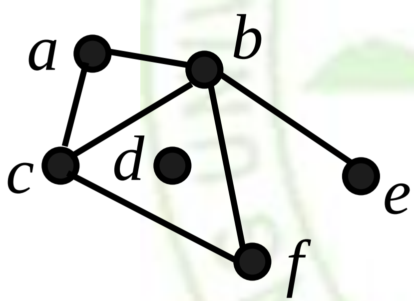
§ 10.3: Graph Representations & Isomorphism

- Graph representations:
  - Adjacency lists.
  - Adjacency matrices.
  - Incidence matrices(undirected graph).
- Graph isomorphism:
  - Two graphs are isomorphic iff they are identical except for their node names.

Adjacency Lists邻接表

- A table with 1 row per vertex, listing its adjacent vertices.



Vertex	*Adjacent Vertices*
a	*b*, *c*
b	*a*, *c*, *e*, *f*
c	*a*, *b*, *f*
d	
e	*b*
f	*c*, *b*

adjacency list
邻接表

Directed Adjacency Lists

- 1 row per node, listing the terminal nodes of each edge incident from that node.
- P669

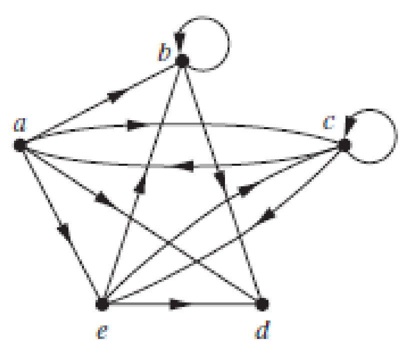


FIGURE 2 A Directed Graph.

TABLE 2 An Adjacency List for a Directed Graph.

Initial Vertex	*Terminal Vertices*
a	*b*, *c*, *d*, *e*
b	*b*, *d*
c	*a*, *c*, *e*
d	
e	*b*, *c*, *d*

Adjacency Matrices邻接矩阵

adjacency matrices邻接矩阵

- A way to represent simple graphs
  - possibly with self-loops.
- Matrix **A**=[a_{ij}], where a_{ij} is 1 if {v_i, v_j} is an edge of *G*, and is 0 otherwise.
- Can extend to pseudographs by letting each matrix elements be the number of links (possibly >1) between the nodes.

Note

- An adjacency matrix of a graph is based on the ordering chosen for the vertices.
- The adjacency matrix of a simple graph is symmetric.
- $a_{ii}=0$
- Sparse matrix(稀疏矩阵)

Example 3

- Use an adjacency matrix to represent the graph shown in Figure 3.

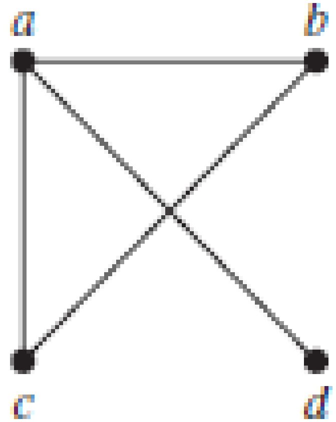


FIGURE 3
Simple Graph.

Example 4

Draw a graph with the adjacency matrix

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

with respect to the ordering of vertices *a, b, c, d.*

Example 5

- Use an adjacency matrix to represent the pseudograph shown in Figure 5.

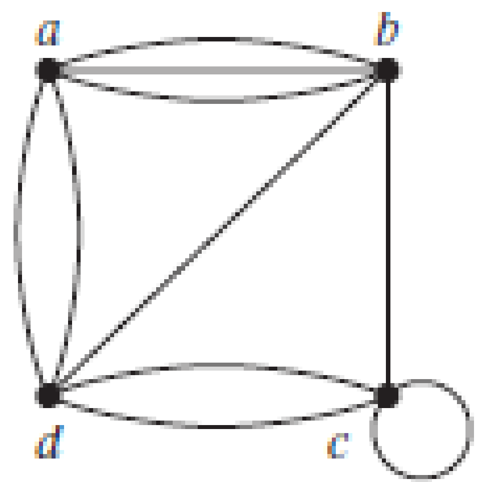


FIGURE 5
A Pseudograph.

有向图邻接矩阵

- Adjacency matrix for a directed graph

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \text{ is an edge of } G, \\ 0 & \text{otherwise.} \end{cases}$$

- Adjacency matrix for a directed multigraph

Incidence matrices关联矩阵

- Let $G=(V,E)$ be an undirected graph. Suppose that $v_1, v_2, \ldots, v_n$ are the vertices and $e_1, e_2, \ldots, e_m$ are the edges of G. Then the incidence matrix with respect to this ordering of V and E is the $n*m$ matrix $M=[m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 \text{ when edge } e_j \text{ is incident with } v_i \\ 0 \text{ otherwise} \end{cases}$$

Example 6

- Represent the graph shown in Figure 6 with an incidence matrix.

