Advanced Counting Techniques

Zhang Yanmei

yanmeizhang@bupt.edu.cn

College of Computer Science & Technology

Beijing University of Posts & Telecommunications

8 Advanced Counting Techniques

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8.1 Recurrence Relations (递推关系)

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yanmeizhang@bupt.edu.cn

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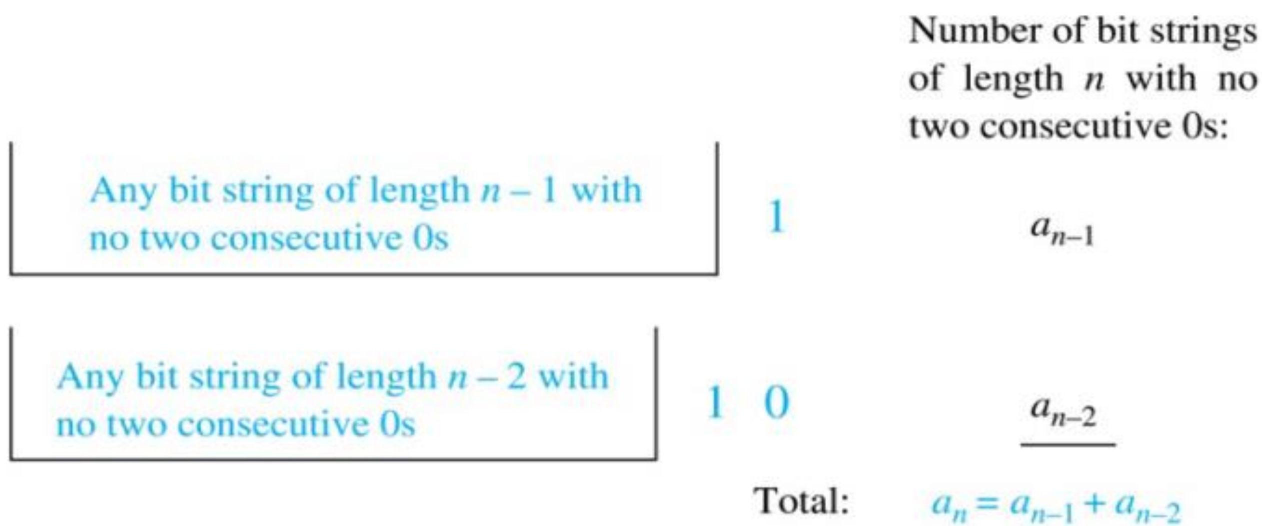
Bit Strings

- Find a recurrence relation and give initial conditions for the number of bit strings of length n that do <u>not have two consecutive 0s</u>. How many such bit strings are there of length five?



Recurrence Relations

- A *recurrence relation* (R.R., or just *recurrence*) for a sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more previous elements $a_0, \ldots, a_{n-1}$ of the sequence, for all $n \geq n_0$.
  - A recursive definition, without the base cases.
- A particular sequence (described non-recursively) is said to *solve* the given recurrence relation if it is consistent with the definition of the recurrence.
  - A given recurrence relation may have many solutions.

Recurrence Relation Example

- Consider the recurrence relation

$$a_n = 2a_{n-1} - a_{n-2} \ (n \geq 2).$$

- Which of the following are solutions?

$a_n = 3n$ Yes

$a_n = 2^n$ No

$a_n = 5$ Yes

- 2*3(n-1)-3(n-2)=6n-6-3n+6=3n.

Remember

- To define a sequence recursively, a recursive formula must be accompanied by information about the beginning of the sequence.
- This information is called the initial condition (初始条件） or conditions for the sequence.

Solving Tower of Hanoi RR

$H_n = 2\,H_{n-1} + 1$

$= 2\,(2\,H_{n-2} + 1) + 1 \qquad = 2^2 H_{n-2} + 2 + 1$

$= 2^2(2\,H_{n-3} + 1) + 2 + 1 \qquad = 2^3 H_{n-3} + 2^2 + 2 + 1$

…

$= 2^{n-1} H_1 + 2^{n-2} + \ldots + 2 + 1$

$= 2^{n-1} + 2^{n-2} + \ldots + 2 + 1$ (since $H_1 = 1$)

$= \sum_{i=0}^{n-1} 2^i$

$= 2^n - 1$ (From $\sum_{j=0}^{n} ar^j = \frac{ar^{n+1} - a}{r - 1}$, where a=1, r=2)

