9.4 Closures of Relations

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Closures of Relations

- Definition
  - The *closure*(闭包) of a relation R with respect to property P is the relation obtained by adding the *minimum number of ordered pairs* to R to obtain property P.

闭包

- 3 elements:
  - R_1 contains R
  - R_1 possesses the property P
  - If R_2 contains R and possesses the property P, then R_2 contains R_1

Closures of Relations

- In terms of the digraph representation of R
  - To find the *reflexive closure* 自反
    - add loops.
  - To find the *symmetric closure* 对称
    - add arcs in the opposite direction.
  - To find the *transitive closure* 传递
    - if there is a path from a to b, add an arc from a to b.

Reflexive Closure

- **Theorem:**
  - Let R be a relation on A.
  - The *reflexive closure* of R, denoted r(R), is $R \cup \Delta$, $\Delta = \{(x,x) \mid x \in A\}$.

自反闭包

- Method:
  - Add loops to all vertices on the digraph representation of R.
  - Put 1's on the diagonal of the connection matrix of R. $M_R \vee M_\Delta$

$r(R)=R\cup\Delta$

- **Proof:**
  - $R\cup\Delta$ is *reflexive, and* $R\subseteq R\cup\Delta$,
  - suppose $S\subseteq A\times A$, be reflexive, and $R\subseteq S$.
  - If $(a,b) \in R\cup\Delta$,
    - case 1: $(a,b) \in R$, *so* $(a,b) \in S$.
    - case 2: $(a,b) \in \Delta$, *so* $a=b$, *S is reflexive, then* $(a,b) \in S$.
  - *So* $R\cup\Delta \subseteq S$, $r(R)=R\cup\Delta$.

Symmetric closure

- **Theorem**
  - Let R be a relation on A.
  - The *symmetric closure* of R, denoted $s(R)$, is the relation $R \cup R^{-1}$, $R^{-1} = \{(b,a) \mid (a,b) \in R\}$.

$S(R)=R\cup R^{-1}$

$s(R)=R\cup R^{-1}$

- **Proof:**
  - *If* (a,b)$\in R$, *then* $(b,a)\in R^{-1}$. *If* (a,b)$\in R^{-1}$, *then* $(b,a)\in R$.
  - *so* (a,b)$\in R\cup R^{-1}$, *and* $(b,a)\in R\cup R^{-1}$.
  - *so* $R\cup R^{-1}$ is symmetric, *and* $R\subseteq R\cup R^{-1}$,
  - suppose $S\subseteq A\times A$, be symmetric, and $R\subseteq S$.
  - If (a,b) $\in R\cup R^{-1}$,
    - case 1: (a,b) $\in R$, *so* (a,b) $\in S$.
    - case 2: (a,b) $\in R^{-1}$, *so* $(b,a)\in R$, $(b,a)\in S$, and S *is symmetric*, *then* (a,b) $\in S$.
  - *So* $R\cup R^{-1}\subseteq S$, $s(R)=R\cup R^{-1}$.

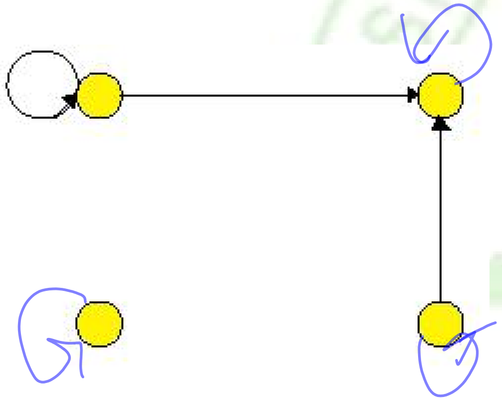
Theorem

- R is symmetric
  - If and only if
- $R = R^{-1}$

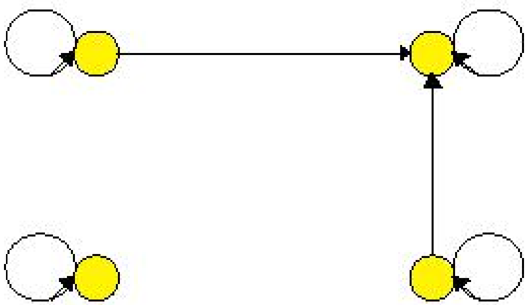
  - Note: in digraph of a symmetric relation, use *undirected edges* instead of *arcs*

