Solutions to test exam part 1 (FDM)

see PDF for instructions

$$(1)\quad \begin{cases} u_t + A u_x = \underbrace{C \cdot u}_{\text{Lower order term}} + F \\ u = f \end{cases} \qquad A = \begin{pmatrix} \alpha & 1 \\ 1 & 0 \end{pmatrix} \quad C = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$$

α constant

$c_{1,2} = c_{1,2}(x)$ bounded.

well-posed if

(i) $\|u\| \le K \cdot e^{\gamma t} \cdot \|f\|$ (For zero data)

where K, γ are constants

(ii) we impose minimal # BC (unique solution).

a) Set $F = 0$ and use Energy method

$$(u, u_t) = -(u, Au_x) + (u, cu) = -u^* A u \Big|_{x_l}^{x_r} + (u_x, Au) + (u, cu)$$

$$+ (u_t, u) = -(Au_x, u) + (cu, u) = -(u_x, A^* u) + (u, c^* u)$$

$$(*)\ \frac{d}{dt}\|u\|^2 = \underbrace{-u^* A u \Big|_{x_l}^{x_r}}_{\substack{BT = 0 \\ \text{Cauchy}}} + \left(u_x (A - A^*) u\right) + \underbrace{(u, (c + c^*) u)}_{\le 2\gamma \|u\|^2}$$

must vanish i.e. $A = A^*$

$\therefore$ (1) is well-posed (Cauchy problem) if A has real eigenvalues and C has bounded coeff.

Eigenvalues to A given by

$$\lambda_{1,2} = \frac{\alpha \pm \sqrt{\alpha^2+4}}{2}$$, hence α has to be real!

b) Energy conservation $\frac{d}{dt}\|u\| = 0$

if α - real and $C + C^* = 0$

$\Downarrow$

$A = A^*$ $\Rightarrow \frac{d}{dt}\|u\|^2 = \underbrace{BT}_{=0}$

c) $BT = -u^*Au\Big|_{x_l}^{x_r}$

$= -u^*Au\Big|^{x_r} + u^*Au\Big|^{x_l}$

pos eigenvalues $=$ # BC at $x = x_l$

neg. eigenvalues $=$ # BC at $x = x_r$

$$\lambda_{1,2} = \frac{\alpha \pm \sqrt{\alpha^2+4}}{2}$$

we have 1 positive and one negative eigenvalue

1 BC at each boundary (2 in total)

d) $\alpha = 2$ $F = 0$, $C = 0$ $\frac{d}{dt}\|u\|^2 < 0$

(**) $BT = -[u^{(1)}\ u^{(2)}]\begin{pmatrix}2 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}u^{(1)}\\ u^{(2)}\end{pmatrix}\Bigg|_{x_l}^{x_r} = -2u^{(1)}u^{(1)} - 2u^{(1)}u^{(2)}\Big|_{x_l}^{x_r}$

For example: $u^{(2)} = 0 \quad x = x_r$?

$\Rightarrow -2\,|u^{(1)}|^2\Big|^{x_r} < 0$. Yes!

($u^{(2)} = 0$ at $x = x_l$ well-posed?

$\Rightarrow 2\,|u^{(1)}|^2\Big|^{x_l} > 0$ not ok)

$$BT = -2\,u^{(1)}\left(u^{(1)} + u^{(2)}\right)\Big|_{x_l}^{x_r}$$

Set $u^{(1)} + \beta\left(u^{(1)} + u^{(2)}\right) = 0 \quad x = x_l \qquad \beta \neq 0$

Insert into BT $\left(u^{(1)} + u^{(2)} = -\frac{u^{(1)}}{\beta} \right)$

$$\Rightarrow 2\,u^{(1)} \cdot \left(-\frac{u^{(1)}}{\beta}\right)\Big|^{x_l} = -\frac{2}{\beta}\,u^{(1)} \cdot u^{(1)} < 0 .$$

chose $\beta > 0$

$$BT\Big|^{x_l} = 2\,u^{(1)}\left(u^{(1)} + u^{(2)}\right) \qquad u^{(2)} = -\gamma \cdot u^{(1)}$$

$$= 2\,u^{(1)}\left(u^{(1)} - \gamma\, u^{(1)}\right) = 2 \cdot (u^{(1)})^2 (1 - \gamma)$$

$\gamma > 1$

$$\alpha_1 u_t^{(1)} + \alpha_2 u^{(2)} + \beta_1 u^{(1)} + \beta_2 u^{(2)} = 0 .$$

Since we did not say decay at both boundaries we can here simply specify $u^{(1)} = 0$ at $x = x_l$

$\therefore \; (**)\begin{cases} u^{(2)} = 0 & x = x_r \\ u^{(1)} = 0 & x = x_l \end{cases}$ well-posed and introduce damping.

f) Let $e^{(1)} = [1 \quad 0]$, $e^{(2)} = [0 \quad 1]$ $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Assume we have $\begin{cases} u^{(2)} = 0 & x = x_r \\ u^{(1)} = 0 & x = x_l \end{cases}$ (2)

semi-discrete approx of (2) $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ $e_m = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

$$\begin{cases} \overbrace{(e^{(2)} \otimes e_m^T)v}^{v_m^{(2)}} = 0 \\ \underbrace{(e^{(1)} \otimes e_1^T)v}_{v_1^{(1)}} = 0 \end{cases} \qquad v = \begin{bmatrix} v_1^{(1)} \\ v_2^{(1)} \\ \vdots \\ v_m^{(1)} \\ v_1^{(2)} \\ v_2^{(2)} \\ \vdots \\ v_m^{(2)} \end{bmatrix} \begin{matrix} v^{(1)} \\ \\ v^{(2)} \end{matrix}$$

