Question 1 (problem 7.3 of text) Weighted least square estimator

Assume, $\vec{\varepsilon} \sim N_n(0, \sigma^2 I)$ $\qquad y = Z\beta + \varepsilon \qquad \therefore \underline{V^{-\frac{1}{2}}y} = \underline{V^{-\frac{1}{2}}Z\beta} + \underline{V^{-\frac{1}{2}}\varepsilon}$

$$y^* = Z^*\beta + \varepsilon^*$$

$$\hat{\beta}_w = (Z^{*T}Z^*)^{-1}Z^{*T}y^*$$
$$= (Z^TV^{-\frac{1}{2}}V^{-\frac{1}{2}}Z)^{-1}Z^TV^{-\frac{1}{2}}V^{-\frac{1}{2}}y$$
$$= (Z^TV^{-1}Z)^{-1}Z^TV^{-1}y$$

$$S(\beta) = \epsilon^{*T}\epsilon^*$$
$$= (y^* - Z^*\beta)^T(y^* - Z^*\beta)$$
$$= (V^{-\frac{1}{2}}y - ZV^{-\frac{1}{2}}\beta)^T(V^{-\frac{1}{2}}y - V^{-\frac{1}{2}}Z\beta)$$

$$\hat{\epsilon}^* = y - \hat{y} = y - Z(Z^TV^{-1}Z)^{-1}Z^TV^{-1}y$$
$$= [I - Z(Z^TV^{-1}Z)^{-1}Z^TV^{-1}]y$$
$$= [I - H^T]y$$

$$E(\hat{\varepsilon}^T\hat{\varepsilon}) = E[Y^T[I-H]^T[I-H]Y]$$
$$= E[Y^T[I-H]Y]$$
$$= E[\epsilon^T[I-H]\varepsilon]$$
$$= tr[[I-H]\underline{E(\epsilon^T\epsilon)}] \qquad E(\varepsilon^T\varepsilon) = \sigma^2$$
$$= \sigma^2 tr[I - Z(Z^TZ)^{-1}Z^T]$$
$$= \sigma^2(n-r-1)$$

$$E(\hat{\varepsilon}^{*T}\hat{\varepsilon}^*) = E[Y^T(I-H^T)[I-H]Y$$
$$= E[\varepsilon^{*T}[I-H]\varepsilon^*]$$
$$= \sigma^2(n-r-1)$$

$$E\left(\frac{1}{n-r-1}(\varepsilon^{*T}\varepsilon^*)\right) = E\left(\frac{1}{n-r-1}[(y^* - Z^*\hat{\beta})^T(y^* - Z^*\hat{\beta})]\right]$$
$$= E\left(\frac{1}{n-r-1}[(y^TV^{\frac{1}{2}} - \hat{\beta}^TZ^TV^{\frac{1}{2}})(V^{\frac{1}{2}}y - V^{\frac{1}{2}}Z\hat{\beta})\right)$$
$$= E\left(\frac{1}{n-r-1}[(Y - Z\hat{\beta})^TV^{\frac{1}{2}}V^{-\frac{1}{2}}(Y - Z\hat{\beta})]\right]$$
$$= \frac{1}{n-r-1}E(\varepsilon^{*T}\varepsilon^*)$$
$$= \frac{1}{n-r-1}(\sigma^2(n-r-1))$$
$$= \sigma^2$$