离散(下)-图论错题讲解

2019-12-16

10.2 12, 42(参考41和握手定理), 64, 72

- 10.2 64:
- 证二分简单图的 $e \le v^2/4$
  - $v=v1+v2$; $\sum deg(v)=2e$;
  - **Bipartite Graph, $deg(v1) \le v1*v2$;$deg(v2) \le v2*v1$;**
  - **$v1*v2 \le v/2*v/2=v^2/4$**
  - **$e=$**$\sum deg(v)/2=(deg(v1)+deg(v2))/2$**$\le v1*v2$**$\le v^2/4$

10.3 28, 46, 52, 60, 68

- 10.3 28: For an undirected graph, the sum of the entries in the ith row is the same as the corresponding column sum, namely the number of edges incident to the vertex i, which is the same as the degree of i minus the number of loops at i (since each loop contributes 2 toward the degree count).
- For a directed graph, the sum of the entries in the ith row is the number of edges that have i as their initial vertex, i.e., the out-degree of i.

10.3 28, 46, 52, 60, 68

- 46: Show G and H is isomorphism,then U-G and U-H is isomorphism.

since $G(V_1,E_1) \cong H(V_2,E_2)$,

exist a f: $V_1 \to V_2$ is bijection, all $(u,v) \in E_1$ iff $(f(u),f(v)) \in E_2$.

$U\text{-}G(V_1,\sim E_1)$, $U\text{-}H(V_2,\sim E_2)$, same f: $V_1 \to V_2$,

since an edge is in U-G iff it is not in G,

$(u,v) \in \sim E_1$ iff $(u,v) \notin E_1$ iff $(f(u),f(v)) \notin E_2$ iff $(f(u),f(v)) \in \sim E_2$

so all $(u,v) \in \sim E_1$ iff $(f(u),f(v)) \in \sim E_2$

Hence $U\text{-}G(V_1,\sim E_1) \cong U\text{-}H(V_2,\sim E_2)$

10.3 28, 46, 52, 60, 68

isomorphism 同构

- 52: if G and U-G is isomorphism,named self-complementary.

Because G and U-G is isomorphism, $e_1=e_2$.

$E_1 \cap E_2=\varnothing$, so $|E_1 \cup E_2| = e_1+e_2 =2e_1$.

since the union of the two graphs is Kn.

Kn: $e=n*(n-1)/2$. so $e_1=e_2=e/2=n*(n-1)/4$.

n must be integer, so $n*(n-1)=4m$, m is a integer.

=> n mod 4 =0, or 1.