弧

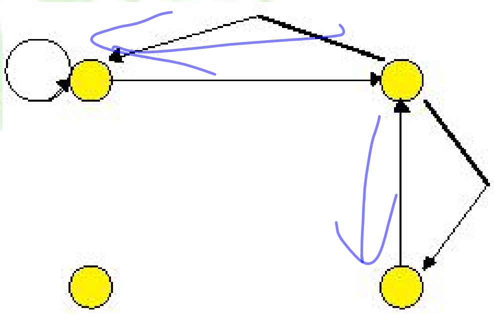
Example



R



$r(R)$



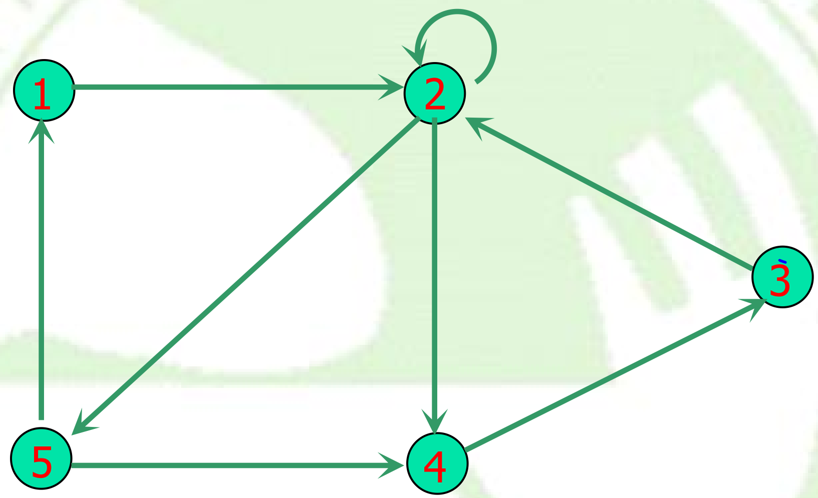
$s(R)$

Paths 路 Paths

- Suppose that R is a relation on a set A. A *path of length n* in R from a to b is a finite sequence $\pi : a, x_1, x_2, \ldots, x_{n-1}, b$, beginning with a and ending with b, such that
  - $a\ R\ x_1, x_1\ R\ x_2, \ldots, x_{n-1}\ R\ b$

Path 路

Example



顶点

- π_1 : 1, 2, 5, 4, 3 is a path of length 4 from vertex 1 to vertex 3
- π_2 : 1, 2, 5, 1 is a path of length 3 from vertex 1 to itself
- π_3 : 2, 2 is a path of length 1 from vertex 2 to itself

Some definitions

- A path that begins and ends at the same vertex is called a *cycle*.

  cycle 回路

- R^n : $x\ R^n\ y$ means that there is a path of length n from x to y in R.

  $x R^n y$ 有条长为n的路径.

  - $R^n(x)$
- R^* : $x\ R^*\ y$ means that there is some path in R from x to y.
  - $R^*(x)$

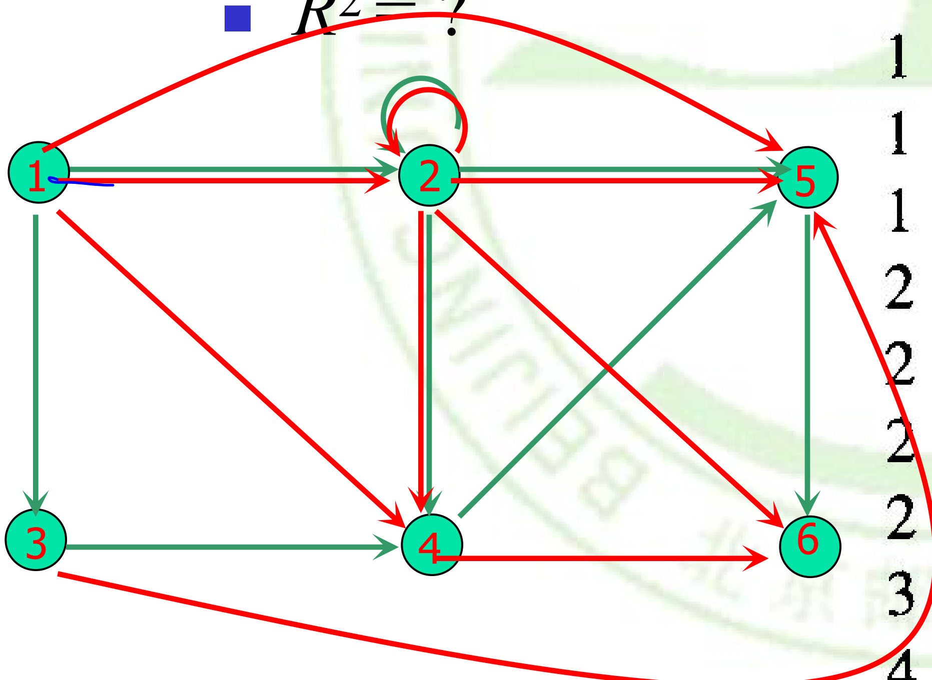
  $R^n(x)$ there is some path in R from x to y

- The relation R^* is sometimes called the *connectivity relation* for R .