Example Applications

- Recurrence relation for growth of a bank account with P% interest per given period:

$$M_n = M_{n-1} + (P/100)M_{n-1}$$

- Growth of a population in which each organism yields 1 new one every period starting 2 periods after its birth.

$$P_n = P_{n-1} + P_{n-2} \quad \text{(Fibonacci relation)}$$

Solving Compound Interest RR

- $M_n = M_{n-1} + (P/100)M_{n-1}$
  $= (1 + P/100)\ M_{n-1}$
  $= r\ M_{n-1}$ (let $r = 1 + P/100$)
  $= r\ (r\ M_{n-2})$
  $= r \cdot r \cdot (r\ M_{n-3})$ …and so on to…
  $= r^n\ M_0$

F(n) 第n月兔子对数

- F（0）＝0
- F（1）＝1
- F（2）＝1
- F（3）＝1＋1＝2
- F（4）＝2＋1＝3（三月数＋二月后代）
- F（5）＝3＋2＝5……
- F（13）＝144＋89＝233

F(0)=0

F(1)=1

F(n)=F(n-1)+F(n-2)

费波那契递推关系

Homework

- § 8.1
  - 14

Divide-and-Conquer Recurrence Relations

- Suppose that a recursive algorithm divides a problem of size n into a subproblems.
- Assume each subproblem is of size n/b.
- Suppose $g(n)$ extra operations are needed in the conquer step.
- Then $f(n)$ represents the number of operations to solve a problem of size n satisisfies the following recurrence relation:

$$f(n) = af(n/b) + g(n)$$

- This is called a *divide-and-conquer recurrence relation*.

Examples

- **Binary search:** Break list into 1 sub-problem (smaller list) (so a=1) of size $\leq\lceil n/2\rceil$ (so b=2).
  - So $T(n) = T(\lceil n/2\rceil)+c$ ($g(n)=c$ constant)
- **Merge sort:** Break list of length n into 2 sublists (a=2), each of size $\leq\lceil n/2\rceil$ (so b=2), then merge them, in $g(n) = \Theta(n)$ time.
  - So $T(n) = 2T(\lceil n/2\rceil) + n$

Solving Divide-and-Conquer Recurrence Relations

$$f(n) = af(n/b) + g(n)$$
$$= a^2 f(n/b^2) + ag(n/b) + g(n)$$
$$= a^3 f(n/b^3) + a^2 g(n/b^2) + ag(n/b) + g(n)$$
$$\ldots$$
$$= a^{\log_b n} f(1) + \sum_{i=0}^{\log_b n - 1} a^i g(n/b^i)$$

Theorem 1

Let f be an increasing function that satisfies the recurrence relation $f(n) = af(n/b) + c$

whenever n is divisible by b, where $a \geq 1$, b is an integer greater than 1, and c is a positive real number. Then $f(n)$ is $\begin{cases} O(n^{\log_b a}) & \text{if} \quad a > 1 \\ O(\log n) & \text{if} \quad a = 1. \end{cases}$

Furthermore, when $n = b^k$ and $a \neq 1$, where k is a positive integer, $f(n) = C_1 n^{\log_b a} + C_2$

where $C_1 = f(1) + c/(a-1)$ and $C_2 = -c/(a-1)$.

proof (1)

- $f(n) = a^{\log_b n} f(1) + \sum_{j=0}^{\log_b n - 1} a^j g(n/b^j)$
- let $g(n)=c$,

$f(n) = a^{\log_b n} f(1) + \sum_{j=0}^{\log_b n - 1} a^i c$

when $a=1$, $f(n)=f(1)+c\log_b n$.

Therefore, $f(n)$ *is* $O(\log_b n)$ when $a=1$.

proof (2)

when $a>1$,

$$f(n) = a^{\log_b n} f(1) + c\sum_{j=0}^{\log_b n - 1} a^j$$

$$= a^{\log_b n} f(1) + c(a^{\log_b n} - 1)/(a-1)$$

$$= a^{\log_b n} (f(1) + c/(a-1)) - c/(a-1)$$

$$= C_1 n^{\log_b a} + C_2 \text{ because } a^{\log_b n} = n^{\log_b a}$$

Therefore, $f(n)$ is $O(n^{\log_b a})$ when $a>1$.

