Securing Distributed Machine Learning in High Dimensions

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Abstract

Standard distributed machine learning frameworks require collecting the training data from data providers and storing it in a datacenter. To ease privacy concerns, alternative distributed maching learning frameworks (such as Federated Learning) have been proposed, wherein the training data is kept confidential by its providers from the learner, and the learner learns the model by communicating with data providers. However, such frameworks suffer from serious security risks, as data providers are vulnerable to adversarial attacks and the learner lacks of enough administrative power. We assume in each communication round, up to q out of the m data providers/workers suffer Byzantine faults. Each worker keeps a local sample of size n and the total sample size is N=nm. Of particular interest is the high-dimensional regime, where the local sample size n is much smaller than the model dimension d.

We propose a secured variant of the gradient descent method and show that it tolerates up to a constant fraction of Byzantine workers. Moreover, we show the statistical estimation error of the iterates converges in $O(\log N)$ rounds to $O(\sqrt{q/N} + \sqrt{d/N})$, which is larger than the minimax-optimal error rate $O(\sqrt{d/N})$ in the failure-free setting by at most an additive term $O(\sqrt{q/N})$. As long as q = O(d), our proposed algorithm achieves the optimal error rate $O(\sqrt{d/N})$. The core of our method is a robust gradient aggregator based on the iterative filtering algorithm proposed by Steinhardt et al. [SCV18]. We establish a uniform concentration of the sample covariance matrix of gradients, and show that the aggregated gradient, as a function of model parameter, converges uniformly to the true gradient function. As a by-product, we develop a new concentration inequality for sample covariance matrices, which might be of independent interest.

1 Introduction

In classical learning frameworks, data is collected from its providers (who may or may not be voluntary) and is stored by the learner. Such data collection immediately leads to data providers' serious privacy concerns, which root in not only pure psychological reasons but also the poor real-world practice of privacy-preserving solutions. In fact, privacy breaches occur frequently, with recent examples including Facebook data leak scandal, iCloud leaks of celebrity photos, and PRISM surveillance program. Putting this privacy risk aside, data providers often benefit from the learning outputs. For example, in medical applications, although participants may be embarrassed about their use of drugs, they might benefit from good learning outputs that can provide high-accuracy predictions of developing diseases.

To resolve this dilemma of data providers, researchers and practitioners have proposed an alternative learning framework wherein the training data is kept confidential by its providers from

the learner and these providers function as workers [KMR15, MR17, DJW14]. This framework has been implemented in many practical systems such as Google's Federated Learning [KMR15, MR17], wherein Google tries to learn a model with the training data kept confidential on the users' mobile devices. We refer to this new learning framework as learning with external workers. In contrast to the traditional learning framework under which models are trained within datacenters, in learning with external workers the learner faces serious security risk. On the one hand, some external workers may be highly unreliable or even be adversarial. For example, in Federated Learning, the users' mobile devices might be unreliable and be easily compromised by hackers. On the other hand, the learner lacks enough administrative power over those external workers. What's worse, each worker/data provider typically has access to a sample of small size comparing to the model dimension, as is often the case in Federated Learning. Two immediate consequences are: (1) to learn an accurate model, the learner has to interact closely with those external workers, and such close interaction gives the adversary more chances to foil the learning process; (2) identifying the adversarial workers based on abnormality is highly challenging, because it is difficult to distinguish the statistical errors from the adversarial errors when the sample sizes are small.

In this paper, we aim to develop strategies to safeguard distributed machine learning against adversarial workers while keeping the following two key practical constraints in mind: ¹

- Small local samples versus high model dimensions: While the total volume of data over all
 workers may be large, individual workers may keep only small samples comparing to model
 dimensions.
- Communication constraints: Transmission between the external workers and the learner typically suffers from high latency and low-throughout. Thus communication between them is a scarce resource. In fact, communication cost is often argued to be the principal constraint in Federated Learning [MMR⁺16, KMY⁺16, KMR15].

Related Work There is a recent flurry of work on securing distributed machine learning algorithms against adversarial attacks [CSX17, CSV17, FXM14, SV16, DKK⁺17, DKK⁺16, BMGS17, AAZL18, YCRB18], among which the most related work is [CSX17, CSV17, FXM14, SV16, BMGS17, AAZL18, YCRB18]. Both [SV16, BMGS17] considered a pure optimization framework and characterizations of statistical performance of the learning outputs are left open; whereas [CSX17, AAZL18, YCRB18] studied the same statistical learning framework as ours.

Among the proposed algorithms, the robust one-shot aggregation algorithms [FXM14, YCRB18] use only one round communication and hence are communication-efficient. The correctness of these algorithms rely crucially on the assumption that the local sample size satisfies $n=N/m\gg d$, where m is the number of workers/machines, n is the local sample size, N=nm is the total sample size, and d is the model dimension. Unfortunately, this assumption excludes the applications of such one-shot algorithms to the high-dimensional regime. There have been attempts to robustify stochastic gradient descent (SGD) [BMGS17, AAZL18]. However, the mean squared error of SGD is only O(1/t) with t iterations [NJLS09] even when the population risk function is strongly convex. This is undesirable for applications where the model dimensions are high and communication is costly [KMY+16, MMR+16].

In contrast to SGD, full gradient descent converges exponentially fast for strongly convex population risk; hence requires only a few communication rounds. In each iteration of the full gradient descent, each worker computes the local gradient based on the *entire* local sample (all n data points). Since n is small, the computational burdens of the workers are reasonable. The previous

¹Depending on the detailed applications, there might be many other constraints.

work [CSX17] proposed a computationally-efficient algorithm – Byzantine-resilient gradient descent (BGD) method, where in each round the learner computes the geometric median of means of the gradients reported by the workers. BGD converges in logarithmic rounds to an estimation error $O(\sqrt{dq/N})$ for $q \ge 1$, which is larger than the minimax-optimal error rate $\sqrt{d/N}$ in the failure-free setting by at most a multiplicative factor of \sqrt{q} . In the low dimensional regime where d = O(1), more recent work [YCRB18] obtains an order-optimal error rate based on coordinate-wise median and trimmed mean, but the dependency of error rate on dimension d is highly suboptimal and is even inferior to the algorithm in [CSX17]. In this work, we focus on improving the estimation error in the more challenging and practical high-dimensional regime.

Contributions We propose a robust full gradient descent method that tolerates up to a constant fraction of adversarial workers and converges to a statistical estimation error $O(\sqrt{q/N} + \sqrt{d/N})$ in $O(\log N)$ communication rounds. This estimation error is larger than the minimax-optimal error rate $O(\sqrt{d/N})$ in the failure-free setting by at most an additive term $O(\sqrt{q/N})$. As long as q = O(d), our proposed algorithm achieves the optimal error rate $O(\sqrt{d/N})$.

On the technical front, to deal with the interplay of the randomness of the data and the iterative updates of θ , we first establish the concentration of sample covariance matrix of gradients uniformly at all possible model parameters; then we prove that our aggregated gradient, as a function of θ , converges uniformly to the population gradient function $\nabla F(\cdot)$. Similar uniform convergence concentration of sample covariance matrix has been derived in [CSV17, Lemma 2.1] under the assumption that the gradients are sub-gaussian. However, even in the simplest linear regression example, the gradients are sub-exponential instead of sub-gaussian. To this end, we develop a new concentration inequality for sample covariance matrices, which might be of independent interest.

2 System Model

Let X denote the input data generated from some unknown distribution P. Let $\Theta \subset \mathbb{R}^d$ denote the set of all possible model parameters. We consider a risk function $f: \mathcal{X} \times \Theta \to \mathbb{R}$, where $f(x,\theta)$ measures the risk induced by a realization x of the data under the model parameter choice θ . A classical example of the above statistical learning framework is linear regression, where $x = (w,y) \in \mathbb{R}^{d-1} \times \mathbb{R}$ is the feature-response pair and $f(x,\theta) = \frac{1}{2} \left(\langle w,\theta \rangle - y \right)^2$. The learner is interested in learning θ^* which minimizes the *population risk* F, i.e.,

$$\theta^* \in \arg\min_{\theta \in \Theta} F(\theta) \triangleq \mathbb{E}\left[f(X, \theta)\right]$$
 (1)

– assuming that $\mathbb{E}[f(X,\theta)]$ is well-defined. The model choice θ^* is optimal in minimizing the expected risk to pay if it is used for prediction over fresh data.

If the population risk F were known, then θ^* might be computed by solving the minimization problem in (1). In statistical learning, however, the distribution P (thus the population risk F) is typically unknown; instead, training data is available for learning θ^* . Formally, we assume that there exist N i.i.d. data points $X_i^{\text{i.i.d.}}P$ in the decentralized learning system wherein the training data is kept locally by data providers and cannot be accessed by the learner directly. The learner can only request those providers to compute gradient-like quantities of their locally kept data, as is the case in Federated Learning. We refer to those data providers as workers, as they can be viewed as "recruited" by the learner. We assume there are m workers, and the N data points are distributed evenly across the m workers. Specifically, the index set [N] is partitioned into m subsets S_i such that $|S_i| = N/m \triangleq n$, and $S_i \cap S_i = \emptyset$ for $i \neq j$. In Federated Learning, Google is the learner

and users' mobile devices, such as smart phones, are the workers. Notably, the local data volume n is often much smaller than the model dimension d, which we refer to as the high-dimensional regime.

The learner communicates with the workers in synchronous communication rounds, but nonfaulty workers do not communicate with each other. We leave the asynchronous communication as one future direction. We use the Byzantine fault model [Lyn96] to capture the unreliability and potential malicious behaviors of the workers. It is assumed that up to q out of the m workers might suffer Byzantine faults and thus behave arbitrarily and possibly maliciously. Those faulty workers are referred to as Byzantine workers. The arbitrarily faulty behavior arises when the workers are reprogrammed by the adversary. We assume the learner knows the upper bound q – a standard assumption in literature [DKK+17, CSV17, SCV18]. Nevertheless, an effective and efficient learning algorithm that does not call for the knowledge of q as input is highly desired. The set of Byzantine workers is allowed to *change* between communication rounds; the adversary can choose different sets of workers to control across communication rounds. Byzantine workers are assumed to have complete knowledge of the system, including the total number of workers m, all N data points over the whole system, the programs that the workers are supposed to run, the program run by the learner, and the realization of the random bits generated by the learner. Moreover, Byzantine workers can collude. Nevertheless, when the adversary gives up the control of a worker, this worker recovers and becomes normal immediately. Note that this mobile Byzantine fault model is more general than the most classic Byzantine fault model, where the set of Byzantine workers is fixed throughout an execution.

As can be seen later, the (mobile) Byzantine faults creates unspecified dependency across communication rounds — a key challenge in our convergence analysis.

3 Our Algorithm and Main Results

A standard approach to estimate θ^* in statistical learning is via empirical risk minimization. Given N independent copies X_1, \dots, X_N of X, the empirical risk function is a random function over Θ defined as $(1/N) \sum_{i=1}^N f(X_i, \theta)$ for all $\theta \in \Theta$. By the functional law of large numbers, the empirical risk function converges uniformly to the population risk function in probability as sample size $N \to \infty$. As a consequence, we expect the minimizer of the empirical risk function (which is random) also converges to the population risk minimizer θ^* in probability. Unfortunately, even a single Byzantine worker can completely skew the empirical risk function and thus foils the whole empirical risk minimization approach [SV16]. While it may be possible to secure the empirical risk minimization using some "robust" versions of empirical risk functions [SV16, CSV17], the characterizations of the estimation error are either unavailable or too loose. Moreover, in our distributed setting, it is costly to transmit the local empirical risk functions.

