Problem

Consider a matrix problem (s.p.d.) of the form

$$A\boldsymbol{u} = \boldsymbol{f}, \quad A \in \mathbb{R}^{n\times n}$$

Suppose we have a multilevel iteration process $\mathcal{M}$

$$I - \mathcal{M}A = \left(I - M^\top A\right)^{\nu_{\text{pre}}} \left(I - P\left(P^\top AP\right)^{-1} P^\top A\right)(I - MA)^{\nu_{\text{post}}}$$

so that

$$\boldsymbol{e} \leftarrow (I - \mathcal{M}A)\boldsymbol{e}$$

The iteration converges for any $\boldsymbol{f}$ and $\boldsymbol{u}_0$ iff $\rho(I - \mathcal{M}A) < 1$.

we are seeking a method that yields **a bound on the error reduction in each iteration that is independent of n.**

Notation

Fine grid $\Omega = \{1, \ldots, n\} = C \cup F$ and coarse grid $\Omega_c = C$. Interpolation / restriction:

$$P : \Omega_c \to \Omega \quad \text{and} \quad R : \Omega \to \Omega_c$$

A is s.p.d., $D = \text{diag}(A)-$ defining an inner product:

$$\begin{aligned}
\langle \boldsymbol{u}, \boldsymbol{v} \rangle_A &= \langle A\boldsymbol{u}, \boldsymbol{v} \rangle \\
\langle \boldsymbol{u}, \boldsymbol{v} \rangle_D &= \langle D\boldsymbol{u}, \boldsymbol{v} \rangle \\
\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{AD^{-1}A} &= \left\langle D^{-1}A\boldsymbol{u}, A\boldsymbol{v} \right\rangle
\end{aligned}$$

Define S as post-relaxation
(and $\mathcal{S}$ as the affine version $\boldsymbol{u} \leftarrow \mathcal{S}(\boldsymbol{u}, \boldsymbol{f})$):

$$\boldsymbol{u} \leftarrow S\boldsymbol{u} + (I - S)A^{-1}\boldsymbol{f} \quad \text{or} \quad \boldsymbol{e} \leftarrow S\boldsymbol{e}$$

Algorithm

$$\begin{aligned}
&\boldsymbol{u} \leftarrow \hat{\mathcal{S}}(\boldsymbol{u}, \boldsymbol{f}) \\
&\boldsymbol{r}_c \leftarrow R\boldsymbol{r} \\
&\boldsymbol{e}_c \leftarrow A_c^{-1}\boldsymbol{r}_c \\
&\hat{\mathbf{e}} \leftarrow P\boldsymbol{e}_c \\
&\boldsymbol{u} \leftarrow \boldsymbol{u} + \hat{\mathbf{e}} \\
&\boldsymbol{u} \leftarrow \mathcal{S}(\boldsymbol{u}, \boldsymbol{f})
\end{aligned}$$

In general, consider relaxation as

$$S = I - MA$$

Assumptions:

- M is norm convergent (in A): $\|S\|_A < 1$
- P is full rank

- A is s.p.d.

Let the coarse grid correction step be

$$T = I - P\big(P^\top AP\big)^{-1}P^\top A$$

T is an A-orthogonal projection onto the range of P: After coarse grid correction, the error is minimized in the energy norm over $\mathrm{range}(P)$

Focus on $V(0,1)$

The A-adjoint of ST is

$$TS^+ \quad S^+ = I - M^\top A$$

The symmetric $V(1,1)$ cycle is

$$\begin{aligned}(I - MA)\left(I - P\big(P^TAP\big)^{-1}P^TA\right)(I - M^TA) &= STS^+ \\ &= STTS^+\end{aligned}$$

Since $\|ST\|_A = \|TS^+\|_A$ (A-adjoints) we have

$$\|STS^+\|_A = \|ST\|_A^2$$

Ok, so we can focus focus on the $V(0,1)$ cycle, the other cycles follow.