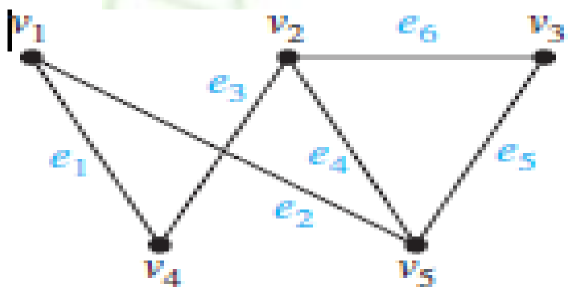


FIGURE 6 An Undirected Graph.

$$
\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array}
\begin{array}{c}
\begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0
\end{bmatrix}
\end{array}.
$$

Example 7

- Represent the pseudograph shown in Figure 7 using an incidence matrix.

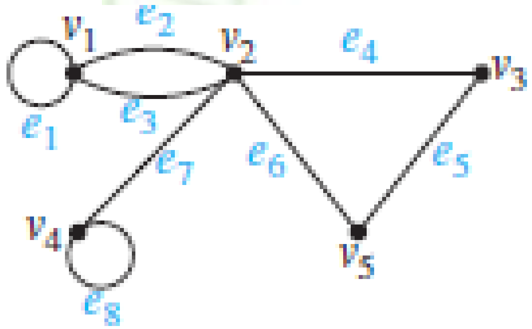
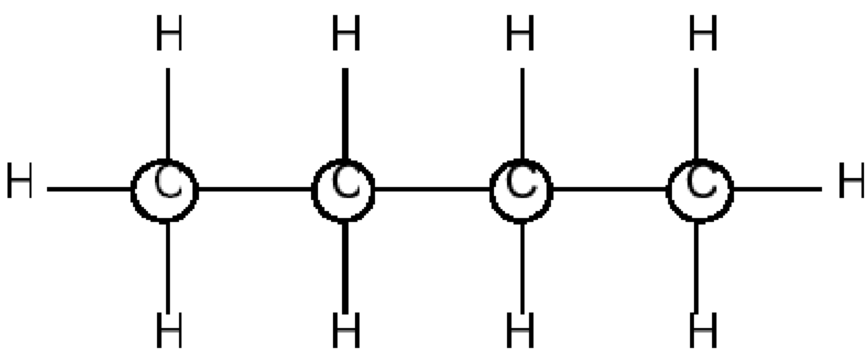


FIGURE 7
A Pseudograph.

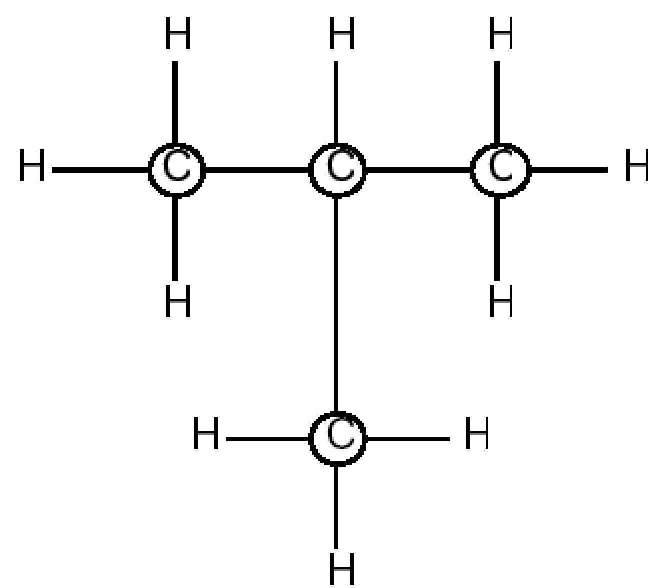
$$
\begin{array}{c}
 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0
\end{bmatrix}
\end{array}.
$$

Graph Isomorphism

- The Greek root "iso" means "same". The Greek root "morphism" means "form".
- Two molecules with the same chemical formula are called *isomers*.



