Euler Paths and Circuits （欧拉路径与欧拉回路）

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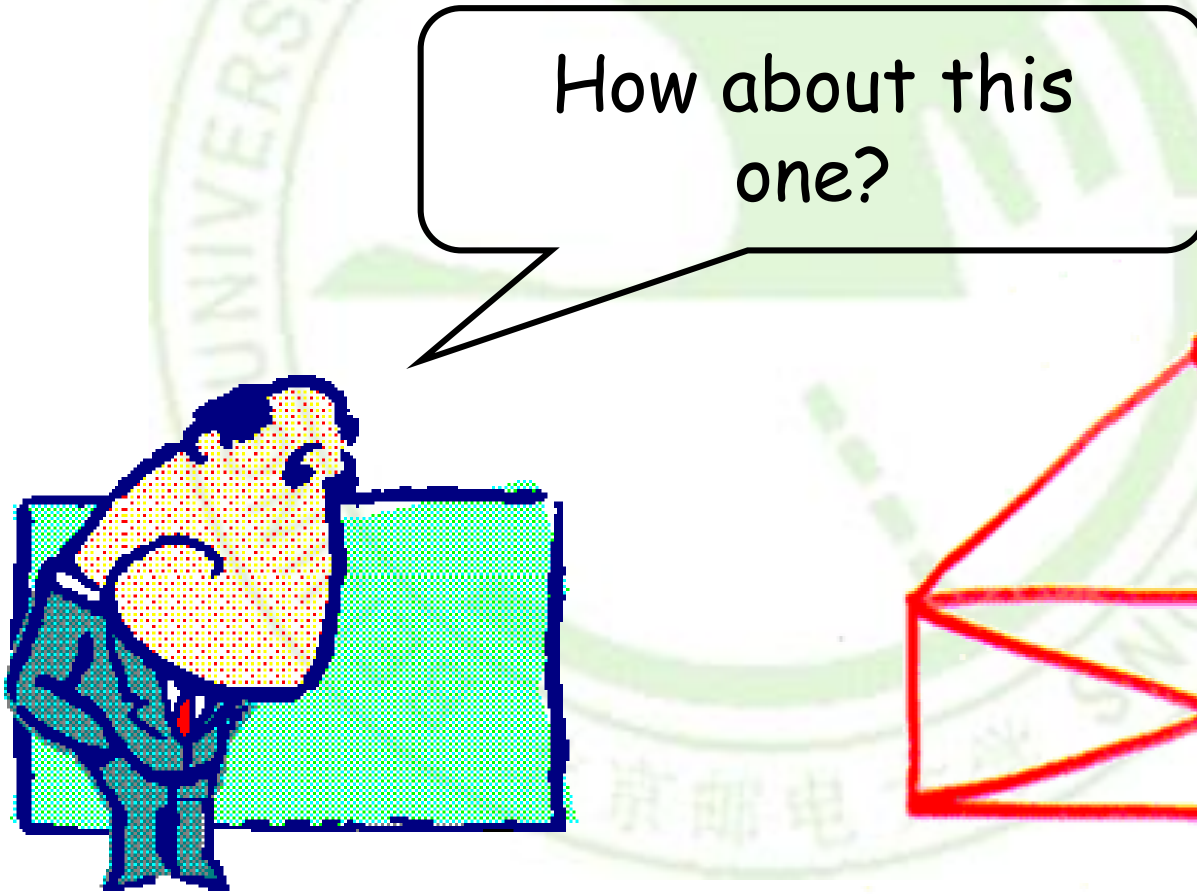
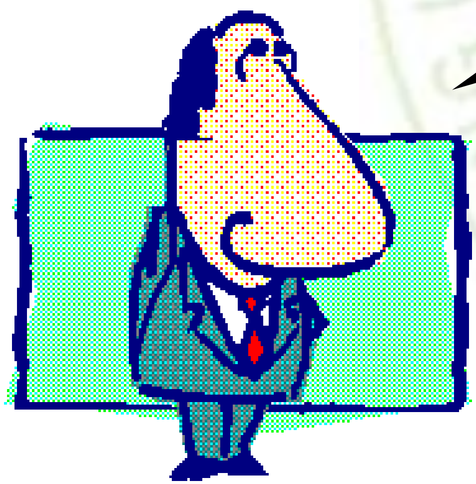
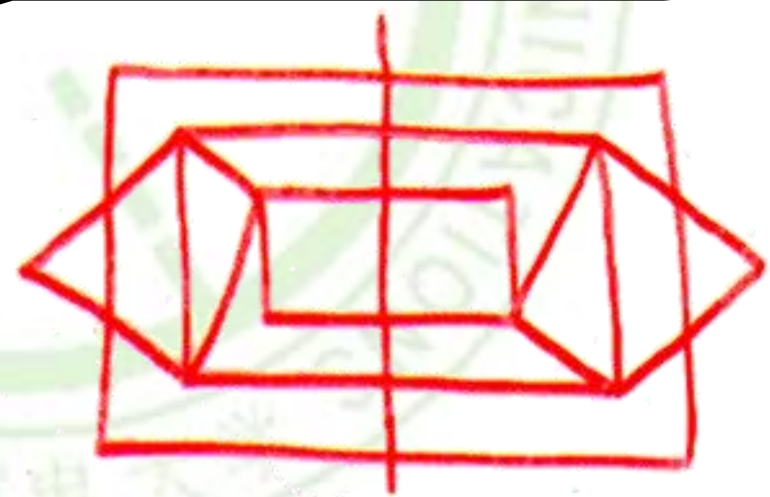
Problem



Problem

Can I draw the figure in one continuous trace with no line being drawn twice?





Eulerian graph（欧拉图）

- An ***Euler circuit*** in a graph G is a simple circuit containing every edge of G.
- An ***Euler path*** in G is a simple path containing every edge of G.
- A walk in a graph is called an ***Euler tour*** if it starts and ends in the same place and uses each edge exactly once. 欧拉 tour
- A walk in a graph is called an ***Euler trail*** if it uses each edge exactly once. 欧拉 trail
- If a graph has an Euler tour, it is said to be an ***Eulerian graph***.

Example 1

- Which of the undirected graphs in Figure 3 have an Euler circuit? Of those that do not, which have an Euler path?