<u>SBP-SAT</u>

$$v_t + \underset{2\times 2}{A} \otimes \underset{m\times m}{D_1} v = \tau_l \otimes H^{-1} e_1 (e^{(1)} \otimes e_1^T v - g)$$

$$\tau_r \otimes H^{-1} e_m (e^{(2)} \otimes e_m^T v - g) \qquad (3)$$

$$\tau_l = \begin{bmatrix} \tau_l^{(1)} \\ \tau_l^{(2)} \end{bmatrix} \quad \tau_r = \begin{bmatrix} \tau_r^{(1)} \\ \tau_r^{(2)} \end{bmatrix}$$

$(A \otimes B)(C \otimes D) = AC \otimes BD$ rule

$$\underset{2\times 2}{\tau_l e^{(1)}} \otimes \underset{m\times m}{H^{-1} e_1 e_1^T}$$

g) Show stability for (3).

Multiply (3) by $v^T I_2 \otimes H$, add transpose

$$v^T I_2 \otimes H v_t = -v^T A \otimes H D_1 v + \overbrace{v^T (I_2 \otimes H)(\tau_l e^{(1)} \otimes H^{-1} e_1 e_1^T) v + v^T (I_2 \otimes H)(\tau_r e^{(2)} \otimes H^{-1} e_m e_m^T) v}$$

$$= -v^T A \otimes (Q + \tfrac{1}{2} B) v + \underbrace{\tau_l^{(1)} v_1^{(1)} \cdot v_1^{(1)} + \tau_l^{(2)} v_1^{(1)} v^{(2)} + \tau_r^{(1)} v_m^{(1)} \cdot v_m^{(2)} + \tau_r^{(2)} v_m^{(2)} v_m^{(2)}}_{= SAT}$$

$$+ v_t^T I_2 \otimes H v = -v^T A \otimes (Q^T + \tfrac{1}{2} B) v + SAT$$

$$\frac{d}{dt} \|v\|_H^2 = -v^T B v + 2 \cdot SAT$$

$$= +2 v_1^{(1)} \cdot v_1^{(1)} + 2 v_1^{(1)} \cdot v_1^{(2)} - 2 v_m^{(1)} v_m^{(1)} - 2 v_m^{(1)} v_m^{(2)}$$

$$+ 2\tau_l^{(1)} v_1^{(1)} \cdot v_1^{(1)} + 2\tau_r^{(1)} v_m^{(1)} v_m^{(2)}$$

$$+ 2 \tau_l^{(2)} v^{(1)} v^{(2)} + 2 \tau_r^{(2)} v_m^{(2)} v_m^{(2)}$$

$\tau_l^{(1)} \leq -1$

$\tau_l^{(2)} = -1$

$\tau_r^{(1)} = 1$

$\tau_r^{(2)} \leq 0$

For stability

$$\Longrightarrow \frac{d}{dt} \|v\|_H^2 = \underbrace{2(v_1^{(1)})^2 (1 + \tau_l^{(1)})}_{\leq 0} - \underbrace{2(v_m^{(1)})^2}_{<0} + \underbrace{2\tau_r^{(2)} (v_m^{(2)})^2}_{\leq 0}$$

additional damping from SAT

additional damping from SAT.

Assume BC given by (2)

h)
$$L^T = \begin{bmatrix} e^{(2)} \otimes e_m^T \\ e^{(1)} \otimes e_1^T \end{bmatrix}$$
Boundary operator

$$\Rightarrow P = I - \bar{H}^{-1} L (L^T \bar{H}^{-1} L)^{-1} L^T$$

where $\bar{H} = I_2 \otimes H$

SBP-Projection

$$V_t = -P A \otimes D_1 P V \qquad (4)$$

(with timedep. boundary data we have

$$V_t = -P A \otimes D_1 (P V + \tilde{B} g) + \tilde{B} g_t$$

where $\tilde{B} = \bar{H}^{-1} L (L^T \bar{H}^{-1} L)^{-1}$)

i) Multiply (4) by $V^T \bar{H}$ and add transp.

$$V^T \bar{H} V_t = -V^T \bar{H} P (A \otimes D_1) P V = -V^T P^T \bar{H} A \otimes D_1 P V$$

$$= (PV)^T A \otimes (Q + \tfrac{1}{2} B) PV$$

$$+ \; V_t^T \bar{H} V = (PV)^T A \otimes (Q^T + \tfrac{1}{2} B) PV$$

$$\Downarrow \quad \frac{d}{dt} \|V\|_H^2 = (PV)^T (A \otimes B)(PV) = -2 (PV)_m^{(1)} (PV)_m^{(1)} - 2 (PV)_m^{(1)} \underbrace{(PV)_m^{(2)}}_{= 0}$$

$$+ 2 \underbrace{(PV)_1^{(1)}}_{= 0} \underbrace{(PV)_1^{(1)}}_{= 0} + 2 \underbrace{(PV)_1^{(1)}}_{= 0} (PV)_1^{(2)}$$

$$= -2 (PV)_m^{(1)} (PV)_m^{(1)} = -2 \left(V_m^{(1)}\right)^2$$

$\therefore$ SBP-Projection exactly mimic underlying continuous energy estimate

j) Plot eigenvalues to semi-discrete problem and verify that eigenvalues are in the left halfplane (complex plane) i.e. no positive eigenvalues and here also some negative eigenvalues (due to damping).

($V_t = MV$ Plot eigenvalues to M)

<u>SBP-SAT:</u>

$$M = -A \otimes D_1 + \tau_l \otimes H^{-1} e_1 e^{(1)} \otimes e_1^T + \tau_r \otimes H^{-1} e_m e^{(2)} \otimes e_m^T$$

<u>SBP-Projection:</u>

$$M = -PA \otimes D_1 P$$