Example 7

Give a big-O estimate for the number of comparisons used by a binary search.

Solution: Since the number of comparisons used by binary search is $f(n) = f(n/2) + c$ where n is even, by Theorem 1, it follows that $f(n)$ is $O(\log_2 n)$.

The Master Theorem

Consider a function $f(n)$ that, for all $n=b^k$ for all $k \in \mathbf{Z}^+$, satisfies the recurrence relation:

$$f(n) = af(n/b) + cn^d$$

with $a \geq 1$, integer $b>1$, real $c>0$, $d \geq 0$. Then:

$$f(n) \in \begin{cases} O(n^d) & \text{if } a < b^d \\ O(n^d \log n) & \text{if } a = b^d \\ O(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

Proof: $f(n) = af(n/b) + cn^d$

- $f(n)=a^{\log_b n} f(1) + \sum_{j=0}^{\log_b n-1} a^j g(n/b^j)$
- let $g(n)=cn^d$, $k=\log_b n$,

$$f(n)=a^k f(1) + c\sum_{j=0}^{k-1} a^j \left(\frac{n}{b^j}\right)^d$$

$$=a^k f(1) + c\sum_{j=0}^{k-1} n^d \left(\frac{a}{b^d}\right)^j$$

$$=a^k f(1) + cn^d \sum_{j=0}^{k-1} \left(\frac{a}{b^d}\right)^j$$

Proof(2)

- If $a = b^d$, then $\log_b a = d$,

$$f(n)=a^k f(1)+ c\sum_{j=0}^{k-1} n^d$$

$$= a^k f(1)+kcn^d$$

$$= n^{\log_b a} f(1)+ c(\log_b n)n^d$$

$$=n^d f(1)+c\, n^d \log_b n.$$

Hence, f(n) is $O(n^d \log n)$.

Proof(3)

- If $a \neq b^d$,

$$f(n) = a^k f(1) + c\, n^d \sum_{j=0}^{k-1} \left(\frac{a}{b^d}\right)^j$$

$$= a^k f(1) + cn^d \frac{(a/b^d)^k - 1}{(a/b^d) - 1} \qquad (\, b^{dk} = b^{\log_b n^d} = n^d .)$$

$$= a^k f(1) + cn^d \frac{b^d (a^k/n^d - 1)}{a - b^d}$$

$$= a^k f(1) + \frac{b^d c}{a - b^d} a^k - \frac{b^d c}{a - b^d} n^d$$

Hence, $f(n) = C_1 n^d + C_2\, n^{\log_b a}$.

Proof(4)

- If $a < b^d$, then $log_b\, a < d$,

  so $O(C_1 n^d + C_2\, n^{log_b a}) \Rightarrow O(n^d)$.

- If $a > b^d$, then $log_b\, a > d$,

  so $O(C_1 n^d + C_2\, n^{log_b a}) \Rightarrow O(n^{log_b a})$.

Example: Mergesort

- $T(n) = 2*T(n/2)+n$
  - $a=2,b=2,d=1;$
  - As $a=b^d$, $T(n)=O(n\log_2 n)$
  - Check：
  - $2\,T(n/2) + n = 2\,(n/2\,\log_2(n/2)) + n$

    $= n\log_2(n/2) + n$

    $= n\log_2(n) - n\log_2 2 + n$

    $= n\log_2(n) - n + n$

    $= n\log_2(n)$

Homework

- § 8.3
  - 14, 28

8.2: Solving Recurrences

- A *linear homogeneous recurrence of degree k with constant coefficients* ("*k*-LiHoReCoCo",k阶定常系数线性齐次递推关系) is a recurrence of the form

  $$a_n = c_1 a_{n-1} + \ldots + c_k a_{n-k},$$

  where the c_i, $(i = 1, .., k)$ are all real, and $c_k \neq 0$.
- Note
  - The solution is uniquely determined if k initial conditions $a_0 \ldots a_{k-1}$ are provided.

Examples

- $P_n = (1.11)P_{n-1}$ linear homogeneous recurrence relation of degree one
- $f_n = f_{n-1} + f_{n-2}$ linear homogeneous recurrence relation of degree two
- $a_n = a_{n-1} + a_{n-2}^2$ not linear
- $H_n = 2H_{n-1} + 1$ not homogeneous
- $B_n = nB_{n-1}$ coefficients are not constants

Solving LiHoReCoCos

- Basic idea: Look for solutions of the form

$a_n = r^n$, where r is a constant.