butane



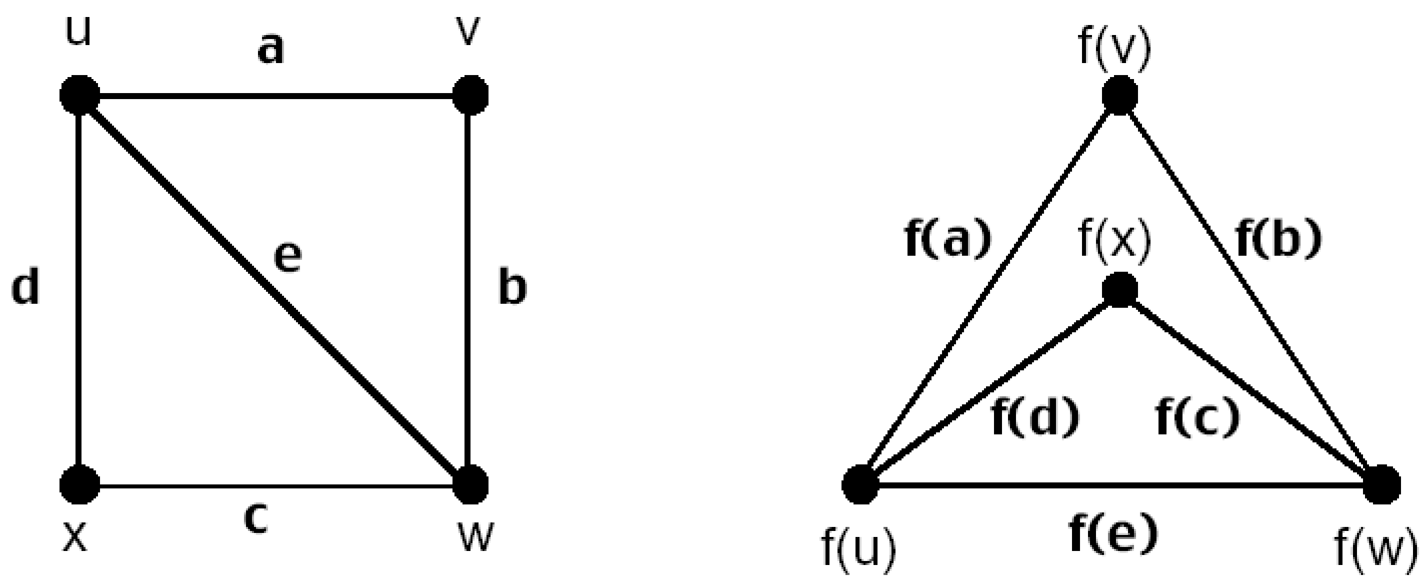
isobutane

Graph Isomorphism

- Formal definition:
  - Simple graphs $G_1=(V_1, E_1)$ and $G_2=(V_2, E_2)$ are *isomorphic* iff $\exists$ a bijection $f: V_1 \rightarrow V_2$ such that $\forall\ a,b \in V_1$, a and b are adjacent in G_1 iff $f(a)$ and $f(b)$ are adjacent in G_2.
  - f is the "renaming" function between the two node sets that makes the two graphs identical.
  - This definition can easily be extended to other types of graphs.

Example

- The graph mapping *f* is an isomorphism.



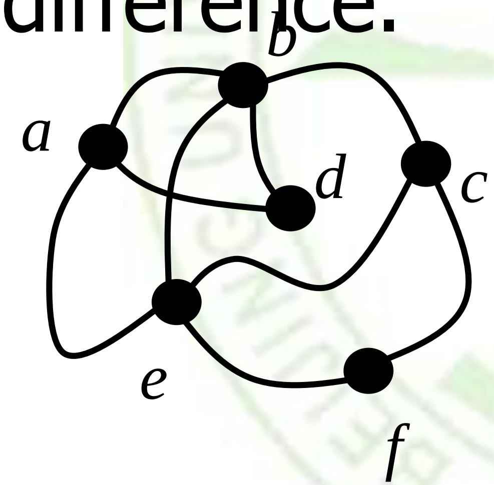
Graph Invariants under Isomorphism图形不变量

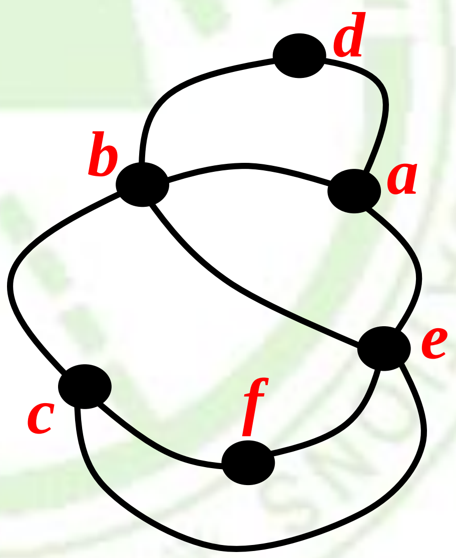
Necessary but not *sufficient* conditions for $G_1=(V_1, E_1)$ to be isomorphic to $G_2=(V_2, E_2)$:

- We must have that $|V1|=|V2|$, and $|E1|=|E2|$.
- The degree sequence is the same in both graphs.
- For every proper subgraph G_1' of G_1, there is a proper subgraph G_2' of G_2 that is isomorphic to G_1'.

