§ 8.4: Generating Functions

生成函数

- Definition: The generating function for the sequence $a_0, a_1 \ldots. a_k$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + \cdots a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k$$

Example 1

- The generating functions for the sequences $\{a_k\}$ with $a_k = 3$, $a_k = k + 1$, and $a_k = 2^k$ are

$$\sum_{k=0}^{\infty} 3x^k, \sum_{k=0}^{\infty} (k+1)x^k, \text{and } \sum_{k=0}^{\infty} 2^k x^k$$

Generating Functions for Finite Sequences

- Generating functions for finite sequences of real numbers can be defined by extending a finite sequence $a_0, a_1, \ldots, a_n$ into an infinite sequence by setting $a_{n+1} = 0$, $a_{n+2} = 0$, and so on.
- The generating function $G(x)$ of this infinite sequence $\{a_n\}$ is a polynomial of degree n because no terms of the form $a_j x^j$ with $j > n$ occur, that is,

$$G(x) = a_0 + a_1 x + \cdots + a_n x^n.$$

Example 2

- What is the generating function for the sequence 1, 1, 1, 1, 1, 1?

Solution: The generating function of 1,1,1,1,1,1 is

$1 + x + x^2 + x^3 + x^4 + x^5$.

By Theorem 1 of Section 2.4, we have

$(x^6 - 1)/(x - 1) = 1 + x + x^2 + x^3 + x^4 + x^5$

when $x \neq 1$.

Consequently $G(x) = (x^6 - 1)/(x - 1)$ is the generating function of the sequence.

Useful Facts About Power Series

- Example 4
  - The generating function of the sequence 1,1,1,1…
  - $G(x)= 1/(1-x) \quad |x|<1$
- Example 5
  - The generating function of the sequence $1,a,a^2, a^3\ldots$
  - $G(x)=1+ax+a^2x^2+\ldots=1/(1-ax) \quad |x|<1/|a|$ 或 $|ax|<1$

- Using Generating Functions to Solve Counting Problems.

Counting Problems and Generating Functions

Example: Find the number of solutions of

$$e_1 + e_2 + e_3 = 17,$$

where e_1, e_2, and e_3 are nonnegative integers with $2 \le e_1 \le 5$, $3 \le e_2 \le 6$, and $4 \le e_3 \le 7$.

Solution: The number of solutions is the coefficient of x^{17} in the expansion of

$$(x^2 + x^3 + x^4 + x^5)\,(x^3 + x^4 + x^5 + x^6)\,(x^4 + x^5 + x^6 + x^7).$$

This follows because a term equal to is obtained in the product by picking a term in the first sum x^{e_1}, a term in the second sum x^{e_2}, and a term in the third sum x^{e_3}, where $e_1 + e_2 + e_3 = 17$.

There are three solutions since the coefficient of x^{17} in the product is 3.

Counting problems and Generating Functions

- Example 11: In how many different ways can eight identical cookies be distributed among three distinct children if each child receives at least two cookies and no more than four cookies?
- *Solution:: Because each child receives at least two but no more than four cookies, for each child there is a factor equal to* $(x^2 + x^3 + x^4)$
- $(x^2 + x^3 + x^4)^3 = x^6 + \ldots + 6x^8 + \ldots + x^{12}$
- *answer: 6*

Example 12

- Use generating functions to determine the number of ways to insert tokens worth \$1, \$2,and \$5 into a vending machine to pay for an item that costs *r* dollars in both the cases when the order in which the tokens are inserted does not matter and when the order does matter.

- *solution:* $(1+x+x^2+x^3+\cdots)(1+x^2+x^4+x^6+\cdots)(1+x^5+x^{10}+x^{15}+\cdots)$.
  - $r=7$, *result*=6.

$e_1+2e_2+5e_3=r.$

Counting Problems and Generating Functions (*continued*)

Example: Use generating functions to find the number of k-combinations of a set with n elements, i.e., $C(n,k)$.