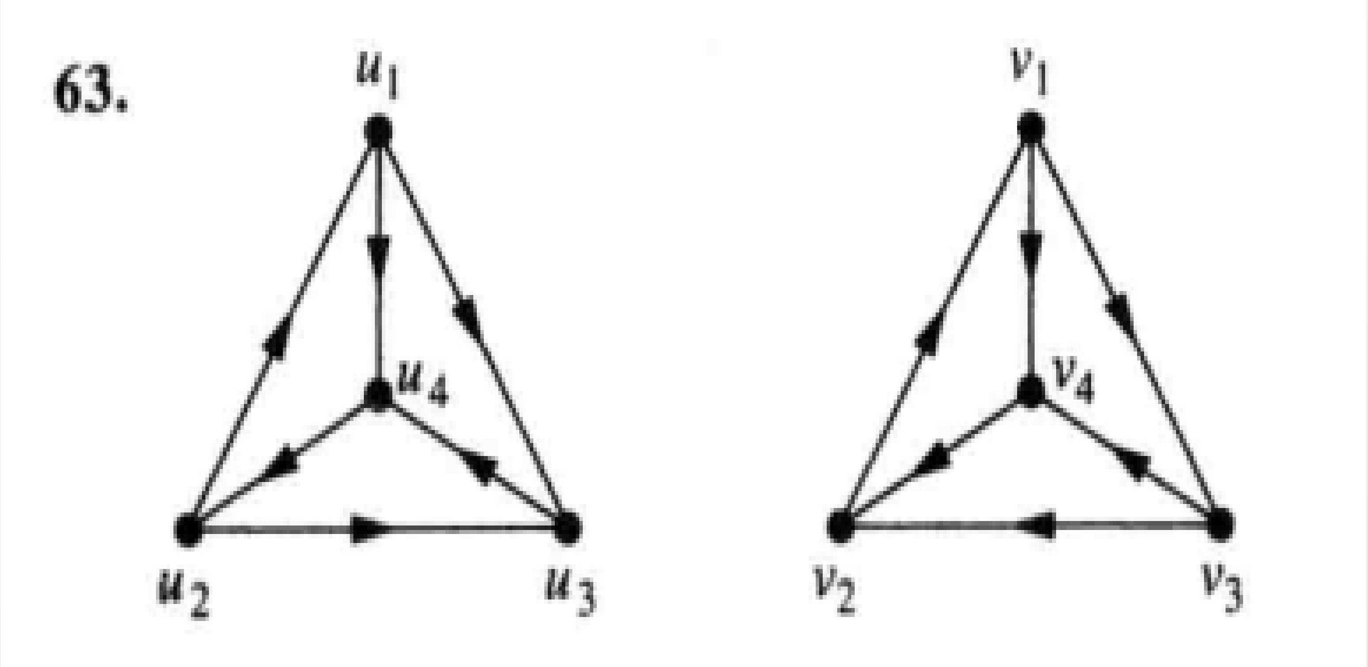
10.3 60,68

- 60:The directed graphs $G_1=(V_1, E_1)$ and $G_2=(V_2, E_2)$ are isomorphic if there is a one-to-one and onto function $f: V_1 \to V_2$ such that for all pairs of vertices a and b in V_1, $(a, b) \in E_1$ if and only if $(f(a), f(b)) \in E_2$.
- 68: nonisomorphic directed simple graphs
- n=2, s(n)=1(e0)+1(e1)+1(e2)=3;
- n=3, s(n)=1(e0)+1+4+4+4+1+1(e6)=16;
- n=4, s(n)=1(e0)+1+4+8+10+2+3+1+1=31;

10.3 有向图的同构

63. u1->v3, u2->v1, u3->v4, u4->v2.

u1,u2,u3,u4	v3,v1,v4,v2
0 0 1 1	0 0 1 1
1 0 1 0	1 0 1 0
0 0 0 1	0 0 0 1
0 1 0 0	0 1 0 0



同构 (原书答案错误!)

10.4: 14,28,36,60

- 28: 证明n点的连通图至少有n-1条边
- We show this by induction on n. For $n = 1$ there is nothing to prove.
- Now assume the inductive hypothesis, and let G be a connected graph with $n+1$ vertices and fewer than n edges, where $n \geq 1$. $\sum deg(v)=2e<2n<2(n+1)$;
- Therefore some vertex has degree less than 2. Since G is connected, this vertex is not isolated, so it must have degree 1.
- Remove this vertex and its edge. Clearly the result is still connected,
- and it has n vertices and fewer than $n-1$ edges,
- contradicting the inductive hypothesis. Therefore the statement holds for G, and the proof is complete.

10.4: 14,28,36,60

- 36: 连通简单图的割点c iff any path u…v must contains c.
- Prove:
- if c is a cut vertex, Since the removal of c increases the number of components, there must be two vertices in different components. Then every path between these two vertices has to pass through c.
- if u and v are as specified, then they must be in different components of the graph with c removed. Therefore the removal of c resulted in at least two components, so c is a cut vertex.

10.4: 14,28,36,60

- 60：证明长度为k的简单回路是图形不变量

Suppose that f is an isomorphism from graph G to graph H. If G has a simple circuit of length k, say u1,u2,...,uk,u1. since each edge u_iu_{i+1} (and u_ku_1) in G corresponds to an edge $f(u_i)f(u_{i+1})$ (and $f(u_k)f(u_1)$) in H.

