Uppsala university, Division of Scientific computing

Sample problems

1TD050: Advanced Numerical Methods, 10.0 hp

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1 Convection dominated problems

Problem 1. Consider the continuous Galerkin finite element method (GFEM) for the one-dimensional problem $-\varepsilon u'' + u' = 0$ in $(0, 1)$ with $u(0) = g, u(1) = 0$.

a) Write down the discrete equations for the GFEM approximation computed on a uniform mesh with M interior nodes.

b) Assume $g = 0$. Prove *a posteriori* and *a priori* error estimates in the energy norm when $\varepsilon = 1$.

c) Assume $g \neq 0$. Prove *a posteriori* error estimates using the goal-oriented argument. For this you need to derive a dual problem with appropriate boundary conditions.

d) What happens with the GFEM approximation for small values of viscosity, i.e.,. $\varepsilon \to 0$? Motivate your answer.

e) Write out the discrete equations when $\varepsilon = h/2$. Explain why this scheme is called the *upwind method* for the reduced problem. How is the convection term approximated by the Galerkin's method?

f) Perform a standard stability estimate for the GFEM approximation and discuss the stability with respect to the value of ε.

Problem 2. Consider a linear problem of the form $\mathcal{L}u = f$ in $\Omega \in \mathbb{R}^d$, where $\mathcal{L}$ is a linear differential operator and $f \in L^2(\Omega)$. Assume that $(\mathcal{L}v, v) \geq c\|v\|^2$ for some positive constant c.

a) Write down weak formulation and GFEM for this problem and obtain the Galerkin orthogonality.

b) Derive the Least-Squares finite element method (LSFEM) and explain why it is stable.

c) Derive and prove stability estimates for the Galerkin Least-Squares method (GLS). Note that LSFEM and GLS are not the same, what is the difference?

d) Is there a difference between Galerkin Least Squares and Petrov-Galerkin methods?

e) Streamline Diffusion method (SD) is a stabilized finite element approximation that is obtained by adding a residual based artificial viscosity to the GLS. Derive the SD method and perform a stability estimate.

f) Prove that LSFEM is well-posed, i.e., it satisfies Lax-Milgram Lemma.

Problem 3. Consider the following convection-reaction-diffusion problem in an open domain $\Omega \in \mathbb{R}^d$:

$$\begin{aligned} \boldsymbol{\beta}\cdot\nabla u + \alpha u - \nabla\cdot(\varepsilon\nabla u) &= f \quad \text{in } \Omega, \\ u &= g_1 \quad \text{on } \Gamma_{\text{inflow}}, \\ \varepsilon \boldsymbol{n}\cdot\nabla u &= g_2 \quad \text{on } \Gamma_{\text{outflow}}, \end{aligned} \tag{1}$$

where $\partial\Omega = \Gamma_{\text{inflow}} \cup \Gamma_{\text{outflow}}$, $f \in L^2(\Omega;\mathbb{R})$, $g_1 \in L^2(\partial\Omega;\mathbb{R})$ and $g_2 \in L^2(\partial\Omega;\mathbb{R})$ are given functions, $\boldsymbol{\beta}(\boldsymbol{x}) \in L^\infty(\Omega;\mathbb{R}^d)$ is a convection vector, ε and α are given constants, $\boldsymbol{n}(\boldsymbol{x})$ is the outward normal to the boundary at point $\boldsymbol{x}$. Moreover, inflow and outflow boundaries are defined by the convection vector:

$$\Gamma_{\text{inflow}} := \{\boldsymbol{x} \in \partial\Omega : \boldsymbol{\beta}\cdot\boldsymbol{n} < 0\} \text{ and } \Gamma_{\text{outflow}} := \{\boldsymbol{x} \in \partial\Omega : \boldsymbol{\beta}\cdot\boldsymbol{n} \geq 0\}.$$

a) Write down the weak and Galerkin finite element approximation for this problem.

b) Prove that

$$(\boldsymbol{\beta}\cdot\nabla u, u) \geq -\frac{1}{2}(u\nabla\cdot\boldsymbol{\beta}, u).$$

c) Give a necessary condition under which the Lax-Milgram Lemma states the well-posedness of the GFEM for the convection-reaction-diffusion problem. What would be the condition if the convection field is a divergence free?

d) Assume that $\alpha - \frac{1}{2}\nabla\cdot\boldsymbol{\beta} \geq c > 0$, for some constant c. Prove that

$$\|\varepsilon^{\frac{1}{2}}\nabla u\|^2 + \|u\|^2 \leq C\|f\|^2,$$

where $C > 0$ is a constant that depends on c and the boundary data.

e) Formulate the Streamline Diffusion method for this problem.

f) Assume that $\alpha - \frac{1}{2}\nabla\cdot\boldsymbol{\beta} \geq c > 0$, for some constant c. Prove that the stability estimate for the SD method applied to this problem is

$$\|\hat{\varepsilon}^{\frac{1}{2}}\nabla u\|^2 + \|\delta^{\frac{1}{2}}(\boldsymbol{\beta}\cdot\nabla u + \alpha u)\|^2 + \|u\|^2 \leq C\|f\|^2,$$

where $\delta = \frac{1}{2}\frac{h}{|\boldsymbol{\beta}|}$, h is the local mesh-size, $\hat{\varepsilon} = \max(\varepsilon, c_1 h^2 |\boldsymbol{\beta}\cdot\nabla u + \alpha u|)$, $C > 0$ is a constant that depends on c and the boundary data.