- This requires the *characteristic equation*:

$$r^n = c_1 r^{n-1} + \ldots + c_k r^{n-k},\ i.e.,$$

$$r^k - c_1 r^{k-1} - \ldots - c_{k-1} r - c_k = 0$$

- The solutions (*characteristic roots*) can yield an explicit formula for the sequence.

Solving 2-LiHoReCoCos

- Consider an arbitrary 2-LiHoReCoCo:

  $$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

- It has the characteristic equation (C.E.):

  $$r^2 - c_1 r - c_2 = 0$$

- **Theroem 1:** If this CE has 2 roots $r_1 \neq r_2$, then

  $$\boldsymbol{a_n = \alpha_1 r_1^{\,n} + \alpha_2 r_2^{\,n}} \text{ for } n \geq 0$$

  for some constants α_1, α_2.

Proof(1)

- Suppose that r_1 and r_2 are roots of $x^2 - c_1 x - c_2 = 0$, so $r_1^2 - c_1 r_1 - c_2 = 0$, $r_2^2 - c_1 r_2 - c_2 = 0$ and then $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$, for $n \geq 1$.
- We show that this definition of a_n defines the same sequence as $a_n = c_1 a_{n-1} + c_2 a_{n-2}$
- First we note that α_1 and α_2 are chosen so that $a_1 = \alpha_1 r_1 + \alpha_2 r_2$ and $a_2 = \alpha_1 r_1^2 + \alpha_2 r_2^2$ and so the initial conditions are satisfied. Then

Proof(2)

$$
\begin{aligned}
a_n &= \alpha_1 r_1^n + \alpha_2 r_2^n \\
&= \alpha_1 r_1^{n-2} r_1^2 + \alpha_2 r_2^{n-2} r_2^2 \\
&= \alpha_1 r_1^{n-2} (c_1 r_1 + c_2) + \alpha_2 r_2^{n-2} (c_1 r_2 + c_2) \\
&= c_1 \alpha_1 r_1^{n-1} + c_2 \alpha_1 r_1^{n-2} + c_1 \alpha_2 r_2^{n-1} + c_2 \alpha_2 r_2^{n-2} \\
&= c_1 (\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}) + c_2 (\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2}) \\
&= c_1 a_{n-1} + c_2 a_{n-2}
\end{aligned}
$$

Proof(3)

$a_0 = C_0 = \alpha_1 + \alpha_2, \qquad a_1 = C_1 = \alpha_1 r_1 + \alpha_2 r_2.$

$\alpha_2 = C_0 - \alpha_1.$

$C_1 = \alpha_1 r_1 + (C_0 - \alpha_1) r_2.$

Hence, $C_1 = \alpha_1 (r_1 - r_2) + C_0 r_2.$

$$\alpha_1 = \frac{C_1 - C_0 r_2}{r_1 - r_2}$$

$$\alpha_2 = C_0 - \alpha_1 = C_0 - \frac{C_1 - C_0 r_2}{r_1 - r_2} = \frac{C_0 r_1 - C_1}{r_1 - r_2}$$

Example 3

- Solve the recurrence $a_n = a_{n-1} + 2a_{n-2}$ given the initial conditions $a_0 = 2$, $a_1 = 7$.
- Solution: Use theorem 1
  - $c_1 = 1$, $c_2 = 2$
  - Characteristic equation:
    $r^2 - r - 2 = 0$
  - Solutions: $r = [-(-1) \pm ((-1)^2 - 4 \cdot 1 \cdot (-2))^{1/2}] / 2 \cdot 1$
    $= (1 \pm 9^{1/2})/2 = (1 \pm 3)/2$, so $r = 2$ or $r = -1$.
  - So $a_n = \alpha_1 2^n + \alpha_2 (-1)^n$.