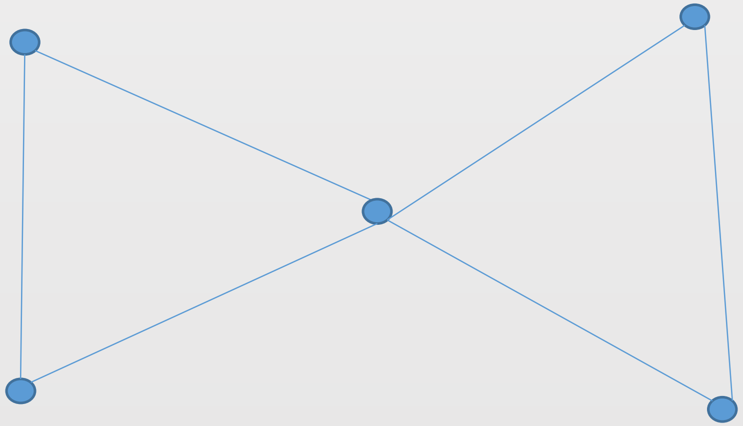
Furthermore, since no edge was repeated in this circuit in G, no edge will be repeated when we use f to move over to H.

10.5 8, 10, 16, 26, 34, 48, 58

- 16：有向图具有欧拉回路 iff all $deg^{+}(v_i)=deg^{-}(v_i)$.
- First suppose that the directed multigraph has an Euler circuit. the graph must be strongly connected. as the circuit passes through a vertex, it adds one to the count of both the in-degree and the out-degree.
- Conversely, suppose that the graph meets the conditions stated. Then we can proceed as in the proof of Theorem 1 and construct an Euler circuit.

10.5 8, 10, 16, 26, 34, 48, 58

- 48：找反例，all $deg(v_i)>=(n-1)/2$, 没有哈密顿回路.
- One way to avoid having a Hamilton circuit is to have a cut vertex.

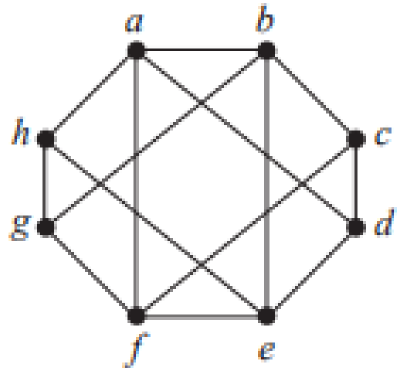


10.6: 8(shortest), 16, 18(not be unique), 26(salesman)

- 16. 扩展Dijkstra算法，求出顶点间最短通路。
- we add an array P to the algorithm, where P(v) is the previous vertex in the best known path to v.
- We modify Algorithm 1 so that when L is updated by the statement L(v) := L(u) + w(u,v), we also set P(v) := u.

10.7: 6, 8, 12, 18, 24, 30

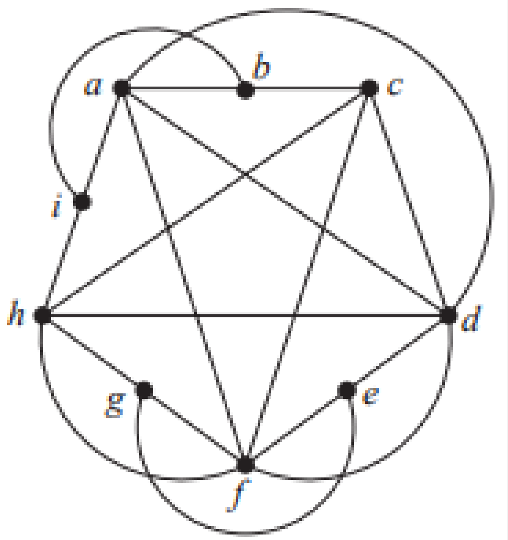
8.



8: 不是平面图，判定理由只能是包含$K_{3,3}$或K_5同胚子图。

{a,c,e}-{b,d,f}构成$K_{3,3}$子图

24.



24: 不是平面图， 和K_5同胚

plannar 平面图、

10.8: 10, 16, 20, 23, 34

16. 证明简单图G，若包含奇数个顶点的回路，着色数>2.

Let the circuit be v1, v2, ..., vn, v1, where n is odd. let the color of v1 be red. Then v2 must be blue, v3 must be red, and so on, until finally vn must be red (since n is odd). But this is a contradiction, since vn is adjacent to v1. Therefore at least three colors are needed.