连通

Example

- Let $A = \{1, 2, 3, 4, 5, 6\}$
- R is shown as in figure
- $R^2 = ?$

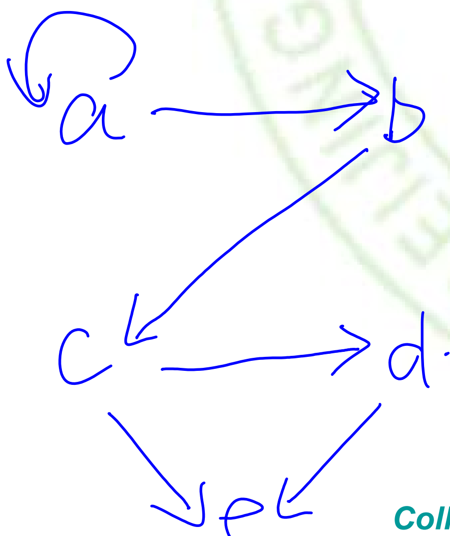


$1\ R^2\ 2$ since $1\ R\ 2$ and $2\ R\ 2$
$1\ R^2\ 4$ since $1\ R\ 2$ and $2\ R\ 4$
$1\ R^2\ 5$ since $1\ R\ 2$ and $2\ R\ 5$
$2\ R^2\ 2$ since $2\ R\ 2$ and $2\ R\ 2$
$2\ R^2\ 4$ since $2\ R\ 2$ and $2\ R\ 4$
$2\ R^2\ 5$ since $2\ R\ 2$ and $2\ R\ 5$
$2\ R^2\ 6$ since $2\ R\ 5$ and $5\ R\ 6$
$3\ R^2\ 5$ since $3\ R\ 4$ and $4\ R\ 5$
$4\ R^2\ 6$ since $4\ R\ 5$ and $5\ R\ 6$.

Example

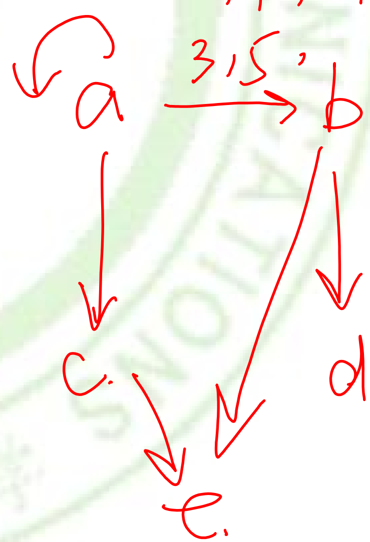
- Let $A = \{a, b, c, d, e\}$
  - $R = \{(a, a), (a, b), (b, c), (c, e), (c, d), (d, e)\}$.
- Compute (a) R^2 ; (b) R^*