In this paper, we take a different approach: Instead of robustifying the empirical risk functions, we aim at robustifying the *learning process*. Specifically, we focus on securing the gradient descent method against the interruption caused by the Byzantine workers during model training. As commented in the introduction, due to the fact that (1) the local sample sizes are typically small, and that (2) the communication is a scarce resource (in terms of rounds), we consider full gradient descent – having each non-faulty worker compute gradient based on the entire local sample. Our approximate (full) gradient descent method is given by Algorithm 1.

If worker j is non-faulty at round t, on receipt of θ_{t-1} , it computes its local gradient at θ_{t-1} and reports the computed gradient to the learner; if worker j is Byzantine faulty at round t, it may

Algorithm 1 Approximate Gradient Descent Method: Round $t \geq 1$

The learner:

- 1: Initialization: Let θ_0 be an arbitrary point in Θ . Let $\eta = \frac{M}{2L^2}$.
- 2: Broadcast the current model iterate θ_{t-1} to all workers;
- 3: Wait to receive all the gradients reported by the m workers; Let $g_j(\theta_{t-1})$ denote the value received from worker j.

If no message from worker j is received, set $g_i(\theta_{t-1})$ to be some arbitrary value;

4: Aggregate gradients: Pass the received gradients to a gradient aggregator \mathcal{R} to obtain an aggregated gradient $G(\theta_{t-1})$, i.e.,

$$G(\theta_{t-1}) \leftarrow \mathcal{R}(g_1(\theta_{t-1}), \cdots, g_j(\theta_{t-1}), \cdots, g_m(\theta_{t-1})). \tag{2}$$

5: Update: $\theta_t \leftarrow \theta_{t-1} - \eta \times G(\theta_{t-1});$

Worker j:

- 1: On receipt of θ_{t-1} , compute the gradient at θ_{t-1} , i.e., $\frac{1}{n} \sum_{i \in \mathcal{S}_i} \nabla f(X_i, \theta_{t-1})$;
- 2: Send $\frac{1}{n} \sum_{i \in S_j} \nabla f(X_i, \theta_{t-1})$ back to the learner;

report arbitrary value to the learner. Formally,

$$g_j(\theta_{t-1}) = \begin{cases} \frac{1}{n} \sum_{i \in \mathcal{S}_j} \nabla f(X_i, \theta_{t-1}), & \text{if worker } j \text{ is non-faulty at round } t; \\ \star, & \text{otherwise,} \end{cases}$$

where \star denotes arbitrary value. Notably, the Byzantine workers might use all the information of the system to determine what value to report. The learner aggregates the received gradients via a gradient aggregator \mathcal{R} (an algorithmic function) to obtain an approximate gradient (line 4 of Algorithm 1).

In the failure-free setting, one can simply use the naïve averaging as a gradient aggregator \mathcal{R} . Unfortunately, averaging is not resilient to even a single Byzantine fault. This is because Byzantine workers have complete information of the system, including the gradients reported by other workers and the realization of the random bits generated by the learner. The previous work [CSX17] chooses its gradient aggregator \mathcal{R} to be the geometric median of means and obtains an estimation error $O(\sqrt{dq/N})$ for $q \geq 1$. In the low dimensional regime where d = O(1), more recent work uses coordinate-wise median and trimmed mean [YCRB18] and obtains an order-optimal error rate. However, its dependency of error rate on dimension d is highly suboptimal and even inferior to that in [CSX17].

To secure the full gradient descent in high dimensions, we use a gradient aggregator that originates from robust mean estimation [SCV18]. However, the existing performance guarantees and analysis are far from being applicable to the general statistical learning problems directly. This is because: (1) The literature on robust mean estimation [SCV18, CSV17, DKK+17] often focuses on the case $q/m = \Theta(1)$, whereas we are interested in the more general case q/m = O(1), including the practically important diminishing error regime; (2) For robust mean estimation [SCV18], only a concentration of random data points is needed, whereas in our setting, we need to establish a

concentration of random functions. For ease of exposition, we postpone the presentation of this gradient aggregator after the statements of our main results.

3.1 Main Results

To characterize the statistical estimation error rate of our proposed algorithm, we need to adopt a set of assumptions. Note that this set of assumptions are quite standard and are satisfied by the standard linear regression as shown in [CSX17]; similar assumptions are also made in [CSX17] and [YCRB18].

Recall that $\nabla F(\theta_{t-1})$ is the gradient of the population risk at θ_{t-1} , η is some fixed stepsize, and θ_0 is the given initial guess of θ^* . For the perfect gradient descent method, i.e.,

$$\theta_t = \theta_{t-1} - \eta \times \nabla F(\theta_{t-1}),\tag{3}$$

to converge exponentially fast, the following standard assumption is often adopted [BV04].

Assumption 1. The population risk function $F : \Theta \to \mathbb{R}$ is M-strongly convex, and differentiable over Θ with L-Lipschitz gradient. That is, for all $\theta, \theta' \in \Theta$,

$$F(\theta') \ge F(\theta) + \langle \nabla F(\theta), \ \theta' - \theta \rangle + \frac{M}{2} \|\theta' - \theta\|^2$$
, and $\|\nabla F(\theta) - \nabla F(\theta')\| \le L \|\theta - \theta'\|$.

Note that both M and L may scale in d – the dimension of θ . Let $S^{d-1} = \{v \in \mathbb{R}^d : ||v|| = 1\}$ denote the unit Euclidean sphere.

Assumption 2. The sample gradient at the optimal model parameter θ^* , i.e., $\nabla f(X, \theta^*)$, is sub-exponential with constants (σ_1, α_1) , i.e., for every unit vector $v \in S^{d-1}$,

$$\mathbb{E}\left[\exp\left(\lambda\langle\nabla f(X,\theta^*),v\rangle\right)\right] \le e^{\sigma_1^2\lambda^2/2}, \quad \forall |\lambda| \le \frac{1}{\alpha_1}.$$

We further assume the Lipschitz continuity of the sample gradient functions.

Assumption 3. There exists an L' such that

$$\|\nabla f(X,\theta) - \nabla f(X,\theta')\| \le L' \|\theta - \theta'\| \quad \forall \ \theta, \theta' \in \Theta.$$

For applications where Assumption 3 does not hold deterministically, it suffices to have Assumption 3 hold with high probability. Notably, L' may be much larger than L and even scale polynomially in N and d. However, L' affect our results only by logarithmic factors $\log L'$.

Next define the gradient difference function

$$h(X,\theta) = \nabla f(X,\theta) - \nabla f(X,\theta^*) - (\nabla F(\theta) - \nabla F(\theta^*)). \tag{4}$$

Note that $h(X,\theta)/\|\theta-\theta^*\|$ characterizes the change of $f(X,\theta)-\nabla F(\theta)$ from $f(X,\theta^*)-\nabla F(\theta^*)$ in terms of $\|\theta-\theta^*\|$; hence it can be viewed as a local Lipschitz parameter with respect to θ^* .

Assumption 4. The local Lipschitz parameter $h(X, \theta) / \|\theta - \theta^*\|$ is sub-exponential with constants (σ_2, α_2) , i.e.., for fixed θ and v,

$$\mathbb{E}\left[\exp\left(\frac{\lambda\langle h(X,\theta),v\rangle}{\|\theta-\theta^*\|}\right)\right] \leq e^{\sigma_2^2\lambda^2/2}, \quad \forall |\lambda| \leq \frac{1}{\alpha_2}.$$

Notably, Assumption 4 assumes a concentration of the *local* Lipschitz parameter with respect to θ^* , instead of a *global* Lipschitz parameter. Now we are ready to present our main results.

Theorem 1 (Main results). Suppose Assumptions 1, 2, 3, and 4 hold. Assume that $\log(L+L') = O(\log(Nd))$ and $\Theta \subset \{\theta : \|\theta - \theta^*\| \le r\}$ for some positive parameter r such that $\log r = O(\log(Nd))$. Suppose that $N \ge cd^2 \log^8(Nd)$ for a sufficiently large constant c, and that $m \le e^{\sqrt{d}}$. Further assume that $N \ge c'q$ for a sufficiently large constant c', and that $M \ge 1$. Then there exists an algorithm of the form in Algorithm 1 such that with probability at least $1 - 2e^{-\sqrt{d}}$, the generated iterates $\{\theta_t\}$ with $\eta = L/(2M^2)$ satisfy

$$\|\theta_t - \theta^*\| \lesssim \left(1 - \frac{M^2}{16L^2}\right)^t \|\theta_0 - \theta^*\| + \left(\sqrt{\frac{q}{N}} + \sqrt{\frac{d}{N}}\right).$$

Note that in Theorem 1, the Lipschitz parameters L, L', and the size of the search space r are allowed to scale even polynomially in N and d. The set of assumptions (Assumptions 1, 2, 3, and 4) is similar to (indeed is more relaxed than) that adopted in [CSX17, CSV17].

Theorem 1 says that the estimation error $O(\sqrt{q/N} + \sqrt{d/N})$ is achieved, significantly improving the previous results $(O(\sqrt{dq/N}))$ for $q \ge 1$ in the high-dimensional regime [CSX17]. Recall that even in the failure-free setting the minimax-optimal error rate is $O(\sqrt{d/N})$. Thus, an immediate consequence of Theorem 1 is that as long as q = O(d), our proposed algorithm achieves the optimal error rate $O(\sqrt{d/N})$. Recall that the sample gradient at each θ is sub-exponential. The sample complexity $N \ge cd^2 \log^8(Nd)$ appears to be suboptimal at first glance. However, it turns out that if we rely on uniform concentration of the sample covariance matrices, this sample complexity is order optimal up to logarithmic factors. See Remark 1 for details.

More recent work uses coordinate-wise median and trimmed mean [YCRB18] and obtains an estimation error $rd(q/(m\sqrt{n}) + 1/\sqrt{N})$ up to logarithmic factors – noting that the radius of the model parameter space r is typically on the order of \sqrt{d} . This error rate is shown to be order-optimal in the low dimensional regime where d = O(1) [YCRB18], but turns out to scale very poorly in dimension d. In comparison, our result significantly improves the state-of-the-art in the more challenging and practical high dimensional regimes.

As an important ingredient of our proof of Theorem 1, we establish a uniform concentration of the sample covariance matrix of gradients. Recall that in our problem local sample gradients $\frac{1}{n}\sum_{i\in\mathcal{S}_j}\nabla f(X_i,\theta)$'s are sub-exponential random vectors. Standard routine to bounding the spectral norm of the sample covariance matrix is available, see [Ver10, Theorem 5.44] and [ALPTJ10, Corollary 3.8] for example. However, it turns out that using these existing results, the uniform concentration bound obtained is far from being optimal. To this end, we develop a new concentration inequality for matrices with i.i.d. sub-exponential column vectors. As can be seen later, this new inequality leads to a near-optimal uniform bound.

Theorem 2. Let A be a $d \times m$ matrix whose columns A_j are independent and identically distributed sub-exponential, zero-mean random vectors in \mathbb{R}^d with parameters (σ, α) . Assume that

$$\sigma/\alpha = \Omega(1). \tag{5}$$

Then with probability at least $1 - \delta$,

$$||A|| \le c \left(\sigma \sqrt{m} + \sigma \phi \left(d + \log \frac{1}{\delta}\right) + \alpha \phi^2 \left(d + \log \frac{1}{\delta}\right)\right),$$

where c is a universal positive constant and $\phi(x): \mathbb{R} \to \mathbb{R}$ is a function given by $\phi(x) = \sqrt{x} \log^{3/2}(x)$.

