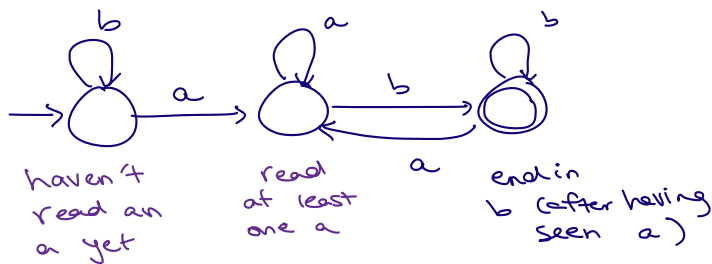


Monday: Automata constructions

Warmup: Design a DFA (deterministic finite automaton) and an NFA (nondeterministic finite automaton) that each recognize each of the following languages over $\{a, b\}$

$\{w \mid w \text{ has an } a \text{ and ends in } b\}$

DFA



Labels of arrows of NFA are characters in alphabet, Σ

NFA



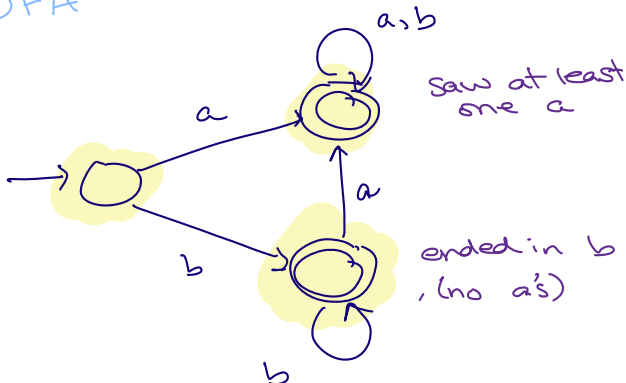
Trace computations on string aab.

$q_0 \xrightarrow{a} q_1 \xrightarrow{a} q_1 \xrightarrow{b} q_2$ ACC

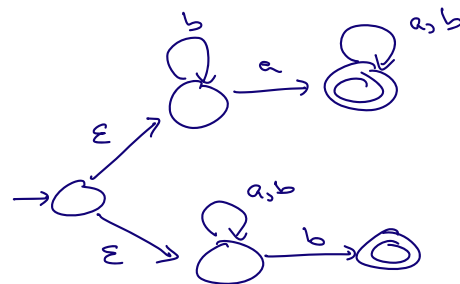
NOW TRY THIS ONE...

$\{w \mid w \text{ has an } a \text{ or ends in } b\}$

DFA



NFA



NOTICE: THERE'S A 2 state DFA that recognizes this languages

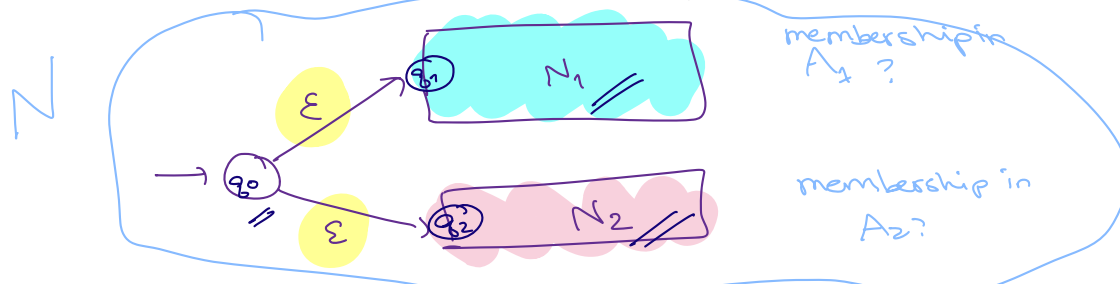
Strategy: To design DFA or NFA for a given language, identify patterns that can be built up as we process strings and create states for intermediate stages. Or: decompose the language to a simpler one that we already know how to recognize with a DFA or NFA.

Recall (from Wednesday of last week, and in textbook Exercise 1.14): if there is a DFA M such that $L(M) = A$ then there is another DFA, let's call it M' , such that $L(M') = \bar{A}$, the complement of A , defined as $\{w \in \Sigma^* \mid w \notin A\}$.

Let's practice defining automata constructions by coming up with other ways to get new automata from old.

Suppose A_1, A_2 are languages over an alphabet Σ . **Claim:** if there is a NFA N_1 such that $L(N_1) = A_1$ and NFA N_2 such that $L(N_2) = A_2$, then there is another NFA, let's call it N , such that $L(N) = A_1 \cup A_2$.

Proof idea: Use nondeterminism to choose which of N_1, N_2 to run.



