Planar Graphs

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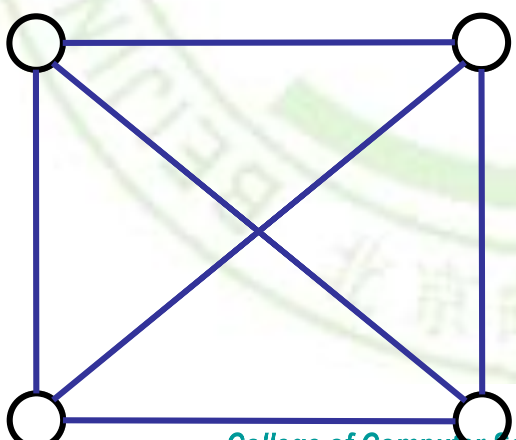
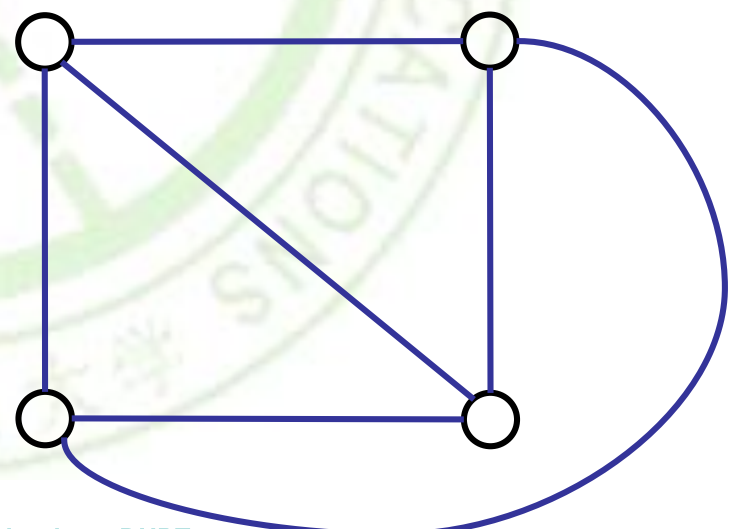
planar graphs

Planar Graphs – 平面图

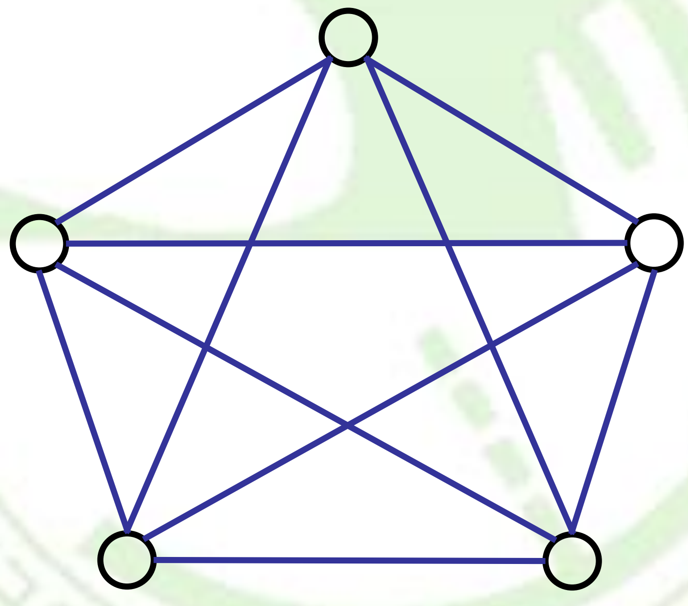
- A graph is called ***planar*** if it can be drawn in the plane in such a way that no two edges cross.
- Example of a planar graph: The clique on 4 nodes.

K_4

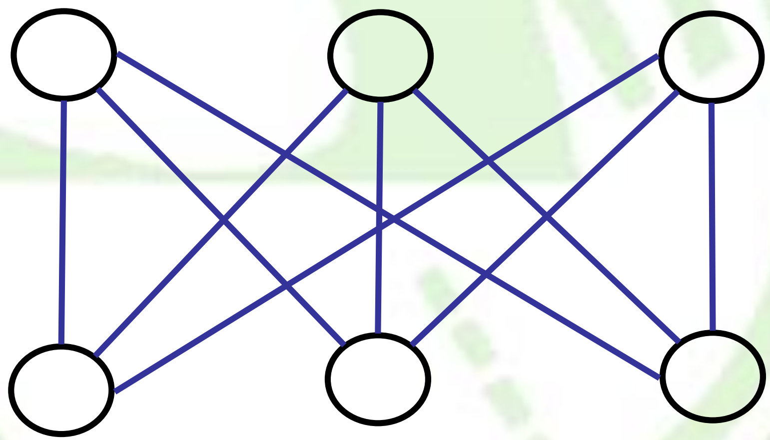




Is K_5 planar?



What about $K_{3,3}$?

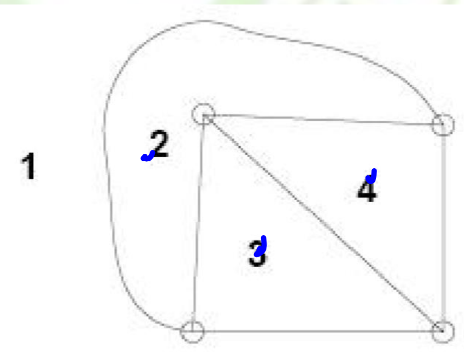


Why Planar?

- The problem of drawing a graph in the plane arises frequently in VLSI layout problems.

Regions, faces — 面

- A drawing of a planar graph divides the plane into ***faces***, regions bounded by edges of the graph.
- For example, the drawing below has four faces, Face 1, which extends off to infinity in all directions, is called the outside face.
  - Degree of face: number of edges in the boundary of the face.

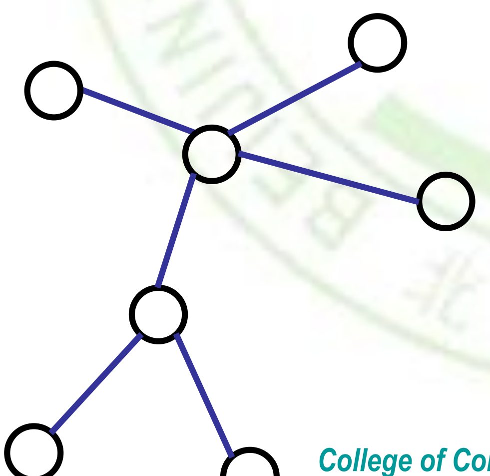


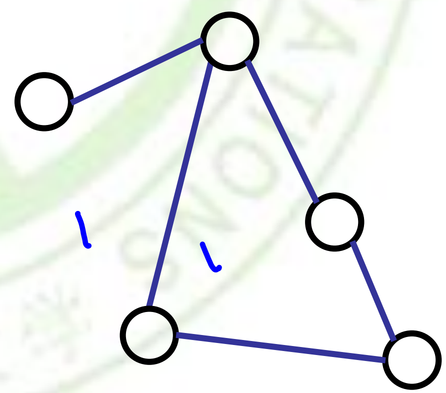
Theorem

- A planar graph G, total degrees of faces are double of edges. $\Sigma deg(R_i)=2e$.
- Proof: Every edge lies on the boundary of at most 2 faces.

