#### **Problem**

Consider a matrix problem (s.p.d.) of the form

$$A\boldsymbol{u} = \boldsymbol{f}, \quad A \in \mathbb{R}^{n \times n}$$

Suppose we have a multilevel iteration process  ${\cal M}$ 

$$I - \mathcal{M}A = ig(I - M^ op Aig)^{
u_{ ext{pre}}} \, \Big(I - Pig(P^ op APig)^{-1} P^ op A\Big) (I - MA)^{
u_{ ext{post}}}$$

so that

$$oldsymbol{e} \leftarrow (I - \mathcal{M}A)oldsymbol{e}$$

The iteration converges for any  $m{f}$  and  $m{u}_0$  iff  $ho(I-\mathcal{M}A)<1$ .

we are seeking a method that yields a bound on the error reduction in each iteration that is independent of n.

#### **Notation**

Fine grid  $\Omega=\{1,\ldots,n\}=C\cup F$  and coarse grid  $\Omega_c=C$ . Interpolation / restriction:

$$P:\Omega_c o\Omega$$
 and  $R:\Omega o\Omega_c$ 

A is s.p.d.,  $D = \operatorname{diag}(A)$  – defining an inner product:

$$egin{aligned} \langle oldsymbol{u}, oldsymbol{v} 
angle_A &= \langle Aoldsymbol{u}, oldsymbol{v} 
angle \ \langle oldsymbol{u}, oldsymbol{v} 
angle_D &= \langle Doldsymbol{u}, oldsymbol{u} 
angle \ \langle oldsymbol{u}, oldsymbol{v} 
angle_{AD^{-1}A} &= \langle D^{-1}Aoldsymbol{u}, Aoldsymbol{v} 
angle \end{aligned}$$

Define S as post-relaxation

(and  ${\mathcal S}$  as the affine version  ${m u} \leftarrow {\mathcal S}({m u},{m f})$  ):

$$oldsymbol{u} \leftarrow Soldsymbol{u} + (I-S)A^{-1}oldsymbol{f} \quad ext{or} \quad oldsymbol{e} \leftarrow Soldsymbol{e}$$

## **Algorithm**

$$egin{aligned} oldsymbol{u} \leftarrow \hat{\mathcal{S}}(oldsymbol{u}, oldsymbol{f}) \ oldsymbol{r}_c \leftarrow Roldsymbol{r} \ oldsymbol{e}_c \leftarrow A_c^{-1}oldsymbol{r}_c \ \hat{oldsymbol{e}} \leftarrow Poldsymbol{e}_c \ oldsymbol{u} \leftarrow U + \hat{oldsymbol{e}} \ oldsymbol{u} \leftarrow \mathcal{S}(oldsymbol{u}, oldsymbol{f}) \end{aligned}$$

In general, consider relaxation as

$$S = I - MA$$

Assumptions:

- ullet M is norm convergent (in A ):  $\|S\|_A < 1$
- $\bullet$  P is full rank

ullet A is s.p.d.

Let the coarse grid correction step be

$$T = I - P(P^{\top}AP)^{-1}P^{\top}A$$

T is an A-orthogonal projection onto the range of P: After coarse grid correction, the error is minimized in the energy norm over  $\mathrm{range}(P)$ 

# Focus on V(0,1)

The A-adjoint of ST is

$$TS^+$$
  $S^+ = I - M^{ op} A$ 

The symmetric V(1,1) cycle is

$$(I - MA) \left( I - P(P^T A P)^{-1} P^T A \right) \left( I - M^T A \right) = STS^+$$

$$= STTS^+$$

Since  $\|ST\|_A = \|TS^+\|_A$  (A-adjoints) we have

$$\left\|STS^+\right\|_{A} = \left\|ST\right\|_{A}^{2}$$

Ok, so we can focus focus on the V(0,1) cycle, the other cycles follow.