Formal construction: Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ and $N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ and assume $Q_1 \cap Q_2 = \emptyset$ and that $q_0 \notin Q_1 \cup Q_2$. Construct $N = (Q, \Sigma, \delta, q_0, F_1 \cup F_2)$ where

- $Q = \{q_0\} \cup Q_1 \cup Q_2$

- $\delta : Q \times \Sigma_\epsilon \rightarrow \mathcal{P}(Q)$ is defined by, for $q \in Q$ and $x \in \Sigma_\epsilon$:

↑ curr state ↑ label of arrow

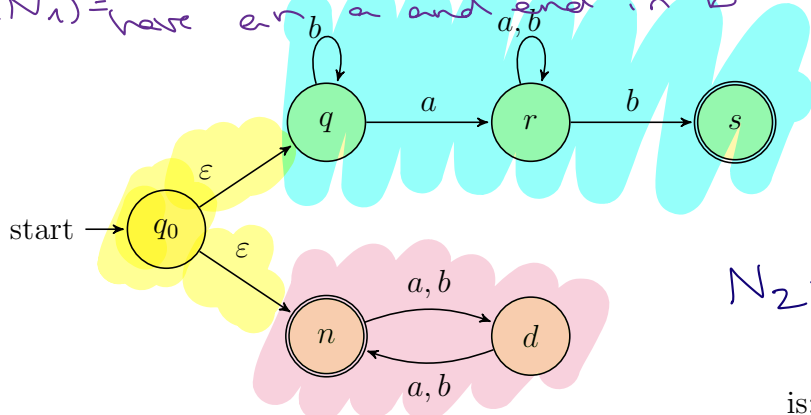
$$\delta(q, x) = \begin{cases} \{q_1, q_2\} & \text{if } q = q_0 \text{ and } x = \epsilon \\ \emptyset & \text{if } q = q_0 \text{ and } x \in \Sigma \\ \delta_1(q, x) & \text{if } q \in Q_1 \text{ and } x \in \Sigma_\epsilon \\ \delta_2(q, x) & \text{if } q \in Q_2 \text{ and } x \in \Sigma_\epsilon \end{cases}$$

labels of arrows are characters in Σ or the label ϵ

* Proof of correctness would prove that $L(N) = A_1 \cup A_2$ by considering an arbitrary string accepted by N , tracing an accepting computation of N on it, and using that trace to prove the string is in at least one of A_1, A_2 ; then, taking an arbitrary string in $A_1 \cup A_2$ and proving that it is accepted by N . Details left for extra practice.

Example: The language recognized by the NFA over $\{a, b\}$ with state diagram

$L(N_1) =$ the set of strings that have an a and end in b



$$N_1 = (\{q, r, s\}, \{a, b\}, \delta_1, q, \{s\})$$

$\delta_1 \dots$

$$N_2 = (\{n, d\}, \{a, b\}, \delta_2, n, \{n, d\})$$

$\delta_2 \dots$

is: the set of strings that either have an a and end in b or are even length, or both.

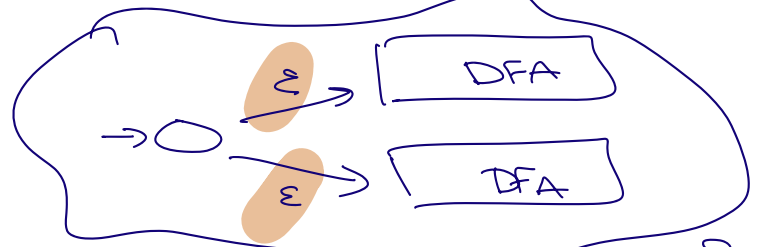
$L(N_2) =$ the set of even length strings

Could we do the same construction with DFA?

No! Because

spontaneous moves not allowed in DFA.

DFA?



Happily, though, an analogous claim is true!

Suppose A_1, A_2 are languages over an alphabet Σ . **Claim:** if there is a DFA M_1 such that $L(M_1) = A_1$ and DFA M_2 such that $L(M_2) = A_2$, then there is another DFA, let's call it M , such that $L(M) = A_1 \cup A_2$.

Theorem 1.25 in Sipser, page 45

Proof idea: Run M_1 and M_2 in parallel to process input.

Formal construction:

states need to capture information in both M_1 and M_2 !

Data structure: ordered pair!