34. 证明W_4不是着色3关键的(本身着色3，删任一边后变2)。

Although the chromatic number of W_4 is 3, if we remove one edge then the graph still contains a triangle, so its chromatic number remains 3. Therefore W_4 is not chromatically 3-critical.

10.8: 10, 16, 20, 23, 34

23. 求C_n和W_n的边着色数.

a) 环图中每个点关联两条边，所以边交替着色，但2色要求奇数个边；否则3色。

2 if n is even, 3 if n is odd.

b) 轮图的中继点关联n条边，所以需要n色，剩余环图着色2或3，但要避开三角形中两条中继边颜色。

因为n一定>2,所以环路部分的着色在n之内选择足够。

结果，W_n的边着色数为n。

10,14@305; <离散数学结构>最大流量算法

- 10: 注意增广路径的发现顺序。

  N1={2,3,4} N2={5,6,7}, path 1-4-7, -3.

  N1={2,3} N2={4,5,6} N3={7}, path 1-2-4-7, -3

  N1={2,3} N2={4,5,6} N3={7}, path 1-2-5-7, -3

  N1={2,3} N2={4,6} N3={5,7}, path 1-3-6-7, -4

  N1={2,3} N2={4,6} N3={5} stop. maxflow=13

14: minimal cut K={(1,2),(1,3)}, maxflow=5.

11.4: 50

- 50-Suppose that G is a directed graph and T is a spanning

tree constructed using breadth-first search. Show that every edge of G has endpoints that are at the same level or one level higher or lower.

Suppose that $uv \in$G, but $uv \notin$T.

(1)Assume that the algorithm processed u before it processed v .

Since $uv \notin$T, then v must be already in the list L.

So the parent p of v must have already been processed before u.

level(v)=level(p)+1<=level(u)>=level(p).

(2)Assume that the algorithm processed v before it processed u.

Since $uv \notin$T, then u must be already in the list L. parent p of u have been in L.
level(u)=level(p)+1<=level(v)>=level(p).

11.5 4(Prim's), 8(Kruskal’s), 10.

- 10: spanning forest of minimum weight.

Kruskal’s algorithm, do until no such edges.

Prim's algorithm, when no vertex adjacent to S, grow a new tree from a shortest edge not in S.

8.1 :14

14. a) Find a recurrence relation for the number of ternary strings of length n that contain two consecutive 0s.

$a_n = 2a_{n-1} + 2a_{n-2} + 3^{n-2}$.

注意：长度n-2的任意串有3^{n-2}，加00都为合格串。

长度n-1合格串后面加0，会和"n-2串+00"处理重复，故去掉，只留+1/+2两种；

长度n-2合格串后面加01，02，会和长度n-1末尾0的合格串处理重复，故去掉；

长度n-2合格串后面加1或2，会形成n-1合格串，重复去掉，只剩再加1或2两种；

b) What are the initial conditions?

$a_0 = a_1 = 0.$

c) How many ternary strings of length six contain two consecutive 0s?

$a_6 = 2a_5 + 2a_4 + 3^4 = 2 \cdot 79 + 2 \cdot 21 + 81 = 281$

8.2: 10, 42, 46

- 10. 证明2阶线性齐次递推关系在重根时的解，讲义上有。
- 42. $a_n = a_{n-1} + a_{n-2}, a_0 = s, a_1 = t$. => $a_n = sf_{n-1} + tf_n$.
- If n = 1, $a_1 = s \cdot f_0 + t \cdot f_1 = s \cdot 0 + t \cdot 1 = t$, which is given;
- if n = 2, $a_2 = s \cdot f_1 + t \cdot f_2 = s \cdot 1 + t \cdot 1 = s + t$, since $a_2 = a_1 + a_0 = t + s$.
- we assume the inductive hypothesis, that the statement is true for values less

than n. Then $a_n = a_{n-1} + a_{n-2} = (sf_{n-2} + tf_{n-1}) + (sf_{n-3} + tf_{n-2})$

$$= s(f_{n-2} + f_{n-3}) + t(f_{n-1} + f_{n-2})$$

$$= sf_{n-1} + tf_n$$

as desired.