$\{a,b,c,d,e\}$



aR^2a
aR^2b
aR^2c
bR^2d
bR^2e
cR^2e

1,1 , 1,2 1,3
2,4 , 2,5
3,5,



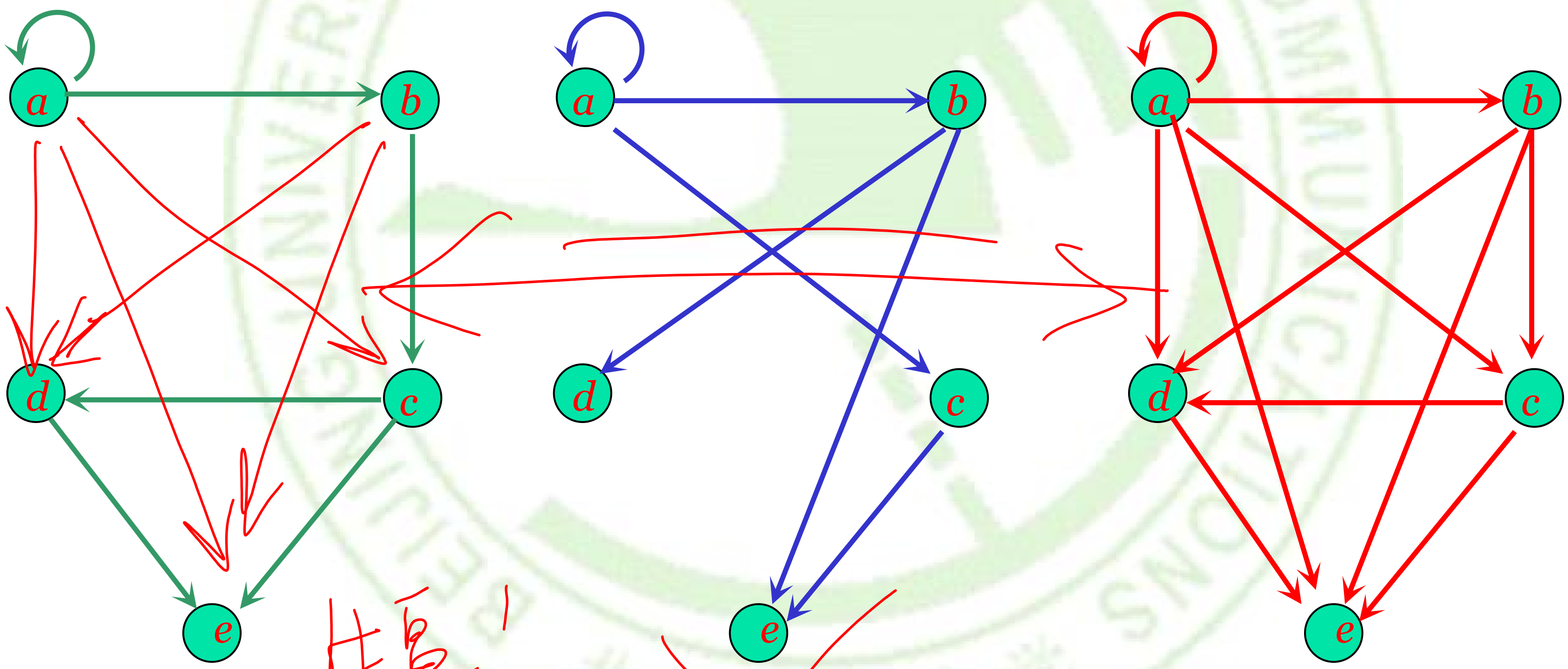
R^2.

Solution

R^*

只要有路的连通，都加

- $R = \{(a, a), (a, b), (b, c), (c, e), (c, d), (d, e)\}.$



R R^2 R^*

过两个

Theorem

- If R is a relation on $A = \{a_1, a_2, \ldots, a_n\}$, then

$$M_{R^2} = M_R \odot M_R$$

$$M_{R^2} = M_R \odot M_R \triangleq (M_R)^2_\odot$$

$$M_{R^2} = M_R \odot M_R \triangleq (M_R)^2_\odot$$

Proof

- Let $M_R = [m_{ij}]$ and $M_{R^2} = [n_{ij}]$.
  - the i, jth element of $M_R \otimes M_R$ is equal to 1
  - $m_{ik} = 1$ and $m_{kj} = 1$ for some k, $1 \leq k \leq n$.
- By definition of the matrix M_R
  - $a_i\ R\ a_k$ and $a_k\ R\ a_j$
  - $a_i\ R^2\ a_j$, and so $n_{ij} = 1$.
- Therefore
  - position i, j of $M_R \otimes M_R$ is equal to 1
  - $n_{ij} = 1$.
- So $M_R \otimes M_R = M_{R^2}$
- QED

Example

- Let $A = \{a, b, c, d, e\}$
  - $R = \{(a, a), (a, b), (b, c), (c, e), (c, d), (d, e)\}$.
- Compute R^2

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

画图

Example cont.

$$M_{R^2} = M_R \odot M_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

好玩!

可!

可!

Theorem

- For $n \geq 2$ and R a relation on a finite set A, we have

$$M_{R^n} = M_R \odot M_R \odot \cdots \odot M_R \quad (n \text{ factors})$$

$$\triangleq (M_R)_{\odot}^{n}$$

Proof by induction

- Let $P(n)$ be the assertion that the statement holds for an integer $n \geq 2$.
- *Basis Step*: $P(2)$ is true by Theorem 1.

什m个JB

Induction Step

- Consider the matrix $M_{R^{k+1}}$. Let $M_{R^{k+1}} = [x_{ij}]$, $M_{R^k} = [y_{ij}]$, and $M_R = [m_{ij}]$
- If $x_{ij} = 1$, we must have a path of length $k + 1$ from a_i to a_j.
- If we let a_s be the vertex that this path reaches just before the last vertex a_j, then there is a path of length k from a_i to a_s and a path of length 1 from a_s to a_j.
- Thus $y_{is} = 1$ and $m_{sj} = 1$, so $M_{R^k} \odot M_R$ has a 1 in position i, j.
- similarly, if $M_{R^k} \odot M_R$ has a 1 in position i, j, then $x_{ij} = 1$.
- So $M_{R^{k+1}} = M_{R^k} \odot M_R$

Induction Step

$\because$ P(k): $M_{R^k} = M_R \odot \cdots \odot M_R$ (k factors)

$\therefore M_{R^{k+1}} = M_{R^k} \odot M_R = (M_R \odot M_R \odot \cdots \odot M_R) \odot M_R$

hence

P(k+1): $M_{R^{k+1}} = M_R \odot \cdots \odot M_R \odot M_R$ ($k+1$ factors)

- is true.
- Thus by the principle of mathematical induction, P(n) is true for all n

- QED

Transitive closure

- The transitive closure of a relation R is the smallest transitive relation containing R.