Each state in M will be an ordered pair

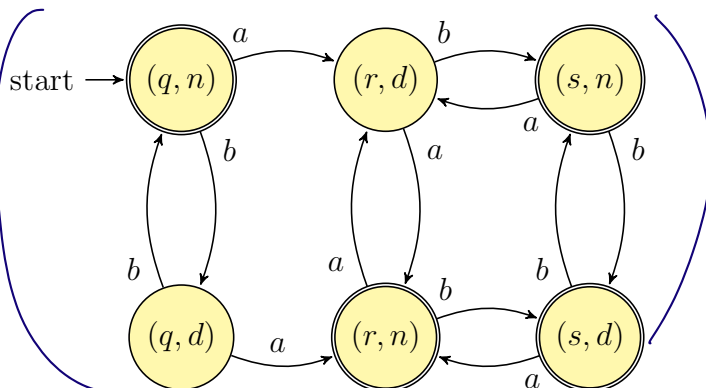
$$(\underbrace{\quad}_{\text{state in } M_1}, \underbrace{\quad}_{\text{state in } M_2})$$

Given $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ DFA, string is in A_1
 $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ DFA string is accepted by M_1
 M_1 landed in an accept state

Define $M = (\underbrace{Q_1 \times Q_2}_{\text{Cartesian product}}, \Sigma, \delta, (q_1, q_2), \underbrace{F_1 \times Q_2 \cup Q_1 \times F_2}_{\substack{M_2 \text{ landed in an accept state} \\ \text{i.e. string is in } A_2}})$

where $\delta: Q_1 \times Q_2 \times \Sigma \rightarrow Q_1 \times Q_2$ is given by
 $\delta((r, s), x) = (\delta_1(r, x), \delta_2(s, x))$ $r \in Q_1, s \in Q_2, x \in \Sigma$

Example: When $A_1 = \{w \mid w \text{ has an } a \text{ and ends in } b\}$ and $A_2 = \{w \mid w \text{ is of even length}\}$.



$$Q_1 = \{q, r, s\}$$

$$Q_2 = \{n, d\}$$

$$= A_1 \cup A_2$$

Suppose A_1, A_2 are languages over an alphabet Σ . **Claim:** if there is a DFA M_1 such that $L(M_1) = A_1$ and DFA M_2 such that $L(M_2) = A_2$, then there is another DFA, let's call it M , such that $L(M) = A_1 \cap A_2$.
Sipser Theorem 1.25, page 45

Proof idea: Still want to run in parallel, be more strict about set of accept states.

Formal construction:

Given $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ DFA,
 $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ DFA

Define $M = (\overset{Q}{Q_1 \times Q_2}, \Sigma, \delta, (q_1, q_2), F_1 \times F_2)$
 Cartesian product

where $\delta: \overset{Q}{Q_1 \times Q_2} \times \Sigma \rightarrow \overset{Q}{Q_1 \times Q_2}$ is given by
 $\delta((r, s), x) = (\delta_1(r, x), \delta_2(s, x))$ $r \in Q_1, s \in Q_2, x \in \Sigma$

Wednesday: Regular languages

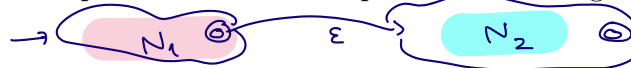
So far we have that:

- If there is a DFA recognizing a language, there is a DFA recognizing its complement.
- If there are NFA recognizing two languages, there is a NFA recognizing their union.
- If there are DFA recognizing two languages, there is a DFA recognizing their union.
- If there are DFA recognizing two languages, there is a DFA recognizing their intersection.

Our goals for today are (1) prove similar results about other set operations, (2) prove that NFA and DFA are equally expressive, and therefore (3) define an important class of languages.

Suppose A_1, A_2 are languages over an alphabet Σ . **Claim:** if there is a NFA N_1 such that $L(N_1) = A_1$ and NFA N_2 such that $L(N_2) = A_2$, then there is another NFA, let's call it N , such that $L(N) = A_1 \circ A_2$.

Proof idea: Allow computation to move between N_1 and N_2 "spontaneously" when reach an accepting state of N_1 , guessing that we've reached the point where the two parts of the string in the set-wise concatenation are glued together.



Formal construction: Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ and $N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ and assume $Q_1 \cap Q_2 = \emptyset$. Construct $N = (Q, \Sigma, \delta, q_0, F)$ where

- $Q = Q_1 \cup Q_2$
- $q_0 = q_1$
- $F = F_2$
- $\delta : Q \times \Sigma_\epsilon \rightarrow \mathcal{P}(Q)$ is defined by, for $q \in Q$ and $a \in \Sigma_\epsilon$:

$$\delta((q, a)) = \begin{cases} \delta_1((q, a)) & \text{if } q \in Q_1 \text{ and } q \notin F_1 \\ \delta_1((q, a)) & \text{if } q \in F_1 \text{ and } a \in \Sigma \\ \delta_1((q, a)) \cup \{q_2\} & \text{if } q \in F_1 \text{ and } a = \epsilon \\ \delta_2((q, a)) & \text{if } q \in Q_2 \end{cases}$$

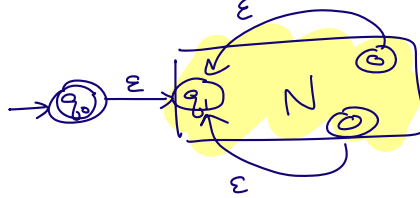
simulating N_1
simulating N_1
simulating N_2

Proof of correctness would prove that $L(N) = A_1 \circ A_2$ by considering an arbitrary string accepted by N , tracing an accepting computation of N on it, and using that trace to prove the string can be written as the result of concatenating two strings, the first in A_1 and the second in A_2 ; then, taking an arbitrary string in $A_1 \circ A_2$ and proving that it is accepted by N . Details left for extra practice.