Isomorphism Example

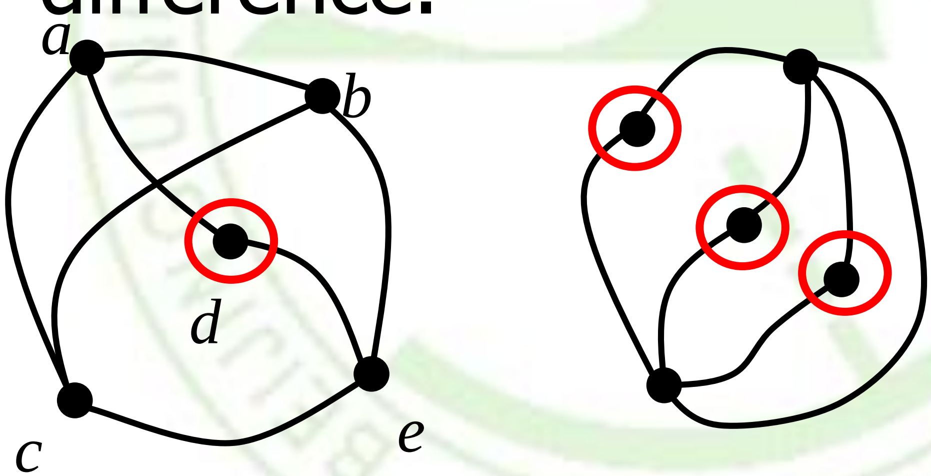
- If isomorphic, label the 2nd graph to show the isomorphism, else identify difference.





Are These Isomorphic?

- If isomorphic, label the 2nd graph to show the isomorphism, else identify difference.



- Same # of vertices
- Same # of edges
- Different # of verts of degree 2! (1 vs 3)

Example 10

- Determine whether the graphs shown in Figure 10 are isomorphic.

note:degree=3 subgraph

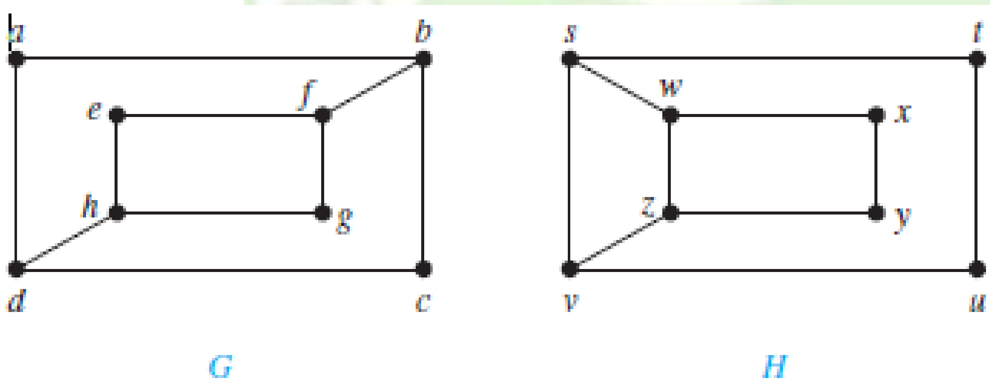
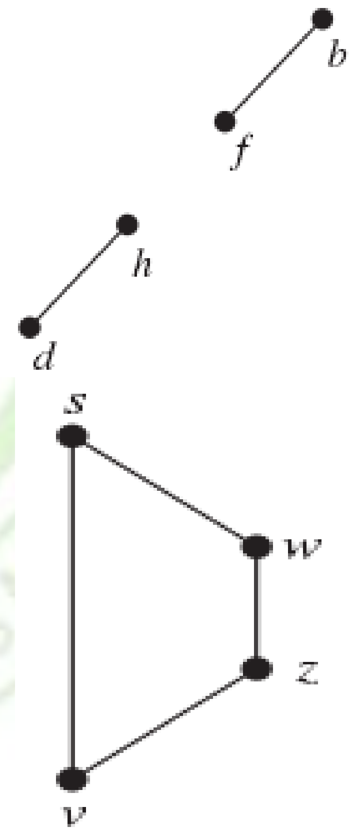


FIGURE 10 The Graphs G and H.



Example 11

- Determine whether the graphs *G* and *H* displayed in Figure 12 are isomorphic.