## **Upper Bound**

We will measure convergence or reduction in  $\|e\|_A$ . Note:

$$\|m{e}\|_A^2 = \|(I-T)m{e}\|_A^2 + \|Tm{e}\|_A^2$$

For a V(0,1) cycle, the reduction in  $\boldsymbol{e}$  is

$$||STe||_A^2 \le (1 - \delta^*) ||e||_A^2$$

we seek a sharp bound

$$\|ST\|_A^2 := \sup_{e 
eq 0} rac{\|STe\|_A^2}{\|e\|_A^2} = 1 - \delta^*$$

#### **Sufficient conditions**

What should we assume on relaxation and interpolation?

• relaxation is effective on the range of coarse grid correction

There exists  $\delta>0$  such that

$$||STe||_A^2 \le (1-\delta)||Te||_A^2 \text{ for all } e.$$

Then, since T is an A-orthogonal projector

$$||STe||_A^2 \leq (1-\delta)||e||_A^2$$
 for all  $e$ 

• No side effects on the range of interpolation

$$\|Soldsymbol{v}\|_A^2 \leq \|oldsymbol{v}\|_A^2 \quad ext{ for all } oldsymbol{v} \perp ext{range}(T)$$

Combine these into an assumption

$$\|Sv\|_A^2 \leq \|STv\|_A^2 + \|S(I-T)v\|_A^2 \leq (1-\delta)\|Tv\|_A^2 + \|(I-T)v\|_A^2 = \|v\|_A^2 - \delta\|Tv\|_A^2$$

#### **Theorem**

If there exists  $\delta>0$  so that

$$||Se||_A^2 \le ||e||_A^2 - \delta ||Te||_A^2$$
 for all  $e$ ,

then

$$\|ST\|_A^2 \le 1 - \delta$$

To be sharp, the largest  $\delta$ , say  $\hat{\delta}$ 

$$\hat{\delta} = \inf_{e: Te 
eq 0} rac{\|e\|_A^2 - \|Se\|_A^2}{\|Te\|_A^2},$$

should be  $\delta^*$ .

Proof

Since  $Toldsymbol{e} = oldsymbol{0}$  gives  $\|SToldsymbol{e}\|_A = 0$  :

$$\|ST\|_A^2 = \sup_{e:Te 
eq 0} rac{\|STe\|_A^2}{\|e\|_A^2} = \sup_{e:Te 
eq 0} rac{\|STe\|_A^2}{\|Te\|_A^2 + \|(I-T)e\|_A^2}$$

Let  $\hat{e}$  be the  $\arg\sup$ 

Then  $T\hat{e}$  is also at the supremum.

Thus we have an error at the supremum with  $(I-T)\hat{e}=0$ 

$$\|ST\|_A^2 = \sup_{e:Te 
eq 0} rac{\|STe\|_A^2}{\|e\|_A^2} = \sup_{e:Te 
eq 0} rac{\|S(Te + (I-T)e)\|_A^2}{\|Te\|_A^2} = \sup_{e:Te 
eq 0} rac{\|Se\|_A^2}{\|Te\|_A^2},$$

And

$$1 - \|ST\|_A^2 = \inf_{e: Te \neq \mathbf{0}} \frac{\|Te\|_A^2 - \|Se\|_A^2}{\|Te\|_A^2} = \inf_{e: Te \neq \mathbf{0}} \frac{\|e\|_A^2 - \|Se\|_A^2}{\|Te\|_A^2} = \hat{\delta}$$

The worst  $\delta$  is sharp

$$\hat{\delta} = \inf_{e: Te 
eq 0} rac{\|e\|_A^2 - \|Se\|_A^2}{\|Te\|_A^2},$$

The early theory split this in two part

For some  $g(\boldsymbol{e})$  define  $\delta, \alpha_g$ , and  $\beta_g$  as in

$$\delta(oldsymbol{e}) = \underbrace{rac{\|oldsymbol{e}\|_A^2 - \|Soldsymbol{e}\|_A^2}{g(oldsymbol{e})}}_{lpha_g(oldsymbol{e})} \underbrace{rac{g(oldsymbol{e})}{\|Toldsymbol{e}\|_A^2}}_{1/eta_g(oldsymbol{e})}$$