If A has sub-Gaussian columns, i.e., $\alpha=0$, then the upper bound in Theorem 2 matches the sub-Gaussian matrix concentration inequality [Ver18][Theorem 5.39] up to logarithmic factors. If $\sigma, \alpha=\Theta(1)$ and $\log(1/\delta)=d$, Theorem 2 implies that with probability at least $1-e^{-d}$, $||A||\lesssim \sqrt{m}+d\log^3 d$, which is tight up to logarithmic factors; whereas the analogous bound implied by standard concentration inequality [ALPTJ10, Corollary 3.8] is on the order of $\sqrt{md}+d$. See Remark 6 in Section 4.1 for details.

With the performance guarantee of our robust aggregator \mathcal{R} (formally stated in the next subsection) and Theorem 2, it can be shown that the approximate gradients used in the robust full gradient descent update (Algorithm 1 with the chosen robust aggregator \mathcal{R}) are good uniformly over $\theta \in \Theta$.

Theorem 3. Suppose Assumptions 1, 2, 3, and 4 hold. Assume that $\log(L + L') = O(\log(Nd))$ and $\Theta \subset \{\theta : \|\theta - \theta^*\| \le r\}$ for some positive parameter r such that $\log r = O(\log(Nd))$. Suppose $N \ge cd^2 \log^8(Nd)$ for a sufficiently large constant c and $m \le e^{\sqrt{d}}$. Let $G(\theta)$ (for each $\theta \in \Theta$) be the aggregated gradient returned by the chosen robust gradient aggregator. Then with probability at least $1 - 2e^{-\sqrt{d}}$,

$$||G(\theta) - \nabla F(\theta)|| \lesssim \left(\sqrt{\frac{q}{N}} + \sqrt{\frac{d}{N}} \log^2(Nd)\right) ||\theta - \theta^*|| + \left(\sqrt{\frac{q}{N}} + \sqrt{\frac{d}{N}}\right)$$

holds for all $\theta \in \Theta$.

Remark 1. Theorem 3 requires the total sample size $N \gtrsim d^2$ (ignoring the logarithmic factors), which is due to our sub-exponential assumption of local Lipschitz parameter $h(X,\theta)/\|\theta-\theta^*\|$. This sample size requirement $N \gtrsim d^2$ is inevitable as can be seen from the linear regression example. If instead $h(X,\theta)/\|\theta-\theta^*\|$ is assumed to be sub-Gaussian, then $N \gtrsim d$ suffices.

3.2 Robust Gradient Aggregator

In this subsection, we present the robust gradient aggregator \mathcal{R} used in Algorithm 1. Robust gradient aggregation is closely related to robust mean estimation, formally stated next.

Definition 1 (Robust mean estimation). Let $S = \{y_1, \dots, y_m\}$ be a sample of size m, wherein each of the data point y_i is generated independently from an unknown distribution. Among those m data points, up to q of them may be adversarially corrupted. Let $\{\hat{y}_1, \dots, \hat{y}_m\}$ be the observed sample. The goal is to estimate the true mean of the unknown distribution when only corrupted sample $\{\hat{y}_1, \dots, \hat{y}_m\}$ is accessible.

A very interesting recent line of work on robust mean estimation [DKK⁺16, DKK⁺17, SCV18] discovers computationally efficient iterative filtering approaches that can achieve a bounded estimation error as long as at most a constant fraction $(q/m = \Theta(1))$ of data is corrupted, irrespective of dimension d. In this paper, we consider an iterative filtering algorithm proposed in [SCV18], formally presented in Algorithm 2. At a high level, by solving (6) and (7) for a saddle point (W, U), Algorithm 2 iteratively finds a direction along which all data points are spread out the most, and filters away data points which have large residual errors projected along this direction. See Appendix B for detailed discussions.

In each iteration in the **while**-loop, the column/row indices of $W \in \mathbb{R}^{|\mathcal{A}| \times |\mathcal{A}|}$ correspond to the data indices remained in \mathcal{A} , and at least one point is removed from set \mathcal{A} . Thus, this loop will be executed at most m times. Given the corrupted sample $\{\hat{y}_1, \dots, \hat{y}_m\}$, ϵ , and σ , Algorithm 2 deterministically outputs an estimate $\hat{\mu}$ that differs from the true sample mean by at most a bounded distance, formally stated in Lemma 1.

Algorithm 2 Iterative Filtering for Robust Mean Estimation [SCV18]

Input: Corrupted sample $\{\hat{y}_1, \dots, \hat{y}_m\} \subseteq \mathbb{R}^d$, $1 - \alpha \triangleq \epsilon \in [0, \frac{1}{4})$, and $\sigma > 0$. Initialization: $A \leftarrow \{1, \dots, m\}, c_i \leftarrow 1 \text{ and } \tau_i \leftarrow 0 \text{ for all } i \in A$.

1: while true do

Let $W \in \mathbb{R}^{|\mathcal{A}| \times |\mathcal{A}|}$ be a minimizer to the following convex program:

$$\min_{\substack{0 \le W_{ji} \le \frac{4-\alpha}{\alpha(2+\alpha)m} \\ \sum_{i \in A} W_{ji} = 1}} \quad \max_{\substack{U \succeq 0 \\ \mathsf{Tr}(U) \le 1}} \quad \sum_{i \in \mathcal{A}} c_i \left(\widehat{y}_i - \sum_{j \in \mathcal{A}} \widehat{y}_j W_{ji} \right)^\top U \left(\widehat{y}_i - \sum_{j \in \mathcal{A}} \widehat{y}_j W_{ji} \right), \tag{6}$$

and $U \in \mathbb{R}^{d \times d}$ be a maximizer to the following convex program:

$$\max_{\substack{U \succeq 0 \\ \operatorname{Tr}(U) \le 1}} \min_{\substack{0 \le W_{ji} \le \frac{4-\alpha}{\alpha(2+\alpha)m} \\ \sum_{j \in \mathcal{A}} W_{ji} = 1}} \sum_{i \in \mathcal{A}} c_i \left(\widehat{y}_i - \sum_{j \in \mathcal{A}} \widehat{y}_j W_{ji} \right)^\top U \left(\widehat{y}_i - \sum_{j \in \mathcal{A}} \widehat{y}_j W_{ji} \right). \tag{7}$$

3: For
$$i \in \mathcal{A}$$
, $\tau_i \leftarrow \left(\widehat{y}_i - \sum_{j \in \mathcal{A}} \widehat{y}_j W_{ji}\right)^{\top} U\left(\widehat{y}_i - \sum_{j \in \mathcal{A}} \widehat{y}_j W_{ji}\right)$

if $\sum_{i \in A} c_i \tau_i > 8m\sigma^2$ then

5: For
$$i \in \mathcal{A}$$
, $c_i \leftarrow \left(1 - \frac{\tau_i}{\tau_{\max}}\right) c_i$, where $\tau_{\max} = \max_{i \in \mathcal{A}} \tau_i$.
6: $\mathcal{A} \leftarrow \mathcal{A} / \left\{i : c_i \leq \frac{1}{2}\right\}$.

6:
$$\mathcal{A} \leftarrow \mathcal{A}/\left\{i: c_i \leq \frac{1}{2}\right\}$$
.

7:

Break **while**—loop. 8:

end if

10: end while

11: **return**
$$\widehat{\mu} = \frac{1}{|\mathcal{A}|} \sum_{i \in \mathcal{A}} \widehat{y}_i$$
.

Lemma 1. [SCV18] Let $S = \{y_1, \dots, y_m\}$ be the true sample. Define $\mu_S = \frac{1}{m} \sum_{i=1}^m y_i$ as the sample mean on S. Let $\{\hat{y}_1, \dots, \hat{y}_m\} \subseteq \mathbb{R}^d$ be the observed sample, which is obtained from S by adversarially corrupting up to q data points. Suppose that

$$\left\| \frac{1}{m} \sum_{i \in \mathcal{S}} (y_i - \mu_{\mathcal{S}}) (y_i - \mu_{\mathcal{S}})^{\top} \right\| \le \sigma^2.$$
 (8)

Then for $q/m = \epsilon \leq \frac{1}{4}$, Algorithm 2 outputs a parameter $\widehat{\mu}$ such that

$$\|\widehat{\mu} - \mu_{\mathcal{S}}\| = O(\sigma\sqrt{\epsilon}). \tag{9}$$

Note that the statement of Lemma 1 is different from that in [SCV18] – in (8) the summation is taken over the entire true sample \mathcal{S} rather than a subset of sample. We make this modification in order to work with the diminishing error regime, i.e., q/m = O(1). For completeness, we present the proof of Lemma 1 in Appendix B.

In this work, we use Algorithm 2 as our robust gradient aggregator \mathcal{R} with inputs

$$\widehat{y}_1(\theta) = g_1(\theta), \cdots, \widehat{y}_m(\theta) = g_m(\theta),$$

where $g_1(\theta), \dots, g_m(\theta)$ are the gradients reported by the m workers, among which up to q reported gradients may not be the true local gradients. The true m local gradients are

$$y_1(\theta) = \frac{1}{n} \sum_{i \in \mathcal{S}_1} \nabla f(X_i, \theta), \quad \cdots, \quad y_m(\theta) = \frac{1}{n} \sum_{i \in \mathcal{S}_m} \nabla f(X_i, \theta).$$

Remark 2 (Alternative Termination Condition). The termination of Algorithm 2 relies on the knowledge of σ . Translating to our statistical learning problem, to guarantee the termination of robust gradient aggregation, the learner needs to know $\|\theta - \theta^*\|$ for all θ , which is impossible. Nevertheless, it turns out that the termination condition of Algorithm 2 can be replaced by checking the cardinality of set $|\mathcal{A}|$. Formally stated next: If

$$\left| \mathcal{A} \setminus \left\{ i : \left(1 - \frac{\tau_i}{\tau_{\max}} \right) c_i \le \frac{1}{2} \right\} \right| \ge \frac{\alpha (2 + \alpha) m}{4 - \alpha},$$

we update $c_i \leftarrow \left(1 - \frac{\tau_i}{\tau_{\text{max}}}\right) c_i$ and remove $\left\{i: c_i \leq \frac{1}{2}\right\}$ from \mathcal{A} ; otherwise, we break the **while**-loop. Similar to the original Algorithm 2, in the modified Algorithm 2, in each iteration of the **while**-loop at least one point will be removed. Thus, the modified Algorithm 2 terminates in at most m iterations. The correctness of this code modification can be found in Appendix B.3.

Next, we discuss two important implications of Lemma 1 in Remarks 3 and 4, respectively.

Remark 3. In Lemma 1, the estimation error bound (9) is in terms of $\|\widehat{\mu} - \mu_{\mathcal{S}}\|$. Let μ be the true mean of the unknown underlying distribution. We can easily deduce an estimation error bound in terms of $\|\widehat{\mu} - \mu\|$ from the following triangle inequality

$$\|\widehat{\mu} - \mu\| \le \|\widehat{\mu} - \mu_{\mathcal{S}}\| + \|\mu_{\mathcal{S}} - \mu\| = O\left(\sigma\sqrt{\epsilon}\right) + \|\mu_{\mathcal{S}} - \mu\|.$$

Thus, to characterize the estimation error $\|\widehat{\mu} - \mu\|$, it is enough to control the spectral norm of the true sample covariance matrix $\left\|\frac{1}{m}\sum_{i\in\mathcal{S}}(y_i - \mu_{\mathcal{S}})(y_i - \mu_{\mathcal{S}})^{\top}\right\|$ and the deviation of the empirical average $\|\mu_{\mathcal{S}} - \mu\|$ – the latter of which is standard.