two faces

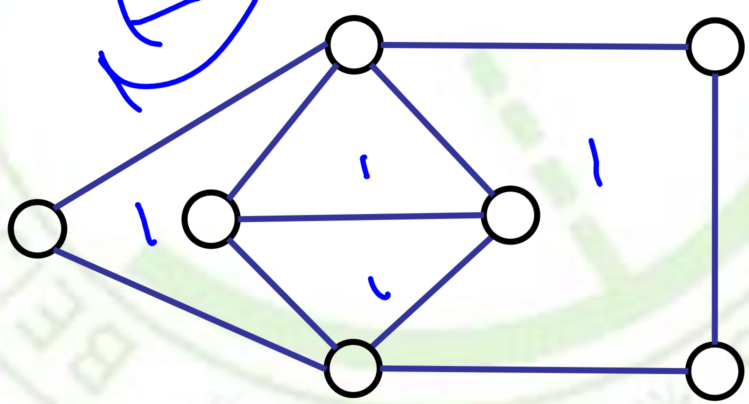
one face





Question

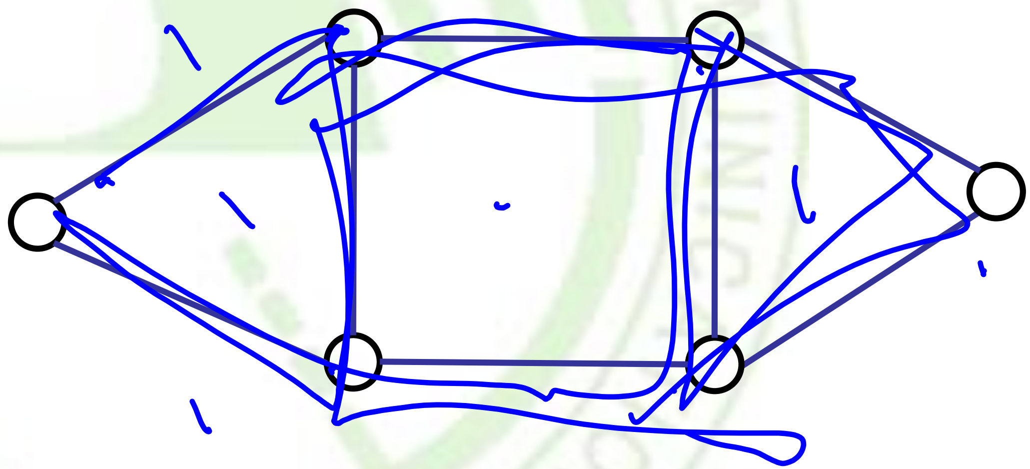
- Can you redraw this graph as a planar graph so as to alter the number of its faces?



Example

8−6+2=4面

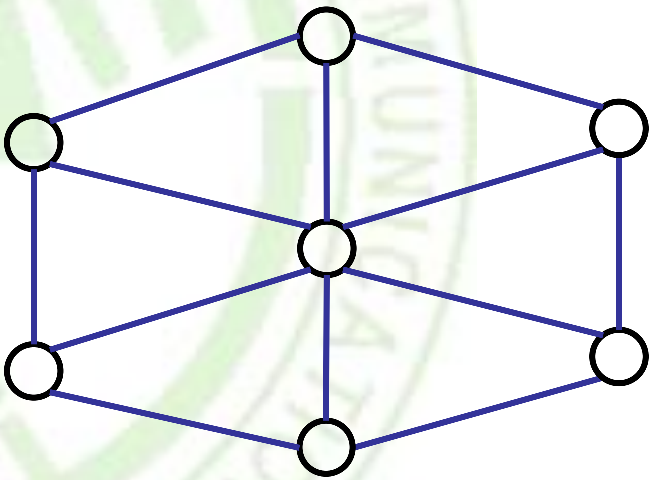
- This graph has
  - 6 vertices
  - 8 edges and
  - 4 faces



- vertices – edges + faces = 2

Example

- This graph has
  - 7 vertices
  - 12 edges and
  - 7 faces



- vertices – edges + faces = 2

Euler Theorem 1752

- Let G be a connected planar graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

$r=e-v+2$

connected planar graph

平面图。

Proof:

- By induction on the edges of *G*.
- ***Base case***:
- *e*=0, *G* is connected,so *v*=1, *r*=1=e-v+2.
- *e*=1, *G* is connected,so *v*=1, or 2.
  - v=1, r=2=e-v+2;
  - v=2, r=1=e-v+2;

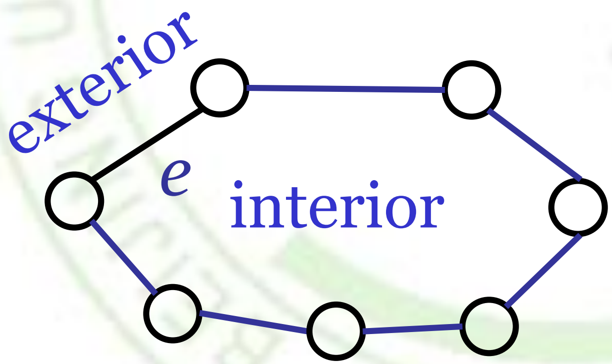
Proof:

- ***Introduction***: G has no cycles.
- G is connected so it must be a tree.
- Thus, $e = v - 1$ and $r = 1$.

$$\begin{aligned} v - e + r &= v - [v - 1] + 1 \\ &= 1 + 1 \\ &= 2 \end{aligned}$$

Inductive Step

- Suppose G has at least one cycle C containing edge e.
- Let $G' = G - e$



v' = # of vertices,
e' = # of edges,
r' = # of regions

Inductive step

- G' is connected since e was on a cycle.
  - $r' = r$-1 and G' has fewer cycles than G.
  - $v' = v$
  - $e' = e$-1
- By induction hypothesis:

$$v' - e' + r' = 2$$

$$v - [e - 1] + [r - 1] = 2$$

$$v - e + r = 2$$

Corollary

- No matter how we redraw a planar graph it will have the same # of regions.
- Proof:
  - $r = 2 - v + e$ is determined by v and e, neither of which change when we redraw the graph.

Theorem

k连通.

- Let G be a planar graph with k connected component and e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + k+1$.
- proof: Suppose component $G_1 G_2 G_k$,
- so $e_i - v_i + 2 = r_i$. => $2k = \Sigma r_i + \Sigma v_i - \Sigma e_i$
- and $r = \Sigma r_i - k+1$;(外部面只有一个)
- so $2k = r + k - 1 + v - e$ => $r = e - v + k + 1$

COROLLARY 1

- Every connected planar simple graph G with n-node($n \geq 3$) has at most $3n-6$ edges.
- *Proof*: $n = 3$: Clearly true.
- $n \geq 3$: G is simple graph, every face has at least 3 edges on its boundary.
- Every edge lies on the boundary of at most 2 faces. Thus $2e>=3r$.

Proof

- because G is connected planar graph,
- thus $n - e + r = 2$

$$n - e + r = 2$$

$$3n - 3e + 3r = 6$$

$$3n - 3e + 2e \geq 6$$

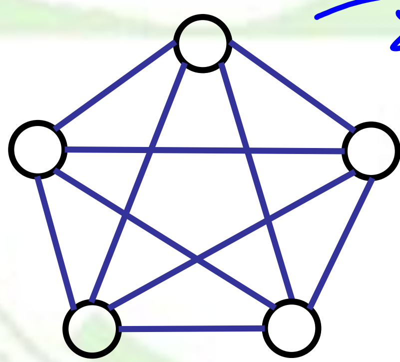
$$e \leq 3n - 6$$

K_5 IS NOT PLANAR

- A connected planar simple graph on n = 5 nodes can have at most 3n-6 = 9 edges.