Example 3 Continued…

- To find α_1 and α_2, solve the equations for the initial conditions a_0 and a_1:

$$a_0 = 2 = \alpha_1 2^0 + \alpha_2 (-1)^0$$
$$a_1 = 7 = \alpha_1 2^1 + \alpha_2 (-1)^1$$

  Simplifying, we have the pair of equations:

$$2 = \alpha_1 + \alpha_2$$
$$7 = 2\alpha_1 - \alpha_2$$

  which we can solve easily by substitution:

$$\alpha_2 = 2 - \alpha_1; \quad 7 = 2\alpha_1 - (2 - \alpha_1) = 3\alpha_1 - 2;$$
$$9 = 3\alpha_1; \ \ \alpha_1 = 3; \quad \alpha_2 = -1.$$

- Final answer: $a_n = 3 \cdot 2^n - (-1)^n$

Check: $\{a_{n \geq 0}\} = 2, 7, 11, 25, 47, 97 \ldots$

Example 4

- The Fibonacci sequence is defined by a linear homogeneous recurrence relation of degree 2, so by Theorem l, the roots of the associated equation are needed to describe the explicit formula for the sequence.
- From $f_n=f_{n-1}+f_{n-2}$ and $f_1=f_2=1$, we have $x^2 - x - 1 = 0$.
- Using the quadratic formula to obtain the roots.

$$r_1=\frac{1+\sqrt{5}}{2} \qquad r_2=\frac{1-\sqrt{5}}{2}$$

Fibonacci Numbers (*continued*)

Therefore by Theorem 1

$$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

for some constants α_1 and α_2.

Using the initial conditions $f_0 = 0$ and $f_1 = 1$, we have

$$f_0 = \alpha_1 + \alpha_2 = 0$$

$$f_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1$$

Solving, we obtain $\alpha_1 = \frac{1}{\sqrt{5}}$, $\alpha_2 = -\frac{1}{\sqrt{5}}$

Hence,

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$$

The Case of Degenerate Roots

- Now, what if the C.E. $r^2 - c_1 r - c_2 = 0$ has only 1 root r_0?
- **Theorem 2:** Then,

  $$\boldsymbol{a_n = \alpha_1 r_0^{\,n} + \alpha_2 n r_0^{\,n}}, \quad \text{for all } n \geq 0,$$

  for some constants α_1, α_2.

Theroem 2 proof(1)

- Suppose that r_0 is only root of $x^2 - c_1 x - c_2 = 0$, so $r_0^2 - c_1 r_0 - c_2 = 0$ and then $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$, for $n \geq 1$.
- $\alpha_1 r_0^n + \alpha_2 n r_0^n = \alpha_1 r_0^2 r_0^{n-2} + \alpha_2 n r_0^2 r_0^{n-2}$

$$= \alpha_1 (c_1 r_0 + c_2) r_0^{n-2} + \alpha_2 n (c_1 r_0 + c_2) r_0^{n-2}$$

$$= \alpha_1 c_1 r_0^{n-1} + \alpha_1 c_2 r_0^{n-2} + \alpha_2 n c_1 r_0^{n-1} + \alpha_2 n c_2 r_0^{n-2}$$

$$= c_1(\alpha_1 r_0^{n-1} + \alpha_2 n r_0^{n-1}) + c_2(\alpha_1 r_0^{n-2} + \alpha_2 n r_0^{n-2})$$

$$= c_1 a_{n-1} + c_2 a_{n-2}$$

Theroem 2 proof(2)

- $a_0 = C_0 = \alpha_1 + \alpha_2 n, \quad a_1 = C_1 = \alpha_1 r_0 + \alpha_2 n r_0.$
- $\alpha_2 = (C_0 - \alpha_1)/n$.
- $C_1 = \alpha_1 r_0 + (C_0 - \alpha_1) r_0 / n.$
- $Hence, C_1 = (n\text{-}1)\alpha_1 r_0 / n + C_0 r_0 / n.$
- $\alpha_1 = (nC_1 - C_0 r_0) / (n\text{-}1) r_0.$
- $\alpha_2 = (n(n\text{-}2)C_0 r_0 - n^2 C_1) / (n\text{-}1) r_0.$

Example 5

- Find solution of recurrence relation

  $a_n = 6a_{n-1} - 9a_{n-2}$ with initial condition $a_0=1$, $a_1=6$

  The only root of $r^2 - 6r + 9 = 0$ is $r = 3$.

  Hence $a_n = \alpha_1 3^n + \alpha_2 n3^n$

  From initial conditions, $a_0=1=\alpha_1$,

  $a_1=6=\alpha_1*3+\alpha_2*3$

  $\alpha_1 = 1$ and $\alpha_2 = 1$

  Therefore, $a_n = 3^n + n3^n$

k-LiHoReCoCos

- Consider a *k*-LiHoReCoCo:
- It's C.E. is: $$r^k - \sum_{i=1}^{k} c_i r^{k-i} = 0$$
- **Thm.3:** If this has *k* distinct roots r_i, then the solutions to the recurrence $a_n = \sum_{i=1}^{k} c_i a_{n-i}$ are of the form:

$$a_n = \sum_{i=1}^{k} \alpha_i r_i^n$$

for all $n \geq 0$, where the α_i are constants.

