9.6 Partial orderings

偏序关系

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- Hasse Diagram（哈斯图）
- Topological Sorting（拓扑排序）
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Review

- Properties of relations on a set *A*:
  - *Reflexive* $\forall x[x \in A \rightarrow (x, x) \in R]$
  - *Irreflexive* $\forall x[x \in A \rightarrow (x, x) \notin R]$
  - *Symmetric* $\forall x \forall y[(x, y) \in R \rightarrow (y, x) \in R]$
  - *Asymmetric* $\forall x \forall y[(x, y) \in R \rightarrow (y, x) \notin R]$
  - *Antisymmetric*

$$\forall x \forall y[(x, y) \in R \wedge (y, x) \in R \rightarrow x = y]$$

  - *Transitive*

传递

$$\forall x \forall y \forall z[(x, y) \in R \wedge (y, z) \in R \rightarrow (x, z) \in R]$$

Partial Orderings

- **Partial order（偏序关系）**:A relation *R* on a set *A* is called a *partial order* if *R* is
反身的 reflexive, antisymmetric（反对称）, and transitive. 传递的
- **Partially ordered set**:The set *A* together with the partial order *R* is called a *partially ordered set*, or simply a *poset*, and we will denote this poset by (*A*, *R*).

偏序集 — 偏序关系

Partial Orderings (*continued*)

Example 1: Show that the "greater than or equal" relation ($\geq$) is a partial ordering on the set of integers.

- *Reflexivity*: $a \geq a$ for every integer a. reflexivity
- *Antisymmetry*: If $a \geq b$ and $b \geq a$, then $a = b$.
- *Transitivity*: If $a \geq b$ and $b \geq c$, then $a \geq c$.

These properties all follow from the order axioms for the integers. (*See Appendix* 1).

Partial Orderings (*continued*)

Example 2: Show that the divisibility relation (|) is a partial ordering on the set of integers.

- *Reflexivity*: $a \mid a$ for all integers a. (*see Example* 9 *in Section* 9.1)
- *Antisymmetry*: If a and b are positive integers with $a \mid b$ and $b \mid a$, then $a = b$. (*see Example* 12 *in Section* 9.1)
- *Transitivity*: Suppose that a divides b and b divides c. Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c. Therefore, the relation is transitive.

- $(\mathbf{Z}^+, \mid)$ is a poset.

Partial Orderings (*continued*)

Example 3: Show that the inclusion relation ($\subseteq$) is a partial ordering on the power set of a set S.

- *Reflexivity*: $A \subseteq A$ whenever A is a subset of S.
- *Antisymmetry*: If A and B are positive integers with $A \subseteq B$ and $B \subseteq A$, then $A = B$.
- *Transitivity*: If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

- $(P(S), \subseteq)$

The properties all follow from the definition of set inclusion.

Comparability

Definition 2: The elements a and b of a poset $(S, \preccurlyeq)$ are *comparable* if either $a \preccurlyeq b$ or $b \preccurlyeq a$. When a and b are elements of S so that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, then a and b are called *incomparable*.

The symbol $\preccurlyeq$ is used to denote the relation in any poset.

Definition 3: If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a *totally ordered* or *linearly ordered set*, and $\preccurlyeq$ is called a *total order* or a *linear order.* A totally ordered set is also called a *chain.*

Definition 4: $(S, \preccurlyeq)$ is a well-ordered set if it is a poset such that $\preccurlyeq$ is a total ordering and every nonempty subset of S has a least element.

Theorem 1 良序归纳原理

- The principle of well-ordered induction: Suppose $(S,\preccurlyeq)$ is a well-ordered set. Then $P(x)$ is true for all $x \in S$, if 良序集
- *INDUCTIVE STEP:* for every $y \in S$, if $P(x)$ is true for all $x \in S$ with $x \preccurlyeq y$, Then $P(y)$ is true.
- 意义:良序集的命题证明简化.

Quasiorder(拟序关系)

- *Quasiorder:* A relation R on a set A is called *quasiorder* if it is *transitive* and *irreflexive*.
  - Example:
    - $(P(S), \subset)$

Product partial order（乘积偏序）

- If $(A, \preccurlyeq_1)$ and $(B, \preccurlyeq_2)$ are posets, then $(A\times B,\preccurlyeq)$ is a poset, with partial order $\preccurlyeq$ defined by
  - $(a,b) \preccurlyeq (a',b')$ if $a \preccurlyeq_1 a'$ *in* A and $b \preccurlyeq_2 b'$ *in* B.
- This ordering is called *product partial order*.

Lexicographic order （词典顺序）

- If $(A, \preccurlyeq_1)$ and $(B, \preccurlyeq_2)$ are posets, then $(A \times B, \prec)$ is a poset, with partial order $\prec$ defined by
  偏序
  - $(a,b) \prec (a',b')$ either if $a \preccurlyeq_1 a'$ or if $a = a'$ and $b \preccurlyeq_2 b'$.

    $(a,b) \prec (a',b')$ $a \preccurlyeq_1 a'$, $a = a'$ $b \preccurlyeq_2 b'$
- This ordering is called *lexicographic*, or *"dictionary" order*.

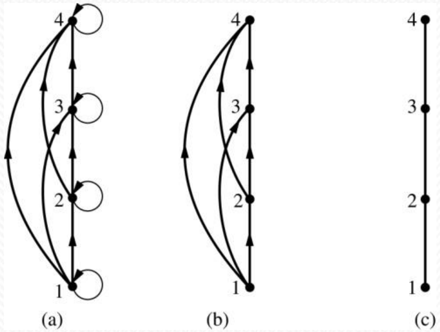
  字典排序

Example: Consider strings of lowercase English letters. A lexicographic ordering can be defined using the ordering of the letters in the alphabet. This is the same ordering as that used in dictionaries.

- *discreet* $\prec$ *discrete*, because these strings differ in the seventh position and $e \prec t$.
- *discreet* $\prec$ *discreetness*, because the first eight letters agree, but the second string is longer.

Hasse Diagrams 哈塞图

Definition: A *Hasse diagram* is a visual representation of a partial ordering that leaves out edges that must be present because of the reflexive and transitive properties.