$\frac{4\times5}{2}=10$ 条边

$n=5$



$$e=\binom{5}{2}=10$$

- Thus: K_5 is not planar.

$e \leq 3v-6$

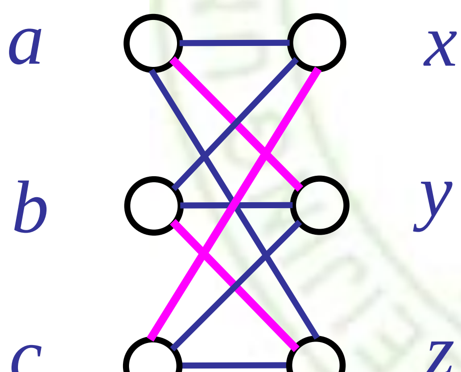
至多9条边

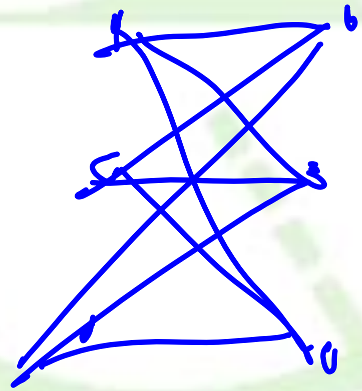
COROLLARY 3

- If a <u>connected planar simple graph</u> has e edges and v vertices with $v \geq 3$ and no circuit of length three, then $e \leq 2v-4$.
- *Proof*: $n \geq 3$: G is simple graph, no circuit of length three, so <u>every face has at least 4 edges on its boundary</u>. Thus $\Sigma\deg(Ri) = 2e >= 4r$.
- *Euler Theorem*: $r = e - v + 2$.
- $4r-4e+4v=8 \Rightarrow 4v-2e \geq 8 \Rightarrow e \leq 2v-4$.

$K_{3,3}$ IS NOT PLANAR

- A connected planar simple graph on $v = 6$ nodes, no circuit of length three, then $e \le 2v-4=8$.
- but $K_{3,3}$ have 9 edges.





- Thus: $K_{3,3}$ is not planar.

COROLLARY 2

- If G is a connected planar simple graph, then G has a vertex of degree not exceeding five.
- proof: v=1,or 2, It is clearly true.
- v≥3, by corollary 1, $e \le 3v-6$, $2e \le 6v-12$.
- If $\delta(G)=6$, $2e=\Sigma deg(v)$(by handshaking theroem) ≥6v.
- contradict, so $\delta(G) \le 5$.

Theroem

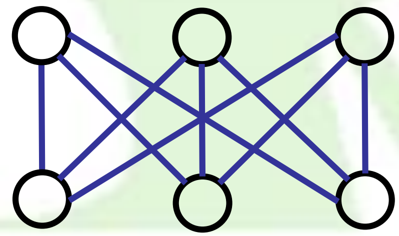
- If G is a connected planar graph, and $\deg(R_i) \geq l$, then $e \leq l*(v-2)/(l-2)$.
- proof: $2e=\Sigma\deg(R_i) \geq lr = l(e-v+2)$.
- $l(v-2) \geq (l-2)e$
- $l(v-2)/(l-2) \geq e$

Theroem

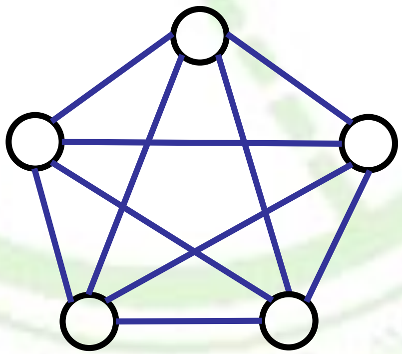
- If G is a planar graph with k connected component, and $\deg(R_i) \geq l$,
- then $e \leq l*(v-k-1)/(l-2)$.
- proof: $2e=\Sigma\deg(R_i) \geq lr = l(e-v+k+1)$.
- $l(v-k-1) \geq (l-2)e$
- $l(v-k-1)/(l-2) \geq e$

The Kuratowski Graphs

$K_{3,3}$

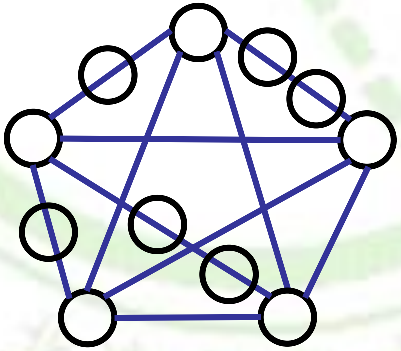


K_5



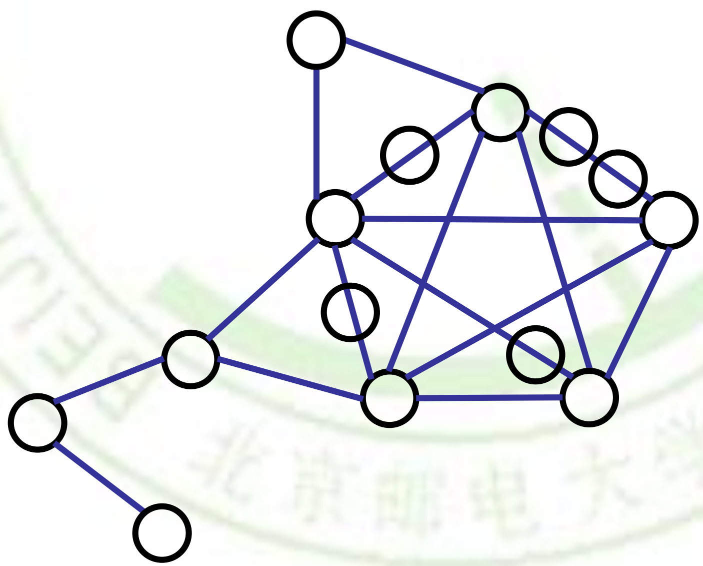
Insight 1

- If we replace edges in a Kuratowski graph by paths of whatever length, they remain non-planar.



Insight 2

- If a graph G contains a subgraph obtained by starting with K_5 or $K_{3,3}$ and replacing edges with paths, then G is non-planar.



Kuratowski's Theorem [1930]

- A graph is planar if and only if it contains no subgraph obtainable from K_5 or $K_{3,3}$ by replacing edges with paths.

