→ Once the optimal coefficients $Y^\star = [\alpha_1^\star, \dots, \alpha_m^\star]^T$ have been determined, we can finally assemble our approximation

$$X^{(m)} = X^{(0)} + \sum_{k=1}^{m} \alpha_m^\star v_m = X^{(0)} + V_m Y^\star$$

→ This is what we call the Generalised Minimal Residual (GMRES) method. The key algorithmic feature of the GMRES method, however, is that the entire process can be done "on the fly", as we build each column of $V_m, \bar{H}_m$, etc, and every factor of $G = G_m G_{m-1} \dots G_1$ for increasing m. An efficient, classical implementation is:

<u>Algorithm</u>: GMRES$(A, x^{(0)}, b, m, tol)$

1. $r^{(0)} = b - Ax^{(0)}$, $v_1 = r^{(0)} / \|r^{(0)}\|$, $g = \|r^{(0)}\| e_1$
2. for $j = 1, \dots, m$
3. $\quad [v_{j+1}, h_j] = \text{getKrylov}(A, v_j)$ (from the previous lecture)

   $\quad$ ↳ j-th column of $\bar{H}_m$

4. for $k = 2, \dots, j$

5. $\quad \begin{bmatrix} h_{k-1,j} \\ h_{k,j} \end{bmatrix} \leftarrow \begin{bmatrix} c_{k-1} & s_{k-1} \\ -s_{k-1} & c_{k-1} \end{bmatrix} \begin{bmatrix} h_{k-1,j} \\ h_{k,j} \end{bmatrix}$

6. end

7. $d = \sqrt{h_{j,j}^2 + h_{j+1,j}^2}\,, \quad c_j = \overbrace{h_{j,j}/d}^{\cos\theta}\,, \quad s_j = \overbrace{h_{j+1,j}/d}^{\sin\theta}$

8. $h_{j,j} \leftarrow d$

9. $\begin{bmatrix} g_j \\ g_{j+1} \end{bmatrix} \leftarrow \begin{bmatrix} c_j & s_j \\ -s_j & c_j \end{bmatrix} \begin{bmatrix} g_j \\ 0 \end{bmatrix}$

10. if $|g_{j+1}| < tol$, exit loop! (because $\|r^{(j)}\| = |g_{j+1}|$)

11. end

12. Solve the upper triangular system $R y^* = g$, where

$$r_{ij} = \begin{cases} \overbrace{h_{ij}}^{\text{overwritten values}}, & \text{if } i \le j \\ 0, & \text{otherwise} \end{cases}$$

13. $x^{(j)} = x^{(0)} + V_j y^*$

→ Notice how we only solve the triangular system (and compute the actual iterate $x^{(j)}$) at the end, but we still know how large the residual is after each iteration ($\|r^{(j)}\| = |g_{j+1}|$)!