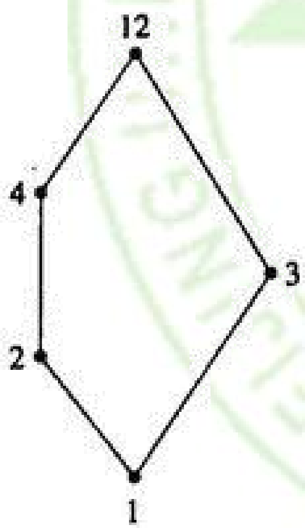
A partial ordering is shown in (a) of the figure above. The loops due to the reflexive property are deleted in (b). The edges that must be present due to the transitive property are deleted in (c). The Hasse diagram for the partial ordering (a), is depicted in (c).

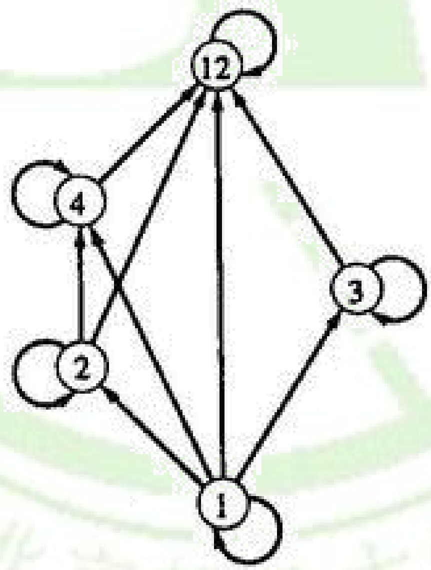
Procedure for Constructing a Hasse Diagram

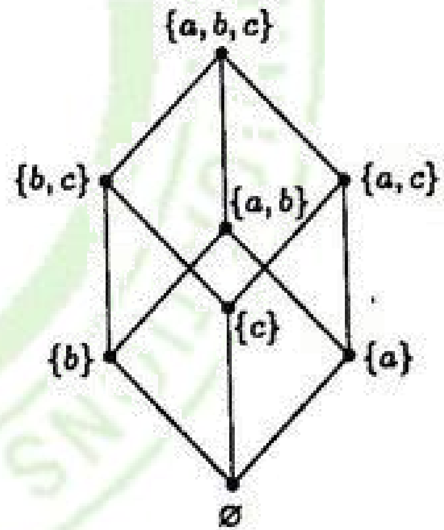
- To represent a finite poset $(S, \preccurlyeq)$ using a Hasse diagram, start with the directed graph of the relation:
  - Remove the loops (a, a) present at every vertex due to the reflexive property.
  - Remove all edges (x, y) for which there is an element $z \in S$ such that $x \prec z$ and $z \prec y$. These are the edges that must be present due to the transitive property.
  - Arrange each edge so that its initial vertex is below the terminal vertex. Remove all the arrows, because all edges point upwards toward their terminal vertex.

Hasse Diagram

- $(\{1, 2, 3, 4, 12\}, |)$
- $(P(\{a, b, c\}), \subseteq)$







Topological Sorting（拓扑排序）

拓扑排序

- If A is a poset with partial order $\leq$, we sometimes need to find *a linear order* $\circ$ for the set A that will merely be an extension of the given partial order in the sense that if $a \leq b$, then $a \circ b$.
- The process of constructing a linear order such as $\circ$ is called *topological sorting*

Topological Sorting Algorithm

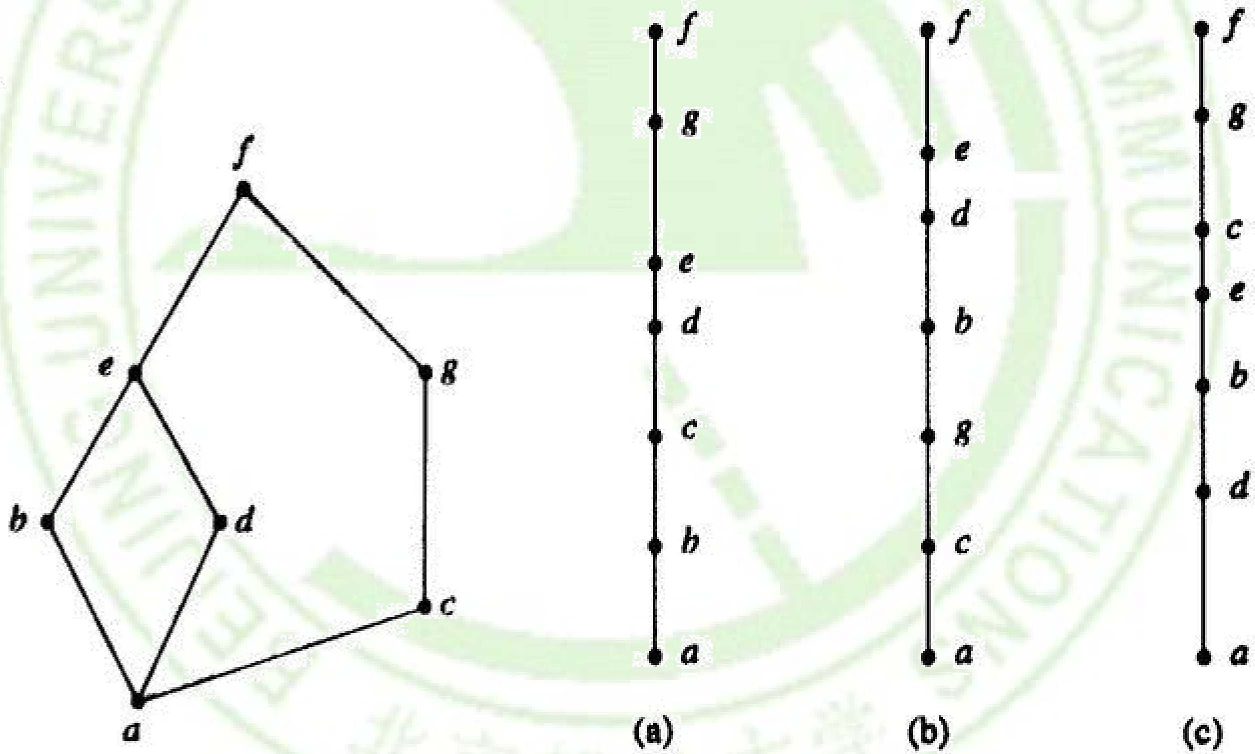
- For finding a topological sorting of a finite poset $(A, \preccurlyeq)$.
- Step1 Choose a minimal element a of A.
- Step2 Make a the next entry of SORT and replace A with A-$\{a\}$.
- Step3 Repeat steps 1 and 2 until A=$\{\}$.
- End of Algorithm