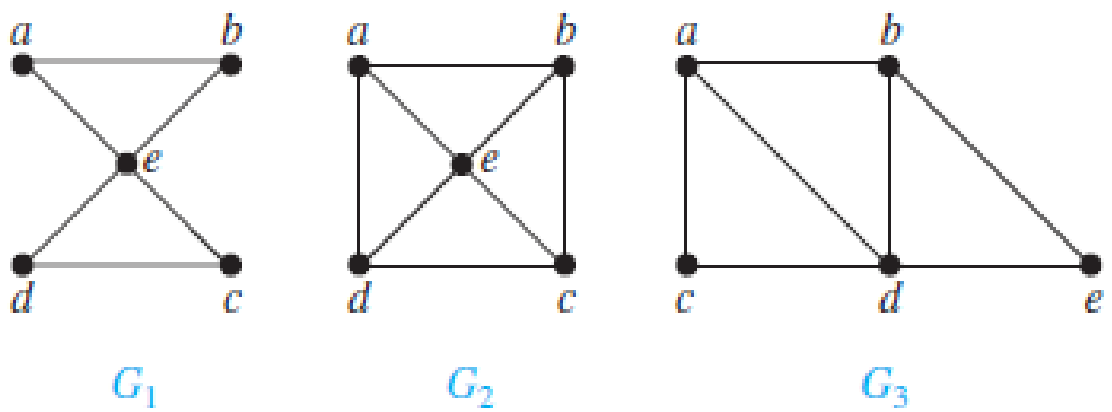


FIGURE 3 The Undirected Graphs G_1, G_2, and G_3.

Example 2

- Which of the directed graphs in Figure 4 have an Euler circuit? Of those that do not, which have an Euler path?

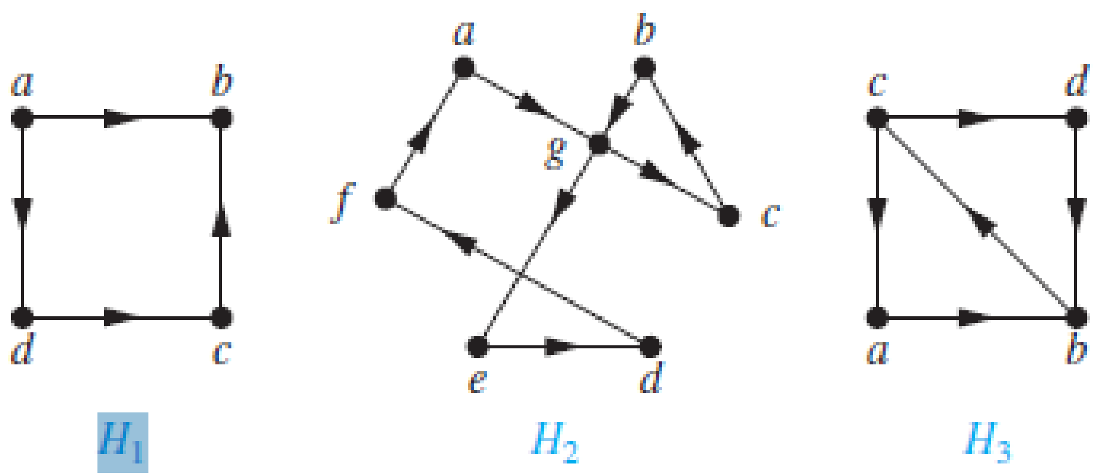


FIGURE 4 The Directed Graphs H_1, H_2, and H_3.

Theorem

- A connected simple graph G is Eulerian iff every graph vertex has even degree.
- A connected direct graph G is Eulerian iff every graph vertex has equal indegree and outdegree.

Theorem

- 1. A connected graph G is Eulerian iff G <u>has no vertices of odd degree</u>

连通多重图具有欧拉回路的充要条件是顶点度均为偶数

- 2. A connected graph G has an Euler trail from node a to some other node b iff <u>$a \neq b$ are the only two nodes of odd degree</u>

连通多重图具有欧拉通路的充要条件是仅有两个度为奇数的顶点

Proof of $\Downarrow$

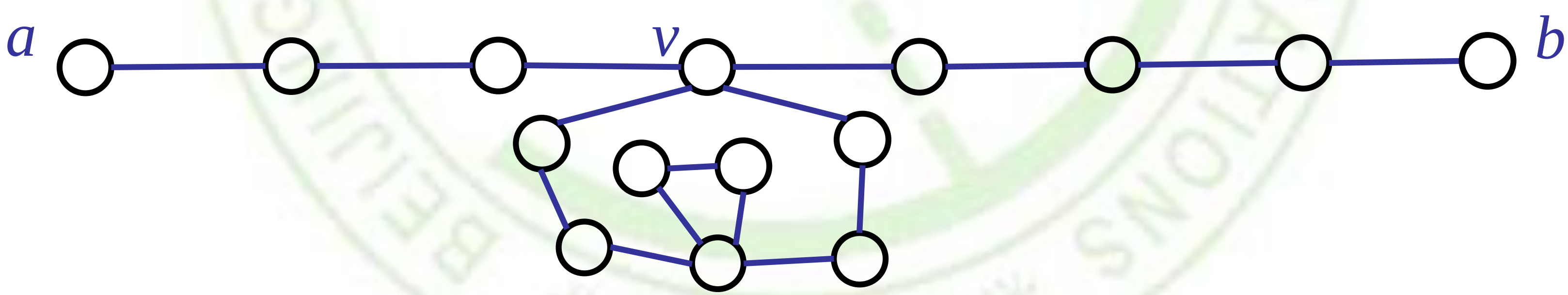
- Assume G has an Euler trail T from node a to node b (a and b not necessarily distinct).
- For every node besides a and b, T uses an edge to exit for each edge it uses to enter. Thus, the degree of the node is even.

1. If $a = b$, then a also has even degree.

2. If $a \neq b$, then a and b both have odd degree.

Proof of $\Uparrow$

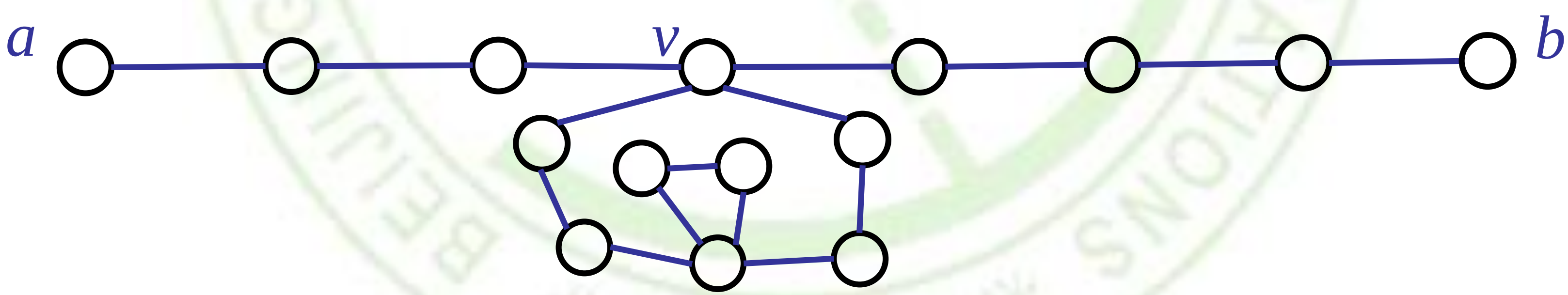
- Assume G is connected. If there are no odd-degree nodes, pick any $a = b$.
- If there are two odd-degree nodes, call these nodes a and b.
- Start at a. Take a walk w_1 until you get stuck. You must be at b.



