Homework 7

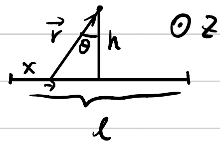
2. A filamentary conductor is formed into an equilateral triangle with sides of length l, carrying current I. Find the magnetic field intensity **H** at the center of the triangle.

Consider one segment with length l

$$d\vec{H} = \frac{I d\vec{l} \times \hat{r}}{4\pi r^2} \Rightarrow d\vec{H} = \frac{I \sin\theta dl}{4\pi r^2} \vec{a}_z$$

Thus evaluate the integral

$$\vec{H} = \vec{a}_z \int_0^l \frac{I\left(\frac{l}{2}-x\right) dx}{4\pi\left[\left(\frac{l}{2}-x\right)^2 + h^2\right]^{3/2}}$$



$$= \frac{I}{4\pi}\left(\frac{1}{h} - \frac{1}{\left(\frac{l^2}{4}+h^2\right)^{1/2}}\right)\vec{a}_z$$

Thus in this setup, $h = \frac{l}{2\sqrt{3}}$, then H field excited by each side is

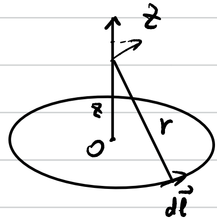
$$\vec{H}_i = \frac{3I}{2\pi l}\vec{a}_z$$

For 3 segments

$$\vec{H} = \frac{9I}{2\pi l}\vec{a}_z$$

3. Two circular current loops are centered on the z axis at $z = \pm h$. Each loop has radius a and carries current I in the $\mathbf{a}_\varphi$ direction. Find $\mathbf{H}$ on the z axis over the range $-h < z < h$.

Biot-Savarts law (for single loop)



$$d\vec{H} = \frac{I}{4\pi} \frac{d\vec{l} \times \hat{r}}{r^2}$$

with the symmtery of rotation, project on z axis and the evaluate the integral

$$d\vec{H} = \frac{I}{4\pi} \frac{a^2 d\varphi}{(z^2+a^2)^{3/2}} \vec{a_z}$$

$$\Rightarrow \vec{H} = \frac{Ia^2 \vec{a_z}}{4\pi (z^2+a^2)^{3/2}} \cdot 2\pi = \frac{Ia^2}{2(z^2+a^2)^{3/2}} \vec{a_z}$$

Thus, for double current loops

$$\vec{H} = \vec{H}(z+h) + \vec{H}(z-h)$$

$$= \frac{Ia^2}{2} \left(\frac{1}{[(z+h)^2+a^2]^{3/2}} + \frac{1}{[(z-h)^2+a^2]^{3/2}} \right) \vec{a_z}$$

4. Let $\mathbf{A} = (3y - z)\mathbf{a}_x + 2xz\mathbf{a}_y$ Wb/m in a certain region of free space. (*a*) Show that $\nabla \cdot \mathbf{A} = 0$. (*b*) At $P(2, -1, 3)$, find **A**, **B**, **H**, and **J**.

(a)

$$\nabla \cdot \vec{A} = \frac{\partial}{\partial x}(3y-z) + \frac{\partial}{\partial y}(2xz) = 0$$

(b) To obtain $\vec{A}$, directly substitude the parameters

$$\vec{A}(2,-1,3) = -6\,\vec{a}_x + 12\,\vec{a}_y \text{ Wb/m}$$

Due to $\vec{B} = \nabla \times \vec{A}$, then

$$\vec{B} = \left.\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & 0 \end{vmatrix}\right|_{(x,y,z)=(2,-1,3)} = -2x\,\vec{a}_x - \vec{a}_y + (2z-3)\vec{a}_z$$

$$= -4\vec{a}_x - \vec{a}_y + 3\vec{a}_z \text{ T}$$

In free space, thus

$$\vec{H} = \frac{\vec{B}}{\mu_0} = -\frac{4}{\mu_0}\vec{a}_x - \frac{1}{\mu_0}\vec{a}_y + \frac{3}{\mu_0}\vec{a}_z \text{ A/m}$$

In free space, thus

$$\vec{J} = 0$$

5. Use an expansion in rectangular coordinates to show that the curl of the gradient of any scalar field G is identically equal to zero.

In rectangular coordinate

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

$$\nabla \times \vec{g} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ g_x & g_y & g_z \end{vmatrix} \quad \text{--- (1)}$$

$$\vec{g} = \nabla G = \frac{\partial G}{\partial x}\vec{i} + \frac{\partial G}{\partial y}\vec{j} + \frac{\partial G}{\partial z}\vec{k}$$

Substitude into (1)

$$\nabla \times \vec{g} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \end{vmatrix} = \left(\frac{\partial}{\partial y}\frac{\partial G}{\partial z} - \frac{\partial}{\partial z}\frac{\partial G}{\partial y}\right)\vec{i}$$

$$+\left(\frac{\partial}{\partial z}\frac{\partial G}{\partial x} - \frac{\partial}{\partial x}\frac{\partial G}{\partial z}\right)\vec{j}$$

$$+\left(\frac{\partial}{\partial x}\frac{\partial G}{\partial y} - \frac{\partial}{\partial y}\frac{\partial G}{\partial x}\right)\vec{k}$$

Order of Calculation of partial derivatives can be changed.
Thus

$$\nabla \times \vec{g} = 0, \text{ where } \vec{g} = \nabla G, \; G \text{ is a scalor field}$$

Q.E.D.