UPPSALA UNIVERSITY
Department of Information Technology
Division of Scientific Computing

Test in Advanced Numerical Methods 2020-09-28

- *Time:* 2 Hours. *Tools:* Pocket calculator, Beta Mathematics Handbook.
- Part A concerns well-posedness for initial-boundary-value-problems and the FDM. Below we present the definitions of the first- and second-derivate SBP operators (D_1 and D_2):

$$D_1 = H^{-1}(Q+B), \quad B = -\tfrac{1}{2}e_1e_1^T + \tfrac{1}{2}e_me_m^T, \quad H = H^T > 0, \quad (Q+Q^T) = 0$$
$$D_2 = H^{-1}(-M - e_1d_1 + e_md_m), \quad M = M^T \geq 0, \quad d_mv \simeq u_x(x_m), \quad e_m^T = [0,\ldots,0,1]$$

- Part B concerns the FEM-part.
- All your answers must be well argued and calculations shall be demonstrated in detail. *Solutions that are not complete can still be of value if they include some correct thoughts.*

Part A, Question 1

Consider the following system of PDE,

$$\begin{array}{ll} \mathbf{u}_t + \mathbf{A}\mathbf{u}_x = \mathbf{C}\mathbf{u} + \mathbf{F} & , \quad x_l \leq x \leq x_r, \quad t \geq 0 \\ \mathbf{u} = \mathbf{f} & , \quad x_l \leq x \leq x_r, \quad t = 0 \end{array}, \tag{1}$$

where $\mathbf{F} = \mathbf{F}(x,t)$ is the forcing function, $\mathbf{f} = \mathbf{f}(x)$ the initial data and

$$\mathbf{A} = \begin{bmatrix} \alpha & 1 \\ 1 & 0 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}, \text{where } \alpha \text{ is a constant, and } c_{1,2} = c_{1,2}(x) \text{ bounded.}$$

a) Derive necessary conditions such that (1) yields a well-posed Cauchy problem (i.e., ignore boundary terms).

b) Derive conditions such that (1) yields a well-posed Cauchy problem, that furthermore leads to energy conservation when $\mathbf{F} = \mathbf{0}$.

c) Derive the correct number of boundary conditions (BC) at each boundary.

d) Let $\alpha = 2$ and derive a set of well-posed BC that leads to energy decay when $\mathbf{F} = \mathbf{0}$, $\mathbf{C} = \mathbf{0}$. (Show the energy estimate.)

e) Let $\alpha = 2$ and derive a set of well-posed BC that leads to energy conservation with zero boundary data, when $\mathbf{F} = \mathbf{0}$, $\mathbf{C} = \mathbf{0}$. (Show the energy estimate.)

f) Write down a consistent SBP-SAT discretization of (1) with the boundary conditions you derived in e). Note: You do not need to show stability.

g) Show stability for your SBP-SAT discretization in f) by choosing appropriate penalty parameters.

h) Write down a consistent SBP-Projection discretization of (1) with the boundary conditions you derived in e). Note: You do not need to show stability.

i) Show stability for your SBP-Projection discretization in h) by choosing appropriate penalty parameters.

j) Explain how you can verify parts g) and i) by using Matlab.

Part B, Question 1

Good luck!
Ken & Murtazo