- Review: R is transitive iff R^n is contained in R for all n.
- Hence, if there is a path from x to y then there must be an arc from x to y, or (x, y) is in R.

Theorem $t(R) = R^* = \bigcup_{i=1}^{\infty} R^i$

- Let R be a relation on a set A. then R^* is the transitive closure of R.

- Proof: we must show that R^*
  - 1) is a transitive relation
  - 2) contains R
  - 3) is the smallest transitive relation which contains R

Proof of Part 1)

- Suppose (x, y) and (y, z) are in R^*. Show (x, z) is in R^*.
  - By definition of R^*, (x, y) is in R^m for some m and (y, z) is in R^n for some n.
  - Then (x, z) is in $R^n \circ R^m = R^{m+n}$ which is contained in R^*.
  - Hence, R^* must be transitive.

Proof of Part 2)

- Easy from the definition of R^*
- $R^* = R \cup R^2 \cup R^3 \cup \ldots$
- So $R \subseteq R^*$

Proof of Part 3)

- Now suppose S is any transitive relation that contains R, show S contains R^* (that is R^* is the smallest such relation).
- $R \subseteq S$, so $R^2 \subseteq S^2 \subseteq S$ since S is transitive.
- Therefore $R^n \subseteq S^n \subseteq S$ for all n. (Why ?)
- Hence S must contain R^* since it must also contain the union of all the powers of R.

Q. E. D.

Useful Results for Transitive Closure

- **Theorem:**
  - If $A \subseteq B$ and $C \subseteq B$, then $A \cup C \subseteq B$.
- **Theorem:**
  - If $R \subseteq S$ and $T \subseteq U$ then $R \circ T \subseteq S \circ U$.
- **Corollary:**
  - If $R \subseteq S$ then $R^n \subseteq S^n$

$R \subseteq$

If $R \subseteq S$ and $T \subseteq U$ then $R \circ T \subseteq S \circ U$

- Proof:
- If (a,b)$\in R \circ T$, *then exist (a,c)*$\in$T and (c,b)$\in$R *for some* c.
- *Because* $R \subseteq S$ *and* $T \subseteq U$, so *(a,c)*$\in$U and (c,b)$\in$S.
- Therefore (a,b)$\in S \circ U$.
- Hence $R \circ T \subseteq S \circ U$.

- Q. E. D.

If $R \subseteq S$ then $R^n \subseteq S^n$

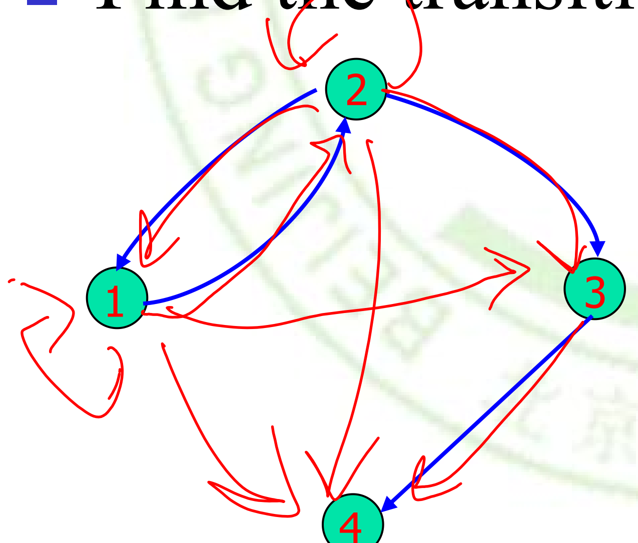
- Proof: Use theroem 1 or a proof by induction:
- *Basis*: Obviously true for $n = 1$.
- *Induction:*
  - The induction hypothesis:
    - assume theorem is true for n. $R^n \subseteq S^n$
  - Show it must be true for $n + 1$.

- $R^{n+1} = R^n \circ R$ so if (x, y) is in R^{n+1} then there is a z such that (x, z) is in R and (z, y) is in R^n.
- But since $R \subseteq S$ *and* $R^n \subseteq S^n$, (x, z) is in S and(z, y) is in S^n.
- $S^{n+1} = S^n \circ S$, (x, y) is in S^{n+1}.
- Hence $R^{n+1} \subseteq S^{n+1}$.

