

Monday

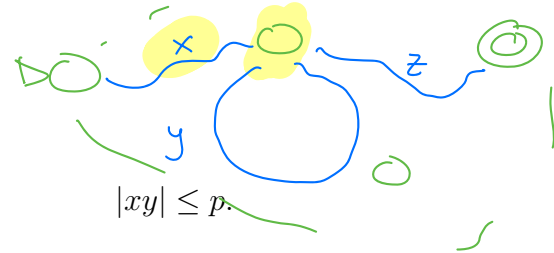
Recap so far: In DFA, the only memory available is in the states. Automata can only “remember” finitely far in the past and finitely much information, because they can have only finitely many states. If a computation path of a DFA visits the same state more than once, the machine can’t tell the difference between the first time and future times it visits this state. Thus, if a DFA accepts one long string, then it must accept (infinitely) many similar strings.

Definition A positive integer p is a **pumping length** of a language L over Σ means that, for each string $s \in \Sigma^*$, if $|s| \geq p$ and $s \in L$, then there are strings x, y, z such that

$$s = xyz$$

and

$$|y| > 0, \quad \text{for each } i \geq 0, xy^iz \in L, \quad \text{and} \quad |xy| \leq p.$$



Negation: A positive integer p is **not a pumping length** of a language L over Σ iff

$$\exists s (|s| \geq p \wedge s \in L \wedge \forall x \forall y \forall z ((s = xyz \wedge |y| > 0 \wedge |xy| \leq p) \rightarrow \exists i (i \geq 0 \wedge xy^iz \notin L)))$$

Informally:

kicked out of L for some i .

Restating **Pumping Lemma**: If L is a regular language, then it has a pumping length.

* **Contrapositive:** If L has no pumping length, then it is nonregular.

To prove L is not regular, we can show that it has no pumping length.

The Pumping Lemma *cannot* be used to prove that a language is regular.

The Pumping Lemma **can** be used to prove that a language *is not* regular.

Extra practice: Exercise 1.49 in the book.

ROADMAP

Proof strategy: To prove that a language L is **not** regular,

- Consider an arbitrary positive integer p
- Prove that p is not a pumping length for L
- Conclude that L does not have *any* pumping length, and therefore it is not regular.

Note: $\Sigma \in L$

Candidate counterexamples "strings that cannot be pumped"

Example: $\Sigma = \{0, 1\}$, $L = \{0^n 1^n \mid n \geq 0\}$.

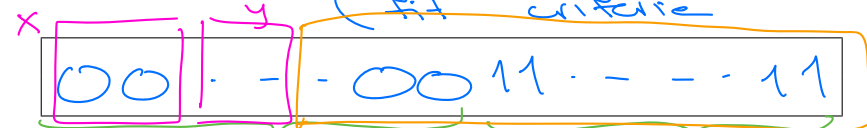
We will prove that L has no pumping length. Want to show L is nonregular.
Fix p an arbitrary positive integer. List strings that are in L and have length greater than or equal to p :
(fixed but unknown) WTS p is not a pumping length for L .

Pick $s = 0^p 1^p$
(depends on p)

Notice $s \in L$ and we will show not pumpable relative to L and p .

Suppose $s = xyz$ with $|xy| \leq p$ and $|y| > 0$.

(arbitrary x, y, z so long as they fit criteria)

S 

$x = 0^k$
 $y = 0^m$
 $z = 0^r 1^p$

Then when $i = 0$, $xy^0z = x y^0 z = 0^k 0^r 1^p = 0^{p-m} 1^p \notin L$

because $m > 0$ so $p-m < p$.

Since we've found a string $s = 0^p 1^p$ that should be pumpable but isn't, we've shown p is not a pumping length for L . But p was arbitrary, so there is no pumping length for L . Hence, L is nonregular.

where
 $k \geq 0$
 $m > 0$
 $k+m \leq p$
 $r = p - (k+m)$

Working out example.

$$x y^0 z = \underbrace{0^k}_x \underbrace{0^r}_z 1^p = 0^{p-m} 1^p$$

$$x y^2 z = \underbrace{0^k}_x \underbrace{0^m}_y \underbrace{0^m}_y \underbrace{0^r}_z 1^p = 0^{p+m} 1^p$$

Summary:

To prove that a language L is nonregular

① Consider arbitrary pos int p .

② Define a specific string s (parametrized by p)
- $s \in L$
- $|s| \geq p$.