Maximal(minimal) element

最大元

- An element $a \in A$ is called a maximal element of A if there is no element c in A such that $a \preccurlyeq c$.
- An element $b \in A$ is called a minimal element of A if there is no element c in A such that $c \preccurlyeq b$.

Example



Extremal elements

maximal 极大元

- *maximal element*（极大元）

minimal 极小元

- *minimal element*（极小元）

greatest 最大元

- *greatest element*（最大元）
  - denoted by *I* and often called the *unit element*（单位元）

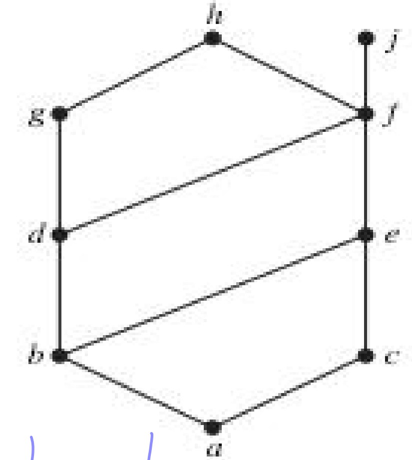
least 最小元

- *least element*（最小元）
  - denoted by *0* and often called the *zero element*（零元）

Greatest (least) element

- An element $a \in A$ is called a **greatest element** of A if $x \preccurlyeq a$ for all $x \in A$.
- An element $a \in A$ is called a **least element** of A if $a \preccurlyeq x$ for all $x \in A$.
- The greatest element of a poset, if it exists, is denoted by I, and is often called the **unit** element. Similarly, the least element is called **zero** element.

Lub（最小上界）
Glb（最大下界）



- Consider a poset and a subset B of A.

  upper bound

  - An element $a \in A$ is called a *upper bound*（上界） of B if $b \preccurlyeq a$ for all $b \in B$.

  lower bound

  - An element $a \in A$ is called a *lower bound*（下界）of B if $a \preccurlyeq b$ for all $b \in B$.

  最小上界 LUB 最小上界

  - An element $a \in A$ is called a *least upper bound* (LUB) of B if a is an upper bound of B and $a \preccurlyeq a'$, whenever a' is an upper bound of B.

  最大下界 最大下界

  - An element $a \in A$ is called a *greatest lower bound* (GLB) of B if a is an lower bound of B and $a' \preccurlyeq a$, whenever a' is an lower bound of B.

Theorem

- Let $(A, \preccurlyeq)$ be a poset
  - If A is a finite nonempty, then A has at least *one* maximal element and at least *one* minimal element. 极大元/极小元
  - A has at most *one* greatest element and at most *one* least element. √
  - A subset B of A has at most *one* LUB and at most *one* GLB. LUB GLB

同构

Isomorphism（同构）

- Let $(A, \preccurlyeq)$ and $(A', \preccurlyeq')$ be posets and let $f: A \rightarrow A'$ be a one-to-one correspondence between A and A'. The function f is called an *isomorphism* from $(A, \preccurlyeq)$ to $(A', \preccurlyeq')$ if, for any a and b in A,
  - $a \preccurlyeq b$ if and only if $f(a) \preccurlyeq' f(b)$.
- If $f: A \rightarrow A'$ is an isomorphism, we say that $(A, \preccurlyeq)$ and $(A', \preccurlyeq')$ are *isomorphic posets*.

Isomorphism: Example

- Let $(Z^+, \leq)$ and $(E^+, \leq)$ be posets. The function $f : Z^+ \rightarrow E^+$ given by
  - $f(a) = 2a$
- is an *isomorphism* from $(Z^+, \leq)$ to $(E^+, \leq)$
- Proof
  - First, is onto and one-to-one
  - Second, $a \leq b$ *iff* $2a \leq 2b$

Principle of correspondence（对应原理）

- Suppose that f: $A \rightarrow A'$ is an isomorphism from a poset $(A, \preccurlyeq)$ to a poset $(A', \preccurlyeq')$. Suppose also that B is a subset of A, and $B' = f(B)$ is the corresponding subset of A'. Then
- If the elements of B have any property relating to one another or to other elements of A, and if this property can be defined entirely in terms of the relation $\preccurlyeq$, then the elements of B' must possess exactly the same property, defined in terms of $\preccurlyeq'$.
  - two finite isomorphic posets must have the same Hasse diagram.

Theorem

Suppose that $(A, \preccurlyeq)$ and $(A', \preccurlyeq')$ are isomorphic posets under the isomorphism $f: A \rightarrow A'$.

- If a is a maximal (minimal) element of $(A, \preccurlyeq)$, then $f(a)$ is a maximal (minimal) element of $(A', \preccurlyeq')$.
- If a is a greatest (least) element of $(A, \preccurlyeq)$, then $f(a)$ is a greatest (least) element of $(A', \preccurlyeq')$.
- If a is an upper bound (lower bound, least upper bound, greatest lower bound) of $(A, \preccurlyeq)$, then $f(a)$ is an upper bound (lower bound, least upper bound, greatest lower bound) of $(A', \preccurlyeq')$.
- If every subset of $(A, \preccurlyeq)$ has a LUB or GLB, then every subset of $(A', \preccurlyeq')$ has a LUB or GLB.

Lattice 格 lattice

- A *lattice* is a poset in which every subset $\{a,b\}$ consisting of two elements has a least upper bound and a greatest lower bound.
  - denote LUB($\{a,b\}$) by $a \vee b$ and call it the *join*（并） of a and b. join
  - denote GLB($\{a,b\}$) by $a \wedge b$ and call it the *meet*（交） of a and b.

meet

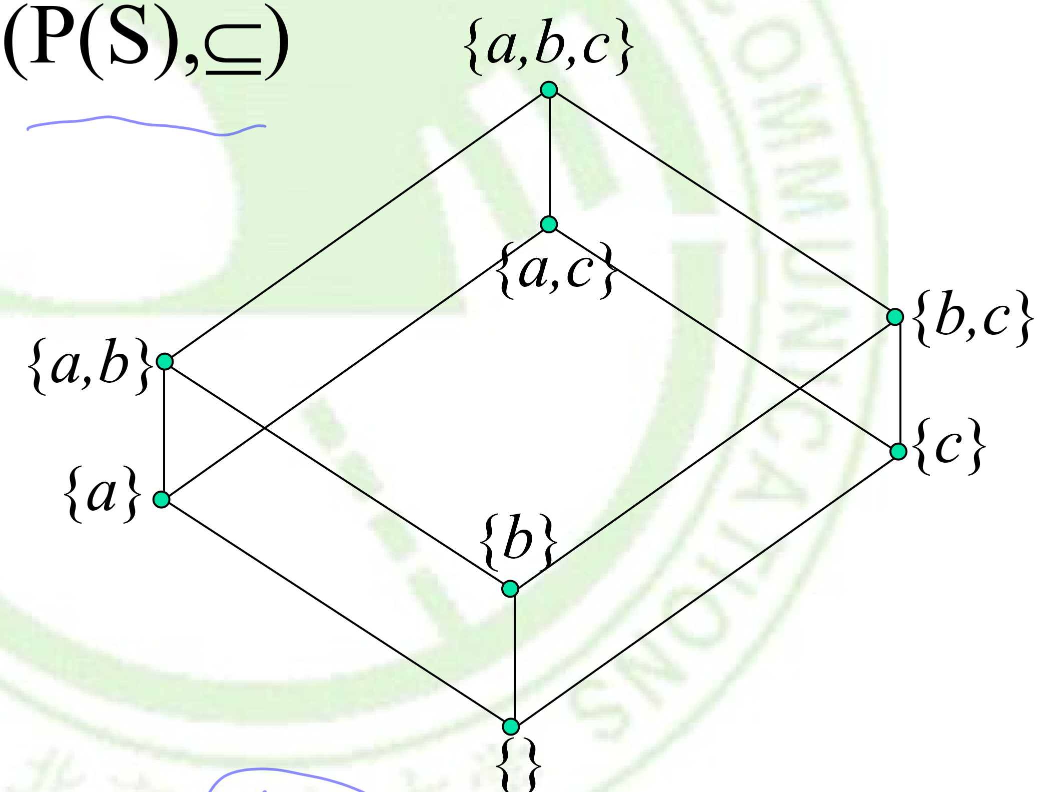
Example

- Let S be a set and let $L = P(S)$. $L = P(S)$
  - As we have seen, $\subseteq$, containment, is a partial order on L. Let A and B belong to the poset $(L, \subseteq)$.
- $A \vee B$ is the set $A \cup B$. $A \cup B$
  - To see that, note that $A \subseteq A \cup B$, $B \subseteq A \cup B$, and if $A \subseteq C$ and $B \subseteq C$, then it follows that $A \cup B \subseteq C$.
- Similarly, $A \wedge B$ is the set $A \cap B$. $A \cap B$
- Thus, L is a lattice.

L is a lattice

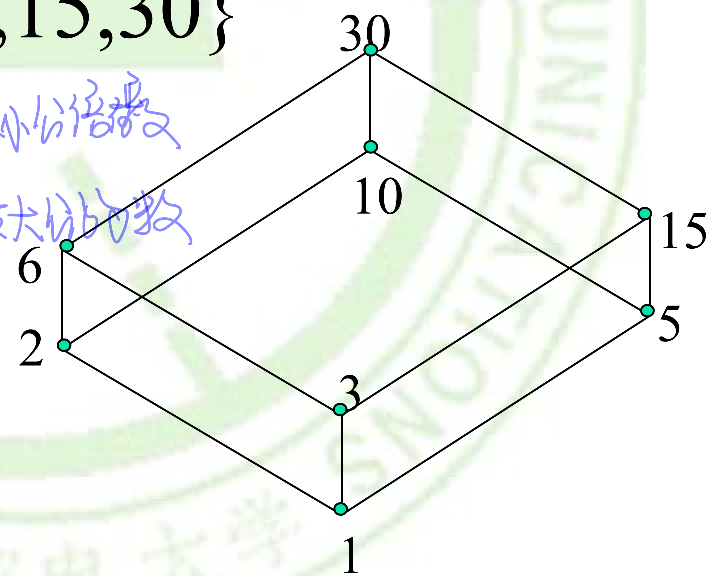
Example

- $S=\{a,b,c\}$, $(P(S),\subseteq)$



Example

- Let n be a positive integer and D_n be the set of all positive divisors of n.
- $D_{30}=\{1,2,3,5,6,10,15,30\}$
  - 最小上界 ⟸ $a \vee b=\mathrm{LCM}(a,b)$ 最小公倍数
  - 最大下界 ⟸ $a \wedge b=\mathrm{GCD}(a,b)$ 最大公约数



$$\begin{cases} a \vee b=\mathrm{LCM}(a,b) \\ a \wedge b=\mathrm{GCD}(a,b) \end{cases}$$

Example

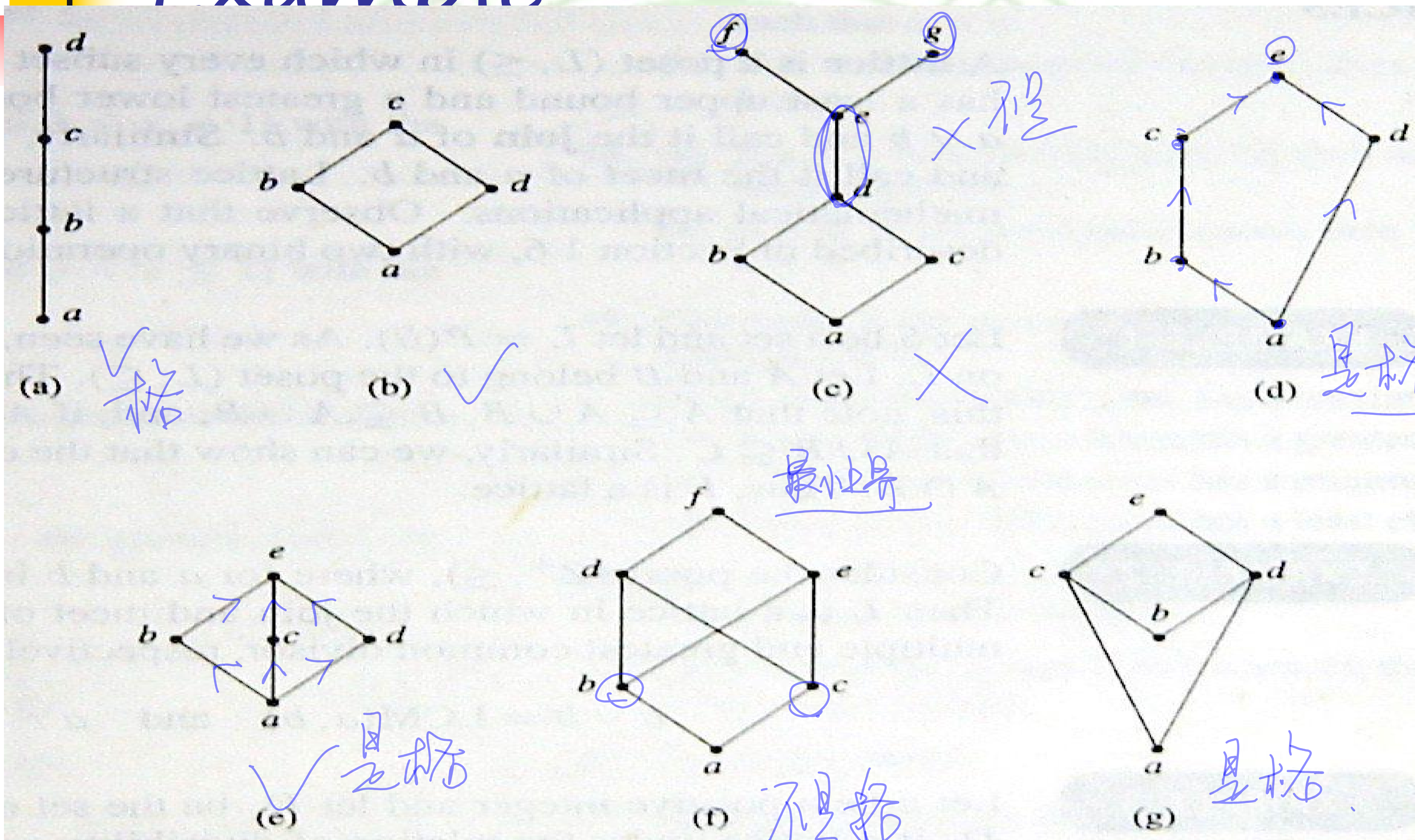


Figure 6.40

Theorem 乘积格

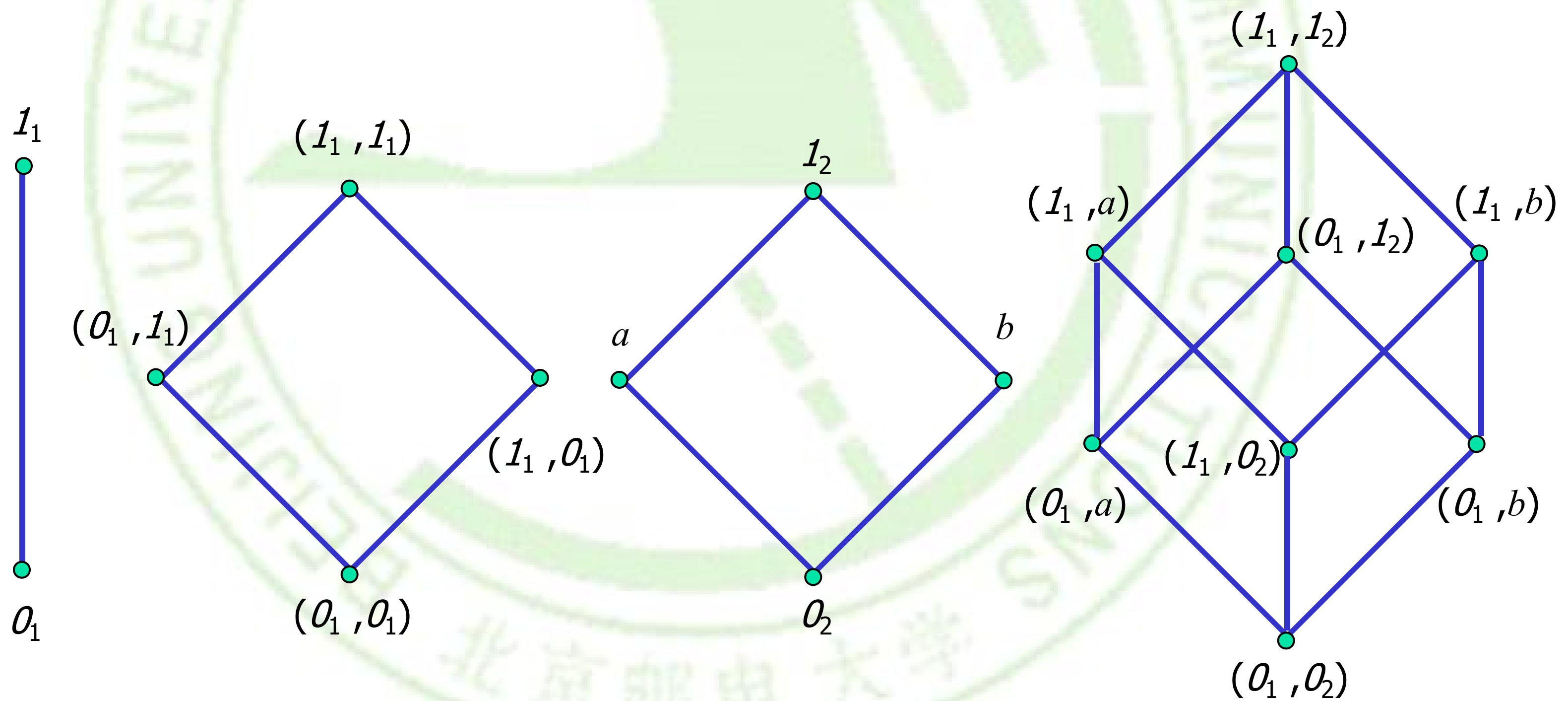
- If $(L_1, \preccurlyeq_1)$ and $(L_2, \preccurlyeq_2)$ are lattices, then $(L, \preccurlyeq)$ is a lattice, where
  - $L = L_1 \times L_2$
  - the partial order $\preccurlyeq$ of L is the product partial order.

Proof

- We denote the join and meet in L_1 by $\vee_1$ and $\wedge_1$, respectively, and the join and meet in L_2 by $\vee_2$ and $\wedge_2$, respectively. Then L is a poset.
- We now need to show that if (a_1, b_1) and $(a_2, b_2) \in L$, then $(a_1, b_1) \vee (a_2, b_2)$ and $(a_1, b_1) \wedge (a_2, b_2)$ exist in L. We leave it as an exercise to verify that
  - $(a_1, b_1) \vee (a_2, b_2) = (a_1 \vee_1 a_2,\ b_1 \vee_2 b_2)$
  - $(a_1, b_1) \wedge (a_2, b_2) = (a_1 \wedge_1 a_2,\ b_1 \wedge_2 b_2)$.
- Thus L is a lattice.

Example

L_1 $\quad L_{2'}=L_1\times L_1$ $\quad L_2$ $\quad L=L_1\times L_2$



Sublattice（子格）

- Let $(L, \preccurlyeq)$ be a lattice. A nonempty subset S of L is called a *sublattice* of L
  - if $a \vee b \in S$ and $a \wedge b \in S$ whenever $a \in S$ and $b \in S$.

Sublattice

- Example : $(D_n, |)$ is a sublattice of $(Z^+, |)$
- Example

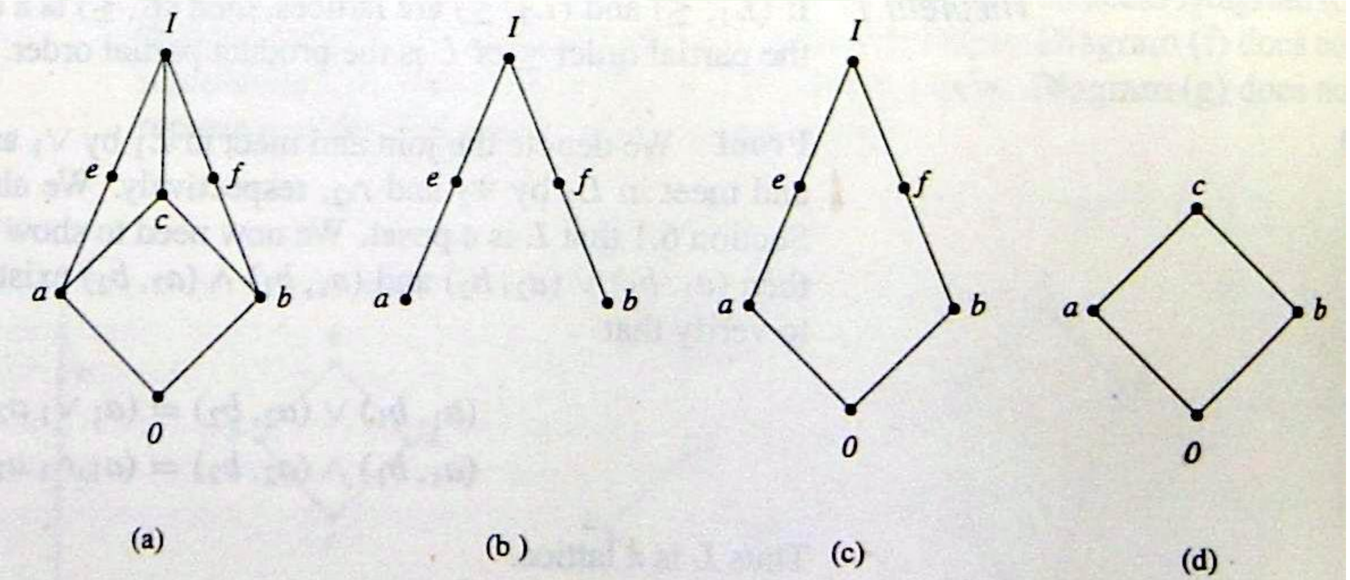


Figure 6.42

Isomorphic Lattices（同构格）

- If f: $L_1 \rightarrow L_2$ is an isomorphism from the poset $(L_1, \preccurlyeq_1)$ to the poset $(L_2, \preccurlyeq_2)$
  - L_1 is a lattice if and only if L_2 is a lattice.
  - If a and b are elements of L_1, then
    - $f(a \wedge b) = f(a) \wedge f(b)$
    - $f(a \vee b) = f(a) \vee f(b)$.
- If two lattices are isomorphic, as posets, we say they are *Isomorphic Lattices*.

Properties of Lattices

- Recall the meaning of $a \vee b$ and $a \wedge b$:
  - $a \preccurlyeq a \vee b$ and $b \preccurlyeq a \vee b$; $a \vee b$ is an *upper bound* of a and b.
  - *any* c, If $a \preccurlyeq c$ and $b \preccurlyeq c$, then $a \vee b \preccurlyeq c$; $a \vee b$ is the *least upper bound* of a and b.
  - $a \wedge b \preccurlyeq a$ and $a \wedge b \preccurlyeq b$; $a \wedge b$ is an *lower bound* of a and b.
  - *any* c, If $c \preccurlyeq a$ and $c \preccurlyeq b$, then $c \preccurlyeq a \wedge b$; $a \wedge b$ is the *greatest lower bound* of a and b.

Theorem

- Let L be a lattice. Then for every a and b in L,
  - $a \vee b = b$ if and only if $a \preccurlyeq b$.
  - $a \wedge b = a$ if and only if $a \preccurlyeq b$.
  - $a \wedge b = a$ if and only if $a \vee b = b$

Proof of $a \vee b = b$ if and only if $a \preccurlyeq b$.

- $\Rightarrow$ only if
  - Suppose $a \vee b = b$, so $a \vee b \preccurlyeq b$ (reflexive)
  - $a \vee b$ is an upper bound of a and b, so $a \preccurlyeq a \vee b$
  - So, $a \preccurlyeq b$ (transitive)
- $\Leftarrow$ if
  - Suppose $a \preccurlyeq b$
  - $b \preccurlyeq b$ (reflexive), b is an upper bound of a and b
  - So $a \vee b \preccurlyeq b$ (by definition of lub)
  - But, $b \preccurlyeq a \vee b$ (by definition of ub)
  - Therefore, $a \vee b = b$ (antisymmetric)

Theorem

- Let L be a lattice. Then
  - Idempotent Properties（等幂律）
    - (a) $a \vee a = a$
    - (b) $a \wedge a = a$
  - Commutative Properties（交换律）
    - (a) $a \vee b = b \vee a$
    - (b) $a \wedge b = b \wedge a$
  - Associative Properties（结合律）
    - (a) $a \vee (b \vee c) = (a \vee b) \vee c$
    - (b) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$
  - Absorption Properties（吸收律）
    - (a) $a \vee (a \wedge b) = a$
    - (b) $a \wedge (a \vee b) = a$

Proof of 4. (a)

- Since $a \wedge b \preccurlyeq a$ and $a \preccurlyeq a$, we see that a is an upper bound of $a \wedge b$ and a; so $a \vee (a \wedge b) \preccurlyeq a$. On the other hand, by the definition of LUB, we have $a \preccurlyeq a \vee (a \wedge b)$, so $a \vee (a \wedge b) = a$.
  - Q.E.D
- It follows from property 3 that we can write $a \vee (b \vee c)$ and $(a \vee b) \vee c$ merely as $a \vee b \vee c$, and similarly for $a \wedge b \wedge c$.

Theorem

- Let L be a lattice. Then for every a, b, and c in L,
- If $a \preccurlyeq b$, then
  - $a \vee c \preccurlyeq b \vee c$.
  - $a \wedge c \preccurlyeq b \wedge c$.
- $a \preccurlyeq c$ and $b \preccurlyeq c$ if and only if $a \vee b \preccurlyeq c$.
- $c \preccurlyeq a$ and $c \preccurlyeq b$ if and only if $c \preccurlyeq a \wedge b$.
- If $a \preccurlyeq b$ and $c \preccurlyeq d$, then
  - $a \vee c \preccurlyeq b \vee d$.
  - $a \wedge c \preccurlyeq b \wedge d$.

Special Types of Lattices

- A lattice L is said to be *bounded（有界的）* if it has a greatest element I and a least element 0 .
- $(Z^+,|)$ is not bounded.
- $(Z,\leq)$ is not bounded.

Example

- The lattice P(S) of all subsets of a set S, as defined in Example 1, is bounded. Its greatest element is S and its least element is $\varnothing$
- If L is a bounded lattice, then for all $a \in A$,
  - $0 \preccurlyeq a \preccurlyeq I$
  - $a \vee 0 = a$
  - $a \wedge 0 = 0$
  - $a \vee I = I$
  - $a \wedge I = a$

Theorem

- Let $L = \{a_1, a_2, \ldots, a_n\}$ be a finite lattice. Then L is bounded.
- **Proof**
  - The greatest element of L is $a_1 \vee a_2 \vee \ldots \vee a_n$,
  - and its least element is $a_1 \wedge a_2 \wedge \ldots \wedge a_n$

Distributive lattice（分配格）

- A lattice is called *distributive* if for any elements a, b, and c in L we have the following distributive properties.
  - $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
  - $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
- If L is not distributive, we say that L is *nondistributive*

Example

- The lattice shown in Figure 6.43 is distributive, as can be seen by verifying the distributive properties for all ordered triples chosen from the elements *a*, *b*, *c*, and *d*.

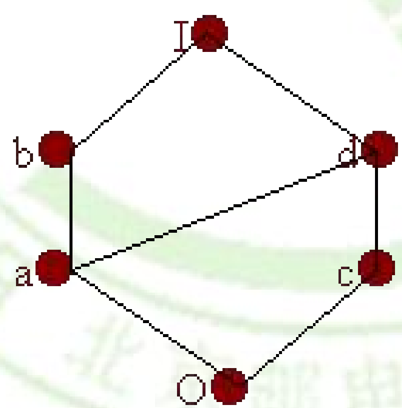


Figure 6.43

nondistributive lattices

- The lattice shown in Figure 6.44 are nondistributive.

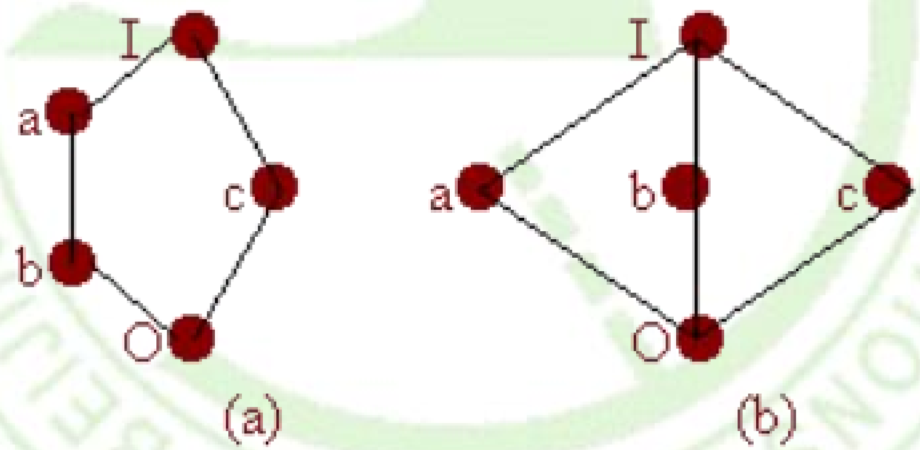


Figure 6.44

Theorem

- A lattic L is nondistributive if and only if it contains a sublattice that is isomorphic to one of the above two lattices.
- Proof:
  - omitted

Complement（补元）

- Let L be a bounded lattice with greatest element I and least element 0, and let $a \in L$. An element a' L is called a *complement of a* if
  - $a \vee a' = I$ and $a \wedge a' = 0$
- Note that
  - $0' = I$ *and* $I' = 0$.
- An element a in a lattice need not have a complement, and it may have more than one complement.

Example

- The lattices in Figure 6.44 each have the property that every element has a complement. The element c in both cases has two complements, a and b.

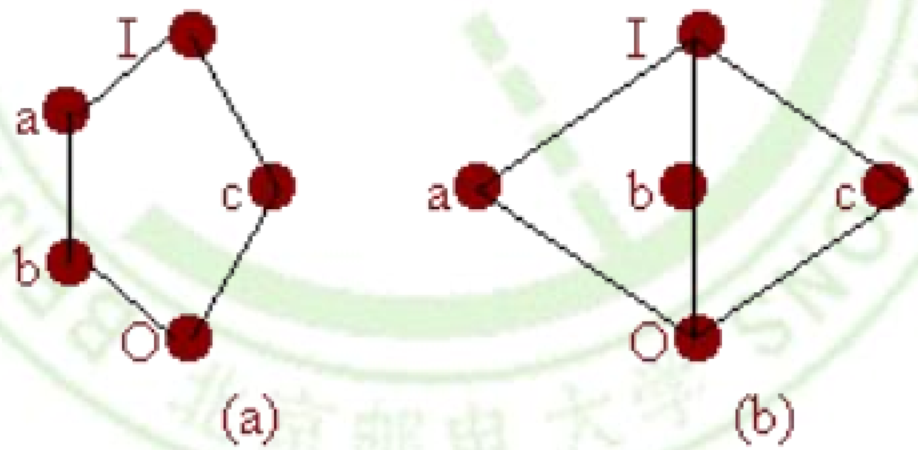
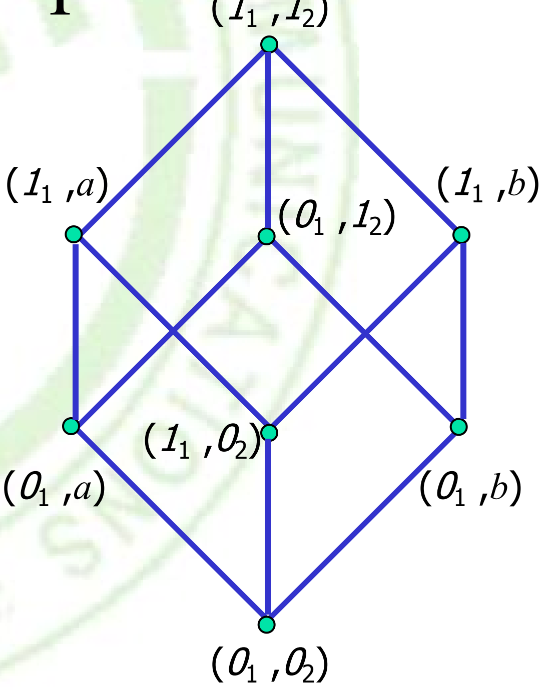


Figure 6.44

Theorem

- Let L be a bounded distributive lattice. If a complement exists, it is unique.



Proof

- Let a' and a'' be complements of the element $a \in L$. Then
  - $a \vee a' = I, \quad a \vee a'' = I, \quad a \wedge a' = 0, \quad a \wedge a'' = 0.$
- Using the distributive laws, we obtain
  - $a' = a' \vee 0 = a' \vee (a \wedge a'') = (a' \vee a) \wedge (a' \vee a'') = I \wedge (a' \vee a'') = a' \vee a''$
- Also
  - $a'' = a'' \vee 0 = a'' \vee (a \wedge a') = (a'' \vee a) \wedge (a'' \vee a') = I \wedge (a'' \vee a') = a' \vee a''$
- Hence $a' = a''$.

Complemented（有补格）

- A lattice is called *complemented* if it is bounded and if every element in L has a complement.
- In this case, the complements are not unique.

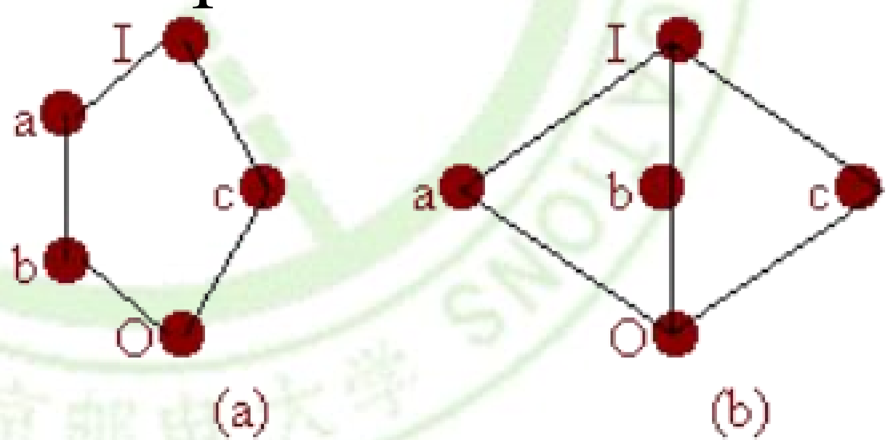


Figure 6.44

Homework

- § 9.6
  - 12, 28, 36, 44