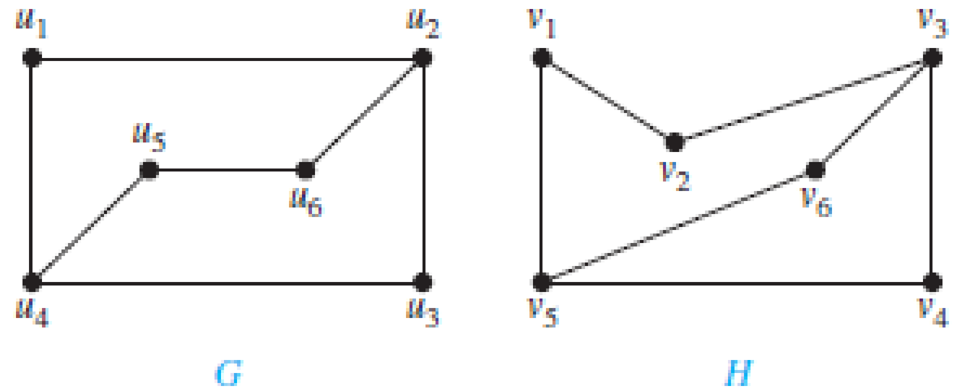
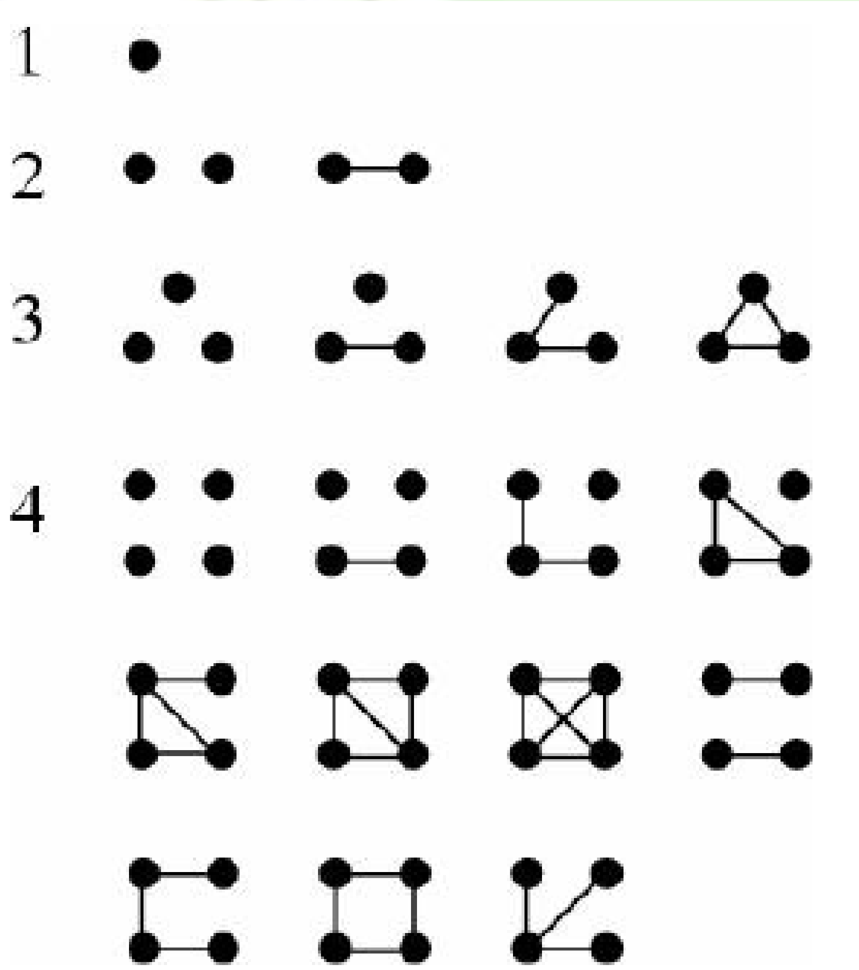


FIGURE 12 Graphs G and H.