Homework

- § 10.7
  - 6, 8, 12, 18, 24, 30

Graph Coloring
图的着色

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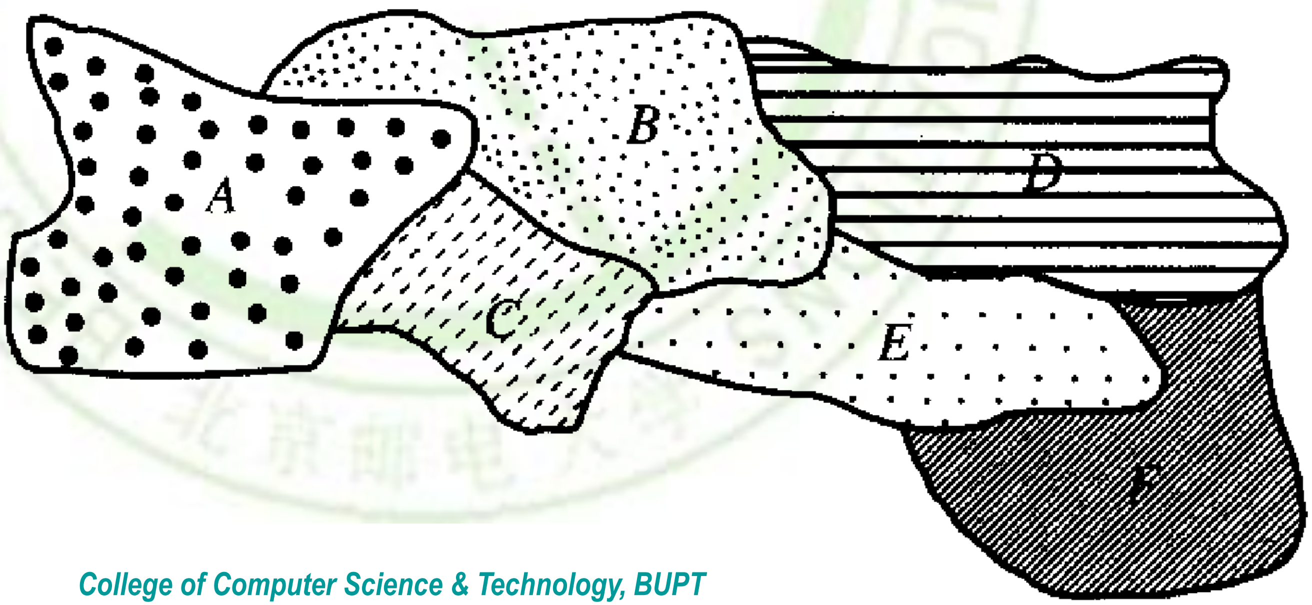
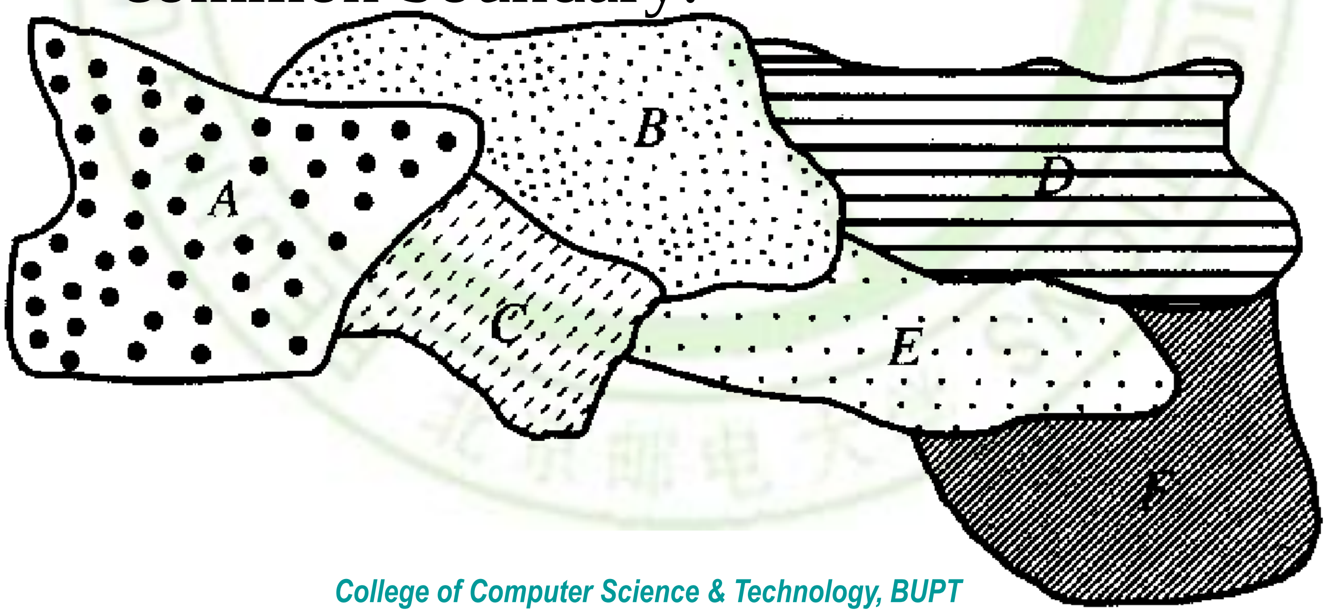
MAP-COLORING PROBLEM

- Of the many problems that can be viewed as graph-coloring problems, one of the oldest is the *map-coloring problem*.(地图着色问题)



DUAL GRAPH对偶图

- Given a map M, construct a graph G_M
  - Vertex $u \leftrightarrow$ region R_u
  - Edge $(u, v) \leftrightarrow$ regions R_u and R_v share a common boundary.



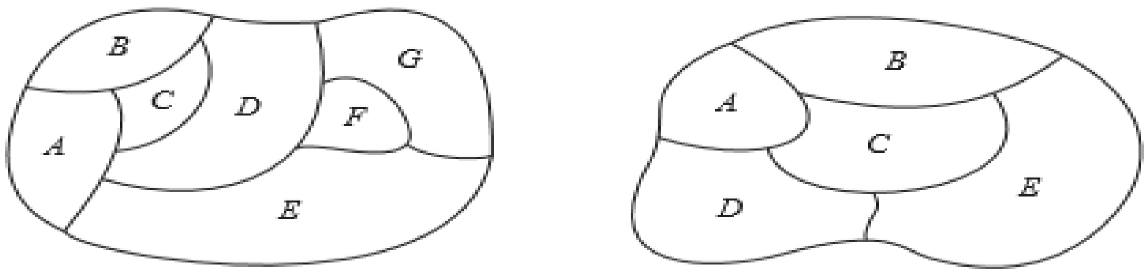


FIGURE 1 **Two Maps.**

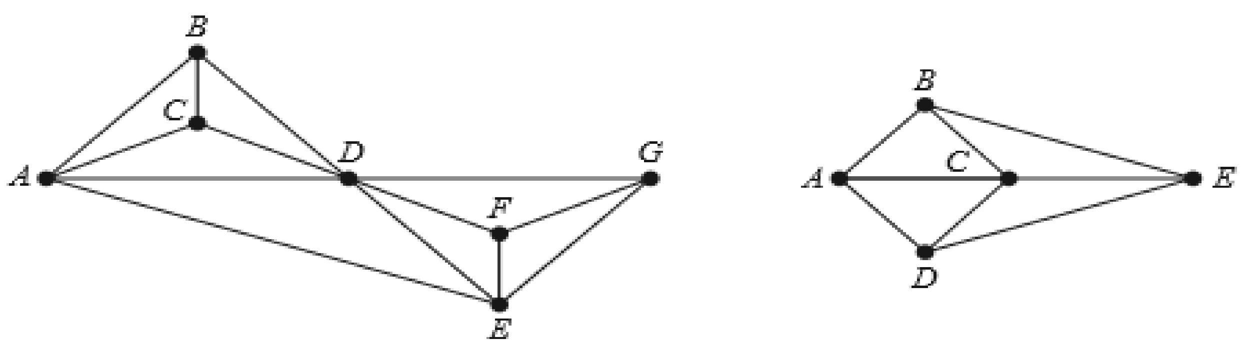


FIGURE 2 **Dual Graphs of the Maps in Figure 1.**

Coloring graphs

- Suppose that $G = (V, E, \gamma)$ is a graph with no multiple edges, and $C = \{c_1, c_2, \ldots, c_n\}$ is any set of n "colors".
- Any function $f: V \rightarrow C$ is called *a coloring of the graph G using n colors* (or using the colors of C).
  - For each vertex v, $f(v)$ is the color of v.

