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Solutions to part A, exam in Advanced Numerical Methods 2020-10-05

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Part A, Question 1

$$\begin{array}{llll} \mathbf{u}_t = \mathbf{A}\mathbf{u}_x + \mathbf{B}\mathbf{u} + \mathbf{F} & , & x_l \leq x \leq x_r, & t \geq 0 \\ \mathbf{u} = \mathbf{f} & , & x_l \leq x \leq x_r, & t = 0 \end{array} , \tag{1}$$

The forcing function $\mathbf{F}$ and the lower order term $\mathbf{Bu}$ has nothing to do with well-posedness since the lower order term is linear ($\mathbf{B}$ is not a function of $\mathbf{u}$). Strictly speaking we should assume that $\mathbf{B}$ is bounded, i.e. $\mathbf{B} + \mathbf{B}^* \leq \alpha$, where α is a constant. The problem (1) is hyperbolic if $\mathbf{A}$ has only real eigenvalues and is diagonalizable. This is a necessary requirement for the Cauchy problem to be well-posed. To see this first assume that $\mathbf{A}$ can be diagonalised, i.e., $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{\Lambda}$, where $\mathbf{\Lambda}$ holds the eigenvalues to $\mathbf{A}$. We now introduce the characteristic variables $\tilde{\mathbf{u}} = \mathbf{S}^{-1}\mathbf{u}$. By multiplying (1) from the left with $\mathbf{S}^{-1}$ we obtain,

$$\begin{array}{llll} \tilde{\mathbf{u}}_t = \mathbf{\Lambda}\tilde{\mathbf{u}}_x + \tilde{\mathbf{B}}\tilde{\mathbf{u}} + \tilde{\mathbf{F}} & , & x_l \leq x \leq x_r, & t \geq 0 \\ \tilde{\mathbf{u}} = \tilde{\mathbf{f}} & , & x_l \leq x \leq x_r, & t = 0 \end{array} , \tag{2}$$

where $\tilde{\mathbf{f}} = \mathbf{S}^{-1}\mathbf{f}$, $\mathbf{S}^{-1}\mathbf{B}\mathbf{S} = \tilde{\mathbf{B}}$ and $\tilde{\mathbf{F}} = \mathbf{S}^{-1}\mathbf{F}$. By multiplying the PDE in (2) by $\tilde{\mathbf{u}}^*$ (where we ignore the forcing and lower order term) IBP and adding the conjugate transpose we obtain,

$$\frac{d}{dt}\|\tilde{\mathbf{u}}\|^2 = \tilde{\mathbf{u}}^*\mathbf{\Lambda}\tilde{\mathbf{u}}|_{x=x_l}^{x=x_r} - \left(\tilde{\mathbf{u}}_x, (\mathbf{\Lambda} - \mathbf{\Lambda}^*)\,\tilde{\mathbf{u}}\right) = BT - \left(\tilde{\mathbf{u}}_x, (\mathbf{\Lambda} - \mathbf{\Lambda}^*)\,\tilde{\mathbf{u}}\right).$$

The boundary terms (BT) are given by

$$BT = \left.\left(\sum_{j=1}^{2} \lambda^{(j)}|\tilde{u}^{(j)}|^2\right)\right|^{x=x_r} - \left.\left(\sum_{j=1}^{2} \lambda^{(j)}|\tilde{u}^{(j)}|^2\right)\right|^{x=x_l},$$

where $\lambda^{(j)}$ is the jth eigenvalue and $\tilde{u}^{(j)}$ the corresponding characteristic variable. Hence, the correct (minimal) number of boundary conditions (BC) at $x = x_r$ equals the number of positive eigenvalues, and the correct (minimal) number of BC at $x = x_l$ equals the number of negative eigenvalues. The eigenvalues to

$$\mathbf{A} = \begin{bmatrix} 2 & \beta \\ 1 & 0 \end{bmatrix}, \text{where } \beta \text{ is a real constant,} \tag{3}$$

is given by $\lambda^{(1,2)} = 1 \pm \sqrt{1+\beta}$. The eigenvalues are real if $\beta \geq -1$. When $\beta = -1$, $\lambda^{(1,2)} = 1$ and we have a double root (this special case requires some special treatment in the analysis of well-posedness, although it is possible to show that the problem is well-posed). When $\beta = 0$, we have one positive and one zero eigenvalue, and only one BC is required (at the right boundary). For $\beta > 0$, we have one positive and one negative eigenvalue, and we need 1 BC at each boundary for well-posedness.

a) The problem is hyperbolic and thus well-posed (for the Cauchy problem) if $\beta \geq -1$, such that $\lambda^{(1,2)}$ are both real.

b) For $\beta = -\frac{1}{2}$, $\lambda^{(1,2)} = 1 \pm \sqrt{\frac{1}{2}} > 0$, and we thus should provide 2 BC at $x = x_r$ for well-posedness.

c) For $\beta = 1$,

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} ,$$

and $\lambda^{(1,2)} = 1 \pm \sqrt{2}$, and we should specify 1 BC at each boundary, and such that $BT = 0$ (here assuming zero boundary data and forcing function). Here $BT = \mathbf{u}^* \mathbf{A} \mathbf{u}|_{x=x_l}^{x=x_r} = 2\, u^{(1)}(u^{(1)} + u^{(2)})|_{x=x_l}^{x=x_r}$. Energy conservation with one BC requires either $u^{(1)} = 0$, or $u^{(1)} + u^{(2)} = 0$ to be specified at each boundary.

d) (*Here using SBP-SAT*) Let $e^{(1)} = [1\,,\ 0]$. A consistent SBP-SAT discretization with for example $u^{(1)} = 0$ specified at both boundaries is given by

$$v_t = A \otimes D_1 v + B \otimes I_m v + F + \tau_l \otimes H^{-1} e_1 \left(e^{(1)} \otimes e_1^T v \right) + \tau_r \otimes H^{-1} e_m \left(e^{(1)} \otimes e_m^T v \right) \ , \quad (4)$$

where $\tau_l = [\tau_l^{(1)},\ \tau_l^{(2)}]^T$ and $\tau_r = [\tau_r^{(1)},\ \tau_r^{(2)}]^T$ are the penalty vectors at the left and right boundaries. Here v is the solution vector.

d) (*Here using SBP-Projection*) Let $e^{(1)} = [1\,,\ 0]$. A consistent SBP-Projection discretization with for example $u^{(1)} = 0$ specified at both boundaries is given by

$$v_t = P\,(A \otimes D_1)\,Pv + P\,(B \otimes I_m)\,Pv + P\,F\ . \quad (5)$$

The boundary operator (using Matlab notation) is given by: $L = [e^{(1)} \otimes e_1\ ; e^{(1)} \otimes e_m]$. Let $\bar{H} = I_2 \otimes H$, where I_2 is the 2×2 identity matrix. The projection operator is given by $P = I - H^{-1} L^T (L H^{-1} L^T)^{-1} L$. Here v is the solution vector. Here $I = I_2 \otimes I_m$.