Let us construct the following finite dimensinal subspace:

$$V_{h,0} = \{ v : v \in C^0(\Omega),\ v|_K \in P_1(K),\ \forall K \in \mathcal{T}_h,\ v=0 \text{ on } \Gamma_i \}$$

(GFEM) Find $u_h \in V_{h,0}$ such that

$$(\bar{\beta}\cdot\nabla u_h, v) + (\alpha u_h, v) + (\varepsilon \nabla u_h, v) = (f, v) \qquad \forall v \in V_{h,0}$$

Note that the integration by parts gave:

$$(-\nabla\cdot(\varepsilon\nabla u), v) = (\varepsilon\nabla u, \nabla v) + \int_{\Gamma_i} \varepsilon \bar{n}\cdot\nabla u \underset{0}{\underset{\shortparallel}{v}} ds + \int_{\Gamma_o} \underbrace{\varepsilon \bar{n}\cdot\nabla u}_{=0} v\, ds$$

(a) Set $v = u_h$ in the Galerkin formulation:

$$\underbrace{(\bar{\beta}\cdot\nabla u_h, u_h)}_{\mathrm{I}} + \alpha\|u_h\|^2 + \varepsilon\|\nabla u\|^2 = \underbrace{(f, u_h)}_{\mathrm{II}}$$

We have:

$$(\bar{\beta}\cdot\nabla u_h, u_h) = -(u_h, \nabla\cdot(\beta u_h)) + \int_{\partial\Omega} \bar{\beta}\cdot\bar{n}\, u^2\, ds$$

$$= -(u_h, \bar{\beta}\cdot\nabla u_h) - (u_h, u_h \overbrace{\nabla\cdot\bar{\beta}}^{=0})$$

$$+ \int_{\Gamma_i} \bar{\beta}\cdot\bar{n}\, \underset{0}{\underset{\shortparallel}{u^2}} ds + \int_{\Gamma_o} \underbrace{\bar{\beta}\cdot\bar{n}}_{\geq 0} u^2 ds$$

$$= -(\bar{\beta}\cdot\nabla u_h, u_h) + \int_{\Gamma_o} \bar{\beta}\cdot\bar{n}\, u^2 ds$$

$$\Rightarrow \quad (\bar{\beta}\cdot\nabla u_h, u_h) = \frac{1}{2}\int_{\Gamma_o} \bar{\beta}\cdot\bar{n}\, u^2 ds \geq 0.$$

Since the term $I \geq 0$ we can drop it. On the other hand

$$(f, u_h) \leq \|f\| \, \|u_h\| \leq \frac{1}{2\alpha} \|f\|^2 + \frac{\alpha}{2} \|u_h\|^2,$$

therefore

$$\alpha \|u_h\|^2 + \varepsilon \|\nabla u_h\|^2 \leq \frac{1}{2\alpha} \|f\|^2 + \frac{\alpha}{2} \|u_h\|^2$$

$$\Rightarrow \quad \frac{\alpha}{2} \|u_h\|^2 + \varepsilon \|\nabla u_h\|^2 \leq \frac{1}{2\alpha} \|f\|^2.$$

Now we see that as $\varepsilon \to 0$ the second term vanishes, thus one can not guarantee that the gradient of u_h stays bounded. The Galerkin solution produces spruous oscillations.

(b) Denote $Lu_h := \bar{\beta} \cdot \nabla u_h + \alpha u_h - \nabla \cdot (\varepsilon \nabla u_h)$, then GLS reads: Find $u_h \in W_{h,0}$ such that

$$(Lu_h, v + \delta Lv) = (f, v + \delta Lv) \quad \forall v \in W_{h,0}$$

Note: * The space $W_{h,0}$ cannot be the same as $V_{h,0}$;

* Due to the term

$$\left(-\nabla \cdot (\varepsilon \nabla u_h), \delta \left(-\nabla \cdot (\varepsilon \nabla v)\right)\right)$$

u_h must be at least a quadratic function and $W_{h,0}$ must be from $H_0^2(\Omega)$, i.e. the first derivative is a continuous function.

Now set $v = u_h$ and get:

$$(Lu_h, u_h + \delta Lu_h) = (f, u_h + \delta Lu_h)$$

$$\Rightarrow \quad \underbrace{(Lu_h, u_h)} + \delta(Lu_h, Lu_h) = \underbrace{(f, u_h)} + \delta(f, Lu_h)$$

The blue terms are Galerkin term, therefore

$$\frac{\alpha}{2}\|u_h\|^2 + \varepsilon\|\nabla u_h\|^2 + \delta\|Lu_h\|^2 \leq \frac{1}{2\alpha}\|f\|^2 + \frac{\delta}{2}\|f\|^2 + \frac{\delta}{2}\|Lu_h\|^2$$

$$\Rightarrow \quad \frac{\alpha}{2}\|u_h\|^2 + \varepsilon\|\nabla u_h\|^2 + \frac{\delta}{2}\|Lu_h\|^2 \leq \left(\frac{1}{2\alpha} + \frac{\delta}{2}\right)\|f\|^2$$

We see that as $\varepsilon \to 0$ all terms in the discretization will be bounded, thus the GLS method is always stable.

(c) We are given

$$\text{(primal problem)} \begin{cases} \underbrace{\bar{\beta}\cdot\nabla u + \alpha u - \nabla\cdot(\varepsilon\nabla u)}_{Lu} = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_i, \\ \varepsilon\bar{n}\cdot\nabla u = 0 & \text{on } \Gamma_o \end{cases}$$

Using the definition of the adjoint operator, we have

$$\begin{aligned}
(Lu, \varphi) &= \left(\bar{\beta}\cdot\nabla u + \alpha u - \nabla\cdot(\varepsilon\nabla u), \varphi\right) \\
&= (u, -\bar{\beta}\cdot\nabla\varphi) + \int_{\Gamma_i} \bar{\beta}\cdot\bar{n}\, \underset{\substack{\parallel \\ 0}}{u}\, \varphi\, ds + \int_{\Gamma_o} \bar{\beta}\cdot\bar{n}\, u\, \underbrace{\varphi}_{=0}\, ds \\
&\quad + (u, \alpha\varphi) + \left(u, -\nabla\cdot(\varepsilon\nabla\varphi)\right) \\
&\quad - \int_{\Gamma_i} \varepsilon\bar{n}\cdot\nabla u\, \underbrace{\varphi}_{=0}\, ds - \int_{\Gamma_o} \underbrace{\varepsilon\bar{n}\cdot\nabla u}_{=0}\, \varphi\, ds \\
&\quad + \int_{\Gamma_i} \varepsilon\bar{n}\cdot\nabla\varphi\, \underset{\substack{\parallel \\ 0}}{u}\, ds + \int_{\Gamma_o} \underbrace{\varepsilon\bar{n}\cdot\nabla\varphi}_{=0}\, u\, ds
\end{aligned}$$

Collecting all terms gives us the followind dual problem

$$\begin{cases} -\bar{\beta}\cdot\nabla\varphi + \alpha\varphi - \nabla\cdot(\varepsilon\nabla\varphi) = \psi_\Omega & \text{in } \Omega, \\ \varphi = 0 & \text{on } \Gamma_i \cup \Gamma_0, \\ \varepsilon\bar{n}\cdot\nabla\varphi = 0 & \text{on } \Gamma_0. \end{cases}$$