8.2: 10, 42, 46

46. 构造山羊数的递推关系a_n, initial a_1=2,

a) $a_n=2a_{n-1}+100$

The associated homogeneous recurrence relation is $a_n=2a_{n-1}$,

$$a^{(h)}_n = \alpha 2^n.$$

The particular solution is a_n= c. c = 2c + 100, c = −100.

$$a_n = \alpha 2^n - 100.$$

$$a_1=2 = 2\alpha-100, \text{ so } \alpha = 51.$$

Hence the desired formula is $a_n = 51\cdot 2^n - 100$.

8.2: 10, 42, 46

46.构造山羊数的递推关系a_n, initial a_1=2,

C) a_n=2 a_{n-1}−n , n>=3, a_2=4,

$a^{(h)}{}_n = \alpha 2^n$. The particular solution is a_n = cn + d.

cn+d-2(c(n-1)+d)+n=(−c+1)n+(2c−d) = 0, c = 1 and d = 2.

$a_n = \alpha 2^n$ + n +2.

a_2= 4 = 4α +4, so α = 0.

Hence a_n= n +2 for all n ≥ 2 (and a1 = 2).

8.3： 14, 28

- 14.求淘汰锦标赛的递推关系
- Suppose that there are $n = 2^k$ teams in an elimination tournament, where there are $n/2$ games in the first round, with the $n/2 = 2^{k-1}$ winners playing in the second round, and so on. Develop a recurrence relation for the number of rounds in the tournament.
  - Since it takes one round to cut the number of teams in half, we have $R(n) = 1+R(n/2)$.
  - If there is only one team, then no rounds are needed, so the base case is $R(1) = 0$.

8.3： 14, 28

- 28.猜数的乌拉姆问题
- a) 2logn+1 (真话需logn次，允许1次谎言，每个问题问2次发现说谎+1).
- b) Divide the set into A, B, C, and D. ask these questions: “Is your number in A∪B?” and “Is your number in A∪C?” If the answers are both “yes,” then we can eliminate D. if both answers are “no,” eliminate A; if the answers are first “yes” and then “no,” eliminate C; and if the answers are first “no” and then “yes,” eliminate B.
- $f(n) = 2 + f(n/(4/3)) = O(\log_{4/3} n)$
  - $f(n)=2 + 2 + f((3/4)^2 n) = 2+2+2+f((3/4)^3 n) =\cdots= 2+2+\cdots+2$, where there are about $\log_{4/3} n$ 2’s in the sum. Noting that $\log_{4/3} n = \log n/\log 4/3 \approx 2.4\log n$, we have that $f(n) \approx 4.8\log n$.

8.4: 16, 24, 36.

a dozen

16-Use generating functions to find the number of ways to choose a dozen bagels from three varieties—egg, salty, and plain—if at least two bagels of each kind but no more than three salty bagels are chosen.

$x1+x2+x3=12$, $x1\geq 2$, $2\leq x2\leq 3$, $x3\geq 2$.

$(x^2 + x^3 + x^4 + \cdots)(x^2 + x^3)(x^2 + x^3 + x^4 + \cdots)$, find the coefficient of x^{12}

$=x^6(1+x+x^2 + x^3 + x^4 + \cdots)^2(1+x)$

$=x^6 (1+x)/(1-x)^2 = x^6 (1/(1-x)^2 + x/(1-x)^2)$,

$x_1+x_2+x_3=12$,

$x_1\geq 2$, $2\leq x_2\leq 3$, $x_3\geq 2$

find the coefficient a_6+a_5 of $1/(1-x)^2$

$$f(x) = \sum_{k=0}^{\infty} x^k, \; f(x)f(x) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} 1\right)x^k = \sum_{k=0}^{\infty}(k+1)x^k$$

$a_6=7, a_5=6$, answer: $a_6+a_5=13$.

$(x^2+x^3+x^4+\cdots)(x^2+x^3)(x^2+x^3+\cdots)$ x^{12}

x^4.. $\frac{1}{1-x}$

8.4: 16, 24, 36.