Incorporate this walk from v into w_1.

Proof of ⇑

- If no vertex along w_1 has an unused edge, we are done.
- Otherwise, call this vertex v. Walk from v until you get stuck. You must be back at v.



Incorporate this walk from v into w_1.

ALGORITHM 1

procedure *Euler*(*G*: connected multigraph with all vertices of even degree)

circuit := a circuit in *G* beginning at an arbitrarily chosen vertex with edges successively added to form a path that returns to this vertex.

H := *G* with the edges of this circuit removed

while *H* has edges

subciruit := a circuit in *H* beginning at a vertex in *H* that also is an endpoint of an edge in circuit.

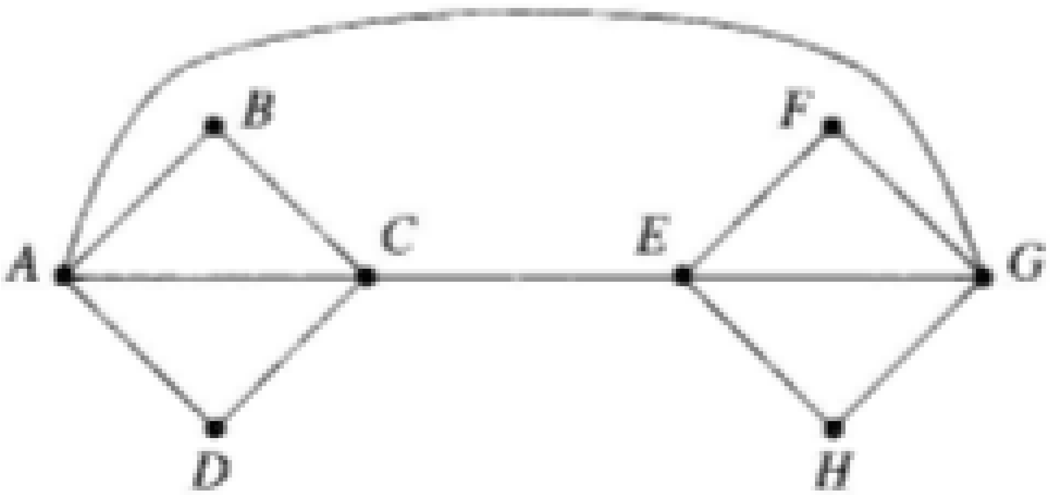
H := *H* with edges of *subciruit* and all isolated vertices removed

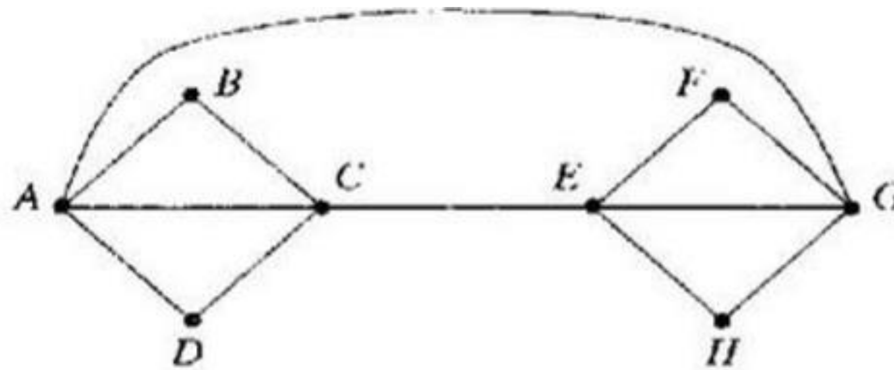
circuit := *circuit* with s*ubcircuit* inserted at the appropriate vertex.

return *circuit*{*circuit* is an Euler circuit}

Fleury's Algorithm

- Fleury's Algorithm for constructing a Eulerian tour.