Example 6

- Find the solution to the recurrence relation $a_n=6a_{n-1}-11a_{n-2}+6a_{n-3}$ with the initial condition $a_0=2, a_1=5$, and $a_2=15$
- $r^3 - 6r^2 + 11r - 6 = 0 \Rightarrow r = 1, r = 2$, and $r = 3$
- $a_n = \alpha_1 \cdot 1^n + \alpha_2 \cdot 2^n + \alpha_3 \cdot 3^n$.
- $a_0 = 2 = \alpha_1 + \alpha_2 + \alpha_3$, $a_1 = 5 = \alpha_1 1 + \alpha_2 \cdot 2 + \alpha_3 \cdot 3$,

$a_2 = 15 = \alpha_1 + \alpha_2 \cdot 4 + \alpha_3 \cdot 9$.

- $\alpha_1 = 1$, $\alpha_2 = -1$, and $\alpha_3 = 2$. Hence, the unique solution to this recurrence relation and the given initial conditions is the sequence $\{an\}$ with $an = 1 - 2^n + 2 \cdot 3^n$.

Example of complex roots

38. a) Find the characteristic roots of the linear homogeneous recurrence relation

$$a_n = 2a_{n-1} - 2a_{n-2}.$$

[Note: These are complex numbers.]

b) Find the solution of the recurrence relation in part (a) with $a_0 = 1$ and $a_1 = 2$.

solution:

a) The characteristic equation is $r^2 - 2r + 2 = 0$, whose roots are, by the quadratic formula, $1 \pm \sqrt{-1}$, in other words, $1 + i$ and $1 - i$.

Solution:

b) The general solution is, by part (a),

$a_n = \alpha 1(1 + i)^n + \alpha 2(1 - i)^n$.

Plugging in the initial conditions gives us $1 = \alpha 1 + \alpha 2$ and $2 = (1 + i)\alpha 1 + (1 - i)\alpha 2$.

Solving these linear equations tells us that

$\alpha 1 = 1/2 - 1/2*i$ and $\alpha 2 = 1/2 + 1/2*i$.

Therefore the solution is

$a_n = (1/2 - 1/2*i)(1 + i)^n + (1/2 + 1/2*i)(1 - i)^n$.

- 39. a) Find the characteristic roots of the linear homogeneous recurrence relation $a_n = a_{n-4}$. [Note: These include complex numbers.]
- b) Find the solution of the recurrence relation in part (a) with $a_0 = 1$, $a_1 = 0$, $a_2 = -1$, and $a_3 = 1$
- solution:
- a) 1, −1, i, −i
- b) $a_n = 1/4 - 1/4 *(-1)^n + (2+i)/4 * i^n + (2-i)/4* (-i)^n$

Degenerate *k*-LiHoReCoCos

- Suppose there are t roots $r_1,\ldots,r_t$ with multiplicities $m_1,\ldots,m_t$. Then: $$a_n = \sum_{i=1}^{t}\left(\sum_{j=0}^{m_i-1}\alpha_{i,j}n^j\right)r_i^n$$

  for all $n\geq 0$, where all the α are constants.

$$\begin{aligned} a_n = \ & (\alpha_{1,0} + \alpha_{1,1}n + \cdots + \alpha_{1,m_1-1}n^{m_1-1})r_1^n \\ & + (\alpha_{2,0} + \alpha_{2,1}n + \cdots + \alpha_{2,m_2-1}n^{m_2-1})r_2^n \\ & + \cdots + (\alpha_{t,0} + \alpha_{t,1}n + \cdots + \alpha_{t,m_t-1}n^{m_t-1})r_t^n \end{aligned}$$

Example 8

- Find the solution to the recurrence relation

$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with the initial condition $a_0 = 1, a_1 = -2$, and $a_2 = -1$.