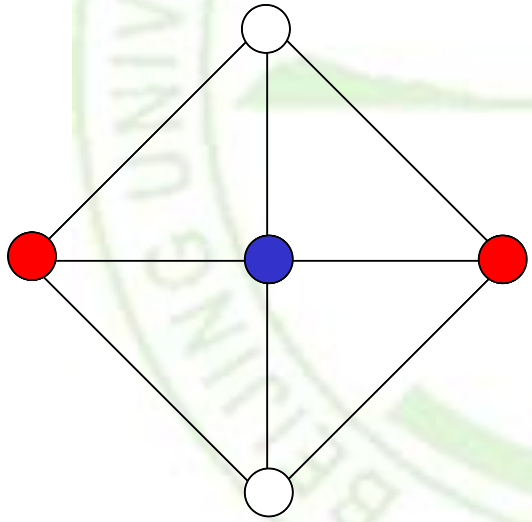
$\chi(G)$

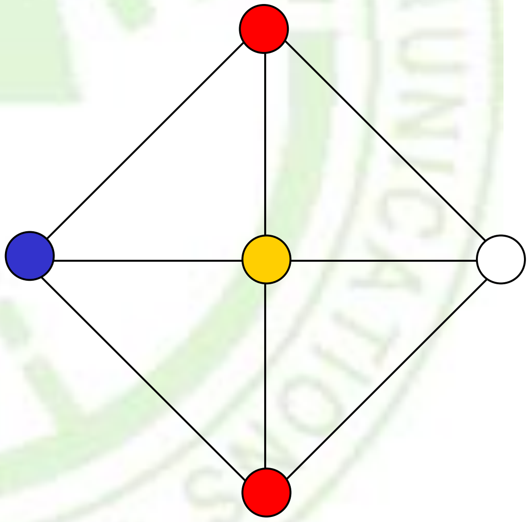
- A coloring is *proper* (合适的) if any two adjacent vertices *v* and *u* have different colors.
- The smallest number of colors needed to produce a proper coloring of a graph *G* is called the *chromatic number of G*(*G*的着色数), denoted by $\chi(G)$.

chromatic number of G $\chi(G)$

Example

- $\chi(G) = 3$





着色数

FOUR COLORS CONJECTURE

- ***Four colors are always enough to color any map drawn on a plane***
  - This conjecture was proved to be true in 1976 with the aid of computer computations performed on almost 2,000 configurations of graphs. There is still no proof known that does not depend on computer checking.

四色定理.

Five Color Theorem

- Any planar graph can be colored with five colors.

Lemma

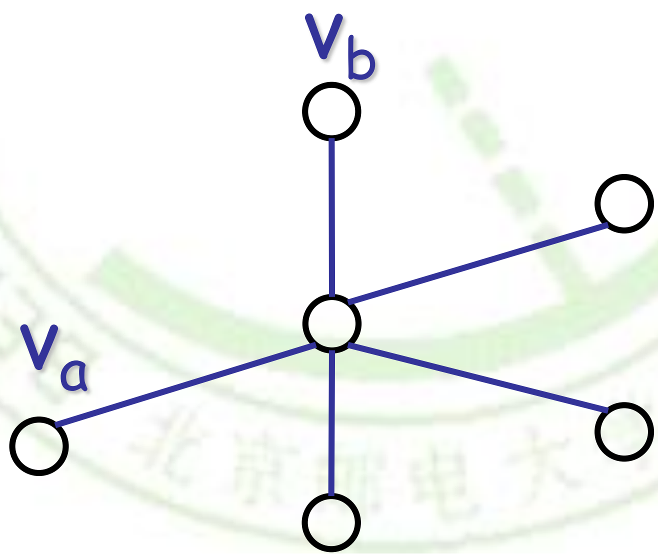
- Every Planar Graph Contains a Node of Degree ≤ 5
- Proof
  - If every node has degree at least 6, then the number of edges would be $3n$, which would contradict our upper bound of $3n$-6 edges in an n-node planar graph.

Proof of 5-color theorem

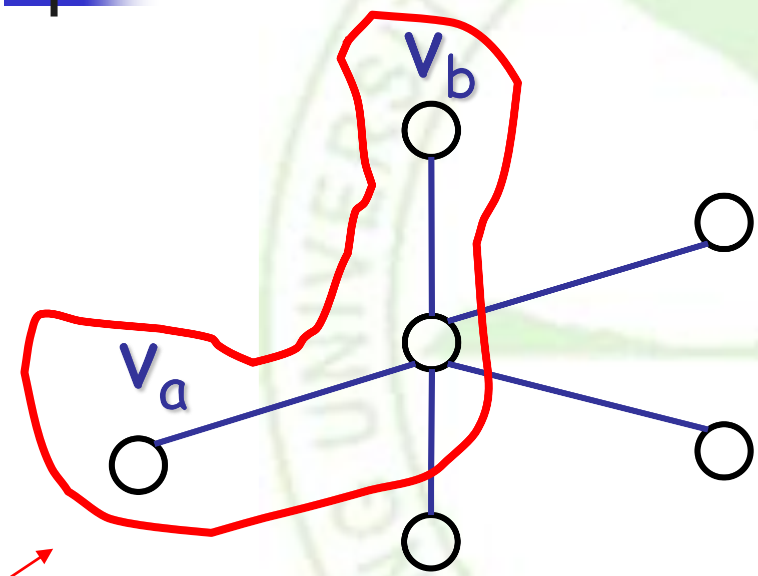
- Let G be a node-minimal counter-example to the theorem, i.e., a planar graph that requires 6 colors.
- By Lemma, G must have a node q with degree ≤ 5. Let the nodes adjacent to q be named v_1, v_2, v_3, v_4, and v_5.

Proof of 5-color theorem

- v_1, v_2, v_3, v_4, v_5 can't form a K_5
- Some two neighbors v_a, and v_b of q must not have an edge between them



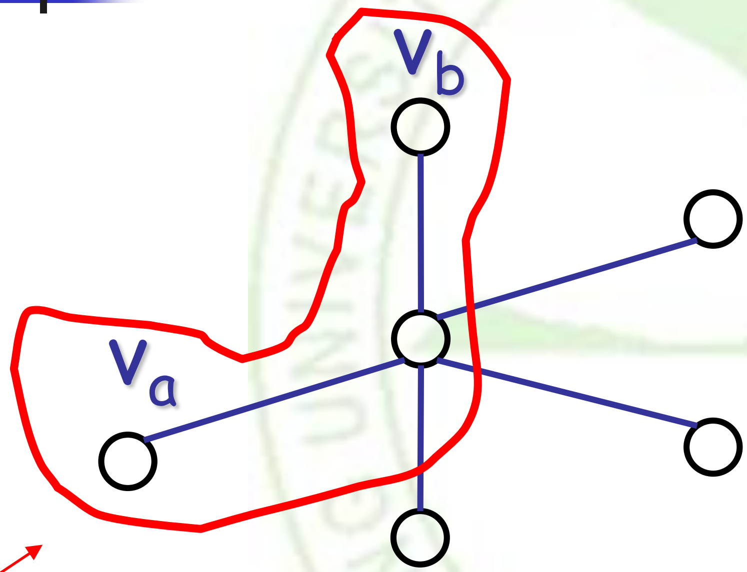
Edge Contraction



- Contract the edges $<q, v_a>$ and $<q,v_b>$ of G to obtain a planar graph G'.
- G' is 5 colorable since it has fewer nodes than G.

v_a, v_b, and q become a single node α in G'

Using G' to 5-color G.