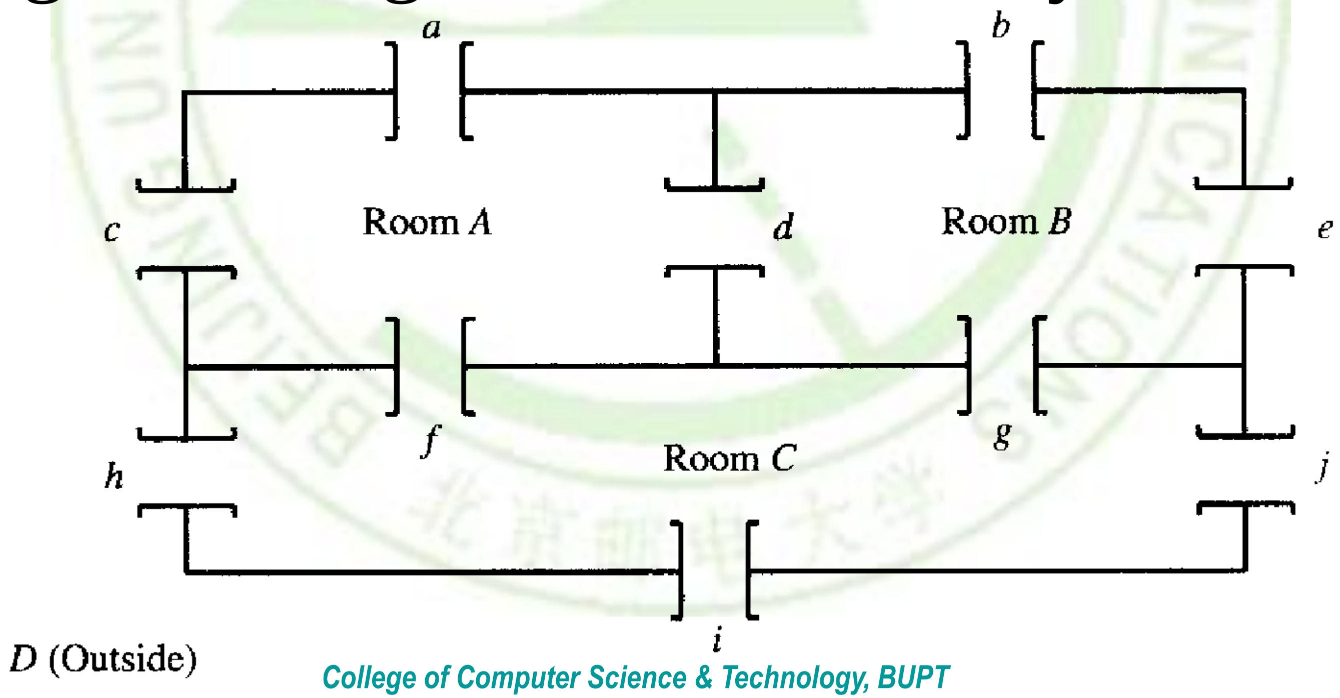


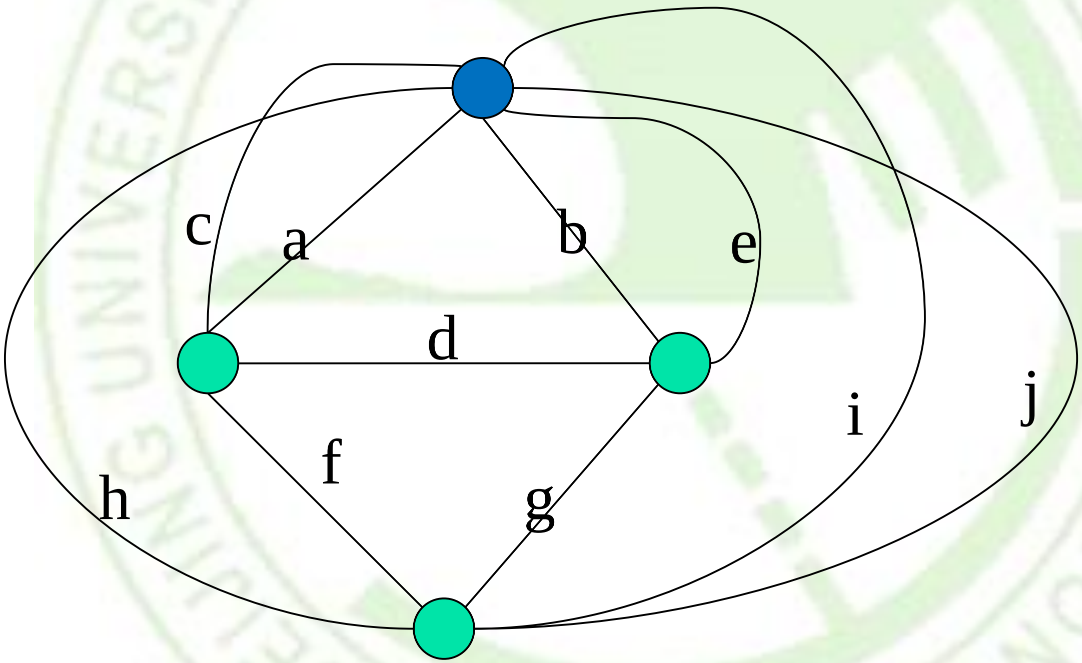


Current Path	Next Edge	Reasoning
$\pi: A$	$\{A, B\}$	No edge from A is a bridge. Choose any one.
$\pi: A, B$	$\{B, C\}$	Only one edge from B remains.
$\pi: A, B, C$	$\{C, A\}$	No edge from C is a bridge. Choose any one.
$\pi: A, B, C, A$	$\{A, D\}$	No edge from A is a bridge. Choose any one.
$\pi: A, B, C, A, D$	$\{D, C\}$	Only one edge from D remains.
$\pi: A, B, C, A, D, C$	$\{C, E\}$	Only one edge from C remains.
$\pi: A, B, C, A, D, C, E$	$\{E, G\}$	No edge from E is a bridge. Choose any one.
$\pi: A, B, C, A, D, C, E, G$	$\{G, F\}$	$\{A, G\}$ is a bridge. Choose $\{G, F\}$ or $\{G, H\}$.
$\pi: A, B, C, A, D, C, E, G, F$	$\{F, E\}$	Only one edge from F remains.
$\pi: A, B, C, A, D, C, E, G, F, E$	$\{E, H\}$	Only one edge from E remains.
$\pi: A, B, C, A, D, C, E, G, F, E, H$	$\{H, G\}$	Only one edge from H remains.
$\pi: A, B, C, A, D, C, E, G, F, E, H, G$	$\{G, A\}$	Only one edge from G remains.
$\pi: A, B, C, A, D, C, E, G, F, E, H, G, A$		

Example

- The problem is this: Is it possible to begin in a room or outside and take a walk that goes through each door exactly once?