24- a) What is the generating function for $\{a_k\}$, where a_k is the number of solutions of $x_1 + x_2 + x_3 + x_4 = k$ when x_1, x_2, x_3, and x_4 are integers with $x_1 \geq 3, 1 \leq x_2 \leq 5, 0 \leq x_3 \leq 4$, and $x_4 \geq 1$?

$(x^3 + x^4 + x^5 + \cdots)(x + x^2 + x^3 + x^4 + x^5)(1 + x + x^2 + x^3 + x^4)(x + x^2 + x^3 + \cdots) = x^5(1 + x + x^2 + x^3 + x^4)^2/(1 - x)^2$

$x(1+x+x^2+x^3+x^4)^2$

b) Use your answer to part (a) to find a_7.

$x^5(1+x+x^2+x^3+x^4)^2 \frac{1}{(1-x)^2}$

$= x^5(1 + 2x + 3x^2 + \ldots)/(1 - x)^2$

$a_7 = b_2 + 2b_1 + 3b_0$ of $1/(1-x)^2$

$= 3+2*2+3*1$

$= 10$

$$f(x) = \sum_{k=0}^{\infty} x^k,\ f(x)f(x) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} 1\right)x^k = \sum_{k=0}^{\infty}(k+1)x^k$$

$2x^6$ 2.

$x^6 (1+x+x^2+\cdots)^2 (1+x)$

$x^6 \dfrac{1+x}{(1-x)^2} = x^6\left[\dfrac{1}{(1-x)^2} + \dfrac{x}{(1-x)^2}\right]$

8.4: 16, 24, 36.

x^6 x^5

36-Use generating functions to solve the recurrence relation $a_k = a_{k-1} + 2a_{k-2} + 2^k$ with initial conditions $a_0 0 = 4$ and $a_1 = 12$.

$$(1)\ G(x) = \sum_{k=0}^{\infty} a_k x^k$$

$$(2)\ G(x) - xG(x) - 2x^2G(x) = \sum_{k=0}^{\infty} a_k x^k - x\sum_{k=0}^{\infty} a_k x^k - 2x^2\sum_{k=0}^{\infty} a_k x^k$$

$$= \sum_{k=0}^{\infty} a_k x^k - \sum_{k=1}^{\infty} a_{k-1} x^k - \sum_{k=2}^{\infty} 2a_{k-2} x^k$$

$$= (a_0 + a_1 x) + (-a_0 x) + \sum_{k=2}^{\infty} (a_k - a_{k-1} - 2a_{k-2}) x^k$$

$$= a_0 + a_1 x - a_0 x + \sum_{k=2}^{\infty} 2^k x^k$$

$$= 4 + 12x - 4x + \left(\sum_{k=0}^{\infty} 2^k x^k - 2^0 x^0 - 2^1 x^1\right)$$

$$= 4 + 8x + \frac{1}{1-2x} - 1 - 2x = \frac{4-12x^2}{1-2x}$$

$\dfrac{1}{(1-x)^2}$

$3+2\times2+3\times1=10$

$\times$

$(1+x+x^2+x^3+x^4)(1+x+x^2+x^3+x^4)$

$=1+2x+3x^2$

8.4: 16, 24, 36.

$$(3)\ G(x)=\frac{\frac{4-12x^2}{1-2x}}{1-x-2x^2}=\frac{4-12x^2}{(1-2x)^2(1+x)}=\frac{-8/9}{1+x}+\frac{38/9}{1-2x}+\frac{2/3}{(1-2x)^2}$$

$$=\sum_{k=0}^{\infty}(-8/9)(-1)^k x^k+\sum_{k=0}^{\infty}(38/9)2^k x^k+(2/3)\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}2^j2^{k-j}\right)x^k$$

$$=\sum_{k=0}^{\infty}\left(\left(\frac{-8}{9}\right)(-1)^k+\left(\frac{38}{9}\right)2^k+\frac{2}{3}(2^k)(k+1)\right)x^k$$

$$So\ a_k=(-8/9)(-1)^k+(38/9)2^k+(2/3)(k+1)2^k$$