数学归纳证

Example

- Let
  - $A=\{1, 2, 3, 4\}$
  - $R=\{(1, 2), (2, 3), (3, 4), (2, 1)\}$
- Find the transitive closure of R.



$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example

$$(M_R)^2_{\odot} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = (M_R)^4_{\odot} = (M_R)^6_{\odot} = \ldots$$

一样

$$(M_R)^3_{\odot} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = (M_R)^5_{\odot} = (M_R)^7_{\odot} = \ldots$$

$$M_{R^\infty} = M_R \vee (M_R)^2_{\odot} \vee (M_R)^3_{\odot} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

有们

我们

成门

列维有路都填上
就完事了

Theorem

- Let A be a set with $|A|=n$, and let R be a relation on A. Then

$$R^* = \bigcup_{i=1}^{n} R^i = R \cup R^2 \cup \ldots \cup R^n$$

$$R^* = \bigcup_{i=1}^{n} R^i = R \cup R^2 \cup \cdots \cup R^n$$

式

Proof

- Let a and $b \in A$ and suppose that $a, x_1, x_2, \ldots, x_{m-1}, b$ is a path of length m from a to b in R.
  - $(a, x_1) \in R$
  - $(x_1, x_2) \in R$
  - …
  - $(x_{m-1}, b) \in R$

a path of length m.

Proof

- There are m+1 elements in the path, but we have only n distinct elements in A.
  - So, there must be some same vertex in the path, say $x_i = x_j = c,\ i<j$
    - $(a, x_1) \in R$
    - $(x_1, x_2) \in R$
    - …
    - $(x_{i-1}, x_i) \in R$
    - $(x_i, x_{i+!}) \in R$
    - …
    - $(x_{j-1}, x_j) \in R$
    - $(x_j, x_{j+1}) \in R$
    - …
    - $(x_{m-1}, b) \in R$
- The red edges form a cycle in the path, we get a new path by deleting the cycle

Proof

- A new path from *a* to *b* by deleting the cycle
  - $(a, x_1) \in R$
  - $(x_1, x_2) \in R$
  - …
  - $(x_{i-1}, x_i) \in R$
  - $(x_j, x_{j+1}) \in R$
  - …
  - $(x_{m-1}, b) \in R$

Proof

- A path from a to b ($x_i = x_j = c$)
  - $a, x_1, x_2, \ldots, x_{i-1}, c, x_{j+1}, \ldots, x_{m-1}, b$
- The length is $k = m - j + i$.
- The process can continue until $k \le n$, so we have
  - $R^m \subseteq R^k$
  - $\forall m\ (m > n \wedge (a, b) \in R^m \rightarrow \exists k\ (k \le n \wedge (a, b) \in R^k))$
- Therefore

$$R^* = \bigcup_{i=1}^{n} R^i = R \cup R^2 \cup \ldots \cup R^n$$

- QED

Algorithm 1 for t(R)

- procedure transitive closure (M_R: zero-one n×n matrix)
  - $A := M_R$
  - $B := A$
  - *for i:=2 to n*
    - $A := A \odot M_R$
    - $B := B \vee A$
  - *return B {B is the zero-one matrix for R*}*

Analysis

- Complexity of Algorithm
  - $M_{R^*} = M_R \vee (M_R)^2_{\odot} \vee \ldots \vee (M_R)^n_{\odot}$
    - $(n-1)*(n^2*2n+n^2)$ is $O(n^4)$.

Some definitions

- W_k: a Boolean matrix, for $1 \le k \le n$
  - W_k has a 1 in position i, j
    - If and only if
  - there is a path from a_i to a_j in R whose interior vertices, if any, come from the set $\{a_1, a_2, \ldots, a_k\}$
- What about W_0 W_n ?
  - Let $W_0 = W_R$
  - $W_n = W_R^*$
  - $W_0, W_1, W_2, \ldots, W_n$

Composition of paths

- Let
  - $\pi_1: a, x_1, x_2, \ldots, x_{n-1}, b$
  - $\pi_2: b, y_1, y_2, \ldots, y_{m-1}, c$
- The composition of π_1 and π_2 is the path
  - $\pi_2 \circ \pi_1: a, x_1, x_2, \ldots, x_{n-1}, b, y_1, y_2, \ldots, y_{m-1}, c$