Suppose A is a language over an alphabet Σ . **Claim:** if there is a NFA N such that $L(N) = A$, then there is another NFA, let's call it N' , such that $L(N') = A^*$.

Proof idea: Add a fresh start state, which is an accept state. Add spontaneous moves from each (old) accept state to the old start state.

Formal construction: Let $N = (Q, \Sigma, \delta, q_1, F)$ and assume $q_0 \notin Q$. Construct $N' = (Q', \Sigma, \delta', q_0, F')$ where



- $Q' = Q \cup \{q_0\}$
- $F' = F \cup \{q_0\}$
- $\delta' : Q' \times \Sigma_\epsilon \rightarrow \mathcal{P}(Q')$ is defined by, for $q \in Q'$ and $a \in \Sigma_\epsilon$:

$$\delta'((q, a)) = \begin{cases} \delta((q, a)) & \text{if } q \in Q \text{ and } q \notin F \\ \delta((q, a)) & \text{if } q \in F \text{ and } a \in \Sigma \\ \delta((q, a)) \cup \{q_1\} & \text{if } q \in F \text{ and } a = \epsilon \\ \{q_1\} & \text{if } q = q_0 \text{ and } a = \epsilon \\ \emptyset & \text{if } q = q_0 \text{ and } a \in \Sigma \end{cases}$$

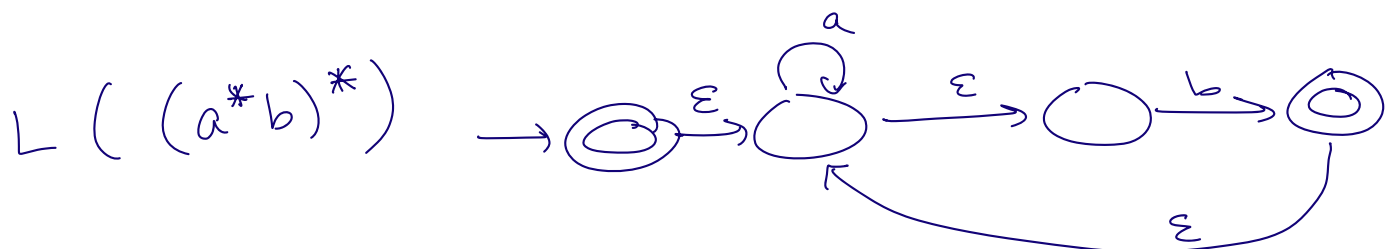
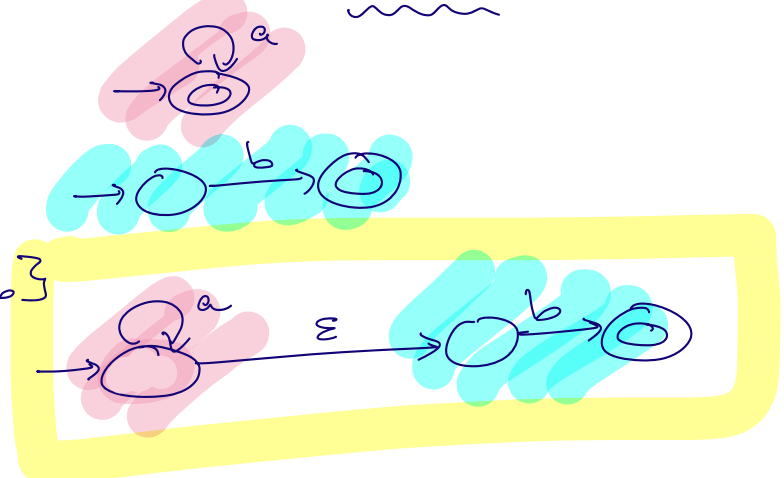
Proof of correctness would prove that $L(N') = A^$ by considering an arbitrary string accepted by N' , tracing an accepting computation of N' on it, and using that trace to prove the string can be written as the result of concatenating some number of strings, each of which is in A ; then, taking an arbitrary string in A^* and proving that it is accepted by N' . Details left for extra practice.*

Application: A state diagram for a NFA over $\Sigma = \{a, b\}$ that recognizes $L((a^*b)^*)$:

$$L(a^*) = \{a^i \mid i \geq 0\}$$

$$L(b) = \{b\}$$

$$L(a^*b) = \{a^i \mid i \geq 0\} \circ \{b\}$$



Suppose A is a language over an alphabet Σ . **Claim:** if there is a NFA N such that $L(N) = A$ then there is a DFA M such that $L(M) = A$.