③ show s is not pumpable relative to p, L

- Consider arbitrary x, y, z satisfying $s = xyz, |y| > 0, |xy| \leq p$

- Prove $\exists i (xy^i z \notin L)$ NEVER PICK $i=1$.

Example: $\Sigma = \{0, 1\}$, $L = \{ww^R \mid w \in \{0, 1\}^*\}$.

$0^p 0^p, 1^p 1^p, 0^p 11 0^p$

Fix p an arbitrary positive integer. List strings that are in L and have length greater than or equal to p :

Pick $s = 0^p 11 0^p$

$(01)^p (10)^p$

Suppose $s = xyz$ with $|xy| \leq p$ and $|y| > 0$. There are $k \geq 0, m > 0$, let $m = p - k - r$
 then $x = 0^k$ $y = 0^m$ $z = 0^r 11 0^p$

Then when $i = 0$, $xy^i z = 0^k 0^r 11 0^p = 0^{p-m} 11 0^p \notin L$
 because either zz is odd length (when m is odd) and so not an even length palindrome hence not in L , or zz is even length and first half has two ones while second half has no ones, so can't be a palindrome.

Example: $\Sigma = \{0, 1\}$, $L = \{0^j 1^k \mid j \geq k \geq 0\}$.

Fix p an arbitrary positive integer. List strings that are in L and have length greater than or equal to p :

Pick $s =$

Suppose $s = xyz$ with $|xy| \leq p$ and $|y| > 0$.

Then when $i =$, $xy^i z =$

Example: $\Sigma = \{0, 1\}$, $L = \{0^n 1^m 0^n \mid m, n \geq 0\}$.

Fix p an arbitrary positive integer. List strings that are in L and have length greater than or equal to p :

Pick $s =$

Suppose $s = xyz$ with $|xy| \leq p$ and $|y| > 0$.

Then when $i =$, $xy^i z =$

Extra practice:

Language	$s \in L$	$s \notin L$	Is the language regular or nonregular?
$\{a^n b^n \mid 0 \leq n \leq 5\}$			
$\{b^n a^n \mid n \geq 2\}$			
$\{a^m b^n \mid 0 \leq m \leq n\}$			
$\{a^m b^n \mid m \geq n + 3, n \geq 0\}$			
$\{b^m a^n \mid m \geq 1, n \geq 3\}$			
$\{w \in \{a, b\}^* \mid w = w^R\}$			
$\{ww^R \mid w \in \{a, b\}^*\}$			

Review: Week 4 Monday

Recall: Review quizzes based on class material are assigned each day. These quizzes will help you track and confirm your understanding of the concepts and examples we work in class. Quizzes can be submitted on Gradescope as many times (with no penalty) as you like until the quiz deadline: the three quizzes each week are all due on Friday (with no penalty late submission open until Sunday).

Please complete the review quiz questions on [Gradescope](#) about pumping lemma and nonregular sets.

Pre class reading for next time: Page 112

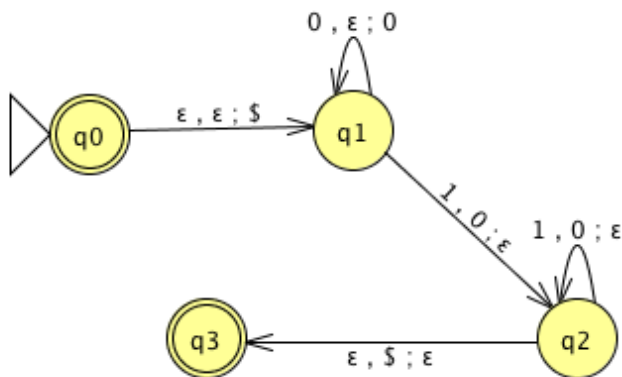
Wednesday

Regular sets are not the end of the story

- Many nice / simple / important sets are not regular
- Limitation of the finite-state automaton model: Can't "count", Can only remember finitely far into the past, Can't backtrack, Must make decisions in "real-time"
- We know actual computers are more powerful than this model...

The **next** model of computation. Idea: allow some memory of unbounded size. How?

- To generalize regular expressions: **context-free grammars**
- To generalize NFA: **Pushdown automata**, which is like an NFA with access to a stack: Number of states is fixed, number of entries in stack is unbounded. At each step (1) Transition to new state based on current state, letter read, and top letter of stack, then (2) (Possibly) push or pop a letter to (or from) top of stack. Accept a string iff there is some sequence of states and some sequence of stack contents which helps the PDA processes the entire input string and ends in an accepting state.