Solution: Each of the n elements in the set contributes the term $(1 + x)$ to the generating function

Hence $f(x) = (1 + x)^n$ where $f(x)$ is the generating function for $\{a_k\}$, where a_k represents the number of k-combinations of a set with n elements.

$$f(x) = \sum_{k=0}^{n} \binom{n}{k} x^k, \qquad C(n,k) = \frac{n!}{k!(n-k)!}.$$

Example 14

- Use generating functions to find the number of r-combinations from a set with *n* elements when repetition of elements is allowed.
- solution: $G(x) = (1+x+x^2+\cdots)^n$.
- *As long as* $|x| < 1$, *we have* $1+x+x^2+\cdots=1/(1-x)$,
- *then* $G(x) = (1-x)^{-n}$
- $a_k = ?$

The extended binomial theorem

- Theorem 2
  - Let x be a real number with $|x|<1$ and let u be a real number. Then

$$(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k$$

- Example 9

$$(1+x)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} x^k. = \sum_{k=0}^{\infty} (-1)^k C(n+k-1,k) x^k.$$

$$(1-x)^{-n} = \sum_{k=0}^{\infty} C(n+k-1,k) x^k.$$

EXTENDED BINOMIAL COEFFICIENT

- Definition 2: Let ***u* be a real number** and k a nonnegative integer. Then the *extended binomial coefficient* $\binom{u}{k}$ is defined by

$$\binom{u}{k} = \begin{cases} u(u-1)\cdots(u-k+1)/k! & \text{if } k>0 \\ 1 & \text{if } k=0 \end{cases}$$

Find the value of the extended binomial coefficient

- Example 7

$$\binom{-2}{3} = \frac{(-2)(-3)(-4)}{3!} = -4.$$

$$\binom{1/2}{3} = \frac{(1/2)(1/2-1)(1/2-2)}{3!}$$

$$= (1/2)(-1/2)(-3/2)/6$$

$$= 1/16.$$

$$\binom{-n}{r} = \frac{(-n)(-n-1)\cdots(-n-r+1)}{r!}$$
$$= \frac{(-1)^r n(n+1)\cdots(n+r-1)}{r!}$$
$$= \frac{(-1)^r (n+r-1)(n+r-2)\cdots n}{r!}$$
$$= \frac{(-1)^r (n+r-1)!}{r!(n-1)!}$$
$$= (-1)^r \binom{n+r-1}{r}$$
$$= (-1)^r C(n+r-1, r).$$

R-COMBINATIONS FROM A SET WITH *N* ELEMENTS WHEN REPETITION

- Use generating functions to find the number of r-combinations from a set with *n* elements when repetition of elements is allowed.
- solution: $G(x) = (1+x+x^2+\cdots)^n$.
- As long as $|x| < 1$, we have $1+x+x^2+\cdots=1/(1-x)$, so $G(x) = (1-x)^{-n}$.

$$(1-x)^{-n} = \sum_{k=0}^{\infty} C(n+k-1,k)x^k.$$

- $a_r = C(n+r-1,r)$.

Example 15

- Use generating functions to find the number of ways to select *r* objects of *n* different kinds if we must select at least one object of each kind.
- *solution:* $(x + x^2 + x^3 + \cdots)^n = x^n(1 + x^2 + x^4 + x^6 + \cdots)^n$
- $= x^n(1-x)^{-n}$.

$$
\begin{aligned}
&= x^n \sum_{r=0}^{\infty} \binom{-n}{r} (-x)^r \\
&= x^n \sum_{r=0}^{\infty} (-1)^r C(n+r-1, r)(-1)^r x^r \\
&= \sum_{r=0}^{\infty} C(n+r-1, r) x^{n+r} \\
&= \sum_{t=n}^{\infty} C(t-1, t-n) x^t \\
&= \sum_{r=n}^{\infty} C(r-1, r-n) x^r .
\end{aligned}
$$

ADDITION AND MULTIPLICATION OF GENERATING FUNCTIONS

- **Theorem 1**

Let $f(x)=\sum_{k=0}^{\infty}a_k x^k$ and $g(x)=\sum_{k=0}^{\infty}b_k x^k$. Then

$$f(x)+g(x)=\sum_{k=0}^{\infty}(a_k+b_k)x^k \text{ and } f(x)g(x)=\sum_{k=0}^{\infty}\Big(\sum_{j=0}^{k}a_j b_{k-j}\Big)x^k$$

Example 6

- **Let** $f(x) = 1/(1-x)^2$ **Find the coefficients** a_0, a_1, a_2, a_3… **in the expansion** $f(x) = \sum_{k=0}^{\infty} a_k x^k$

$$1/(1-x)^2 = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} 1 \right) x^k = \sum_{k=0}^{\infty} (k+1) x^k.$$

- Using Generating Functions to Solve Recurrence Relations.