Remark 4. Note that

$$\left\| \frac{1}{m} \sum_{i \in \mathcal{S}} (y_i - \mu_{\mathcal{S}}) (y_i - \mu_{\mathcal{S}})^\top \right\| = \frac{1}{m} \left\| \left([y_1, \dots, y_m] - \mu_{\mathcal{S}} \mathbf{1}_m^\top \right) \left([y_1, \dots, y_m] - \mu_{\mathcal{S}} \mathbf{1}_m^\top \right)^\top \right\|$$

$$= \frac{1}{m} \left\| [y_1, \dots, y_m] - \mu_{\mathcal{S}} \mathbf{1}_m^\top \right\|^2$$

$$\leq \frac{1}{m} \left(\left\| [y_1, \dots, y_m] - \mu \mathbf{1}_m^\top \right\| + \sqrt{m} \left\| \mu - \mu_{\mathcal{S}} \right\| \right)^2,$$

where $\mathbf{1}_m \in \mathbb{R}^m$ is an all-ones vector. Therefore, to bound $\left\|\frac{1}{m}\sum_{i\in\mathcal{S}}(y_i - \mu_{\mathcal{S}})(y_i - \mu_{\mathcal{S}})^\top\right\|$, it is enough to bound $\|\mu - \mu_{\mathcal{S}}\|$ and $\frac{1}{\sqrt{m}}\|[y_1, \cdots, y_m] - \mu \mathbf{1}_m^\top\|$.

4 Main Analysis

In this section, we provide the missing proofs of our main results.

4.1 New Matrix Concentration Inequality: Theorem 2 and its Proof

Recall that we use Algorithm 2 as our robust gradient aggregator \mathcal{R} with inputs

$$\widehat{y}_1(\theta) = g_1(\theta), \cdots, \widehat{y}_m(\theta) = g_m(\theta),$$

where $g_1(\theta), \dots, g_m(\theta)$ are the gradients reported by the m workers, among which up to q reported gradients may not be the true local gradients. The true m local gradients are

$$y_1(\theta) = \frac{1}{n} \sum_{i \in \mathcal{S}_1} \nabla f(X_i, \theta), \quad \cdots, \quad y_m(\theta) = \frac{1}{n} \sum_{i \in \mathcal{S}_m} \nabla f(X_i, \theta).$$

- recalling that $|S_j| = n = \frac{N}{m}$ for $j \in [m]$. The sample mean $\mu_{\mathcal{S}}(\theta)$ is

$$\frac{1}{m} \sum_{j=1}^{m} \left(\frac{1}{n} \sum_{i \in \mathcal{S}_j} \nabla f(X_i, \theta) \right) = \frac{1}{N} \sum_{i=1}^{N} \nabla f(X_i, \theta),$$

and the population mean $\mathbb{E}[Y(\theta)]$ is $\nabla F(\theta)$.

As discussed in Remarks 3 and 4, to guarantee that the aggregated gradient is close to the true gradient, it suffices to bound

$$\frac{1}{\sqrt{m}} \left\| \left[\frac{1}{n} \sum_{i \in \mathcal{S}_1} \nabla f(X_i, \theta) - \nabla F(\theta), \cdots, \frac{1}{n} \sum_{i \in \mathcal{S}_m} \nabla f(X_i, \theta) - \nabla F(\theta) \right] \right\|$$
(10)

and

$$\left\| \frac{1}{N} \sum_{i=1}^{N} \nabla f(X_i, \theta) - \nabla F(\theta) \right\| \tag{11}$$

uniformly over all $\theta \in \Theta$. Bounding (11) involves standard concentration of sum of i.i.d. random vectors and is relatively easy. The main challenge is to bound (10).

For any fixed θ , the matrix in (10) is of independent columns. Standard routine to bound the spectral norm of (10) is available, see [Ver10, Theorem 5.44] and [ALPTJ10, Corollary 3.8] for example. To get a uniform concentration result, we can use ϵ -net argument to extend the concentration of a fixed θ to uniform over all $\theta \in \Theta$. Nevertheless, using these standard matrix concentration results, the uniform concentration bound obtained is far from being optimal.

The following theorem is a state-of-the-art concentration inequality for matrices with sub-exponential columns [ALPTJ10, Corollary 3.8].

Theorem 4. Let A be a $d \times m$ matrix whose columns A_j are i.i.d., zero-mean, sub-exponential random vectors in \mathbb{R}^d with the scaling parameters σ and α . Assume that $\sigma, \alpha = O(1)$ and $m \leq e^{\sqrt{d}}$. There are absolute positive constants C and c such that for every $K \geq 1$, with probability at least $1 - e^{-cK\sqrt{d}}$,

$$||A|| \le CK\left(\sqrt{m} + \sqrt{d}\right).$$

Note that assuming $m \leq e^{\sqrt{d}}$ only loses minimal generality in the high-dimensional regime. The above theorem is tight up to constant factors when the tail probability is on the order of $e^{-\sqrt{d}}$, i.e., when $K = \Theta(1)$ [ALPTJ10][Remark 3.7]. However, in our problem, to guarantee a uniform bound of the spectral norm of (10), we need a tail probability on the order of e^{-d} , i.e., $K \approx \sqrt{d}$.

In this case, Theorem 4 yields an upper bound on the order of $\sqrt{md} + d$. Using [Ver10, Theorem 5.44] instead, we can obtain an alternative upper bound $O(\sqrt{m} + d^{3/2})$. Both of these two upper bounds are not tight. To this end, we develop a new matrix concentration inequality, proving a nearly-tight upper bound on the order of $\sqrt{m} + d$ up to logarithmic factors.

A key step in deriving a concentration inequality for matrices with sub-exponential random vectors is to obtain a large deviation inequality for the sum of independent random variables whose tails decay slower than sub-exponential random variables. Note that in this case, the moment generating function may not exist and thus we cannot follow the standard approach to obtain a large deviation inequality by invoking the Chernoff bound. To circumvent this, we partition the support of a real-valued random variable Y into countably many finite segments, and write Y as a summation of component random variables, each of which is supported on its corresponding segment. Due to the fact that each segment is of finite length, we can apply Bennett's inequality for bounded random variables (cf. Lemma 5). Then we take a union bound to arrive at a concentration result of the original Y. Some additional care is needed in choosing the partition. Our proof is inspired by Proposition 2.1.9 and Excercise 2.1.7 in [Tao12].

Lemma 2. Let Y be a random variable whose tail probability satisfies

$$\mathbb{P}\left\{ \left|Y\right| \geq t\right\} \leq \exp\left(-E(t)\right),$$

where $E(t): \mathbb{R}_+ \to [0,\infty]$ is a non-decreasing function. Suppose that there exists $t_0 \geq e^2$ such that

$$E(e^{k-1}) \ge 2(2k + 4\log(k+1) + \log 2 - \log t), \quad \forall k \text{ with } 4(k+1)^2 e^k \ge t, \ \forall t \ge t_0,$$
 (12)

and

$$E(t)/t$$
 is monotone in t. (13)

Let Y_1, \dots, Y_m be m independent copies of Y. If E(t)/t is non-decreasing, then

$$\mathbb{P}\left\{\left|\sum_{j=1}^{m} Y_{j} - m\mathbb{E}\left[Y\right]\right| \ge mt\right\} \le 2\log(mt)\exp\left(-\frac{m}{4(\log(mt) + 1)^{2}}E\left(\frac{t}{4e\log^{2}t}\right)\right) + \exp\left(-\frac{1}{2}E\left(\frac{mt}{e}\right)\right); \tag{14}$$

if E(t)/t is non-increasing, then

$$\mathbb{P}\left\{\left|\sum_{j=1}^{m} Y_{j} - m\mathbb{E}\left[Y\right]\right| \ge mt\right\} \le 2\log(mt)\exp\left(-\frac{1}{4e(\log(mt) + 1)^{2}}E\left(\frac{mt}{e}\right)\right) + \exp\left(-\frac{1}{2}E\left(\frac{mt}{e}\right)\right). \tag{15}$$

Remark 5. Let us consider the following special cases:

Suppose Y is sub-Gaussian. In this case, $E(t) = ct^2$ for a universal constant c > 0. Thus, there exists a universal constant $t_0 \ge e^2$ such that both (12) and (13) hold. It follows from Lemma 2 that for all $t \ge t_0$,

$$\mathbb{P}\left\{\left|\sum_{j=1}^{m} Y_j - m\mathbb{E}\left[Y\right]\right| \ge mt\right\} \le 2\log(mt)\exp\left(-\frac{cmt^2}{64e^2\log^2(emt)\log^4t}\right) + \exp\left(-\frac{cm^2t^2}{2e^2}\right),$$

which gives the desired sub-Gaussian tail bound up to logarithmic factors.

Suppose Y is sub-exponential. In this case, E(t) = ct for a universal constant c. Thus, there exists $t_0 \ge e^2$ that only depends on c such that both (12) and (13) hold. It follows from Lemma 2 that for all $t \ge t_0$ (large deviation region),

$$\mathbb{P}\left\{ \left| \sum_{j=1}^{m} Y_j - m\mathbb{E}\left[Y\right] \right| \ge mt \right\} \le 2\log(mt) \exp\left(-\frac{cmt}{16e \log^2(emt) \log^2 t}\right) + \exp\left(-\frac{cmt}{2e}\right),$$

which gives the desired sub-exponential tail bound up to logarithmic factors.

Suppose $Y = Z^2$, where Z is sub-exponential. In this case, $E(t) = c\sqrt{t}$ for a universal constant c > 0. Thus, there exists $t_0 \ge e^2$ that only depends on c such that both (12) and (13) hold. It follows from Lemma 2 that for all $t \ge t_0$ (large deviation region),

$$\mathbb{P}\left\{\left|\sum_{j=1}^{m} Y_j - m\mathbb{E}\left[Y\right]\right| \ge mt\right\} \le 2\log(mt)\exp\left(-\frac{c\sqrt{mt}}{4e\sqrt{e}\log^2(emt)}\right) + \exp\left(-\frac{c\sqrt{mt}}{2\sqrt{e}}\right).$$

Despite the fact that Lemma 2 is loose up to logarithmic factors comparing to the standard sub-gaussian and sub-exponential random variables, Lemma 2 applies to much larger family than the sub-gaussian distributions, and requires much less structures on the distributions. In particular, Lemma 2 does not even require the existence of moment generating function.

The proof of Lemma 2 can be found in Appendix A. Lemma 2 is our key machinery to obtain the concentration inequality for matrices with i.i.d. sub-exponential random vectors given in Theorem 2. We restate the theorem below for ease of reference.

Theorem (Theorem 2). Let A be a $d \times m$ matrix whose columns A_j are independent and identically distributed sub-exponential, zero-mean random vectors in \mathbb{R}^d with parameters (σ, α) . Assume that

$$\sigma/\alpha = \Omega(1). \tag{16}$$

Then with probability at least $1 - \delta$,

$$||A|| \le c \left(\sigma \sqrt{m} + \sigma \phi \left(d + \log \frac{1}{\delta}\right) + \alpha \phi^2 \left(d + \log \frac{1}{\delta}\right)\right),$$

where c is a universal positive constant and $\phi(x): \mathbb{R} \to \mathbb{R}$ is a function given by $\phi(x) = \sqrt{x} \log^{3/2}(x)$.

Remark 6. We discuss two consequences of Theorem 2.

Suppose $\alpha = 0$. In this case, A has sub-Gaussian columns, and Theorem 2 implies that

$$||A|| \lesssim \sigma \left(\sqrt{m} + \sqrt{d + \log \frac{1}{\delta}} \log^{3/2} \left(d + \log \frac{1}{\delta} \right) \right),$$

which matches the sub-Gaussian matrix concentration inequality [Ver18][Theorem 5.39] up to logarithmic factors.