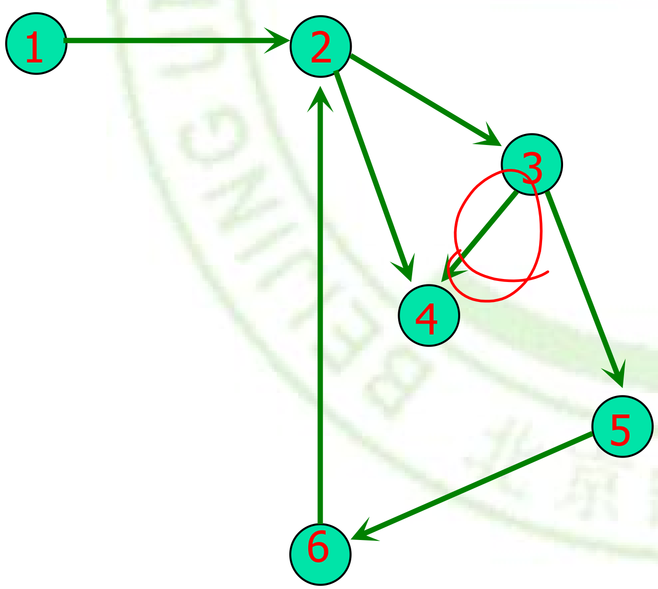
$\pi_2 \circ \pi_1$

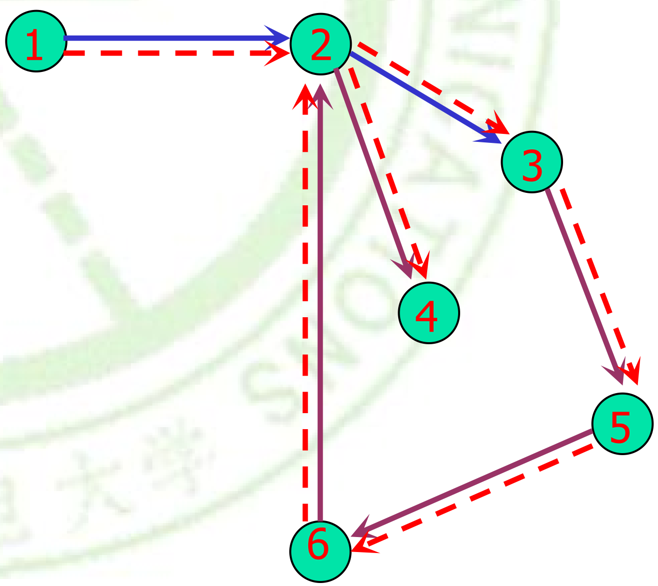
  - *Note the order of composition!*

注意顺序

Example

- Consider the relation whose digraph is given in Figure and the paths
  - π_1: 1, 2, 3 π_2: 3, 5, 6, 2, 4 $\pi_2 \circ \pi_1$





Warshall's Algorithm

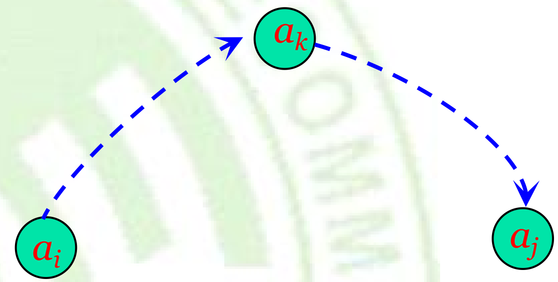
- Procedure 沃舍尔算法
  - begin with the matrix of R, and
  - compute each matrix W_k from the previous matrix W_{k-1}, and,
  - reach W_R^* in n steps,

沃舍尔算法

Warshall's Algorithm

- Procedure Warshall(M_R: zero-one n×n matrix)
  - $W := M_R$
  - *for k:=1 to n /* 下面直接更新W*/*
    - *for i := 1 to n*
      - *for j :=1 to n*
        - $W[i,j] := W[i,j] \vee (W[i,k] \wedge W[k,j])$
  - *return W {W_n is the zero-one matrix for R*}*

W_n is $t(R)$ Proof (1)



- Proof: Use a proof by induction.
- Suppose
  - $W_{k+1} = [t_{ij}]$
  - $W_k = [s_{ij}]$
- *Basis*: k = 1, $W_1 := M[i,j] \vee (M[i,1] \wedge M[1,j])$.
  - *case 1*: a_1 is not an interior vertex, so T[i,j] = M*[i,j]*.
  - *case 2*: a_1 is an interior vertex, so T[i,j] = 1.
  - *So* W_1 has a 1 in position *i, j* iff there is a path from a_i to a_j in *R* whose interior vertices come from the set $\{a_1\}$.