c
a
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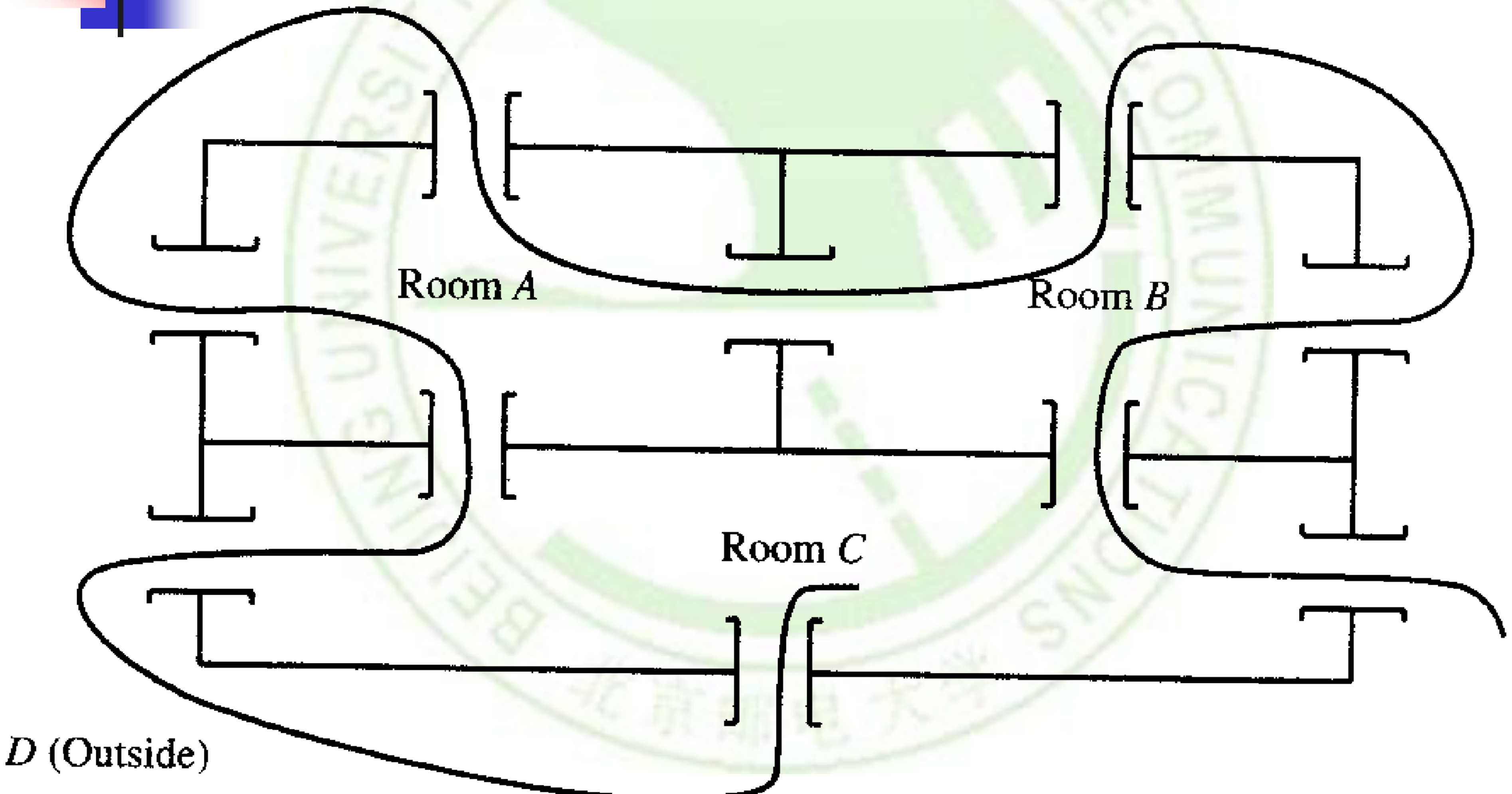
Example



Example 3

- Draw a picture in a continuous motion without lifting a pencil so that no part of the picture is retraced.

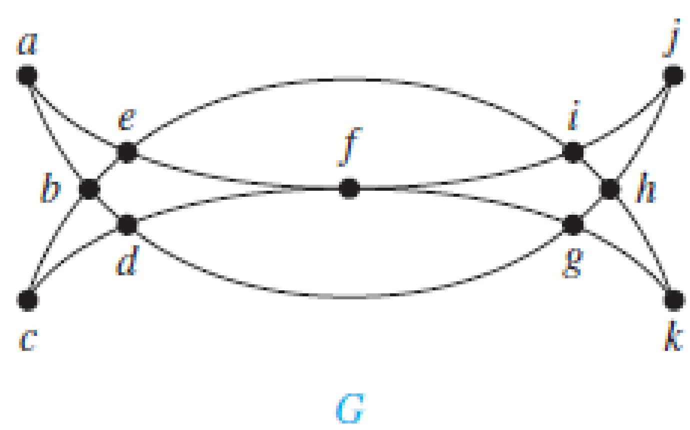


FIGURE 6 **Mohammed's Scimitars.**

Example 4

- Which graphs shown in Figure 7 have an Euler path?

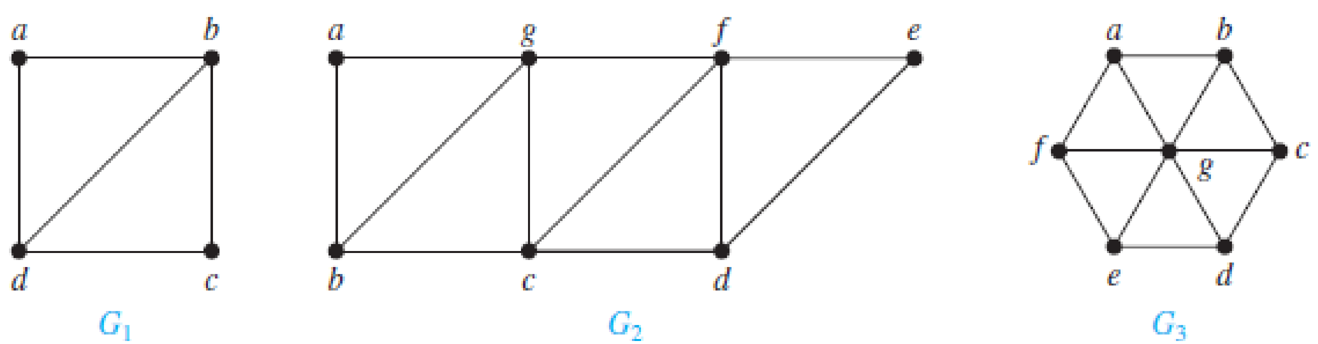


FIGURE 7 Three Undirected Graphs.

Hamilton Paths and Circuits（哈密顿路径与回路）

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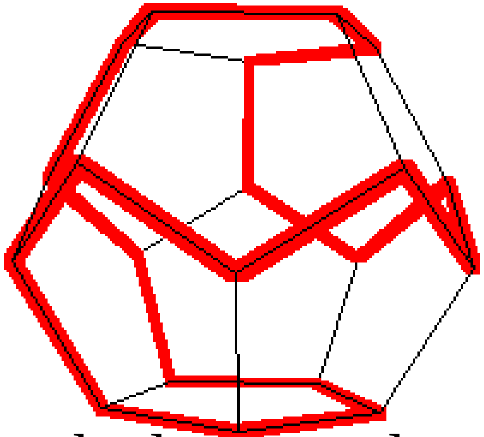
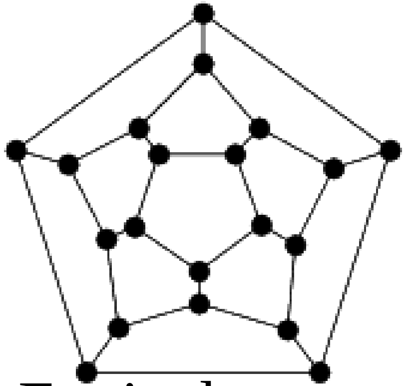
Round-the-World Puzzle