g) Let $\mathcal{L}$ be a linear operator. The *adjoint* operator $\mathcal{L}^*$ of $\mathcal{L}$ is a linear operator that is computed as $(\mathcal{L}u, v) = (u, \mathcal{L}^* v)$. Prove that the adjoint operator to the convection-reaction-diffusion without boundary terms is $\mathcal{L}^* z := -\nabla\cdot(\boldsymbol{\beta} z) + \alpha z - \nabla\cdot(\varepsilon\nabla z)$.

h) Consider a pure convection equation, i.e., $\alpha = \varepsilon = 0$. In addition assume $g_1 = g_2 = 0$. Derive and prove a goal oriented a posteriori error estimate with respect to the target functional $J(u) = \int_\Omega u \,\mathrm{d}\boldsymbol{x}$.

Problem 4. Consider the following time dependent convection-reaction-diffusion problem in an open domain $\Omega \in \mathbb{R}^d$:

$$\begin{aligned} \partial_t u + \boldsymbol{\beta}\cdot\nabla u + \alpha u - \nabla\cdot(\varepsilon\nabla u) &= f \quad \text{in } \Omega \times [0,T], \\ u &= g_1 \quad \text{on } \Gamma_{\text{inflow}} \times [0,T], \\ \varepsilon \boldsymbol{n}\cdot\nabla u &= g_2 \quad \text{on } \Gamma_{\text{outflow}} \times [0,T], \\ u(\boldsymbol{x},0) &= u_0(\boldsymbol{x}) \quad \text{in } \Omega,\ t = 0. \end{aligned} \tag{2}$$

where $\partial\Omega = \Gamma_{\text{inflow}} \cup \Gamma_{\text{outflow}}$, $f \in L^2(\Omega \times [0,T];\mathbb{R})$, $g_1 \in L^2(\partial\Omega \times [0,T];\mathbb{R})$,$g_2 \in L^2(\partial\Omega \times [0,T];\mathbb{R})$ and $u_0 \in L^2(\Omega;\mathbb{R})$ are given functions, $\boldsymbol{\beta}(\boldsymbol{x}) \in L^\infty(\Omega \times [0,T];\mathbb{R}^d)$ is a convection vector, ε and α are given constants, $\boldsymbol{n}(\boldsymbol{x})$ is the outward normal to the boundary at point $\boldsymbol{x}$. Moreover, inflow and outflow boundaries are defined by the convection vector:

$$\Gamma_{\text{inflow}} := \{\boldsymbol{x} \in \partial\Omega : \boldsymbol{\beta}\cdot\boldsymbol{n} < 0\} \text{ and } \Gamma_{\text{outflow}} := \{\boldsymbol{x} \in \partial\Omega : \boldsymbol{\beta}\cdot\boldsymbol{n} \geq 0\}.$$

We assume that the convection term is divergence free.

a) Derive a weak and Galerkin finite element formulations with appropriate spaces.

b) Write down the Galerkin approximation in the matrix form and explain what are the matrices. Also show how the boundary conditions are implemented.

c) Prove the standard stability estimates.

d) Looking at the stability estimate from the part **c)** discuss why the Galerkin approximation is unstable for vanishing ε.

e) Derive the GLS method for this problem and prove the standard stability estimates.

f) Derive the SD method for this problem and prove the standard stability estimates.

g) Derive the RV method for this problem and prove the standard stability estimates.

h) Looking at parts **e)**, **f)**, **g)** what conclusion can you draw in terms of stability, efficiency and implementation?

i) Show that the dual problem in this case can be defined as follows:

$$
\begin{aligned}
-\partial_t z - \boldsymbol{\beta}\cdot\nabla z + \alpha z - \nabla\cdot(\varepsilon\nabla z) = \psi_\Omega \quad & \text{in } \Omega\times[T,0], \\
z = 0 \quad & \text{on } \Gamma_{\text{outflow}}\times[T,0], \\
\varepsilon\boldsymbol{n}\cdot\nabla z = 0 \quad & \text{on } \Gamma_{\text{outflow}}\times[T,0], \\
z(\boldsymbol{x},T) = 0 \quad & \text{in } \Omega,\ t=T.
\end{aligned}
\tag{3}
$$

j) Assume $J(u) = \int_{\Omega\times[0,T]} u\psi_\Omega \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$ is a linear target functional. Prove the following a posteriori estimate for the error on the target functional:

$$
J(u) - J(u_h) = \int_{\Omega\times[0,T]} -R(u_h)(z-\pi_h z)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,
$$

where $R(u_h) := f - \partial_t u_h - \boldsymbol{\beta}\cdot\nabla u_h - \alpha u_h + \nabla\cdot(\varepsilon\nabla u_h)$, is the finite element residual and $\pi_h z$ is an interpolation operator in the finite element space V_h.

k) Write down an adaptive algorithm that uses the above goal oriented error estimate and refines 10% of the cells with the largest error indicator.

2 Nonlinear problems

Solve all problems at page 239 in the Larsson-Bengzon's book.

3 Abstract finite element formulation

Solve problems 7.1, 7.2, 7.3, 7.4, 7.8, 7.9, 7.10 at pages 200-201 in the Larsson-Bengzon's book.