Suppose $\sigma, \alpha = \Theta(1)$, and $\log(1/\delta) = d$. In this case, we get that with probability at least $1 - e^{-d}$,

$$||A|| \lesssim \sqrt{m} + d\log^3 d$$
 implied by Theorem 2, (17)

whereas the analogous bound implied by Theorem 4 is on the order of $\sqrt{md} + d$. The upper bound (17) is tight up to logarithmic factors. To see this, consider an example, where A_j 's are i.i.d. isotropic Laplace distribution with the density function given by $f(x) = \prod_{i=1}^{d} (1/\sqrt{2}) \exp(-\sqrt{2}x_i)$ for $x \in \mathbb{R}^d$. In this case, note that

$$\left\{ \|A\| \ge \max\{\sqrt{m/2}, d\} \right\} \supseteq \left\{ |A_{11}| \ge d \text{ and } \sum_{j=1}^m A_{2j}^2 \ge m/2 \right\}.$$

Since

$$\mathbb{P}\left\{|A_{11}| \ge d\right\} = \int_{|t| > d} \frac{1}{\sqrt{2}} \exp\left(-\sqrt{2}t\right) dt = \exp\left(-\sqrt{2}d\right),$$

and by Chebyshev's inequality,

$$\mathbb{P}\left\{\sum_{j=1}^{m} A_{2j}^{2} \ge m/2\right\} \ge 1 - O(1/m) \ge \frac{1}{2}$$

for m sufficiently large, and A_{11} is independent of $\sum_{j=1}^{m} A_{2j}^2$, it follows that

$$\mathbb{P}\left\{\|A\| \geq \max\{\sqrt{m/2}, d\}\right\} \geq \mathbb{P}\left\{|A_{11}| \geq d \text{ and } \sum_{j=1}^{m} A_{2j}^2 \geq m/2\right\} \geq \frac{1}{2} \exp\left(-\sqrt{2}d\right).$$

The proof of Theorem 2 also uses the following lemma presents the standard and well-known concentration inequality for sum of independent sub-exponential random variables.

Lemma 3. [Ver18] Let Y_1, \ldots, Y_m denote a sequence of independent random variables, where Y_j 's are sub-exponential with scaling parameters (σ_j, α_j) and mean 0. Then $\sum_{j=1}^m Y_j$ is sub-exponential with scaling parameters (σ_*, α_*) , where $\sigma_*^2 = \sum_{j=1}^m \sigma_j^2$ and $\alpha_* = \max_{1 \le j \le m} \alpha_j$. Moreover,

$$\mathbb{P}\left\{\sum_{j=1}^{m} Y_j \ge t\right\} \le \begin{cases} \exp\left(-\frac{t^2}{2\sigma_*^2}\right) & \text{if } 0 \le t \le \sigma_*^2/\alpha_*; \\ \exp\left(-\frac{t}{2\alpha_*}\right) & \text{o.w.} \end{cases}$$

The following lemma gives an upper bound to the spectral norm of the covariance matrix of a sub-exponential random vector.

Lemma 4. Let $Y \in \mathbb{R}^d$ denote a zero-mean, sub-exponential random vector with scaling parameters (σ, α) , and Σ denote its covariance matrix $\Sigma = \mathbb{E}[YY^\top]$. Then

$$\|\Sigma\| \le 4\sigma^2 + 16\alpha^2.$$

Proof. First recall that

$$\|\Sigma\| = \sup_{v \in S^{d-1}} v^{\top} \Sigma v = \sup_{v \in S^{d-1}} v^{\top} \mathbb{E} \left[Y Y^{\top} \right] v = \sup_{v \in S^{d-1}} \mathbb{E} \left[\langle Y, v \rangle^2 \right].$$

For each unit vector v, from [Ver18, Exercise 1.2.3], we have

$$\mathbb{E}\left[\langle Y, v \rangle^{2}\right] = \int_{0}^{\infty} 2t \, \mathbb{P}\left\{\left|\langle Y, v \rangle\right| \ge t\right\} dt$$

$$\le \int_{0}^{\infty} 4t \, \exp\left(-\frac{1}{2} \min\left\{\frac{t^{2}}{\sigma^{2}}, \frac{t}{\alpha}\right\}\right) dt$$

$$\le 4\sigma^{2} + 16\alpha^{2}. \tag{18}$$

Note that the above upper bound is independent of v. The lemma follows by combining the last two displayed equations.

Proof of Theorem 2. Recall $\Sigma = \mathbb{E}\left[A_1 A_1^{\top}\right]$. Then

$$||A||^2 = ||AA^\top|| \le ||AA^\top - m\Sigma|| + m ||\Sigma||.$$

In view of Lemma 4, we have $\|\Sigma\| \le 4\sigma^2 + 16\alpha^2$. It remains to bound $\|AA^\top - m\Sigma\|$. Note that

$$\left\|AA^{\top} - m\Sigma\right\| = \sup_{v \in S^{d-1}} \left|v^{\top} \left(AA^{\top} - m\Sigma\right)v\right| = \sup_{v \in S^{d-1}} \left|\sum_{j=1}^{m} \left(\langle A_j, v \rangle^2 - \mathbb{E}\left[\langle A_j, v \rangle^2\right]\right)\right|.$$

Fix a $v \in S^{d-1}$. Note that $\langle A_j, v \rangle$ is zero-mean sub-exponential random variable with parameter (σ, α) . For $j = 1, \dots, m$, define

$$Y_j = \langle A_j, v \rangle^2 / \sigma^2. \tag{19}$$

It follows from Lemma 3 that

$$\mathbb{P}\left\{|Y_j| \ge t\right\} = \mathbb{P}\left\{|\langle A_j, v \rangle| \ge \sigma \sqrt{t}\right\} \le 2 \exp\left(-\min\left\{\frac{t}{2}, \frac{\sigma \sqrt{t}}{2\alpha}\right\}\right),$$

We apply Lemma 2 to Y_1, \dots, Y_m with

$$E(t) = \min\left\{\frac{t}{2}, \frac{\sigma\sqrt{t}}{2\alpha}\right\} - \log 2,$$

which is non-decreasing in t. By assumption $\sigma/\alpha = \Omega(1)$, it follows that E(t) scales as \sqrt{t} in t. Thus there exits $t_0 \geq e^2$ such that (12) holds. In addition, E(t)/t is non-increasing. Therefore, (15) in Lemma 2 applies, i.e., for all $t \geq t_0$,

$$\mathbb{P}\left\{\left|\sum_{j=1}^{m} (Y_j - \mathbb{E}[Y_j])\right| \ge mt\right\} \\
\le 2\log(mt)\exp\left(-\frac{1}{4e\log^2(emt)}E\left(\frac{mt}{e}\right)\right) + \exp\left(-\frac{1}{2}E\left(\frac{mt}{e}\right)\right) \\
\le 4\log(mt)\exp\left(-\frac{1}{4e\log^2(emt)}E\left(\frac{mt}{e}\right)\right). \tag{20}$$

Next, we apply ϵ -net argument. Let $\mathcal{N}_{\frac{1}{4}}$ be the $\frac{1}{4}$ -net of the unit sphere S^{d-1} . From [Ver10, Lemma 5.2], we know that $\left|\mathcal{N}_{\frac{1}{4}}\right| \leq 9^d$. In addition, it follows from [Ver10, Lemma 5.4] that

$$\left\|AA^{\top} - \Sigma\right\| \le 2 \sup_{v \in \mathcal{N}_{\frac{1}{4}}} \left| \sum_{j=1}^{m} \left(\langle A_j, v \rangle^2 - \mathbb{E}\left[\langle A_j, v \rangle^2 \right] \right) \right|.$$

Hence,

$$\mathbb{P}\left\{\left\|AA^{\top} - \Sigma\right\| \geq 2\sigma^{2}mt\right\} \\
\leq \mathbb{P}\left\{\sup_{v \in \mathcal{N}_{\frac{1}{4}}} \left| \sum_{j=1}^{m} \left(\langle A_{j}, v \rangle^{2} - \mathbb{E}\left[\langle A_{j}, v \rangle^{2}\right]\right) \right| \geq \sigma^{2}mt\right\} \\
\leq \left|\mathcal{N}_{\frac{1}{4}}\right| \mathbb{P}\left\{\left| \sum_{j=1}^{m} \left(\langle A_{j}, v \rangle^{2} - \mathbb{E}\left[\langle A_{j}, v \rangle^{2}\right]\right) \right| \geq \sigma^{2}mt\right\} \\
\leq 9^{d}\mathbb{P}\left\{\left| \sum_{j=1}^{m} \left(Y_{j} - \mathbb{E}\left[Y_{j}\right]\right) \right| \geq mt\right\} \quad \text{by definition of } Y_{j} \text{ in } (19) \\
\leq \exp\left(-\frac{1}{4e\log^{2}(emt)}E\left(\frac{mt}{e}\right) + \log 4 + \log\log(mt) + d\log 9\right) \quad \text{by } (20).$$

To complete the proof, we need to choose mt so that the right hand side of the last inequality is smaller than δ . In other words, we need to find $x \geq mt_0$ such that

$$\frac{1}{4e\log^2(ex)}E(x/e) - \log\log x \ge \log\frac{4}{\delta} + d\log 9 \triangleq a.$$

One such x is given by

$$x = c \left(a \log^3 a + \frac{\alpha^2}{\sigma^2} a^2 \log^6 a + m \right),$$

where c is a sufficiently large constant. Therefore, we choose

$$mt = c\left(\left(d + \log\frac{1}{\delta}\right)\log^3\left(d + \log\frac{1}{\delta}\right) + \frac{\alpha^2}{\sigma^2}\left(d + \log\frac{1}{\delta}\right)^2\log^6\left(d + \log\frac{1}{\delta}\right) + m\right)$$

The lemma follows by taking the square root of mt.

4.2 Proof of Theorem 3

With Lemma 1 and Theorem 2, we are ready to prove Theorem 3. Recall that we need to bound (10) and (11) uniformly for all $\theta \in \Theta$. Bounding (11) uniformly is relatively easy and has been done in previous work [CSX17, Proposition 3.8].