Proof idea: States in M are “macro-states” – collections of states from N – that represent the set of possible states a computation of N might be in.

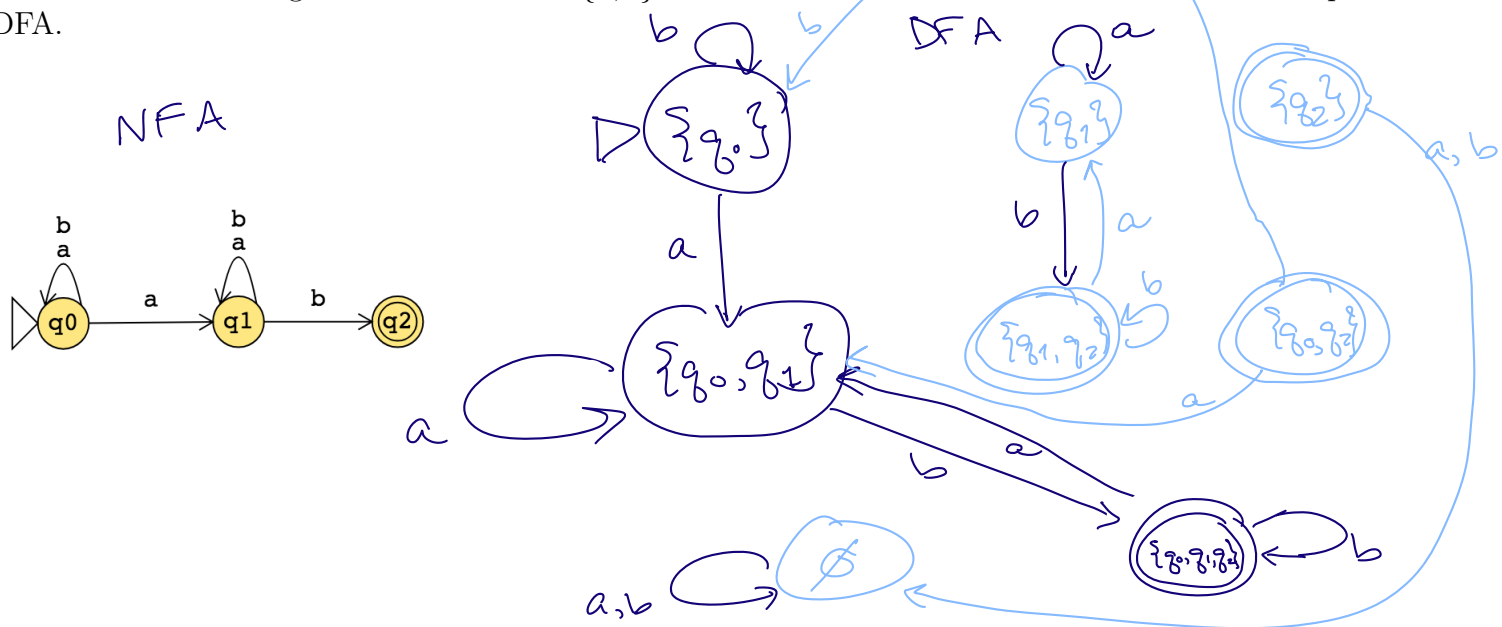
Formal construction: Let $N = (Q, \Sigma, \delta, q_0, F)$. Define

$$M = (\mathcal{P}(Q), \Sigma, \delta', q', \{X \subseteq Q \mid X \cap F \neq \emptyset\})$$