Nonisomorphic Graph for n=1,2,3,4



Homework

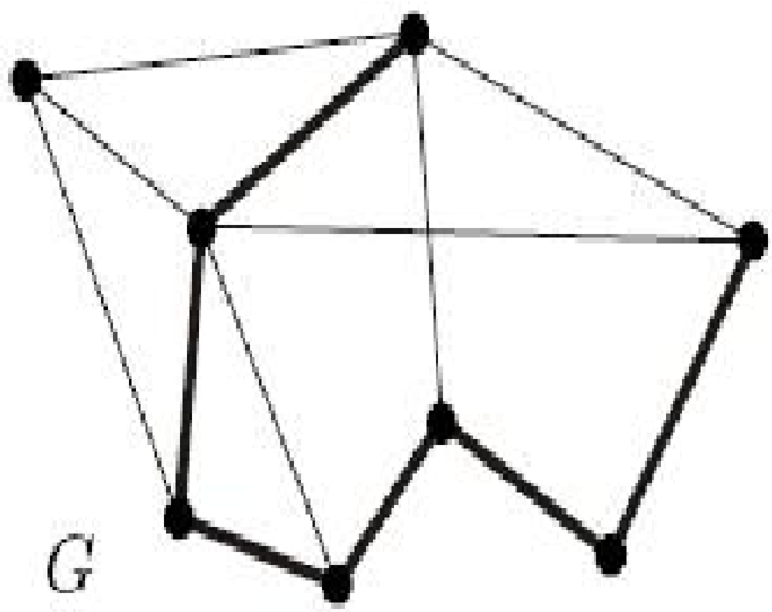
- § 10.3
  - 28, 46, 52, 60, 68

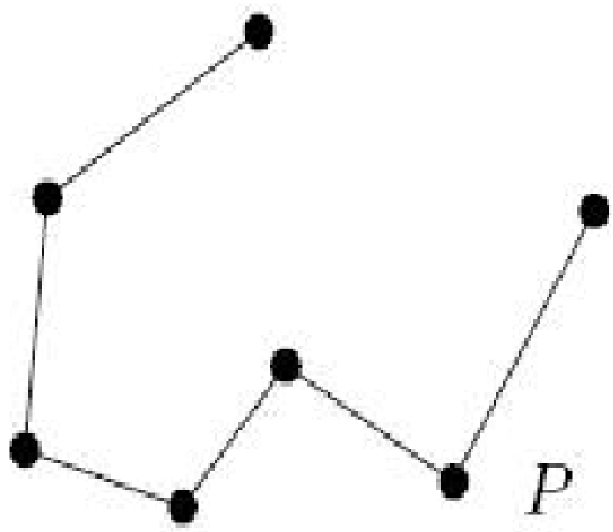
§ 10.4: Connectivity连通性

- Defination:(Paths in Relation)
- Let R be a relation on a set A.
- A path of length *n* in R from a to b is a finite sequence π*: a, $x_1, x_2, \ldots, x_{n-1}$, b*, such that $aRx_1, x_1Rx_2, \ldots, x_{n-1}Rb$.
- A path that begins and ends at the same vertex is called a circle.

§ 10.4: Connectivity连通性

- Defination:(Paths in Graphe Theory)
- In an undirected graph $G=(V, E)$, a *path of length n from* $u(=v_0)$ *to* $v\ (=v_n)$
- is a sequence of adjacent edges $e_1, e_2, \ldots e_n$, going from vertex u to vertex v, *such that* $e_i=\{v_{i-1}, v_i\}$ *for* $1\le i\le n$.
- is a sequence of vertex $v_0, v_1, \ldots, v_n$ such that $\{v_i, v_{i+1}\}\in E$ for $0\le i\le n-1$.


G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§ 10.4: Connectivity连通性

- A path is a *circuit* if *u=v*.回路
- A path *pass through* the vertices *or traverses* the edges. 途经、遍历
- A path is *simple* if it contains no edge more than once.简单通路
- Lemma: The shortest path connecting two vertices is simple.

Paths in Directed Graphs

- Same as in undirected graphs, but the path must go in the direction of the arrows.

Connectedness连通图

- An undirected graph is *connected* if there is a path between every pair of distinct vertices in the graph.连通图
- Example 4, p681

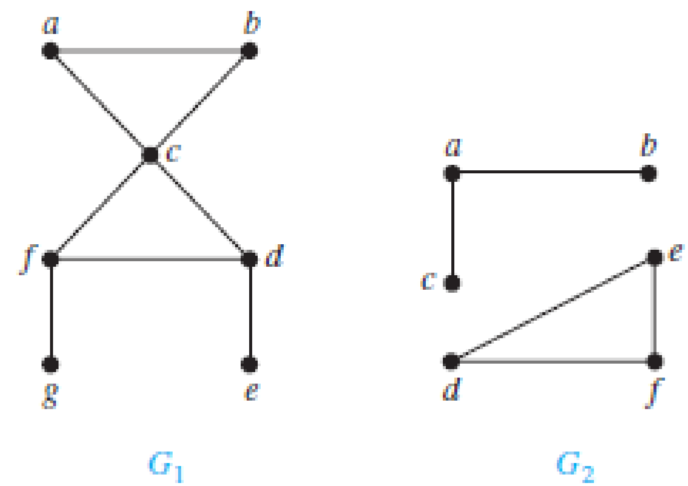
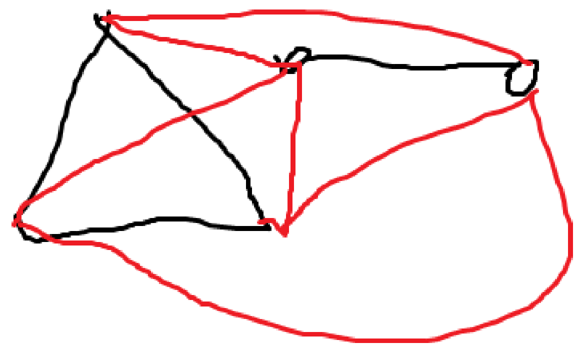


FIGURE 2 The Graphs G_1 and G_2.

- **Theorem:** There is a *simple* path between every pair of distinct vertices of a connected undirected graph.连通图中任意两点间存在简单通路

Theroem

- If G is disconnected, then its complement $\overline{G}$ is connected.



Connected component连通分支/独立子图

- A *Connected component* of a graph *G* is a connected subgraph of *G* that is not a proper subgraph of another connected subgraph of *G*.
- A graph *G* that is not connected has two or more connected components

that are disjoint and

have *G* as their union.