- Can we traverse all the vertices of a dodecahedron, visiting each once?`



Dodecahedron puzzle



Equivalent graph

Pegboard version

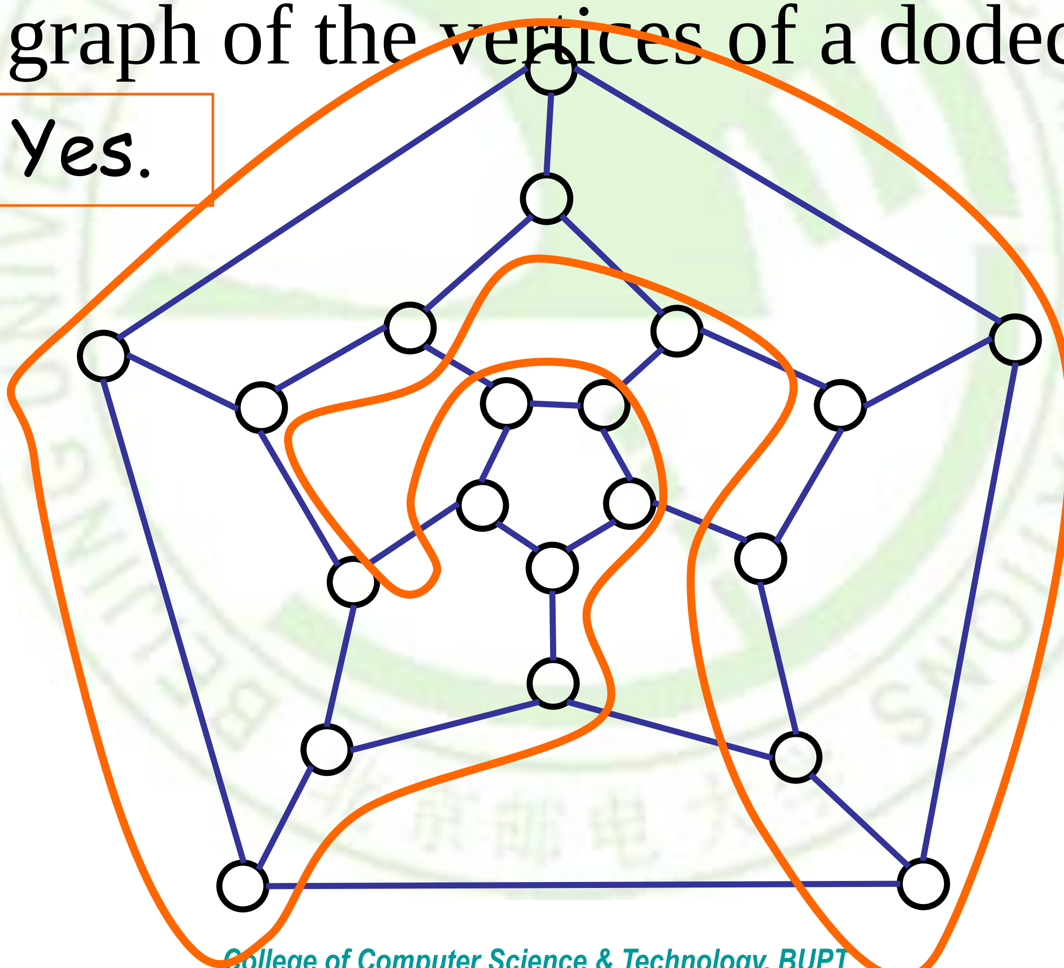
Hamiltonian Graph （哈密顿图）

- A graph has a *Hamiltonian tour* if there is a tour that visits every vertex exactly once (and returns to its starting point).
- A graph with a Hamiltonian tour is called a *Hamiltonian graph*.
- A *Hamiltonian path* is a path that contains each vertex exactly once.
- A ***Hamilton circuit*** is a circuit that traverses each vertex in *G* exactly once.
- A ***Hamilton path*** is a path that traverses each vertex in *G* exactly once

Is it Hamiltonian?

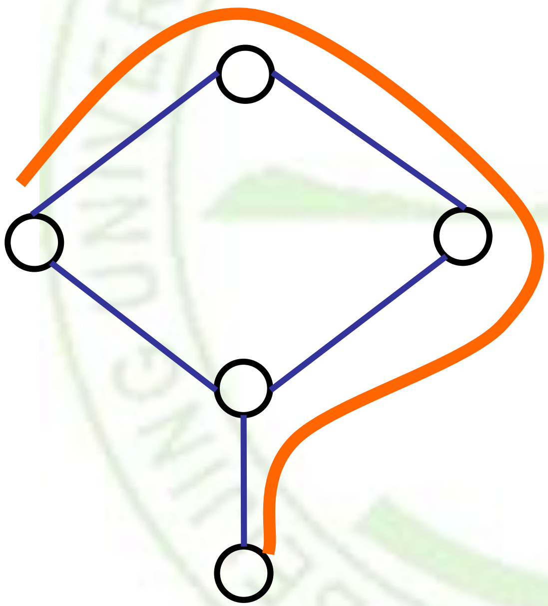
- A graph of the vertices of a dodecahedron.

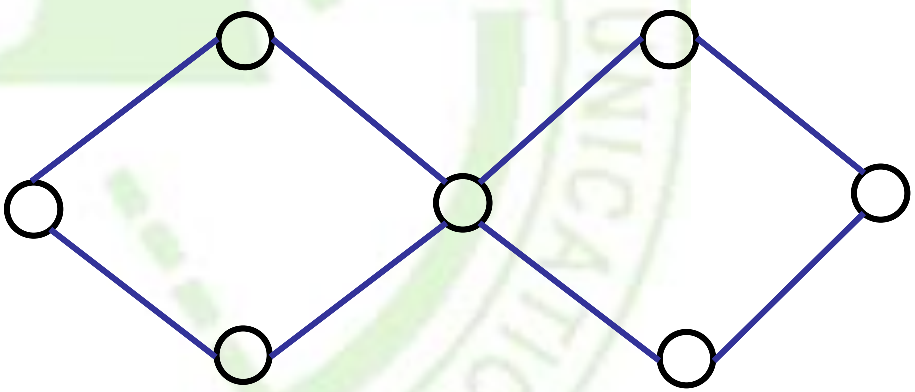
Yes.



Euler tour Hamiltonian tour

- Left one has a Hamiltonian path, but not a Hamiltonian tour.





- Right one has an Euler tour, but no Hamiltonian tour.

No one knows

- There is probably no nice characterization of Hamiltonian graphs the way there was with Eulerian graphs.
  - Deciding if a graph is Hamiltonian is NP–complete.
    - This means, if an algorithm for solving it in polynomial time were found, it could be used to solve *all* NP problems in polynomial time.

Kn has a Hamilton circuit

- Kn has a Hamilton circuit whenever $n \geq 3$. We can form a Hamilton circuit in Kn beginning at any vertex.