Using Generating Functions to Solve Recurrence Relations

- Example 16: Solve the recurrence relation $a_k = 3a_{k-1}$ for $k=1,2,3\ldots$ and initial condition $a_0=2$
- Solution： Let $G(x) = \sum_{k=0}^{\infty} a_k x^k$.
- Find $\displaystyle\sum_{k=1}^{\infty} a_{k-1} x^k = \sum_{k=0}^{\infty} a_k x^{k+1} = xG(x)$.
- Solve G(x)

$$
\begin{aligned}
G(x) - 3xG(x) &= \sum_{k=0}^{\infty} a_k x^k - 3\sum_{k=1}^{\infty} a_{k-1} x^k \\
&= a_0 + \sum_{k=1}^{\infty} (a_k - 3a_{k-1}) x^k \\
&= 2,
\end{aligned}
$$

Using Generating Functions to Solve Recurrence Relations

- Example 16
- $G(x)-3xG(x)=(1-3x)G(x)=2$.
- $G(x) = 2/(1-3x)$.
- *by* $1/(1-ax) = \sum_{k=0}^{\infty} a^k x^k$,

$$G(x) = 2\sum_{k=0}^{\infty} 3^k x^k = \sum_{k=0}^{\infty} 2 \cdot 3^k x^k.$$

- *Hence*, $a_n=2 \cdot 3^n$.

Example 17

- Suppose that a valid codeword is an n-digit number in decimal notation containing an <u>even number of 0s</u>. Let a_n denote the number of valid codewords of length n. In Example 4 of Section 8.1 we showed that the sequence $\{a_n\}$ satisfies the recurrence relation $\mathbf{a_n = 8a_{n-1} + 10^{n-1}}$ and the initial condition $\mathbf{a_1 = 9}$. Use generating functions to find an explicit formula for a_n.

Solve nonhomogeneous recurrence relation

nonhomogeneous recurrence relation

$a_n = 8a_{n-1} + 10^{n-1}$

$a_n \mathbf{x^n} = 8a_{n-1} \mathbf{x^n} + 10^{n-1} \mathbf{x^n}$

$G(x) = \sum_{k=0}^{\infty} a_k x^k$

$= \sum_{k=1}^{\infty}\left(8a_{k-1}x^k + 10^{k-1}x^k\right) + a_0$

$=8x \sum_{k=1}^{\infty} a_{k-1}x^{k-1} + x \sum_{k=1}^{\infty} 10^{k-1}x^{k-1} + a_0$

$=8x \sum_{k=0}^{\infty} a_k x^k + x \sum_{k=0}^{\infty} 10^k x^k + a_0$

- $G(x)-1= 8xG(x)+ x/(1-10x)$.

$$G(x) = \frac{1-9x}{(1-8x)(1-10x)}.$$

$$G(x) = \frac{1}{2}\left(\frac{1}{1-8x} + \frac{1}{1-10x}\right).$$

$$G(x) = \frac{1}{2}\left(\sum_{n=0}^{\infty} 8^n x^n + \sum_{n=0}^{\infty} 10^n x^n\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{2}(8^n + 10^n)x^n.$$

- Hence, $a_n=(8^n+10^n)/2$.

Proving Identities via Generating Functions

- Example 18: Use generating functions to show that

$$\sum_{k=0}^{n} C(n,k)^2 = C(2n,n)$$

When n is a positive integer.

. $C(2n, n)$ is the coefficient of x^n in $(1+x)^{2n}$.

ADDITION AND MULTIPLICATION OF GENERATING FUNCTIONS

- **Theorem 1**

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$$f(x)+g(x)=\sum_{k=0}^{\infty}(a_k+b_k)x^k \text{ and } f(x)g(x)=\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} a_j b_{k-j}\right)x^k$$

$$(1+x)^{2n} = [(1+x)^n]^2$$
$$= [C(n,0) + C(n,1)x + C(n,2)x^2 + \cdots + C(n,n)x^n]^2.$$

The coefficient of x^n in this expression is

$$C(n,0)C(n,n) + C(n,1)C(n,n-1) + C(n,2)C(n,n-2) + \cdots + C(n,n)C(n,0).$$

This equals $\sum_{k=0}^{n} C(n,k)^2$, because $C(n,n-k) = C(n,k)$.

Homework

- § 8.4
  - 16, 24, 36