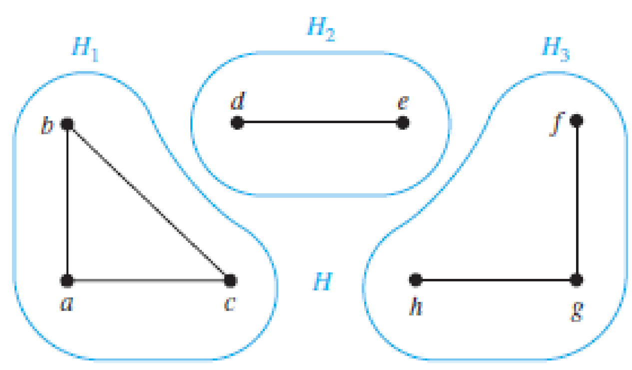
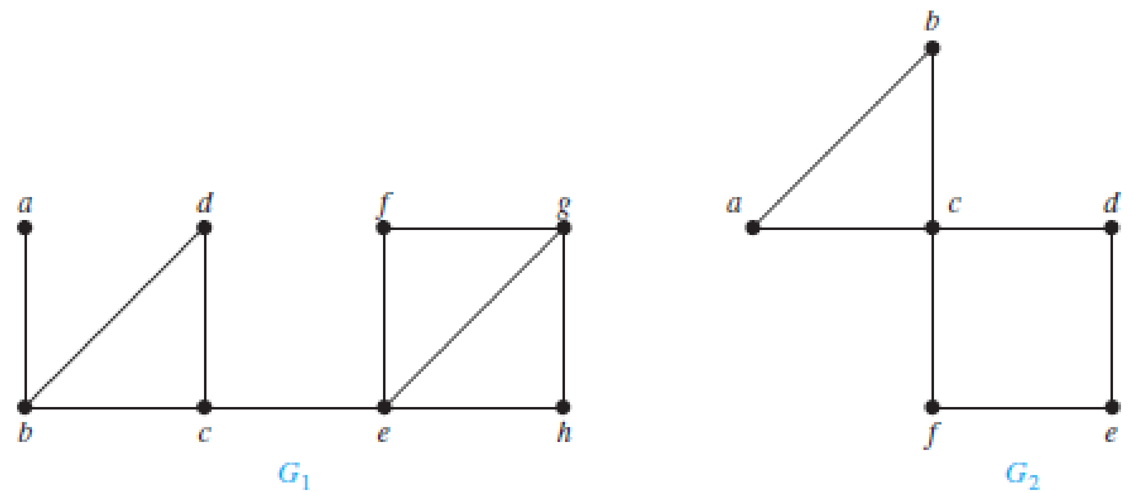


FIGURE 3 The Graph H and Its Connected Components H_1, H_2, and H_3.

割点、割边

- A *cut vertex* or *cut edge (bridge)* separates 1 connected component into 2 if removed.

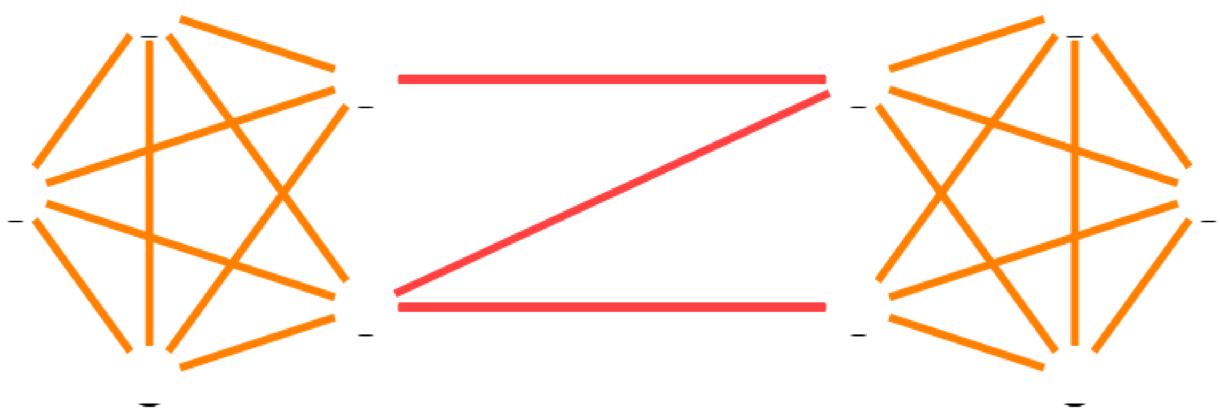


- Vertex Connectivity
  - Nonseparable graphs: without cut vertices.
  - Vertex cut: $G-V'$ is disconnected.
  - Vertex connectivity: $\kappa(G)=\min|V'|$
- Edge Connectivity
  - Edge cut: $G-E'$ is disconnected.
  - Edge connectivity: $\lambda(G)=\min|E'|$

Theroem

- Let $\kappa(G)$ be the vertex connectivity of a graph G, $\lambda(G)$ be the edge connectivity, and $\delta(G)$ be the minumum degree, then for every graph,

$$\kappa(G) \leq \lambda(G) \leq \delta(G)$$



Directed Connectedness

- A directed graph is *strongly connected* (强连通)if there is a path from *a* to *b* and from *b* to *a* whenever *a* and *b* are vertices in the graph.
- It is *weakly connected(弱连通)* if there is a path between every two vertices in the underlying *undirected* graph.
- Note *strongly* implies *weakly* but not vice-versa.