K_n

Partial result

- We now state some partial answers that say if a graph *G* has "*enough*" edges, a Hamiltonian circuit can be found.
- These are again existence statements; no method for constructing a Hamiltonian circuit is given.

Theorem 3,4

- DIRAC'S THEOREM (1952)

G has a Hamiltonian circuit **if** each vertex has degree greater than or equal to $n/2$.

- Corollary: ORE'S THEOREM

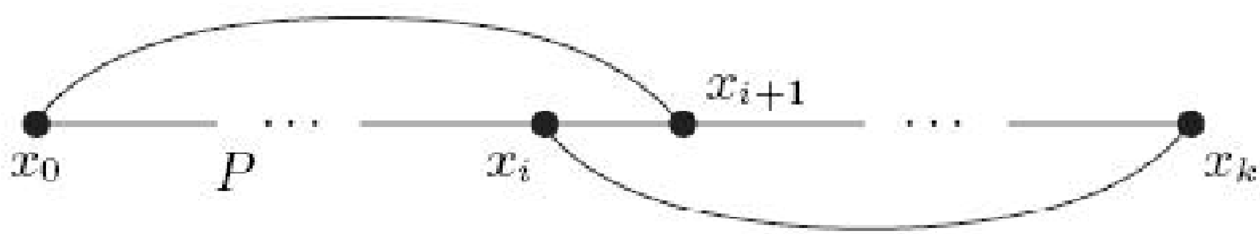
Let *G* be a connected graph with *n* vertices, $n > 2$, and no loops or multiple edges. *G* has a Hamiltonian circuit **if** for any two vertices *u* and *v* of *G* that are not adjacent, the degree of *u* plus the degree of *v* is greater than or equal to *n*.

Proof of Dirac's Theorem

- Let $G=(V, E)$ be a graph with $|G|=n>2$, and $\delta(G) \geq n/2$.
- Then G is connected. otherwise, the degree of any vertex in the smallest component C of G would be less than $|C|<n/2$.
- Let $P=x_0 \ldots x_k$ be a longest path in G. By the maximality of G, all the neighbors of x_0 and all the neighbors of x_k lies on P.

Proof of Dirac's Theorem

- Hence at least $n/2$ of the vertices $x_0 \ldots x_{k-1}$ are adjacent to x_k, and at least $n/2$ of these same k<n vertices x_k are such that $x_0 x_{i+1} \in E$.
- By the pigeon hole principle, there is a vertex x_i that have both properties, so we have $x_0 x_{i+1} \in E$ and $x_i x_k \in E$ for some i<k.



Proof of Dirac's Theorem

- We claim that the cycle $C:=x_0x_{i+1}Px_kx_iPx_0$ is a Hamilton cycle of G.
- indeed, since G is connected, C would otherwise have a neighbor in G-C, which would be combined with a spanning path of C into a path longer than P.

Corollary

- Let the number of edges of G be m.
- Then G has a Hamiltonian circuit if
  - $m \geq (n^2-3n+6)/2$.

$m \geq \frac{n^2-3n+6}{2}$

Proof of Corollary

- Suppose that u and v are any two vertices of G that are not adjacent.
  - We write deg(u) for the degree of u.
- Let H be the graph produced by eliminating u and v from G along with any edges that have u or v as end points.
- Then H has n-2 vertices and m-deg(u)-deg(v) edges (one fewer edge would have been removed if u and v had been adjacent).

Proof

- The maximum number of edges that H could possibly have is $\binom{n-2}{2}$ .This happens when there is an edge connecting every distinct pair of vertices. Thus the number of edges of H is at most

$$ {}_{n-2}C_2 = \frac{(n-2)(n-3)}{2} \quad \text{or} \quad \frac{1}{2}(n^2 - 5n + 6). $$

Proof

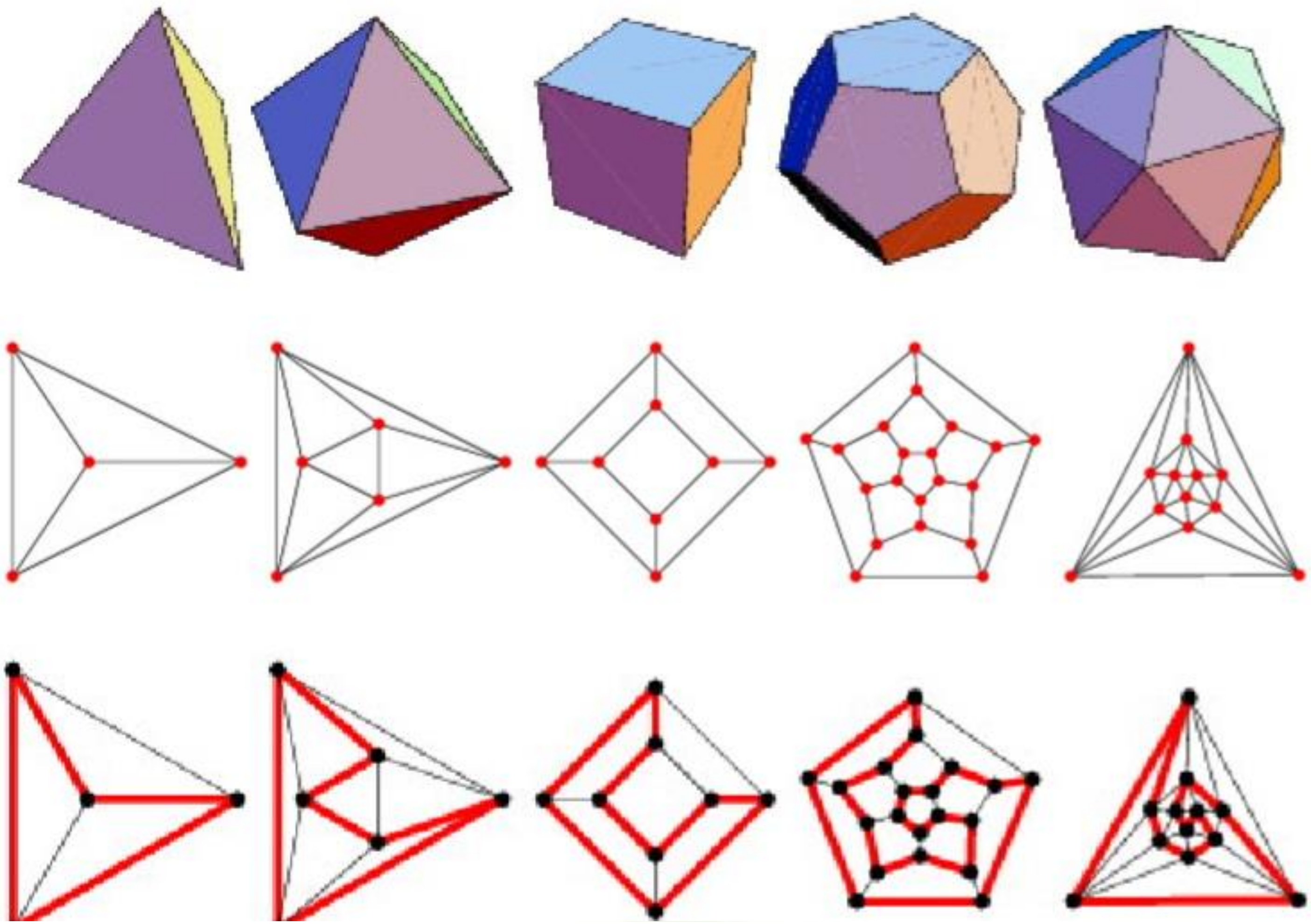
- So
  - m - deg(u) - deg(v) $\leq (n^2 - 5n + 6)/2$.
- Therefore
  - deg(u)+deg(v) $\geq m - (n^2 - 5n + 6)/2$.
- By the hypothesis of the theorem,
  - deg(u)+deg(v) $\geq (n^2-3n+6)/2 - (n^2-5n+6)/2 = n$.
- Thus the result follows from Ore's Theorem.