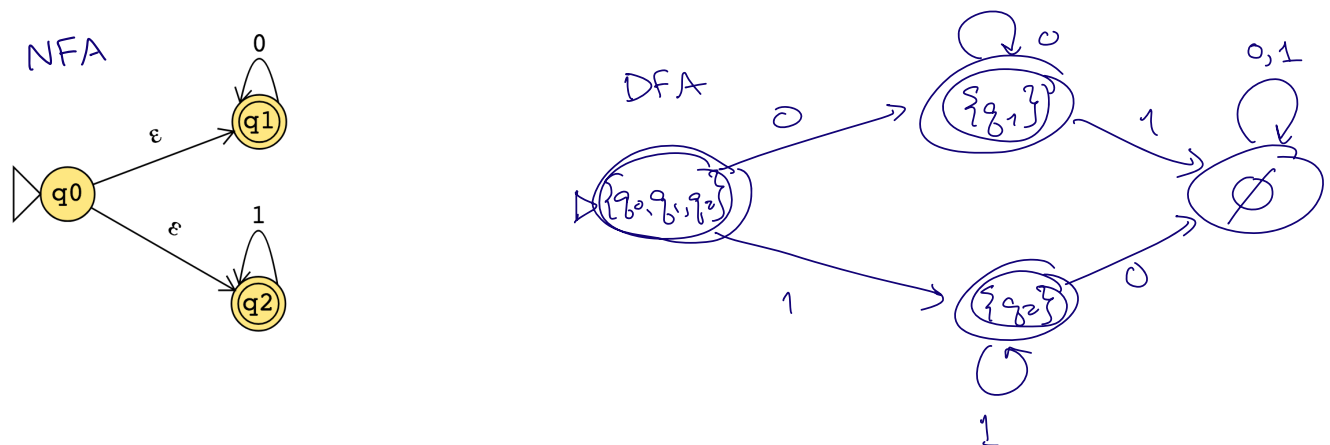
where $q' = \{q \in Q \mid q = q_0 \text{ or is accessible from } q_0 \text{ by spontaneous moves in } N\}$ and

$\delta'((X, x)) = \{q \in Q \mid q \in \delta(r, x) \text{ for some } r \in X \text{ or is accessible from such an } r \text{ by spontaneous moves in } N\}$

Consider the state diagram of an NFA over $\{a, b\}$. Use the “macro-state” construction to find an equivalent DFA.



Consider the state diagram of an NFA over $\{0, 1\}$. Use the “macro-state” construction to find an equivalent DFA.



Note: We can often prune the DFAs that result from the “macro-state” constructions to get an equivalent DFA with fewer states (e.g. only the “macro-states” reachable from the start state).

The class of regular languages

(Friday)

Fix an alphabet Σ . For each language L over Σ :

* There is a DFA over Σ that recognizes L
if and only if

$$\exists M \text{ (} M \text{ is a DFA and } L(M) = A \text{)}$$

* There is a NFA over Σ that recognizes L
if and only if

$$\exists N \text{ (} N \text{ is a NFA and } L(N) = A \text{)}$$

* There is a regular expression over Σ that describes L $\exists R$ (R is a regular expression and $L(R) = A$)

A language is called **regular** when any (hence all) of the above three conditions are met.

We already proved that DFAs and NFAs are equally expressive. It remains to prove that regular expressions are too.

Part 1: Suppose A is a language over an alphabet Σ . If there is a regular expression R such that $L(R) = A$, then there is a NFA, let's call it N , such that $L(N) = A$.

Structural induction: Regular expression is built from basis regular expressions using inductive steps (union, concatenation, Kleene star symbols). Use constructions to mirror these in NFAs.

Application: A state diagram for a NFA over $\{a, b\}$ that recognizes $L(a^*(ab)^*)$:

a^* $(ab)^*$

$$L(a) = L \left(\begin{array}{c} \text{Start} \xrightarrow{a} \text{Accept} \end{array} \right)$$

$$L(b) = L \left(\begin{array}{c} \text{Start} \xrightarrow{b} \text{Accept} \end{array} \right)$$

$$L(a^*) = L \left(\begin{array}{c} \text{Start} \xrightarrow{\varepsilon} \text{State} \xrightarrow{a} \text{State} \xrightarrow{\varepsilon} \text{Accept} \\ \vdots \end{array} \right)$$

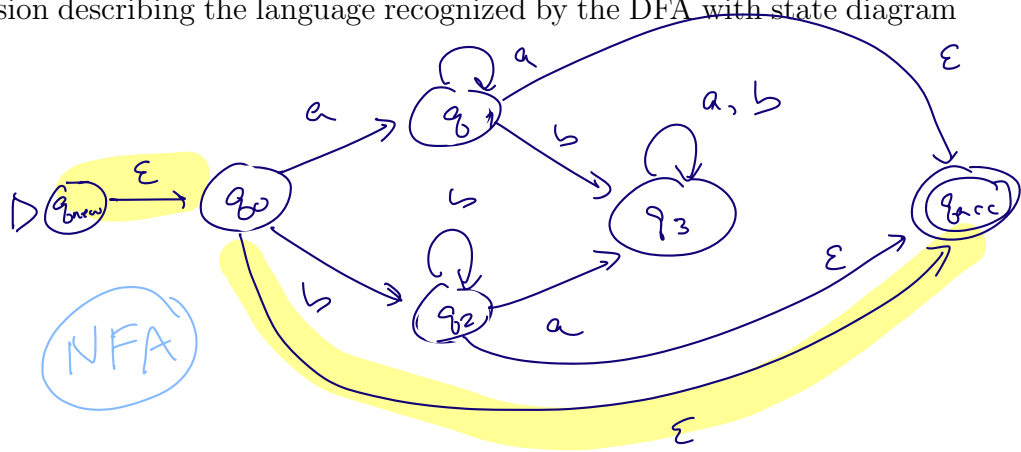
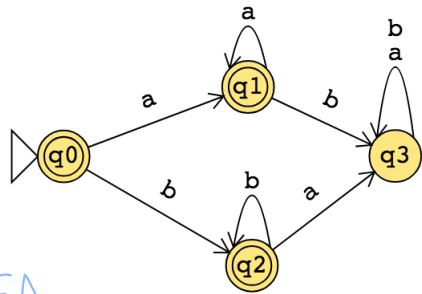
Part 2: Suppose A is a language over an alphabet Σ . If there is a DFA M such that $L(M) = A$, then there is a regular expression, let's call it R , such that $L(R) = A$.