Consider the smallest  $\alpha_g$  and the largest  $\beta_g$ :

$$\hat{lpha}_g = \inf_{oldsymbol{e}: g(oldsymbol{e}) 
eq oldsymbol{a}} lpha_g(oldsymbol{e}) \quad \hat{eta}_g = \sup_{oldsymbol{e}: g(oldsymbol{e}) 
eq oldsymbol{0}} eta_g(oldsymbol{e})$$

For  $m{e}$  such that  $g(Tm{e}) 
eq 0$ ,

$$egin{aligned} \|SToldsymbol{e}\|_A^2 &\leq \|Toldsymbol{e}\|_A^2 - \hat{lpha}_g g(Toldsymbol{e}) \leq \|Toldsymbol{e}\|_A^2 - rac{\hat{lpha}_g}{\hat{eta}_g}\|Toldsymbol{e}\|_A^2 &= \left(1 - rac{\hat{lpha}_g}{\hat{eta}_g}
ight)\|Toldsymbol{e}\|_A^2 \ &\leq \left(1 - rac{\hat{lpha}_g}{\hat{eta}_g}
ight)\|oldsymbol{e}\|_A^2 \end{aligned}$$

Ok, so this is generally worse than the sharp bound

$$\|ST\|_A = \sqrt{1 - \hat{\delta}} \leq \sqrt{1 - rac{\hat{lpha}_g}{\hat{eta}_g}}$$

(  $lpha_g$  and  $eta_g$  are not generally simultaneously satisfied)

Early works, e.g. Ruge-Stüben 1987, use  $g(m{e})=\|m{e}\|_{AD^{-1}A}^2$  Or the weaker form  $g(m{e})=\|Tm{e}\|_{AD^{-1}A}^2$ 

## **Smoothing and Approximation**

if there exists  $\bar{lpha}_g>0$  such that

$$||Se||_A^2 \le ||e||_A^2 - \bar{\alpha}_g g(e)$$
 for all  $e$  (smoothing)

and there exists  $ar{eta}_g>0$  such that

$$\|Toldsymbol{e}\|_A^2 \leq ar{eta}_g g(Toldsymbol{e}) \quad ext{ for all } oldsymbol{e} \quad ext{ (approximation)}$$

then 
$$\|ST\|_A \leq \sqrt{1-ar{lpha}_g/ar{eta}_g}$$

Select  $g(oldsymbol{e}) = \|oldsymbol{e}\|_{AD^{-1}A}^2$ 

Since T is an A-orthogonal projection we have

$$\lVert Te 
Vert_A = \inf_{e_c} \lVert e - Pe_c 
Vert_A$$

(strong approximation) Assume there is a  $ar{eta}_s$  such that

$$\inf_{oldsymbol{e}_c} \left\| oldsymbol{e} - P oldsymbol{e}_c 
ight\|_A^2 \leq ar{eta}_s \| oldsymbol{e} \|_{AD^{-1}A}^2 \quad ext{ for all } oldsymbol{e}.$$

The weaker version looks like (for some  $\hat{eta}$  )

$$||Te||_A^2 \leq \overline{\beta}||Te||_{AD^{-1}A} \quad \text{ for all } e.$$

Weaker, means weaker norm. And we can make this a bit more practical. The range of T is A-orthogonal to the range of P, so

$$egin{aligned} \|Toldsymbol{e}\|_A^2 &= \langle AToldsymbol{e}, Toldsymbol{e} - Poldsymbol{e}_c 
angle \ &\leq \|Toldsymbol{e}\|_{AD^{-1}A} \|Toldsymbol{e} - Poldsymbol{e}_c\|_D. \end{aligned}$$

(weak approximation) Assume that

$$\inf_{oldsymbol{e}} \left\| oldsymbol{e} - P oldsymbol{e}_c 
ight\|_D^2 \leq ar{eta}_w \|oldsymbol{e}\|_A^2 \quad ext{ for all } oldsymbol{e},$$