Proposition 1. [CSX17, Proposition 3.8] Consider the same setup as Theorem 3. Assume that $N = \Omega(d \log(Nd))$. Then with probability at least $1 - e^{-d}$,

$$\left\| \frac{1}{N} \sum_{i=1}^{N} \nabla f(X_i, \theta) - \nabla F(\theta) \right\| \lesssim \Delta_2 \|\theta - \theta^*\| + \Delta_1, \quad \forall \ \theta \in \Theta,$$

where

$$\Delta_1 \triangleq \sqrt{\frac{d}{N}}, \quad and \quad \Delta_2 \triangleq \sqrt{\frac{d \log(Nd)}{N}}.$$

It remains to bound (10) uniformly over all $\theta \in \Theta$. For notational convenience, let

$$G(X_{\mathcal{S}}, \theta) \triangleq \frac{1}{\sqrt{m}} \left[\frac{1}{n} \sum_{i \in \mathcal{S}_1} \nabla f(X_i, \theta) - \nabla F(\theta), \cdots, \frac{1}{n} \sum_{i \in \mathcal{S}_m} \nabla f(X_i, \theta) - \nabla F(\theta) \right]. \tag{21}$$

Proposition 2. Consider the same setup as Theorem 3. With probability at least $1 - 2e^{-\sqrt{d}}$,

$$||G(X_{\mathcal{S}}, \theta^*)|| \lesssim \Delta_3 \quad and \quad ||G(X_{\mathcal{S}}, \theta) - G(X_{\mathcal{S}}, \theta^*)|| \lesssim \Delta_4 ||\theta - \theta^*||, \ \forall \theta \in \Theta,$$
 (22)

where

$$\begin{split} & \Delta_3 \; \triangleq \; \frac{1}{\sqrt{n}} + \sqrt{\frac{d}{N}}, \\ & \Delta_4 \; \triangleq \; \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{N}} \phi \left(d \log \left(r \sqrt{n} (L + L') \right) \right) + \frac{1}{\sqrt{Nn}} \phi^2 \left(d \log \left(r \sqrt{n} (L + L') \right) \right), \end{split}$$

and $\phi(x) = \sqrt{x} \log^{3/2}(x)$. It follows from triangle inequality that

$$||G(X_{\mathcal{S}}, \theta)|| \lesssim \Delta_4 ||\theta - \theta^*|| + \Delta_3, \ \forall \theta \in \Theta.$$

Remark 7. The uniform upper bound Δ_4 in (22) depends linearly in d (ignoring logarithmic factors). Such linear dependency is inevitable as can be seen from the linear regression example. Suppose $y_i = \langle w_i, \theta^* \rangle + \zeta_i$, where θ^* is an unknown true model parameter, $w_i \sim N(0, \mathbf{I})$ is the covariate vector whose covariance matrix is assumed to be identity, and $\zeta_i \sim N(0, 1)$ is i.i.d. additive Gaussian noise independent of w_i 's. The risk function $f(X_i, \theta) = \frac{1}{2} (\langle w_i, \theta \rangle - y_i)^2$. In this case $\nabla f(X_i, \theta) = w_i w_i^{\top} (\theta - \theta^*) - w_i \zeta_i$ and $\nabla F(\theta) = \theta - \theta^*$. For simplicity, assume n = 1 and m = N. Then

$$\sup_{\theta \in S^{d-1}} \|G(X_{\mathcal{S}}, \theta) - G(X_{\mathcal{S}}, \theta^*)\| \ge \sup_{\theta \in S^{d-1}} \frac{1}{\sqrt{N}} \|(w_1 w_1^\top - \mathbf{I})(\theta - \theta^*)\|$$

$$= \frac{1}{\sqrt{N}} (\|w_1\|^2 - 1) \|\theta - \theta^*\|$$

$$= O_P \left(\frac{d}{\sqrt{N}}\right) \|\theta - \theta^*\|,$$

where the first equality holds by choosing $\theta - \theta^*$ parallel to w_1 .

Proof of Proposition 2. We prove the two bounds in (22) individually.

Bounding $||G(X_{\mathcal{S}}, \theta^*)||$: It follows from Assumption 2 that the columns of $G(X_{\mathcal{S}}, \theta^*)$ are i.i.d. sub-exponential random vectors in \mathbb{R}^d with mean 0 and scaling parameters σ_1/\sqrt{nm} and $\alpha_1/(n\sqrt{m})$. The sub-exponential parameters for the scaled matrix $\sqrt{N} G(X_{\mathcal{S}}, \theta^*)$ are σ_1 and α_1/\sqrt{n} – recalling that N = nm. Applying Theorem 4 to $A = \sqrt{N} G(X_{\mathcal{S}}, \theta^*)$, it holds that with probability at least $1 - e^{-\sqrt{d}}$,

$$||G(X_{\mathcal{S}}, \theta^*)|| = \frac{1}{\sqrt{N}} ||A|| \lesssim \frac{1}{\sqrt{N}} \left(\sqrt{m} + \sqrt{d}\right) = \frac{1}{\sqrt{n}} + \sqrt{\frac{d}{N}}.$$
 (23)

Bounding $||G(X_S, \theta) - G(X_S, \theta^*)||$ for a fixed $\theta \in \Theta$: For notational convenience, define

$$H(X_{\mathcal{S}}, \theta) \triangleq G(X_{\mathcal{S}}, \theta) - G(X_{\mathcal{S}}, \theta^*) = \frac{1}{\sqrt{m}} \left[\frac{1}{n} \sum_{i \in \mathcal{S}_1} h(X_i, \theta), \cdots, \frac{1}{n} \sum_{i \in \mathcal{S}_m} h(X_i, \theta) \right], \tag{24}$$

where recall from (4) that the gradient difference function $h(X,\cdot)$ is defined as

$$h(X, \theta) = \nabla f(X, \theta) - \nabla f(X, \theta^*) - (\nabla F(\theta) - \nabla F(\theta^*)).$$

It follows from Assumption 4 that the columns of $H(X_{\mathcal{S}}, \theta)/\|\theta - \theta^*\|$ are i.i.d. sub-exponential random vectors in \mathbb{R}^d with mean 0 and scaling parameters σ_2/\sqrt{nm} and $\alpha_2/(n\sqrt{m})$. Recall that N = nm. Applying Theorem 4.1 to $H(X_{\mathcal{S}}, \theta)/\|\theta - \theta^*\|$, we know that for any fixed θ , with probability at least $1 - \delta$,

$$||H(X_{\mathcal{S}}, \theta)|| \lesssim \left(\frac{\sigma_2}{\sqrt{n}} + \frac{\sigma_2}{\sqrt{N}} \phi \left(d + \log \frac{1}{\delta}\right) + \frac{\alpha_2}{\sqrt{Nn}} \phi^2 \left(d + \log \frac{1}{\delta}\right)\right) ||\theta - \theta^*||$$

$$\lesssim \left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{N}} \phi \left(d + \log \frac{1}{\delta}\right) + \frac{1}{\sqrt{Nn}} \phi^2 \left(d + \log \frac{1}{\delta}\right)\right) ||\theta - \theta^*||, \tag{25}$$

where we used $\phi(x) = \sqrt{x} \log^{3/2}(x)$, and $\sigma_2 = O(1)$, $\alpha_2 = O(1)$.

 ϵ -net argument: We apply ϵ -net argument to extend the point convergence in (25) to the uniform convergence over Θ . In particular, let \mathcal{N}_{ϵ_0} be an ϵ_0 -cover of $\Theta = \{\theta : \|\theta - \theta^*\| \leq r\}$ with

$$\epsilon_0 = \frac{\sigma_1}{\sqrt{n}(L+L')}.$$

By [Ver10, Lemma 5.2], we have

$$\log |\mathcal{N}_{\epsilon_0}| \le d \log (r/\epsilon_0) = d \log \frac{r(L+L')\sqrt{n}}{\sigma_1}.$$

By (25) and the union bound, we get that with probability at least $1 - \delta$, for all $\theta \in \mathcal{N}_{\epsilon_0}$

$$||H(X_{\mathcal{S}}, \theta)|| \lesssim \left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{N}}\phi\left(d + \log\frac{|\mathcal{N}_{\epsilon_0}|}{\delta}\right) + \frac{1}{\sqrt{Nn}}\phi^2\left(d + \log\frac{|\mathcal{N}_{\epsilon_0}|}{\delta}\right)\right) ||\theta - \theta^*||. \tag{26}$$

So far, we have shown the uniform convergence over net \mathcal{N}_{ϵ_0} . Next, we extend this uniform convergence to the entire set Θ .

For any $\theta \in \Theta$, there exists a $\theta_k \in \mathcal{N}_{\epsilon_0}$ such that $\|\theta - \theta_k\| \leq \epsilon_0$. By triangle inequality,

$$||H(X_{\mathcal{S}},\theta)|| \le ||H(X_{\mathcal{S}},\theta_k)|| + ||H(X_{\mathcal{S}},\theta) - H(X_{\mathcal{S}},\theta_k)||.$$

Note that

$$||H(X_{\mathcal{S}}, \theta) - H(X_{\mathcal{S}}, \theta_{k})|| \leq ||H(X_{\mathcal{S}}, \theta) - H(X_{\mathcal{S}}, \theta_{k})||_{F}$$

$$\leq \frac{1}{n} \max_{1 \leq j \leq m} \left\| \sum_{i \in \mathcal{S}_{j}} \left(h(X_{i}, \theta) - h(X_{i}, \theta_{k}) \right) \right\|$$

$$\stackrel{(a)}{\leq} (L + L') ||\theta - \theta_{k}|| \leq (L + L') \epsilon_{0} = \frac{\sigma_{1}}{\sqrt{n}}, \tag{27}$$

where (a) holds because

$$\frac{1}{n} \left\| \sum_{i \in \mathcal{S}_j} \left(h(X_i, \theta) - h(X_i, \theta_k) \right) \right\| \le \frac{1}{n} \sum_{i \in \mathcal{S}_j} \left\| h(X_i, \theta) - h(X_i, \theta_k) \right\| \le \left(L + L' \right) \left\| \theta - \theta_k \right\|,$$

in view of Assumption 1 and Assumption 3.

Combining (26) and (27), we have that with probability at least $1 - \delta$, for any $\theta \in \Theta$,

$$||H(X_{\mathcal{S}}, \theta)|| \leq ||H(X_{\mathcal{S}}, \theta_k)|| + \frac{\sigma_1}{\sqrt{n}}$$

$$\lesssim \left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{N}}\phi\left(d + \log\frac{|\mathcal{N}_{\epsilon_0}|}{\delta}\right) + \frac{1}{\sqrt{Nn}}\phi^2\left(d + \log\frac{|\mathcal{N}_{\epsilon_0}|}{\delta}\right)\right) ||\theta - \theta^*|| + \frac{1}{\sqrt{n}}.$$

Choosing $\delta = e^{-d}$, we get with probability at least $1 - e^{-d}$, for all $\theta \in \Theta$.

$$||H(X_{\mathcal{S}}, \theta)|| \lesssim \left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{N}}\phi\left(d\log\left(r\sqrt{n}(L+L')\right)\right) + \frac{1}{\sqrt{Nn}}\phi^{2}\left(d\log\left(r\sqrt{n}(L+L')\right)\right)\right) ||\theta - \theta^{*}|| + \frac{1}{\sqrt{n}}.$$
(28)

Putting all pieces together Combing (23) and (28), we conclude Proposition 2.

Finish the proof of Theorem 3:

Recall that $\log(L + L') = O(\log(Nd))$, $\Theta \subset \{\theta : \|\theta - \theta^*\| \le r\}$ for some positive parameter r such that $\log r = O(\log(Nd))$, and that $N = \Omega(d^2 \log^8(Nd))$. Then,

$$\Delta_4 \triangleq \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{N}} \phi \left(d \log \left(r \sqrt{n} (L + L') \right) \right) + \frac{1}{\sqrt{Nn}} \phi^2 \left(d \log \left(r \sqrt{n} (L + L') \right) \right)$$
$$\lesssim \frac{1}{\sqrt{n}} + \sqrt{\frac{d}{N}} \log^2(Nd) + \frac{1}{\sqrt{n}} \lesssim \frac{1}{\sqrt{n}} + \sqrt{\frac{d}{N}} \log^2(Nd).$$

Let \mathcal{E}_1 and \mathcal{E}_2 denote the two events on which the conclusions in Proposition 1 and Proposition 2 hold, respectively. On event $\mathcal{E}_1 \cap \mathcal{E}_2$, for each $\theta \in \Theta$, condition (8) in Lemma 1 is satisfied with

$$c_2\left(\frac{1}{\sqrt{n}} + \sqrt{\frac{d}{N}}\log^2(Nd)\right) \|\theta - \theta^*\| + \left(\sqrt{\frac{1}{n}} + \sqrt{\frac{d}{N}}\right)\log(Nd).$$
 (29)

for sufficently large constant c_2 . In view of Lemma 1, Remarks 3 and 4, we have

$$||G(\theta) - \nabla F(\theta)|| \lesssim \left(\sqrt{\frac{q}{m}}\Delta_4 + \Delta_2\right) ||\theta - \theta^*|| + \sqrt{\frac{q}{m}}\Delta_3 + \Delta_1.$$

The proof is complete by invoking Proposition 1 and Proposition 2.