Trace the computation of this PDA on the input string 01.

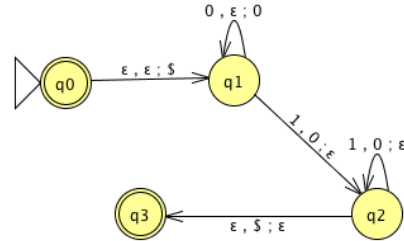
Trace the computation of this PDA on the input string 011.

Definition A **pushdown automaton** (PDA) is specified by a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$ where Q is the finite set of states, Σ is the input alphabet, Γ is the stack alphabet,

$$\delta : Q \times \Sigma_\epsilon \times \Gamma_\epsilon \rightarrow \mathcal{P}(Q \times \Gamma_\epsilon)$$

is the transition function, $q_0 \in Q$ is the start state, $F \subseteq Q$ is the set of accept states.

Formal definition



Draw the state diagram of a PDA with $\Sigma = \Gamma$.

Draw the state diagram of a PDA with $\Sigma \cap \Gamma = \emptyset$.

Review: Week 4 Wednesday

Please complete the review quiz questions on [Gradescope](#) about PDA definitions.

Pre class reading for next time: Page 102

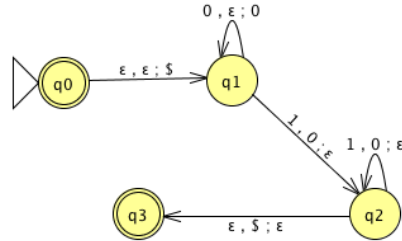
Friday

Definition A **pushdown automaton** (PDA) is specified by a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$ where Q is the finite set of states, Σ is the input alphabet, Γ is the stack alphabet,

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Formal definition



Draw the state diagram of a PDA with $\Sigma = \Gamma$.

Draw the state diagram of a PDA with $\Sigma \cap \Gamma = \emptyset$.

A PDA recognizing the set $\{$ can be informally described as:

Read symbols from the input. As each 0 is read, push it onto the stack. As soon as 1s are seen, pop a 0 off the stack for each 1 read. If the stack becomes empty and there is exactly one 1 left to read, read that 1 and accept the input. If the stack becomes empty and there are either zero or more than one 1s left to read, or if the 1s are finished while the stack still contains 0s, or if any 0s appear in the input following 1s, reject the input.

State diagram for this PDA:

Extra practice: Consider the state diagram of a PDA with input alphabet Σ and stack alphabet Γ .

Label	means
$a, b; c$ when $a \in \Sigma, b \in \Gamma, c \in \Gamma$	
$a, \varepsilon; c$ when $a \in \Sigma, c \in \Gamma$	
$a, b; \varepsilon$ when $a \in \Sigma, b \in \Gamma$	
$a, \varepsilon; \varepsilon$ when $a \in \Sigma$	

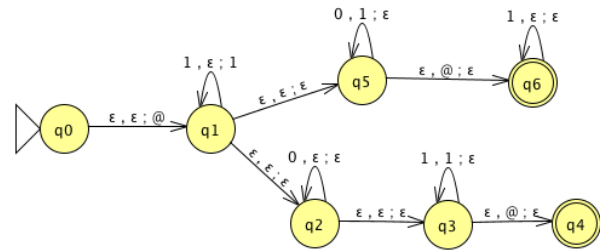
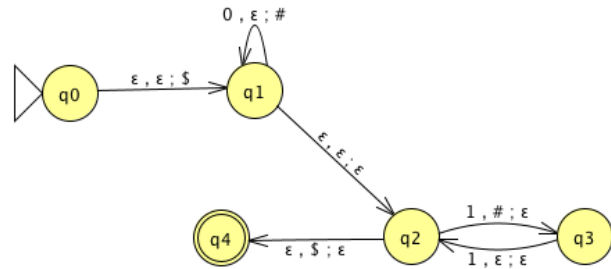
How does the meaning change if a is replaced by ε ?

Note: alternate notation is to replace ; with \rightarrow

For the PDA state diagrams below, $\Sigma = \{0, 1\}$.

Mathematical description of language

State diagram of PDA recognizing language



$$\{0^i 1^j 0^k \mid i, j, k \geq 0\}$$

Review: Week 4 Friday

Please complete the review quiz questions on [Gradescope](#) about PDA construction.