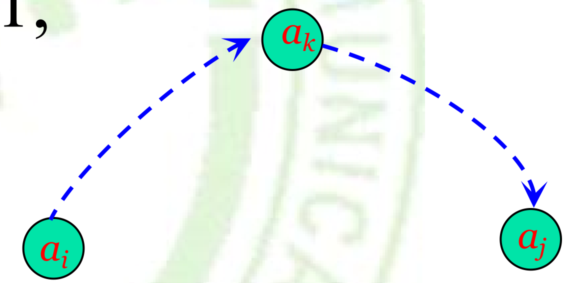
W_n is $t(R)$ Proof (2)

- *Induction:*
  - The induction hypothesis:
- assume W_k has $s_{ij} = 1$, *iff* there is a path from a_i to a_j in R whose interior vertices, if any, come from the set $\{a_1, a_2, \ldots, a_k\}$.
  - Show it must be true for W_{k+1}.
  - $t_{ij} = 1$ if and only if

  either $s_{ij} = 1$ or $s_{i,k+1} = 1$ and $s_{k+1,j} = 1$.

W_n is t(R) Proof (3)

- *case 1:* $s_{ij} = 1$, then all interior vertices must actually come from the set $\{a_1, a_2, \ldots, a_k\}$.
- *case 2:* $s_{i,k+1} = 1$ and $s_{k+1,j} = 1$,
  - So a_{k+1} is an interior vertex.
  - Two subpaths
    - a_i to a_{k+1} and a_{k+1} to a_j



- Therefore W_{k+1} has $t_{ij} = 1$, *iff* there is a path from a_i to a_j in R whose interior vertices, if any, come from the set $\{a_1, a_2, \ldots, a_{k+1}\}$.

W_n is t(R) Proof (4)

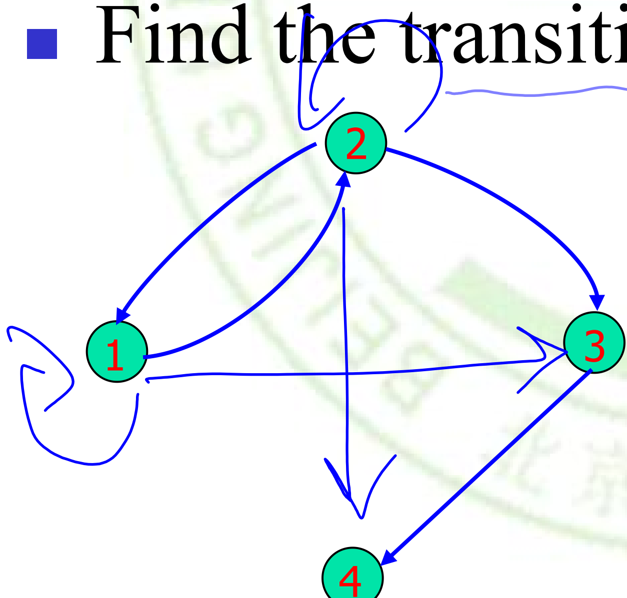
- Hence W_n has $t_{ij} = 1$, *iff* there is a path from a_i to a_j in R whose interior vertices, if any, come from the set $\{a_1, a_2, \ldots, a_n\}$.
- So $W_n = R^* = t(R)$.

manual operation

- *Step1*:
  - First transfer to W_k all 1's in W_{k-1}.
- *Step2*:
  - List the locations p_1, p_2, …, in column k of W_{k-1}, where the entry is 1.
  - List the locations q_1, q_2, …, in row k of W_{k-1}, where the entry is 1.
- *Step3*:
  - Put 1's in all the positions p_i, q_j of W_k (if they are not already there)

Example (1)

- Let
  - $A=\{1, 2, 3, 4\}$
  - $R=\{(1, 2), (2, 3), (3, 4), (2, 1)\}$
- Find the transitive closure of R.



$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example

$W_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $W_1 =$ [

$\begin{matrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{matrix}$

$\begin{matrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{matrix}$

$W_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = W_4 = W^\infty$

$$W_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = M_R, \quad W_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$W_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad W_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = W_4 = W^\infty$$

Please feel free to ask questions!

沃舍算法：

W1，看第1列哪行为1，把这一行与第一行逻辑加.

W2 看第二列哪行为1，把这个行与第二行逻辑加

Wk 看第k列哪行为1，把这一行与第k行逻辑加.

homework

- § 9.4
  - 20, 22, 28

Useful Results for Transitive Closure

- **Theorem:**
  好规律
  - If R is transitive then so is R^n
  - Trick proof: Show $(R^n)^2 = (R^2)^n \subset R^n$ 设计证明
- **Theorem:**
  - If $R^k = R^j$ for some $j > k$, then $R^{j+m} = R^n$ for some $n \geq j$.
  - We don't get any new relations beyond R^j.