Upper Bound

We will measure convergence or reduction in $\|\boldsymbol{e}\|_A$. Note:

$$\|\boldsymbol{e}\|_A^2 = \|(I-T)\boldsymbol{e}\|_A^2 + \|T\boldsymbol{e}\|_A^2$$

For a $V(0,1)$ cycle, the reduction in $\boldsymbol{e}$ is

$$\|STe\|_A^2 \le (1-\delta^*)\|e\|_A^2$$

we seek a sharp bound

$$\|ST\|_A^2 := \sup_{e\neq 0} \frac{\|STe\|_A^2}{\|e\|_A^2} = 1 - \delta^*$$

Sufficient conditions

What should we assume on relaxation and interpolation?

- relaxation is effective on the range of coarse grid correction

There exists $\delta > 0$ such that

$$\|ST\boldsymbol{e}\|_A^2 \le (1-\delta)\|T\boldsymbol{e}\|_A^2 \text{ for all } \boldsymbol{e}.$$

Then, since T is an A-orthogonal projector

$$\|ST\boldsymbol{e}\|_A^2 \le (1-\delta)\|\boldsymbol{e}\|_A^2 \text{ for all } \boldsymbol{e}$$

- No side effects on the range of interpolation

$$\|S\boldsymbol{v}\|_A^2 \leq \|\boldsymbol{v}\|_A^2 \quad \text{for all } \boldsymbol{v} \perp \text{range}(T)$$

Combine these into an assumption

$$\|Sv\|_A^2 \leq \|STv\|_A^2 + \|S(I-T)v\|_A^2 \leq (1-\delta)\|Tv\|_A^2 + \|(I-T)v\|_A^2 = \|v\|_A^2 - \delta\|Tv\|_A^2$$

Theorem
If there exists $\delta > 0$ so that

$$\|Se\|_A^2 \leq \|e\|_A^2 - \delta\|Te\|_A^2 \quad \text{for all } e,$$

then

$$\|ST\|_A^2 \leq 1-\delta$$

To be sharp, the largest δ, say $\hat{\delta}$

$$\hat{\delta} = \inf_{e:Te\neq 0} \frac{\|e\|_A^2 - \|Se\|_A^2}{\|Te\|_A^2},$$

should be δ^*.

Proof

Since $\boldsymbol{T}\boldsymbol{e} = \mathbf{0}$ gives $\|ST\boldsymbol{e}\|_A = 0$:

$$\|ST\|_A^2 = \sup_{e:Te\neq 0} \frac{\|STe\|_A^2}{\|e\|_A^2} = \sup_{e:Te\neq 0} \frac{\|STe\|_A^2}{\|Te\|_A^2 + \|(I-T)e\|_A^2}$$

Let $\hat{\boldsymbol{e}}$ be the $\arg\sup$
Then $\boldsymbol{T}\hat{\boldsymbol{e}}$ is also at the supremum.
Thus we have an error at the supremum with $(I-T)\hat{e} = 0$

$$\|ST\|_A^2 = \sup_{e:Te\neq 0} \frac{\|STe\|_A^2}{\|e\|_A^2} = \sup_{e:Te\neq 0} \frac{\|S(Te + (I-T)e)\|_A^2}{\|Te\|_A^2} = \sup_{e:Te\neq 0} \frac{\|Se\|_A^2}{\|Te\|_A^2},$$

And

$$1 - \|ST\|_A^2 = \inf_{e:Te\neq\mathbf{0}} \frac{\|Te\|_A^2 - \|Se\|_A^2}{\|Te\|_A^2} = \inf_{e:Te\neq\mathbf{0}} \frac{\|e\|_A^2 - \|Se\|_A^2}{\|Te\|_A^2} = \hat{\delta}$$

The worst δ is sharp

$$\hat{\delta} = \inf_{e:Te\neq 0} \frac{\|e\|_A^2 - \|Se\|_A^2}{\|Te\|_A^2},$$

The early theory split this in two part

For some $g(\boldsymbol{e})$ define δ, α_g, and β_g as in

$$\delta(\boldsymbol{e}) = \underbrace{\frac{\|\boldsymbol{e}\|_A^2 - \|S\boldsymbol{e}\|_A^2}{g(\boldsymbol{e})}}_{\alpha_g(\boldsymbol{e})} \underbrace{\frac{g(\boldsymbol{e})}{\|T\boldsymbol{e}\|_A^2}}_{1/\beta_g(\boldsymbol{e})}$$