- Color v_a and v_b the same as α. Color each node besides q, as it is colored in G'.
- Color q whatever color is not used on its 5 neighbors.

- Q.E.D.

v_a, v_b, and q become a single node α in G'

Example

- What is the chromatic number of the complete bipartite graph $k_{m,n}$ where *m* and *n* are positive integers?

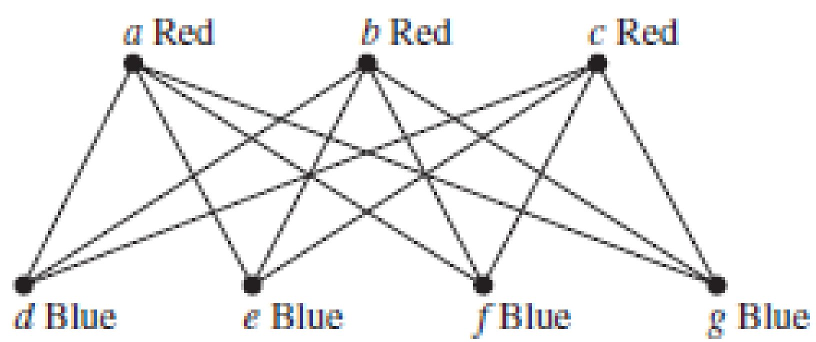


FIGURE 6 A Coloring of $K_{3,4}$.

Example

- What is the chromatic number of the graph C_n?

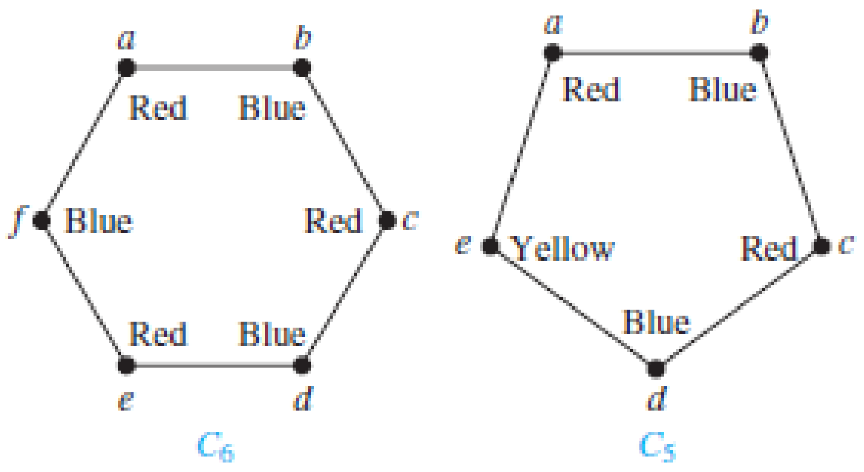


FIGURE 7 Colorings of C_5 and C_6.

Graph-coloring

- Graph-coloring problems also arise from counting problems.

Example

- Fifteen different foods are to be held in refrigerated compartments within the same refrigerator.
- Construct a graph G as follows.
  - Construct one vertex for each food and connect two with an edge if they must be kept in separate compartments in the refrigerator.
  - Then $\chi(G)$ is the smallest number of separate containers needed to store the 15 foods properly.

Applications of graph colorings

- How can the final exams at a university be scheduled so that no student has two exams at the same time? p731

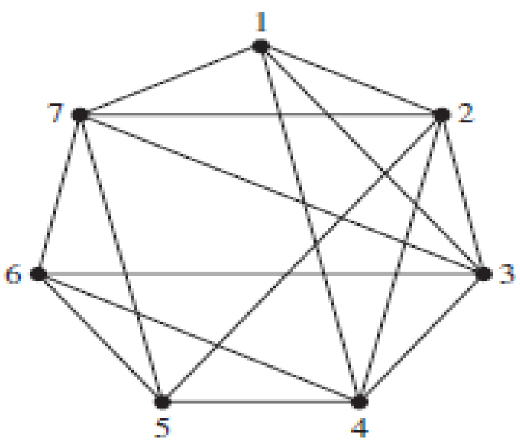
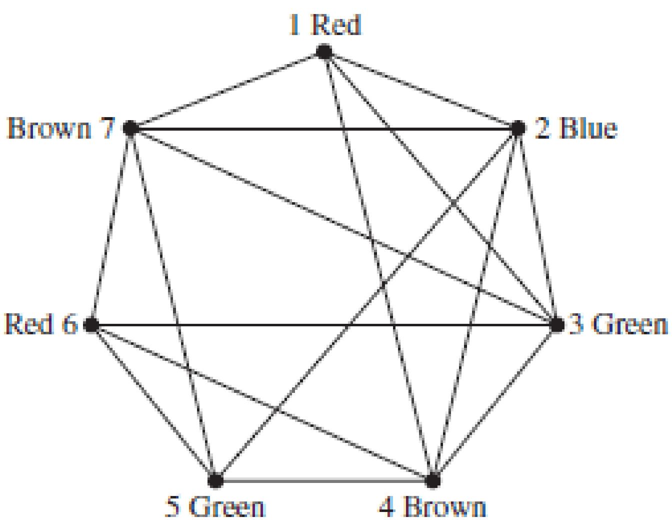


FIGURE 8 The Graph Representing the Scheduling of Final Exams.



Time Period	Courses
I	1, 6
II	2
III	3, 5
IV	4, 7

FIGURE 9 Using a Coloring to Schedule Final Exams.

Applications of graph colorings

- Frequency assignments
  - Television channels 2 through 13 are assigned to stations in North America so that no two stations within 150 miles can operate on the same channel. How can the assignment of channels be modeled by graph coloring?

Applications of graph colorings

- For a given loop, how many index registers are needed?
  - `for (i=0;i<n;i++){}`
  - `while(a<n){a=a+b;...}`

COLORING EDGES

- An edge coloring of a graph is an assignment of colors to edges so that edges incident with a common vertex are assigned different colors.
- The edge chromatic number of a graph is the smallest number of colors that can be used in an edge coloring of the graph. The edge chromatic number of a graph G is denoted by $\chi'(G)$.

COLORING EDGES

- To set up this correspondence, each edge of the map is represented by a vertex. Edges connect two vertices if these edges have a common vertex.

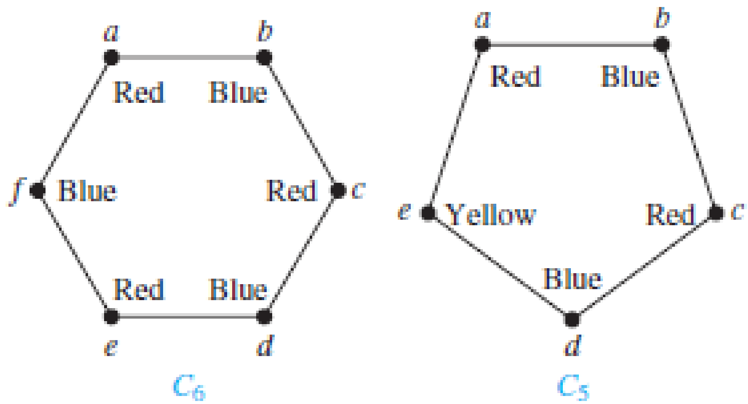


FIGURE 7 Colorings of C_5 and C_6.