Example 11

- Strongly connected components or strong components

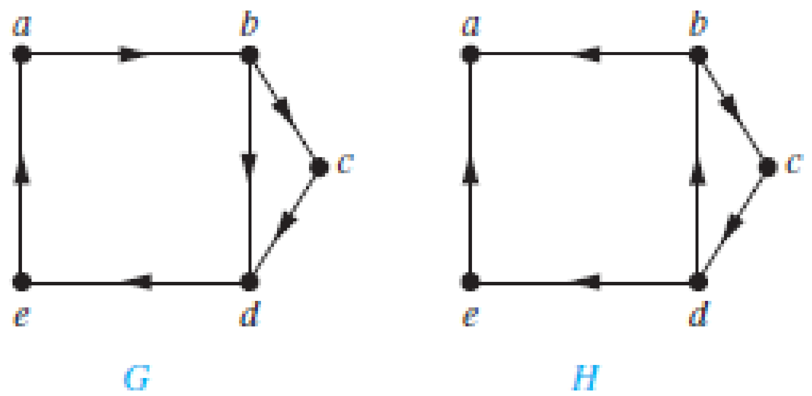


FIGURE 5 The Directed Graphs G and H.

Paths & Isomorphism

- Note that connectedness, and <u>the existence of a circuit or simple circuit of length *k* are graph invariants</u> with respect to isomorphism.
- Example 13

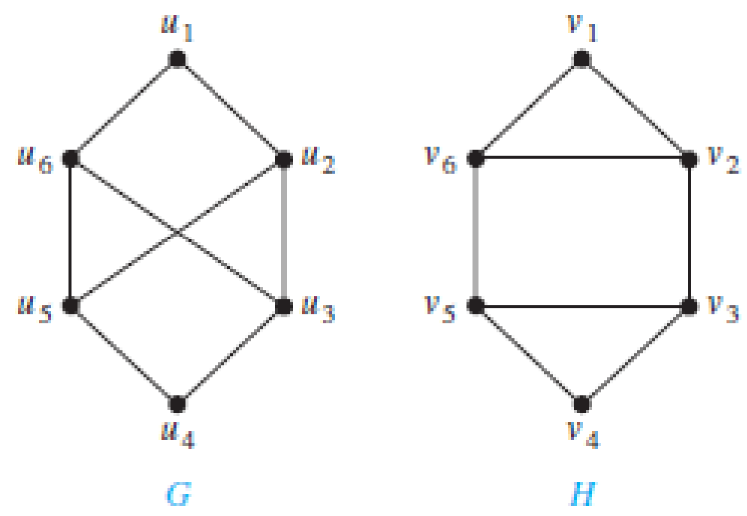


FIGURE 6 The Graphs G and H.

沿着相同的通路（同点数、途经各点同度数）

- Use paths to find mappings
- Example 14,P687

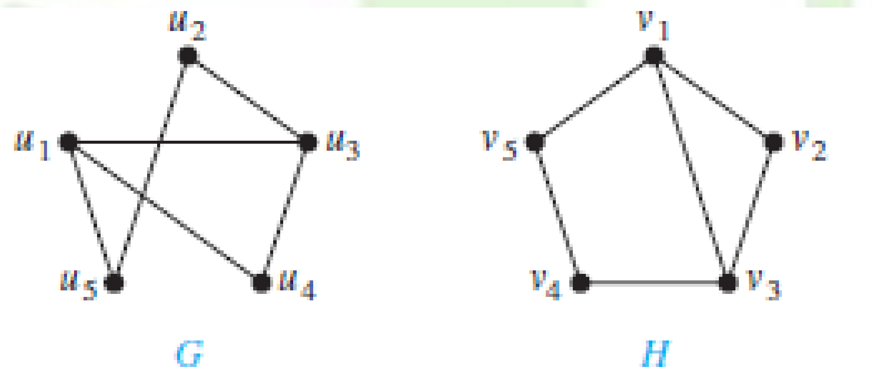


FIGURE 7 The Graphs G and H.

Counting Paths by Adjacency Matrices计算两点间通路数

- Let **A** be the adjacency matrix of graph G.
- The number of paths of length k from v_i to v_j is equal to $(\mathbf{A}^k)_{i,j}$.
  - The notation $(\mathbf{M})_{i,j}$ denotes $m_{i,j}$ where $[m_{i,j}] = \mathbf{M}$.
  - note: $b_{i1}a_{1j}+b_{i2}a_{2j}+\ldots+b_{in}a_{nj}$

Example 15

- How many paths of length four are there from *a* to *d* in the simple graph G in Figure 8?

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

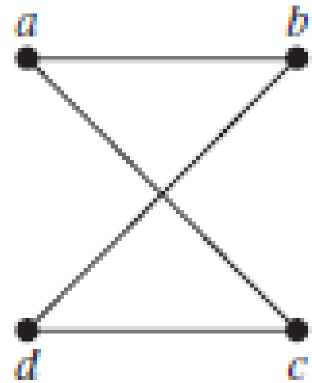


FIGURE 8
Graph *G*.

$$\mathbf{A}^4 = \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix},$$

Homework

- 10.4: 14,28,36,60