and $L(A) = \emptyset\}$
...for NFA	E_{NFA}	$\{\langle A\rangle \mid A$ is a NFA and $L(A) = \emptyset\}$
...for regular expressions	E_{REX}	$\{\langle R\rangle \mid R$ is a regular expression and $L(R) = \emptyset\}$
...for CFG	E_{CFG}	$\{\langle G\rangle \mid G$ is a context-free grammar and $L(G) = \emptyset\}$
...for PDA	E_{PDA}	$\{\langle A\rangle \mid A$ is a PDA and $L(A) = \emptyset\}$
Language equality testing		
...for DFA	EQ_{DFA}	$\{\langle A, B\rangle \mid A$ and B are DFAs and $L(A) = L(B)\}$
...for NFA	EQ_{NFA}	$\{\langle A, B\rangle \mid A$ and B are NFAs and $L(A) = L(B)\}$
...for regular expressions	EQ_{REX}	$\{\langle R, R'\rangle \mid R$ and R' are regular expressions and $L(R) = L(R')\}$
...for CFG	EQ_{CFG}	$\{\langle G, G'\rangle \mid G$ and G' are CFGs and $L(G) = L(G')\}$
...for PDA	EQ_{PDA}	$\{\langle A, B\rangle \mid A$ and B are PDAs and $L(A) = L(B)\}$
Sipser Section 4.1		