4.3 Proof of Theorem 1

Proof. From Theorem 3, we know that there a constant c_0 such that with probability at least $1 - 2e^{-\sqrt{d}}$, for all $\theta \in \Theta$,

$$||G(\theta) - \nabla F(\theta)|| \le c_0 \left(\left(\sqrt{\frac{q}{N}} + \sqrt{\frac{d}{N}} \log^2(Nd) \right) ||\theta - \theta^*|| + \left(\sqrt{\frac{q}{N}} + \sqrt{\frac{d}{N}} \right) \right).$$

Thus, with probability at least $1 - 2e^{-\sqrt{d}}$, we have for each t

$$\|\theta_{t} - \theta^{*}\| = \|\theta_{t-1} - \eta G(\theta_{t-1}) - \theta^{*}\|$$

$$= \|\theta_{t-1} - \eta \nabla F(\theta_{t-1}) - \theta^{*} + \eta \left(\nabla F(\theta_{t-1}) - G(\theta_{t-1})\right)\|$$

$$\leq \|\theta_{t-1} - \eta \nabla F(\theta_{t-1}) - \theta^{*}\| + \eta \|\left(\nabla F(\theta_{t-1}) - G(\theta_{t-1})\right)\|$$

$$\leq \sqrt{1 - \frac{M^{2}}{4L^{2}}} \|\theta_{t-1} - \theta^{*}\|$$

$$+ c_{0} \frac{M}{2L^{2}} \left(\left(\sqrt{\frac{q}{N}} + \sqrt{\frac{d}{N}} \log^{2}(Nd)\right) \|\theta_{t-1} - \theta^{*}\| + \left(\sqrt{\frac{q}{N}} + \sqrt{\frac{d}{N}}\right)\right), \quad (30)$$

where the last inequality follows from [CSX17, Lemma 3.2] and Theorem 3. Let

$$\rho \triangleq \sqrt{1 - \frac{M^2}{4L^2}} + c_0 \frac{M}{2L^2} \left(\sqrt{\frac{q}{N}} + \sqrt{\frac{d}{N}} \log^2(Nd) \right).$$

Recall that, by definition, $M \leq L$. For sufficiently large constants c' and c such that $N \geq c'q$, $N \geq cd^2 \log^8(Nd)$, it holds that

$$c_0 \frac{M}{2L^2} \left(\sqrt{\frac{q}{N}} + \sqrt{\frac{d}{N}} \log^2(Nd) \right) \le \frac{M}{16L^2}.$$

Consequently,

$$\rho \leq \sqrt{1 - \frac{M^2}{4L^2}} + \frac{M}{16L^2} \leq 1 - \frac{M^2}{8L^2} + \frac{M}{16L^2} \leq 1 - \frac{M^2}{16L^2},$$

where the last inequality follows from the assumption that $M \geq 1$. From (30), we have

$$\|\theta_{t} - \theta^{*}\| \leq \rho^{t} \|\theta_{0} - \theta^{*}\| + c_{0} \frac{M}{2L^{2}} \frac{1}{1 - \rho} \left(\sqrt{\frac{q}{N}} + \sqrt{\frac{d}{N}} \right)$$
$$\leq \left(1 - \frac{M^{2}}{16L^{2}} \right)^{t} \|\theta_{0} - \theta^{*}\| + 8c_{0} \left(\sqrt{\frac{q}{N}} + \sqrt{\frac{d}{N}} \right),$$

proving Theorem 1.

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Appendices

A Proof of Lemma 2

We first quote a classical concentration inequality for sum of independent, bounded random variables.

Lemma 5 (Bennett's inequality). Let Y_1, \dots, Y_m be independent random variables. Assume that $|Y_j - \mathbb{E}[Y_j]| \leq B$ almost surely for every j. Then for any t > 0, we have

$$\mathbb{P}\left\{\sum_{j=1}^{m} (Y_j - \mathbb{E}\left[Y_j\right]) \ge t\right\} \le \exp\left(-\frac{\sigma^2}{B^2} \cdot h\left(\frac{Bt}{\sigma^2}\right)\right),\,$$

where $\sigma^2 = \sum_{j=1}^m \text{var}(Y_j)$ is the variance of the sum, and

$$h(u) = (1+u)\log(1+u) - u.$$

Proof of Lemma 2. We use the idea of truncation. In this proof, we adopt the convention that $\frac{1}{0} = +\infty$.

For each copy $j = 1, \dots, m$, we partition Y_j into countably many pieces as follows: Let

$$Y_{j,0} = Y_j \mathbf{1}_{\{|Y_j| \le 1\}}$$

 $Y_{j,k} = Y_j \mathbf{1}_{\{e^{k-1} \le |Y_j| \le e^k\}}, \text{ for } k = 1, 2, ...$

It is easy to see that

$$Y_j = \sum_{k=0}^{\infty} Y_{j,k}, \text{ for } j = 1, \dots, m.$$

Let $S = \sum_{j=1}^{m} Y_j$. We have

$$S = \sum_{j=1}^{m} Y_j = \sum_{j=1}^{m} \left(\sum_{k=0}^{\infty} Y_{j,k} \right) = \sum_{k=0}^{\infty} \sum_{j=1}^{m} Y_{j,k} = \sum_{k=0}^{\infty} S_k,$$

where $S_k \triangleq \sum_{j=1}^m Y_{j,k}$, for $k = 0, 1, \cdots$. Thus,

$$\mathbb{P}\left\{ \left| \sum_{j=1}^{m} Y_j - m\mathbb{E}\left[Y\right] \right| > mt \right\} = \mathbb{P}\left\{ \left| S - \mathbb{E}\left[S\right] \right| > mt \right\} = \mathbb{P}\left\{ \left| \sum_{k=0}^{\infty} \left(S_k - \mathbb{E}\left[S_k\right] \right) \right| > mt \right\}$$

To bound $\mathbb{P}\left\{\left|\sum_{j=1}^{m}Y_{j}-m\mathbb{E}\left[Y\right]\right|>mt\right\}$ for a given t, our plan is to find a sequence of t_{k} (which depends on t) such that

$$\{|S - \mathbb{E}[S]| > mt\} \subseteq \bigcup_{k=0}^{\infty} \{|S_k - \mathbb{E}[S_k]| > mt_k\},$$
(31)

and

$$\mathbb{P}\left\{\left|S_{k}-\mathbb{E}\left[S_{k}\right]\right|>mt_{k}\right\}$$

is small enough to apply the union bound over all k.

In this proof, we choose $t_k = \frac{t}{2(k+1)^2}$ for $k = 0, 1, \dots$. It is easy to see that (31) holds.

Next, we bound $\mathbb{P}\{|S_k - \mathbb{E}[S_k]| > mt_k\}$ for each k. For given $t \geq t_0$, define

$$k_0 \triangleq \inf \left\{ k \in \mathbb{Z} : 4e^k (k+1)^2 \ge t \right\}. \tag{32}$$

We are particularly interested in the setting when $t \ge t_0 \ge e^2$, which implies that

$$1 \le k_0 \le \log t - 1,\tag{33}$$

noting that $4e^{\log t - 1}(\log t - 1 + 1)^2 \ge t$.

Case 1: $0 \le k \le k_0 - 1$. It is easy to see that when $t \ge t_0 \ge e^2$, $k_0 \ge 1$. Thus, case 1 is well posed. As per the definition of (32), for all $0 \le k \le k_0 - 1$, it holds that $4e^k(k+1)^2 < t$. That is,

$$2e^k < \frac{t}{2(k+1)^2} = t_k. (34)$$

On the other hand, by construction of $Y_{i,k}$ we have deterministically

$$|Y_{j,k} - \mathbb{E}[Y_{j,k}]| \le 2e^k$$
, for all k . (35)

Thus

$$|S_k - \mathbb{E}\left[S_k\right]| = \left|\sum_{j=1}^m Y_{j,k} - \mathbb{E}\left[\sum_{j=1}^m Y_{j,k}\right]\right| \le \sum_{j=1}^m |Y_{j,k} - \mathbb{E}\left[Y_{j,k}\right]| \le 2me^k \quad \text{for all } k,$$

i.e.,

$$\mathbb{P}\left\{ |S_k - \mathbb{E}\left[S_k\right]| > 2me^k \right\} = 0 \quad \text{for all } k.$$

By (34), we have when $0 \le k \le k_0 - 1$,

$$\mathbb{P}\left\{\left|S_{k} - \mathbb{E}\left[S_{k}\right]\right| > mt_{k}\right\} \leq \mathbb{P}\left\{\left|S_{k} - \mathbb{E}\left[S_{k}\right]\right| > 2me^{k}\right\} = 0,\tag{36}$$

Case 2: $k_0 \le k \le \log(mt)$. For each k in this range, we will apply Bennett's inequality.

From (35), we know that for any fixed k, the random variable $|Y_{j,k} - \mathbb{E}[Y_{j,k}]| \leq 2e^k$. The variance of $Y_{j,k}$ can be bounded as follows: for $k \geq 1$

$$\operatorname{var}(Y_{j,k}) = \mathbb{E}\left[(Y_{j,k} - \mathbb{E}\left[Y_{j,k}\right])^2 \right] \le \mathbb{E}\left[Y_{j,k}^2 \right] \le e^{2k} \mathbb{P}\left\{ |Y_j| \ge e^{k-1} \right\} \le e^{2k} \exp\left(-E\left(e^{k-1}\right) \right). \tag{37}$$

For notational convenience, define

$$\sigma_k^2 \triangleq e^{2k} \exp\left(-E(e^{k-1})\right). \tag{38}$$

To see that σ_k^2 is well-defined, recall that we adopt the convention that $\frac{1}{0} = \infty$ and $\exp(-\infty) = 0$. For each k in this case, i.e., $k_0 \le k \le \log(mt)$, by Lemma 5, we get

$$\begin{split} \mathbb{P}\left\{|S_k - \mathbb{E}\left[S_k\right]| \geq mt_k\right\} &= \mathbb{P}\left\{\left|\sum_{j=1}^m (Y_{j,k} - \mathbb{E}\left[Y_{j,k}\right])\right| \geq mt_k\right\} \\ &\leq 2\exp\left(-\frac{\sum_{j=1}^m \mathsf{var}(Y_{j,k})}{e^{2(k+1)}} \cdot h\left(\frac{e^{(k+1)}mt_k}{\sum_{j=1}^m \mathsf{var}(Y_{j,k})}\right)\right), \end{split}$$

Note that when u > 0, it holds that $h(u) \ge u \log(u/e)$. So we have

$$\mathbb{P}\left\{|S_{k} - \mathbb{E}\left[S_{k}\right]| \ge mt_{k}\right\} \le 2\exp\left(-\frac{\sum_{j=1}^{m} \operatorname{var}(Y_{j,k})}{e^{2(k+1)}} \cdot \frac{e^{(k+1)}mt_{k}}{\sum_{j=1}^{m} \operatorname{var}(Y_{j,k})} \log\left(\frac{e^{(k+1)}mt_{k}}{e\sum_{j=1}^{m} \operatorname{var}(Y_{j,k})}\right)\right) \\
= 2\exp\left(-\frac{mt_{k}}{e^{(k+1)}} \log\left(\frac{e^{k}mt_{k}}{\sum_{j=1}^{m} \operatorname{var}(Y_{j,k})}\right)\right) \\
\le 2\exp\left(-\frac{mt_{k}}{e^{(k+1)}} \log\left(\frac{e^{k}t_{k}}{\sigma_{k}^{2}}\right)\right), \tag{39}$$

where the last inequality follows from the fact that $\sum_{j=1}^{m} \mathsf{var}(Y_{j,k}) \leq m\sigma_k^2$. We proceed to bound