Proof idea: Trace all possible paths from start state to accept state. Express labels of these paths as regular expressions, and union them all.

1. Add new start state with ε arrow to old start state.
2. Add new accept state with ε arrow from old accept states. Make old accept states non-accept.
3. Remove one (of the old) states at a time: modify regular expressions on arrows that went through removed state to restore language recognized by machine.

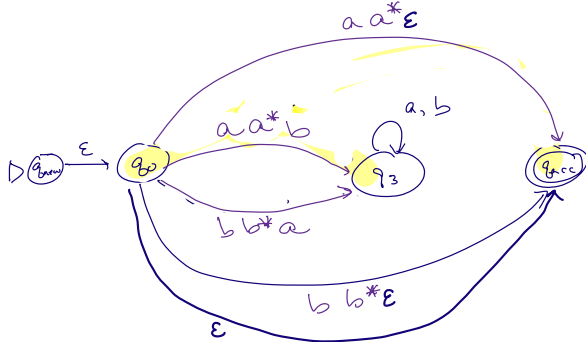
Application: Find a regular expression describing the language recognized by the DFA with state diagram

DFA

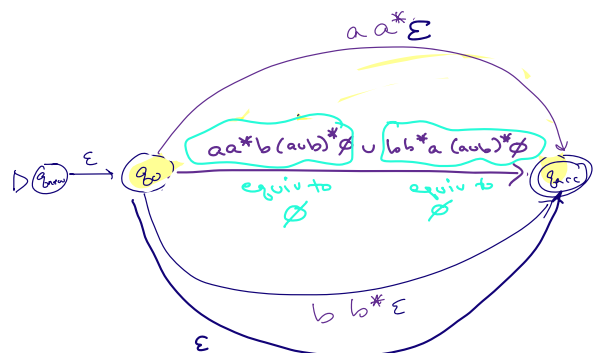
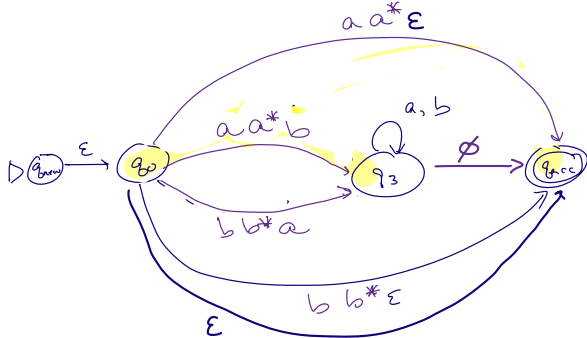


Step 3 Create intermediate representations
"GNFA"

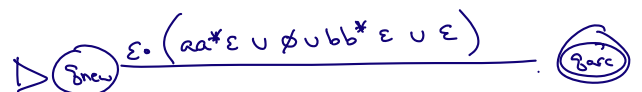
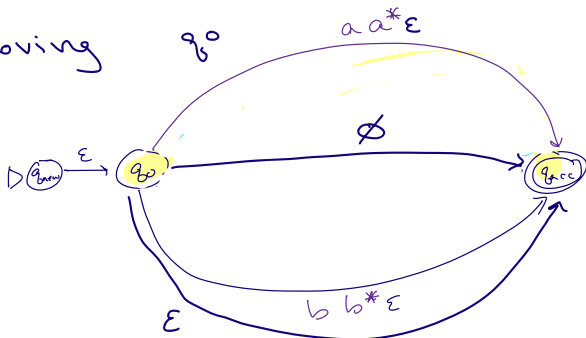
Removing $q1, q2$



Removing $q3$



Removing $q0$



Regular expression describing language is $\epsilon \cdot (aa^*\epsilon \cup \emptyset bb^*\epsilon \cup \epsilon)$
or equivalently, $\epsilon \cdot aa^* \cup bb^*$

Friday: Nonregular languages

Definition and Theorem: For an alphabet Σ , a language L over Σ is called **regular** exactly when L is recognized by some DFA, which happens exactly when L is recognized by some NFA, and happens exactly when L is described by some regular expression

We saw that: The class of regular languages is closed under complementation, union, intersection, set-wise concatenation, and Kleene star.