Consider the smallest α_g and the largest β_g :

$$\hat{\alpha}_g = \inf_{\boldsymbol{e}:g(\boldsymbol{e})\neq\boldsymbol{0}} \alpha_g(\boldsymbol{e}) \quad \hat{\beta}_g = \sup_{\boldsymbol{e}:g(\boldsymbol{e})\neq\boldsymbol{0}} \beta_g(\boldsymbol{e})$$

For $\boldsymbol{e}$ such that $g(T\boldsymbol{e}) \neq 0$,

$$\begin{aligned}\|ST\boldsymbol{e}\|_A^2 \le \|T\boldsymbol{e}\|_A^2 - \hat{\alpha}_g g(T\boldsymbol{e}) \le \|T\boldsymbol{e}\|_A^2 - \frac{\hat{\alpha}_g}{\hat{\beta}_g}\|T\boldsymbol{e}\|_A^2 = \left(1 - \frac{\hat{\alpha}_g}{\hat{\beta}_g}\right)\|T\boldsymbol{e}\|_A^2 \\ \le \left(1 - \frac{\hat{\alpha}_g}{\hat{\beta}_g}\right)\|\boldsymbol{e}\|_A^2\end{aligned}$$

Ok, so this is generally worse than the sharp bound

$$\|ST\|_A = \sqrt{1-\hat{\delta}} \le \sqrt{1 - \frac{\hat{\alpha}_g}{\hat{\beta}_g}}$$

(α_g and β_g are not generally simultaneously satisfied)

Early works, e.g. Ruge-Stüben 1987, use $g(\boldsymbol{e}) = \|\boldsymbol{e}\|_{AD^{-1}A}^2$ Or the weaker form $g(\boldsymbol{e}) = \|T\boldsymbol{e}\|_{AD^{-1}A}^2$

Smoothing and Approximation

if there exists $\bar{\alpha}_g > 0$ such that

$$\|Se\|_A^2 \le \|e\|_A^2 - \bar{\alpha}_g g(e) \quad \text{for all } e \quad \text{(smoothing)}$$

and there exists $\bar{\beta}_g > 0$ such that

$$\|T\boldsymbol{e}\|_A^2 \le \bar{\beta}_g g(T\boldsymbol{e}) \quad \text{for all } \boldsymbol{e} \quad \text{(approximation)}$$

then $\|ST\|_A \le \sqrt{1 - \bar{\alpha}_g/\bar{\beta}_g}$

Select $g(\boldsymbol{e}) = \|\boldsymbol{e}\|_{AD^{-1}A}^2$
Since T is an A-orthogonal projection we have

$$\|Te\|_A = \inf_{e_c} \|e - Pe_c\|_A$$

(strong approximation) Assume there is a $\bar{\beta}_s$ such that

$$\inf_{\boldsymbol{e}_c} \|\boldsymbol{e} - P\boldsymbol{e}_c\|_A^2 \le \bar{\beta}_s \|\boldsymbol{e}\|_{AD^{-1}A}^2 \quad \text{for all } \boldsymbol{e}.$$

The weaker version looks like (for some $\hat{\beta}$)

$$\|Te\|_A^2 \le \bar{\beta}\|Te\|_{AD^{-1}A} \quad \text{for all } e.$$

Weaker, means weaker norm. And we can make this a bit more practical. The range of T is A-orthogonal to the range of P, so

$$\begin{aligned}\|T\boldsymbol{e}\|_A^2 = \langle AT\boldsymbol{e}, T\boldsymbol{e}\rangle = \langle AT\boldsymbol{e}, T\boldsymbol{e} - P\boldsymbol{e}_c\rangle \\ \le \|T\boldsymbol{e}\|_{AD^{-1}A}\|T\boldsymbol{e} - P\boldsymbol{e}_c\|_D.\end{aligned}$$

(weak approximation) Assume that

$$\inf_{\boldsymbol{e}_c} \|\boldsymbol{e} - P\boldsymbol{e}_c\|_D^2 \le \bar{\beta}_w \|\boldsymbol{e}\|_A^2 \quad \text{for all } \boldsymbol{e},$$