SIMPLE GRAPH COLORING ALGORITHM

- 1. list the vertices v1,v2,v3,...,vn in order of decreasing degree so that deg(v1) $\geq$ deg(v2) $\geq \cdots \geq$deg(vn).
- 2. Assign color 1 to v1 and to the next vertex vi in the list not adjacent to v1 (if one exists). and, assign color 1 to the next vertex vj in the list not adjacent{v1,vi}, so on untill no vertex.
- 3. Then assign color 2 to the first vertex in the list not already colored. Continue this process until all vertices are colored.

Another problem

- Computing the total number of different proper colorings of a graph G using a set $\{c_1, c_2, \ldots, c_n\}$ of colors.

Chromatic Polynomials

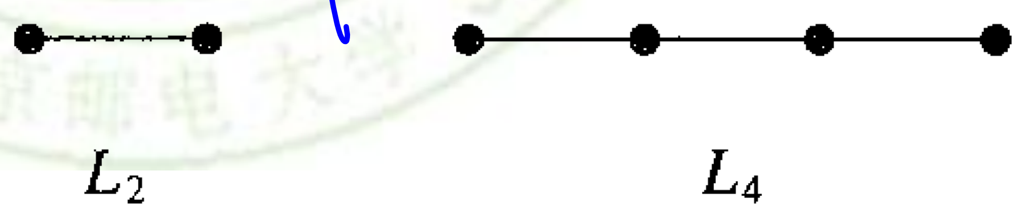
- If G is a graph and $n \geq 0$ is an integer, let $P_G(n)$ be the number of ways to color G properly using n or fewer colors.
- P_G is called the *chromatic polynomial*(着色多项式) of G.

Linear graph L_4

- Suppose we have x colors
  - $P_{L4} = x(x\text{-}1)^3$
  - $P_{L4}(0) = 0$
  - $P_{L4}(1) = 0$
  - $P_{L4}(2) = 2$
  - $P_{L4}(3) = 24$
  - So, $\chi(L_4) = 2$

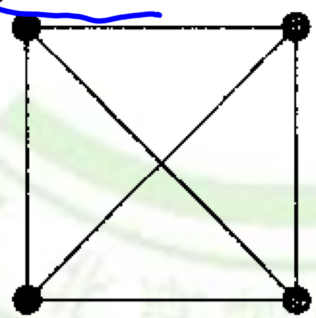
$P_{L4} = x(x-1)^3$

$P_{L5} = 5!$

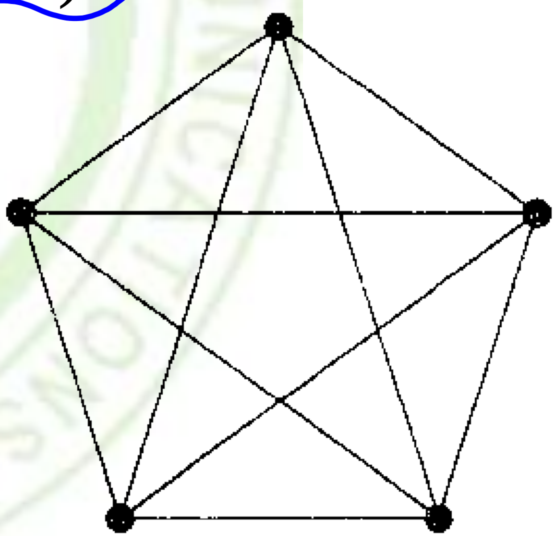


Complete graph K_n

- Suppose we have x colors, if $x < n$, no proper coloring is possible. So let $x \geq n$
  - $P_{K_5}(x) = x(x-1)(x-2) \ldots (x-n+1)$
  - $P_{K_5}(5) = 5!$
  - So, $\chi(K_5) = 5$



K_4



K_5

Theorem 1

- If G is a disconnected graph with components $G_1, G_2, \ldots . G_m$, then $P_G(x)$ is the product of the chromatic polynomials for each component
  - $P_G(x) = P_{G_1}(x)\, P_{G_2}(x) \ldots P_{G_m}(x)$

Subgraph

- One of the most important subgraphs is the one that arises by deleting one edge and no vertices.
- If $G = (V, E, \gamma)$ is a graph and $e \in E$, then we denote by G_e the subgraph obtained by omitting the edge e from E and keeping all vertices.

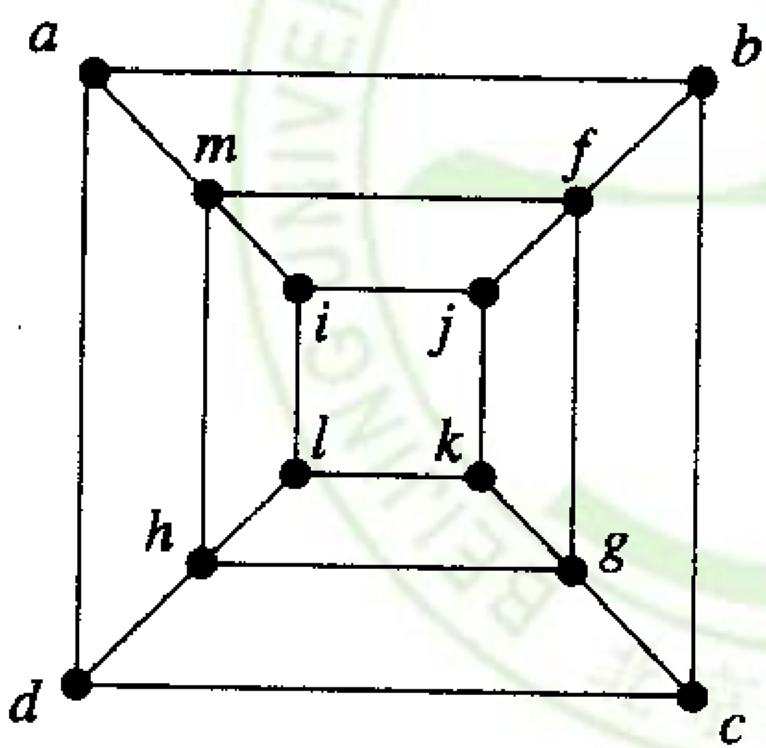
Quotient graph（商图）

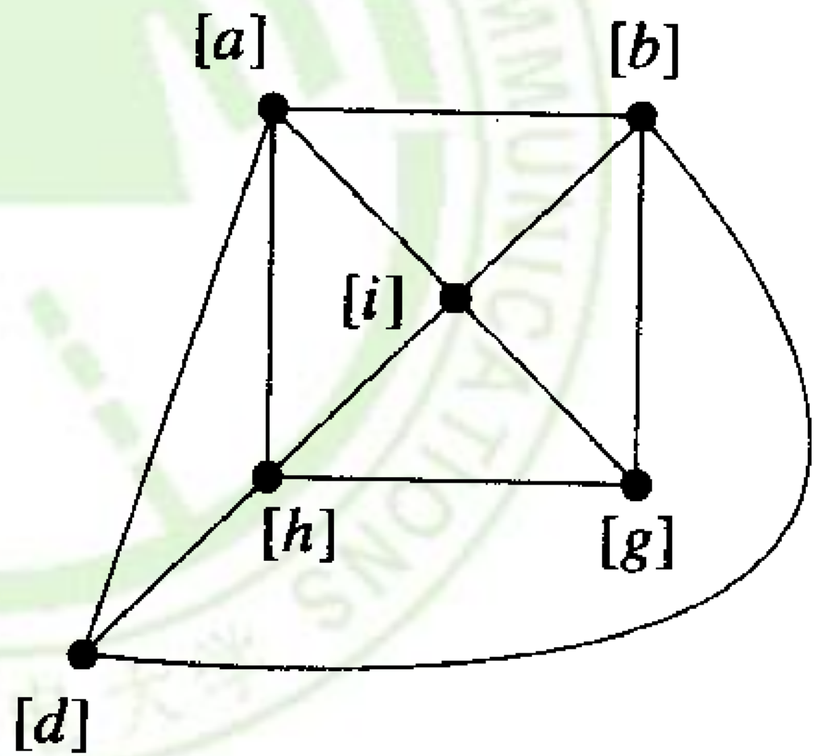
- Suppose that $G = (V, E, \gamma)$ is a graph without multiple edges between the same vertices and that R is an equivalence relation on the set V. Construct the *quotient graph* G^R as follow:
  - The vertices of G^R are the equivalence classes of V produced by R. If $[v]$ and $[w]$ are the equivalence classes of vertices v and w of G, then there is an edge in G^R from $[v]$ to $[w]$ if and only if some vertex in $[v]$ is connected to some vertex in $[w]$ in the graph G.