Prove or Disprove: There is some alphabet Σ for which there is some language recognized by an NFA but not by any DFA.

For any alphabet and arbitrary language L over this alphabet, if there is NFA N with $L(N) = L$ then can create DFA (using macro states) recognizing same language as N , namely recognizing L .

Prove or Disprove: There is some alphabet Σ for which there is some finite language not described by any regular expression over Σ .

By induction: can build up regular expression describing each finite language

Notice $\emptyset = L(\emptyset)$

Prove or Disprove: If a language L is recognized by an NFA then the complement of this language is not recognized by any DFA.

	NFA	N	$L(N) = L$
L	DFA	M	$L(M) = L$
	RegEx	R	$L(R) = L$

so: $\bar{L} = L(\tilde{M})$ where \tilde{M} is DFA w/ flipped accept/reject states compared to M .

Fix alphabet Σ . Is every language L over Σ regular?

Set	Cardinality
$\{0, 1\}$	2
Set of strings over $\{0, 1\}$ $\{0, 1\}^*$	countably infinite
$\{\emptyset, \{0\}, \{1\}, \{0, 1\}\} = \mathcal{P}(\{0, 1\})$	$4 = 2^2$
$\mathcal{P}(\{0, 1\}^*)$ The set of all languages over $\{0, 1\}$	uncountable.
The set of all regular expressions over $\{0, 1\}$	
The set of all regular languages over $\{0, 1\}$	countably infinite.

Regular language \sim regular expression

String order:

$\epsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, \dots$

\uparrow \updownarrow \updownarrow \updownarrow \updownarrow
 \sim 2 3 4 5 - -

Cantor diagonalization:

Powerset of countably infinite set is uncountable

Regular expressions in string order:

$\epsilon, \emptyset, 0, 1, \dots - - -$

Strategy: Find an **invariant** property that is true of all regular languages. When analyzing a given language, if the invariant is not true about it, then the language is not regular.

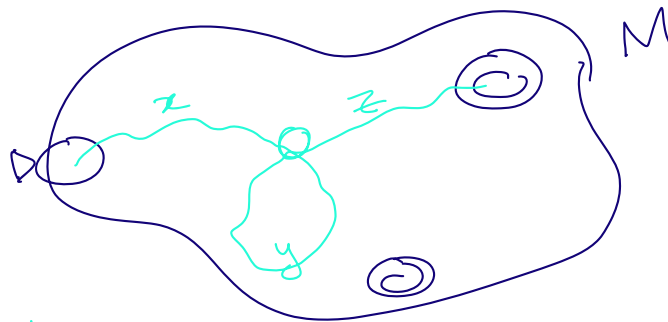
Pumping Lemma (Sipser Theorem 1.70): If A is a regular language, then there is a number p (a pumping length) where, if s is any string in A of length at least p , then s may be divided into three pieces, $s = xyz$ such that

- $|y| > 0$
- for each $i \geq 0$, $xy^iz \in A$
- $|xy| \leq p$.

"any long enough string in our language can be pumped"

Proof illustration

let A be a regular language.
By def, there's a DFA M that recognizes A .



xz
 $S = xyz$
 $xyyz$
 $xyyyz$
 $xyyyyz$

When $s \in A$, ^{and long} computation of M on s starts at start state and ends in an accept state.

True or False: A pumping length for $A = \{0, 1\}^*$ is $p = 5$.

Week 3 at a glance

Textbook reading: Chapter 1

For Monday: Pages 60-61 Theorem 1.47 and Theorem 1.48 (closure proofs).

For Wednesday: Theorem 1.39 “Proof Idea”, Example 1.41, Example 1.56, Example 1.58.

For Friday: Introduction to Section 1.4 (page 77)

For Week 4 Monday: Example 1.75, Example 1.77

Make sure you can:

- Explain nondeterminism and describe tools for simulating it with deterministic computation.
 - Find equivalent DFA for a given NFA
 - Convert between regular expressions and automata
- Use precise notation to formally define the state diagram of DFA, NFA and use clear English to describe computations of DFA, NFA informally.
 - Determine the language recognized by a given NFA
 - Design general constructions for NFA
 - Choose between multiple models to prove that a language is regular
- Classify the computational complexity of a set of strings by determining whether it is regular
 - Explain the limits of the class of regular languages
- Use the pumping lemma to prove that a given language is not regular.
 - Justify why the Pumping Lemma is true

TODO:

Review quizzes based on class material each day.

Homework assignment 2 due next Thursday.

Have you dropped by office hours yet? Find the schedule on the calendar:

