9.5 Equivalence Relations

Equivalence Relations

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Section Summary

- Equivalence Relations
- Equivalence Classes
- Equivalence Classes and Partitions 划分

equiv 等价类

自反 + 对称 + 传递，

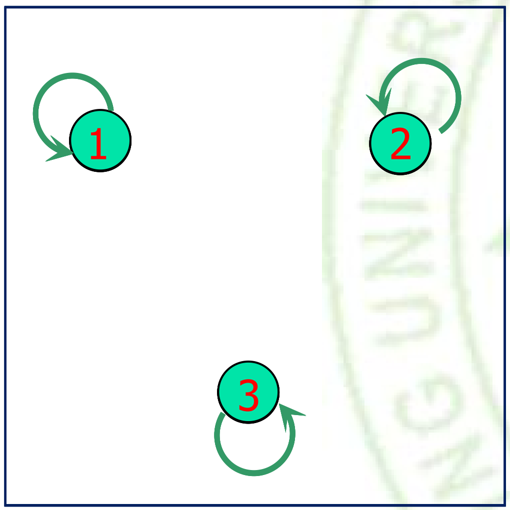
是等价关系

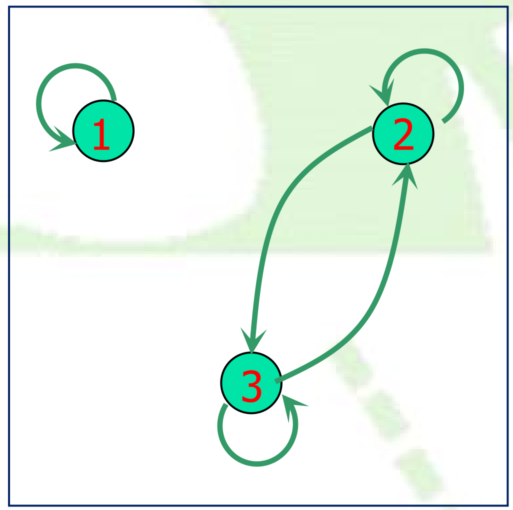
Equivalence Relations

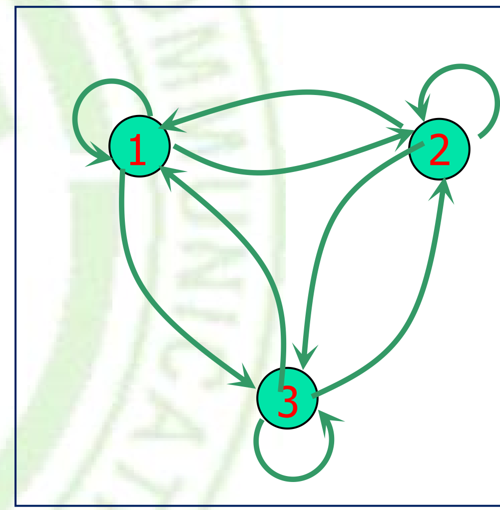
Definition 1: A relation on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Definition 2: Two elements a, and b that are related by an equivalence relation are called *equivalent*. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

Equivalence Relations has 3 elements







- Count the number of equivalence relations on a set A with n elements.
- Can you find a recurrence relation?

Strings

Example: Suppose that R is the relation on the set of strings of English letters such that aRb if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x. Is R an equivalence relation?

Solution: Show that all of the properties of an equivalence relation hold.

- *Reflexivity*: Because $l(a) = l(a)$, it follows that aRa for all strings a.
- *Symmetry*: Suppose that aRb. Since $l(a) = l(b)$, $l(b) = l(a)$ also holds and bRa.
- *Transitivity*: Suppose that aRb and bRc. Since $l(a) = l(b)$,and $l(b) = l(c)$, $l(a) = l(a)$ also holds and aRc.

Congruence Modulo *m*

Example: Let m be an integer with $m > 1$. Show that the relation

$R = \{(a,b) \mid a \equiv b \pmod{m}\}$

is an equivalence relation on the set of integers.

Solution: Recall that $a \equiv b \pmod{m}$ if and only if m divides $a - b$.

Reflexivity: $a \equiv a \pmod{m}$ since $a - a = 0$ is divisible by m since $0 = 0 \cdot m$.

Symmetry: Suppose that $a \equiv b \pmod{m}$. Then $a - b$ is divisible by m, and so $a - b = km$, where k is an integer. It follows that $b - a = (-k)\, m$, so $b \equiv a \pmod{m}$.

Transitivity: Suppose that $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$. Then m divides both $a - b$ and $b - c$. Hence, there are integers k and l with $a - b = km$ and $b - c = lm$. We obtain by adding the equations:

$$a - c = (a - b) + (b - c) = km + lm = (k + l)\, m.$$

Therefore, $a \equiv c \pmod{m}$.

Divides

Example: Show that the "divides" relation on the set of positive integers is not an equivalence relation.

Solution: The properties of reflexivity, and transitivity do hold, but there relation is not symmetric. Hence, "divides" is not an equivalence relation.

- *Reflexivity*: $a \mid a$ for all a.
- *Not Symmetric*: For example, $2 \mid 4$, but $4 \nmid 2$. Hence, the relation is not symmetric.
- *Transitivity*: Suppose that a divides b and b divides c. Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c. Therefore, the relation is transitive.

Equivalence Classes

Definition 3: Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the *equivalence class* of a. The equivalence class of a with respect to R is denoted by $[a]_R$.

When only one relation is under consideration, we can write $[a]$, without the subscript R, for this equivalence class.

Note that $[a]_R = \{s \mid (a,s) \in R\}$.

- If b ∈ [a]R, then b is called a representative(代表元) of this equivalence class. Any element of a class can be used as a representative of the class.

Equivalence Classes

- The equivalence classes of the relation congruence modulo m are called the *congruence classes modulo m*. The congruence class of an integer a modulo m is denoted by $[a]_m$, so $[a]_m = \{\ldots, a-2m, a-m, a+m, a+2m, \ldots\}$. For example,

$[0]_4 = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$ $[1]_4 = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$

$[2]_4 = \{\ldots, -6, -2, 2, 6, 10, \ldots\}$ $[3]_4 = \{\ldots, -5, -1, 3, 7, 11, \ldots\}$

Equivalence Classes and Partitions

$c \in [b] \Leftrightarrow bRc \Leftrightarrow bRa + aRc \Leftrightarrow c \in [a]$

Theorem 1: let R be an equivalence relation on a set A. These statements for elements a and b of A are equivalent:

$aRb \Leftrightarrow [a]=[b] \Leftrightarrow [a] \cap [b] \neq \emptyset$

(*i*) aRb

(*ii*) $[a] = [b]$

(*iii*) $[a] \cap [b] \neq \emptyset$

Proof: We show that (*i*) implies (*ii*). Assume that aRb.

Now suppose that $c \in [a]$. Then aRc. Because aRb and R is symmetric, bRa. Because R is transitive and bRa and aRc, it follows that bRc. Hence, $c \in [b]$. Therefore, $[a] \subseteq [b]$. A similar argument (omitted here) shows that $[b] \subseteq [a]$. Since $[a] \subseteq [b]$ and $[b] \subseteq [a]$, we have shown that $[a] = [b]$.

Proof continued

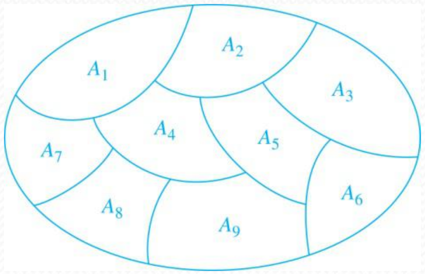
- Let $c \in R(a) \cap R(b)$
  - $a\ R\ c$, $b\ R\ c$
  - $c\ R\ b$, since R is symmetric
  - $a\ R\ b$, since R is transitive
  - $R(a) = R(b)$ by Theroem (ii)
- So, If $R(a) \cap R(b) \neq \varnothing$, then $R(a) = R(b)$.
- else $R(a)$ and $R(b)$ are not identical, then $R(a) \cap R(b) = \varnothing$.

- QED

Partition of a Set

Definition: A *partition* of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets A_i, where $i \in I$ (where I is an index set), forms a partition of S if and only if

- $A_i \neq \emptyset$ for $i \in I$,
- $A_i \cap A_j = \emptyset$ when $i \neq j$,
- and $\bigcup_{i \in I} A_i = S.$



A Partition of a Set

An Equivalence Relation Partitions a Set

- Let R be an equivalence relation on a set A. The union of all the equivalence classes of R is all of A, since an element a of A is in its own equivalence class $[a]_R$. In other words,

$$\bigcup_{a \in A} [a]_R = A.$$

等价类.

$$\bigcup_{a \in A} [a]_R = A$$

- From Theorem 1, it follows that these equivalence classes are either equal or disjoint, so $[a]_R \cap [b]_R = \emptyset$ when $[a]_R \neq [b]_R$.
- Therefore, the equivalence classes form a partition of A, because they split A into disjoint subsets.

The quotient set A/R

- If R is an equivalence relation on A,
- The partition P consists of all equivalence classes of R is denoted by A/R and called
  - the *quotient set (商集)*, or 标志
  - *the partition of A induced by R*, or,
  - *A modulo R*.

商集

$\{\{1,2\}\ \{3,4\}\}$ 划分

Example

这是啥

- Let
  - $A = \{1, 2, 3, 4\}$
  - $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 3), (3, 3), (4, 4)\}$.
- Determine A/R.

A/R

- Solution
  - $R(1) = \{1, 2\} = R(2)$
  - $R(3) = \{3, 4\} = R(4)$.
  - Hence $A/R = \{\{1, 2\}, \{3, 4\}\}$

$R(1) = \{1, 2\} = R(2)$

$R(3) = \{3, 4\} = R(4)$

$A/R = \{\{1,2\}, \{3,4\}\}$

Algorithm of A/R

- *STEP 1*: Choose any element of A and compute the equivalence class $R(a)$.
- *STEP 2*: if $R(a) \neq A$, choose an element b, not included in $R(a)$, and compute the equivalence class $R(b)$.
- *STEP 3*: If A is not the union of previously computed equivalence classes, then choose an element x of A that is not in any of those equivalence classes and compute $R(x)$.
- *STEP 4*: Repeat step 3 until all elements of A are included in the computed equivalence classes. If A is countable, this process could continue indefinitely. In that case, continue until a pattern emerges that allows you to describe or give a formula for all equivalence classes

An Equivalence Relation Partitions a Set (*continued*)

Theorem 2: Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i, $i \in I$, as its equivalence classes.

Proof: We have already shown the first part of the theorem.

For the second part, assume that $\{A_i \mid i \in I\}$ is a partition of S. Let R be the relation on S consisting of the pairs (x, y) where x and y belong to the same subset A_i in the partition.

An Equivalence Relation Partitions a Set (*continued*)

Proof: We must show that R satisfies the properties of an equivalence relation.

- *Reflexivity*: For every $a \in S$, $(a,a) \in R$, because a is in the same subset as itself.
- *Symmetry*: If $(a,b) \in R$, then b and a are in the same subset of the partition, so $(b,a) \in R$.
- *Transitivity*: If $(a,b) \in R$ and $(b,c) \in R$, then a and b are in the same subset of the partition, as are b and c. Since the subsets are disjoint and b belongs to both, the two subsets of the partition must be identical. Therefore, $(a,c) \in R$ since a and c belong to the same subset of the partition.

Example

根据划分求R

划分

- Let $A = \{1, 2, 3, 4\}$ and consider the partition $P = \{\{1, 2, 3\}, \{4\}\}$ of A. Find the equivalence relation R on A determined by P.
- Solution
  - The blocks of P are $\{1, 2, 3\}$ and $\{4\}$.
  - Each element in a block is related to every other element in the same block and only to those elements.
  - Thus $R = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (4, 4)\}$.

R= (1,1)(1,2)(1,3)(2,1)(2,2)(2,3)(3,1)(3,2)(3,3)
(4,4)

Theroem

- *Let R be a relation on set A, then tsr(R) is an equivalence relation on A.*
- *tsr(R) is the reflexive, symmetric, transitive closure of R, is called the equivalence relation induced by R.*

equivalence tsr(R)

equivalence relation included by R.

Theroem

- *If R and S are equivalence relations on a set A, then the smallest equivalence relation containing both R and S is* $(R \cup S)^*$.

$(R \cup S)^*$

Exercise

So that you don't get bored, here are some problems to discuss on your next blind date:

- Do the closure operations commute?
  - Does $st(R) = ts(R)$?
  - Does $rt(R) = tr(R)$?
  - Does $rs(R) = sr(R)$?
- Do the closure operations distribute
  - Over the set operations?
  - Over inverse?
  - Over complement?
  - Over set inclusion?
- Examples:
  - Does $t(R_1 - R_2) = t(R_1) - t(R_2)$?
  - Does $r(R^{-1}) = [r(R)]^{-1}$?

homework

- § 9.5
  - 16, 56, 60, 64