- Solution: $r^3 + 3r^2 + 3r + 1 = 0$. => $r = -1$
- $a_n = \alpha_{1,0}(-1)^n + \alpha_{1,1}n(-1)^n + \alpha_{1,2}n^2(-1)^n$.
- $a_0 = 1 = \alpha_{1,0}$, $a_1 = -2 = -\alpha_{1,0} - \alpha_{1,1} - \alpha_{1,2}$,

$a_2 = -1 = \alpha_{1,0} + 2\alpha_{1,1} + 4\alpha_{1,2}$.

- $\alpha_{1,0} = 1$, $\alpha_{1,1} = 3$, and $\alpha_{1,2} = -2$.
- $a_n = (1 + 3n - 2n^2)(-1)^n$.

Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

Definition: A *linear nonhomogeneous recurrence relation with constant coefficients* is a recurrence relation of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots.. + c_k\, a_{n-k} + F(n)\,,$$

where c_1, c_2, ….,c_k are real numbers, and $F(n)$ is a function not identically zero depending only on n.

The recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots.. + c_k\, a_{n-k}\,,$$

is called the associated homogeneous recurrence relation.

Linear Nonhomogeneous Recurrence Relations with Constant Coefficients (*cont.*)

The following are linear nonhomogeneous recurrence relations with constant coefficients:

$a_n = a_{n-1} + 2^n$,

$a_n = a_{n-1} + a_{n-2} + n^2 + n + 1,$

$a_n = 3a_{n-1} + n3^n,$

$a_n = a_{n-1} + a_{n-2} + a_{n-3} + n!$

where the following are the associated linear homogeneous recurrence relations, respectively:

$a_n = a_{n-1}$,

$a_n = a_{n-1} + a_{n-2},$

$a_n = 3a_{n-1},$

$a_n = a_{n-1} + a_{n-2} + a_{n-3}$

Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

Theorem 5: If $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}.$$

Proof: Because $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous recurrence relation, we know that

$$a_n^{(p)} = c_1 a_{n-1}^{(p)} + c_2 a_{n-2}^{(p)} + \cdots + c_k a_{n-k}^{(p)} + F(n).$$

Now suppose that $\{b_n\}$ is a second solution of the nonhomogeneous recurrence relation, so that

$$b_n = c_1 b_{n-1} + c_2 b_{n-2} + \cdots + c_k b_{n-k} + F(n).$$

Subtracting the first of these two equations from the second shows that

$$b_n - a_n^{(p)} = c_1(b_{n-1} - a_{n-1}^{(p)}) + c_2(b_{n-2} - a_{n-2}^{(p)}) + \cdots + c_k(b_{n-k} - a_{n-k}^{(p)}).$$

It follows that $\{b_n - a_n^p\}$ is a solution of the associated homogeneous linear recurrence, say, $\{a_n^{(h)}\}$. Consequently, $b_n = a_n^{(p)} + a_n^{(h)}$ for all n. ◁

Example 10

- Find all solutions to $a_n = 3a_{n-1}+2n$. Which solution has $a_1 = 3$?
  - Its associated 1-LiHoReCoCo is $a_n = 3a_{n-1}$, whose solutions are all of the form $a_n^{(h)} = \alpha 3^n$.
  - Thus the solutions to the original problem are all of the form $a_n = a_n^{(p)} + \alpha 3^n$. So, all we need to do is find one $a_n^{(p)}$ that works.

Trial Solutions

- If the extra terms $F(n)$ are a degree-t polynomial in n, you should try a degree-t polynomial as the particular solution $a_n^{(p)}$.
- This case: $F(n)$ is linear so try $a_n^{(p)} = cn + d$.

$cn+d = 3(c(n-1)+d) + 2n$ (for all n)

$(-2c-2)n + (3c-2d) = 0$ (collect terms)

then 2+2c=0 and 3c-2d=0.

So $c = -1$ and $d = -3/2$.

So $a_n^{(p)} = -n - 3/2$ is a solution.

Check: $3\ a_{n-1}^{(p)} + 2n = 3(-(n-1)-3/2)+2n = \ldots = -n-3/2$.

Finding a Desired Solution

- From the previous, we know that all general solutions to our example are of the form:

  $$a_n = -n - 3/2 + \alpha 3^n.$$

  Solve this for α for the given case, $a_1 = 3$:

  $$3 = -1 - 3/2 + \alpha 3^1$$

  $$\alpha = 11/6$$

- The answer is $a_n = -n - 3/2 + (11/6)3^n$.

Homework

- § 8.2
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