Note

- The converses of Theorems 3 and 4 given above are not true; that is, the conditions given are sufficient, but not necessary, for the conclusion.
- Example: graph C_n.

Hamiltonian Platonic Cycles

All Platonic solids are Hamiltonian, as illustrated below.

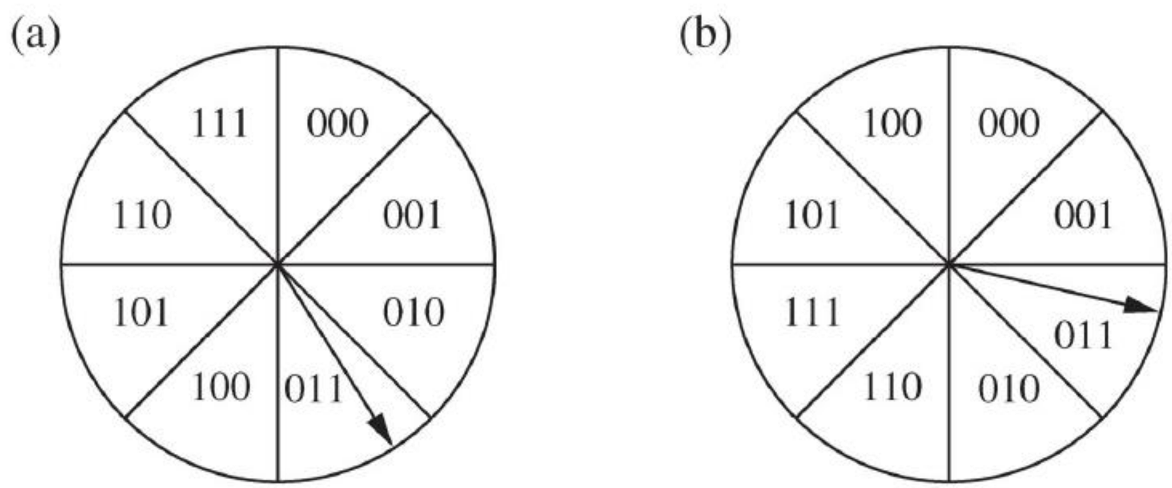


Remarks

- The problem we have been considering has a number of important variations. In one case, the edges may have *weights* representing distance, cost, and the like. The problem is then to find a Hamiltonian circuit (or path) for which the total sum of weights in the path is a minimum.
- For example, the vertices might represent cities; the edges, lines of transportation; and the weight of an edge, the cost of traveling along that edge. This version of the problem is often called *the traveling salesperson problem*.

Rotating memory drum （旋转鼓轮-格雷码）

- A Cray code is a labeling of the arcs of the cycle such that adjacent arcs are labeled with bit strings that differ in exactly one bit.



Rotating memory drum

- A Hamilton circuit in Q_n.

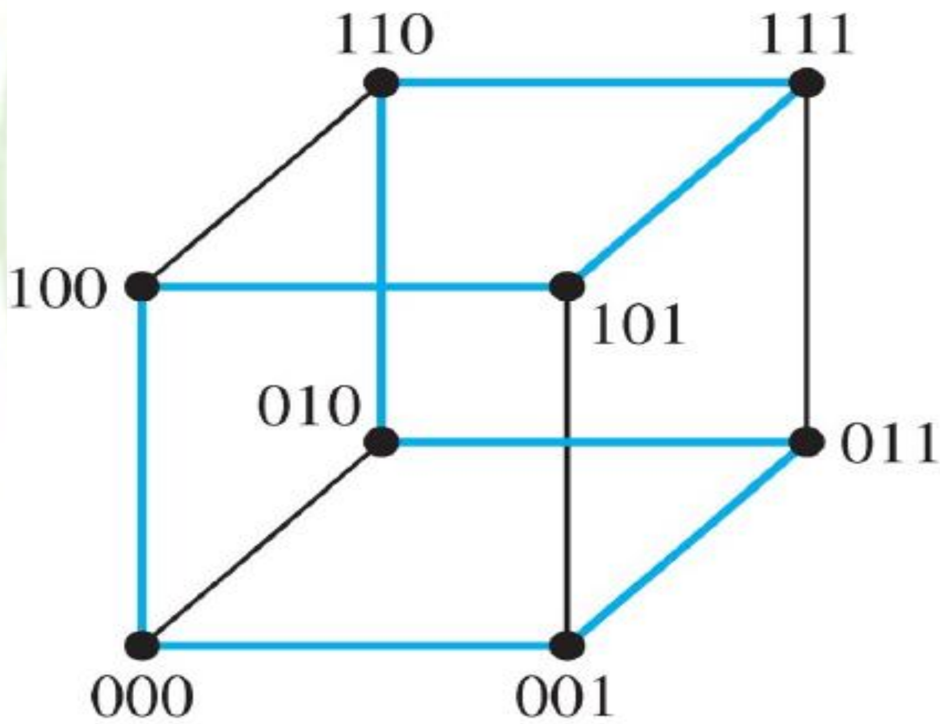


FIGURE 14 **A Hamilton Circuit for Q_3.**

HOMEWORK

- § 10.5
  - 8, 10, 16, 26, 34, 48, 58

Please feel free to ask questions!