Quotient graph

Example

- $R = \{\{i,j,k,l\},\{a,m\},\{f,b,c\},\{d\},\{g\},\{h\}\}$





Theorem 2

- Let $G = (V, E, \gamma)$ be a graph with no multiple edges, and let $e \in E$, say $e = \{a, b\}$.
  - G_e : subgraph of G obtained by deleting e,
  - G^e : the quotient graph of G obtained by merging the end points of e.
- Then with x colors:

$$P_G(x) = P_{G_e}(x) - P_{G^e}(x).$$

Proof

- Consider all the proper colorings of G_e
  - a and b have different colors
  - a and b have the same color
- A coloring of the first type is also a proper coloring for G
  - since a and b are connected in G, and this coloring gives them different colors.

Proof

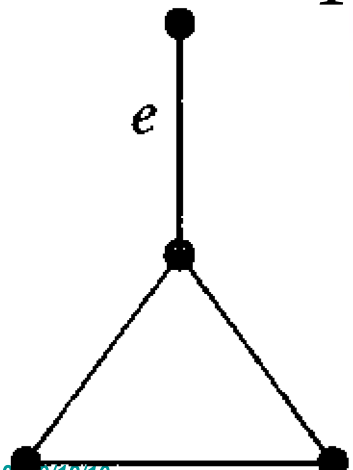
- a and b have different colors
- a and b have the same color

- A coloring of G_e of the second type corresponds to a proper coloring of G^e.
  - In fact, since a and b are combined in G^e, they must have the same color there. All other vertices of G_e have the same connections as in G. Thus
  - $P_{Ge}(x) = P_G(x) + P_G{}^e(x)$
    - or
  - $P_G(x) = P_{Ge}(x) - P_G{}^e(x)$

- Q.E.D

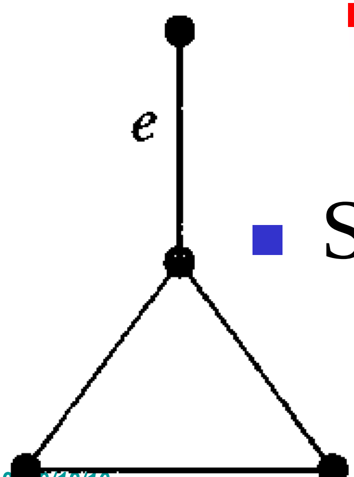
Example 8

- Let us compute $P_G(x)$ for the graph G using the edge e.
- Then G^e is K_3 and G_e has two components, one being a single point and the other being K_3.



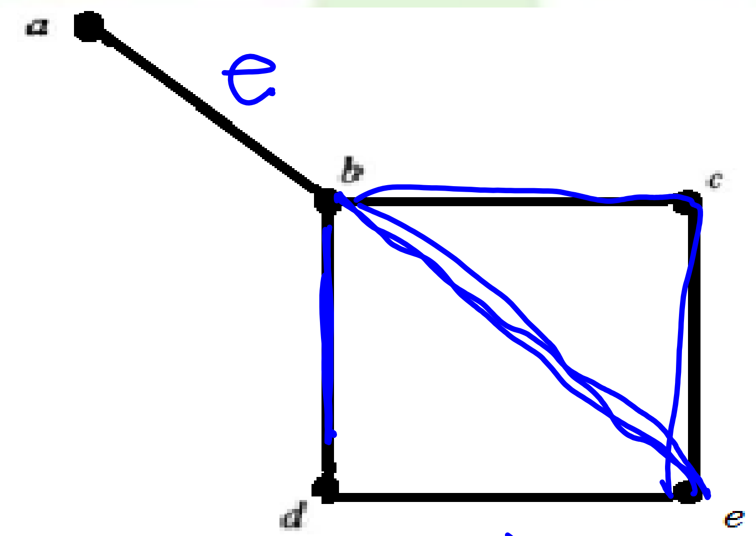
Example 8

- By Theorem 1
  - $P_{G_e}(x) = x(x(x-1)(x-2)) = x^2(x-1)(x-2)$
  - $P_{G^e}(x) = x(x-1)(x-2)$
- By Theorem 2,
  - $P_G(x) = x^2(x-1)(x-2) - x(x-1)(x-2)$
    $= x(x-1)^2(x-2)$
- So, $\chi(G) = 3$



Example

- Find the chromatic polynomial P_G for the given graph and use P_G to find chromatic number $\chi(G)$ of G.



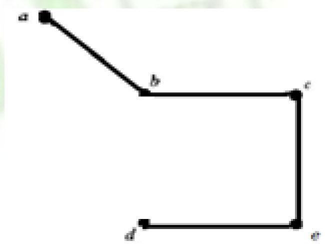
$x(x-1)^3$

$x(x-1)^4 - [x\,x(x-1)(x-2) - x(x-1)(x-1)]$.

式

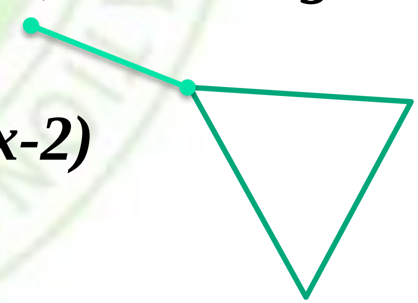
- *G 可以分为bd边和一个链图{a,b,c,e,d}, 令 k={b,d}.*
- *G_k : subgraph of G obtained by deleting k,*

  $$P_{Gk}(x) = x(x-1)^4$$



- *G^k : the quotient graph of G obtained by merging the end points of k. G^k is K_3 and ab edge has two components, one being a single point and the other being K_3.*

  $$P_G^k(x) = x^2(x-1)(x-2) - x(x-1)(x-2) = x(x-1)^2(x-2)$$



$= x(x-1)^4 - x(x-1)^2(x-2)$

$P_G(2) = 2.$

$P_G(x) = P_{G_k}(x) - P_{G^k}(x)$

$= x(x-1)^4 - x(x-1)^2(x-2) = x(x-1)^2((x-1)^2-(x-2))$

$= x(x-1)^2(x^2-2x+1-x+2) = x(x-1)^2(x^2-3x+3)$

$P_G(1) = 0$, $P_G(2)=2$, so $\chi(G) = 2$.

卑了...

HOMEWORK

- § 10.8
  - 10, 16, 20, 23, 34