 $\log\left(\frac{e^k t_k}{\sigma_k^2}\right)$ using the assumption (12).

$$\log\left(\frac{e^{k}t_{k}}{\sigma_{k}^{2}}\right)$$

$$= \log\left(\frac{e^{k}t}{2(k+1)^{2}e^{2k}\exp\left(-E\left(e^{k-1}\right)\right)}\right)$$

$$= \log\left(\frac{t}{2(k+1)^{2}e^{k}\exp\left(-E\left(e^{k-1}\right)\right)}\right)$$

$$= \log t - \left(\log 2 + 2\log(k+1) + k - E\left(e^{k-1}\right)\right)$$

$$= E(e^{k-1}) - (\log 2 + 2\log(k+1) + k - \log t)$$

$$\stackrel{(a)}{\geq} \frac{1}{2}E(e^{k-1}) + (2k+4\log(k+1) + \log 2 - \log t) - (\log 2 + 2\log(k+1) + k - \log t)$$

$$\geq \frac{1}{2}E(e^{k-1}) + 2\log(k+1) + k,$$

$$(40)$$

where inequality (a) holds due to the assumption (12). Combining the last displayed equation with (39) yields

$$\mathbb{P}\left\{|S_{k} - \mathbb{E}\left[S_{k}\right]| \ge mt_{k}\right\} \le 2 \exp\left(-\frac{mt_{k}}{2e^{(k+1)}} E(e^{k-1})\right) \\
= 2 \exp\left(-\frac{mt}{4(k+1)^{2}e^{(k+1)}} E(e^{k-1})\right) \\
\le 2 \exp\left(-\frac{mt}{4(\log(mt) + 1)^{2}e^{(k+1)}} E(e^{k-1})\right), \tag{41}$$

where the last inequality holds because in the case under consideration, $k_0 \le k \le \log(mt)$. To proceed, we use the monotonicity assumption of E(t)/t. If E(t)/t is non-decreasing (increasing), we can bound (41) as

$$\mathbb{P}\left\{|S_{k} - \mathbb{E}\left[S_{k}\right]| \ge mt_{k}\right\} \stackrel{(a)}{\le} 2 \exp\left(-\frac{mt}{4(\log(mt) + 1)^{2}e^{(k_{0} + 1)}} E(e^{k_{0} - 1})\right) \\
\stackrel{(b)}{\le} 2 \exp\left(-\frac{mt}{4(\log(mt) + 1)^{2}t} E\left(\frac{t}{4e(k_{0} + 1)^{2}}\right)\right) \\
\stackrel{(c)}{\le} 2 \exp\left(-\frac{m}{4(\log(mt) + 1)^{2}} E\left(\frac{t}{4e\log^{2}t}\right)\right), \tag{42}$$

where (a) holds because $k_0 \le k \le \log(mt)$; (b) holds because $k_0 \le \log t - 1$, $4e^{k_0}(k_0 + 1)^2 \ge t$, and that $E(\cdot)$ is non-decreasing; (c) follows from $k_0 \le \log t - 1$, and that $E(\cdot)$ is non-decreasing. If E(t)/t is non-increasing, we can bound (41) as

$$\mathbb{P}\left\{|S_k - \mathbb{E}\left[S_k\right]| \ge mt_k\right\} \le 2\exp\left(-\frac{mt}{4(\log(mt) + 1)^2 e^{\log(mt) + 1}} E(e^{\log(mt) - 1})\right) \\
= 2\exp\left(-\frac{1}{4e(\log(mt) + 1)^2} E\left(\frac{mt}{e}\right)\right). \tag{43}$$

Case 3: $k \ge \log(mt)$. In this case, we use the Chebyshev's inequality:

$$\mathbb{P}\left\{|S_k - \mathbb{E}\left[S_k\right]| \ge mt_k\right\} \le \frac{\sigma_k^2}{t_k} = \exp\left(-\log\frac{t_k}{\sigma_k^2}\right) \\
\stackrel{(a)}{\le} \frac{1}{(k+1)^2} \exp\left(-\frac{1}{2}E(e^{k-1})\right) \\
\le \frac{1}{(k+1)^2} \exp\left(-\frac{1}{2}E\left(\frac{mt}{e}\right)\right), \tag{44}$$

where (a) follows from (40); the last inequality follows from the fact that E(u) is increasing (non-decreasing) in u.

For a fix t, summing over all $k \in \mathbb{N}$, we have

$$\begin{split} \mathbb{P}\left\{\left|\sum_{j=1}^{m}Y_{j}-m\mathbb{E}\left[Y\right]\right| \geq mt\right\} \leq \sum_{k=0}^{\infty}\mathbb{P}\left\{\left|S_{k}-\mathbb{E}\left[S_{k}\right]\right| \geq mt_{k}\right\} \\ &= \sum_{k=0}^{k_{0}-1}\mathbb{P}\left\{\left|S_{k}-\mathbb{E}\left[S_{k}\right]\right| \geq mt_{k}\right\} + \sum_{k=k_{0}}^{\log(mt)}\mathbb{P}\left\{\left|S_{k}-\mathbb{E}\left[S_{k}\right]\right| \geq mt_{k}\right\} \\ &+ \sum_{\log(mt)+1}^{\infty}\mathbb{P}\left\{\left|S_{k}-\mathbb{E}\left[S_{k}\right]\right| \geq mt_{k}\right\} \\ &\leq 0 + \exp\left(-\frac{1}{2}E\left(\frac{mt}{e}\right)\right) + \sum_{k=k_{0}}^{\log(mt)}\mathbb{P}\left\{\left|S_{k}-\mathbb{E}\left[S_{k}\right]\right| \geq mt_{k}\right\}. \end{split}$$

Therefore, we have if E(t)/t non-decreasing

$$\mathbb{P}\left\{\left|\sum_{j=1}^{m} Y_{j} - m\mathbb{E}\left[Y\right]\right| \ge mt\right\} \le 2\log(mt)\exp\left(-\frac{m}{4(\log(mt) + 1)^{2}}E\left(\frac{t}{4e\log^{2}t}\right)\right) + \exp\left(-\frac{1}{2}E\left(\frac{mt}{e}\right)\right);$$

if E(t)/t is non-increasing

$$\mathbb{P}\left\{\left|\sum_{j=1}^{m} Y_j - m\mathbb{E}\left[Y\right]\right| \ge mt\right\} \le 2\log(mt)\exp\left(-\frac{1}{4e(\log(mt) + 1)^2}E\left(\frac{mt}{e}\right)\right) + \exp\left(-\frac{1}{2}E\left(\frac{mt}{e}\right)\right).$$

B Robust Mean Estimation

We present the proof of Lemma 1 for completeness. For ease of exposition, in the sequel, we let

$$\alpha \triangleq 1 - \epsilon$$
 and $\tilde{\sigma}^2 = 2\sigma^2$.

We first need a minimax identity between the min-max problem (6) and max-min problem (7). For $W \in \mathbb{R}^{|\mathcal{A}| \times |\mathcal{A}|}$ and $U \in \mathbb{R}^{d \times d}$, define a function $\psi : (W, U) \to \mathbb{R}$ as:

$$\psi(W, U) = \sum_{i \in \mathcal{A}} c_i \left(\widehat{y}_i - \sum_{j \in \mathcal{A}} \widehat{y}_j W_{ji} \right)^{\top} U \left(\widehat{y}_i - \sum_{j \in \mathcal{A}} \widehat{y}_j W_{ji} \right).$$

Also, let \mathcal{W} denote the set of all column stochastic matrices $W \in \mathbb{R}^{|\mathcal{A}| \times |\mathcal{A}|}$ such that $0 \leq W_{ji} \leq \frac{4-\alpha}{\alpha(2+\alpha)m}$, and \mathcal{U} denote the set of all positive semi-definite matrices $U \in \mathbb{R}^{d \times d}$ such that $\mathsf{Tr}(U) \leq 1$. Then the min-max program (6) can be rewritten as

$$W^* \in \arg\min_{W \in \mathcal{W}} \max_{U \in \mathcal{U}} \psi(W, U) = \arg\min_{W \in \mathcal{W}} \left\| \sum_{i \in \mathcal{A}} c_i (\widehat{y}_i - \sum_{j \in \mathcal{A}} \widehat{y}_j W_{ji}) (\widehat{y}_i - \sum_{j \in \mathcal{A}} \widehat{y}_j W_{ji})^\top \right\|. \tag{45}$$

and the max-min program (7) can be rewritten as

$$U^* \in \max_{U \in \mathcal{U}} \min_{W \in \mathcal{W}} \psi(W, U).$$

Note that $\psi(W, U)$ is convex in W for a fixed U and concave (in fact linear) in U for a fixed W. By von Neumann's minimax theorem, we have

$$\min_{W \in \mathcal{W}} \ \max_{U \in \mathcal{U}} \ \psi(W,U) = \max_{U \in \mathcal{U}} \ \min_{W \in \mathcal{W}} \ \psi(W,U) = \psi(W^*,U^*).$$

Moreover, (W^*, U^*) is a saddle point, i.e.,

$$W^* \in \arg\min_{W \in \mathcal{W}} \ \psi(W, U^*), \tag{46}$$

$$U^* \in \arg\max_{U \in \mathcal{U}} \ \psi(W^*, U). \tag{47}$$

The saddle point properties (46) and (47) are crucial to prove Lemma 1.

Moreover, by condition (8), the underlying true sample S (of size m) satisfies the following condition:

$$\left\| \frac{1}{m} \sum_{i=1}^{m} (y_i - \mu_{\mathcal{S}}) (y_i - \mu_{\mathcal{S}})^{\top} \right\| \leq \sigma^2,$$

where $\mu_{\mathcal{S}} = \frac{1}{m} \sum_{i=1}^{m} y_i$. Recall that up to q points in \mathcal{S} are corrupted. Let $\mathcal{S}_0 \subseteq \mathcal{S}$ be a subset of uncorrupted subset of \mathcal{S} of size $m - q = (1 - \epsilon)m = \alpha m$. Notably, since q is only an upper bound on the number of corrupted data points, the choice of subset \mathcal{S}_0 may not be unique. Nevertheless, for any choice of subset \mathcal{S}_0 , the following holds:

$$\left\| \frac{1}{|\mathcal{S}_0|} \sum_{i \in \mathcal{S}_0} (y_i - \mu_{\mathcal{S}}) (y_i - \mu_{\mathcal{S}})^\top \right\| = \frac{1}{|\mathcal{S}_0|} \left\| \sum_{i \in \mathcal{S}_0} (y_i - \mu_{\mathcal{S}}) (y_i - \mu_{\mathcal{S}})^\top \right\|$$

$$\leq \frac{1}{|\mathcal{S}_0|} \left\| \sum_{i=1}^m (y_i - \mu_{\mathcal{S}}) (y_i - \mu_{\mathcal{S}})^\top \right\|$$

$$\leq \frac{1}{\alpha} \sigma^2$$

$$\leq 2\sigma^2, \tag{48}$$

where the last inequality follows because by assumption, $\alpha = 1 - \epsilon \ge \frac{3}{4} \ge \frac{1}{2}$.

As commented in Subsection 3.2, Algorithm 2 terminates within m iterations. For ease of exposition, we use $t = 1, 2, \cdots$ to denote the iteration number in the loop from line 3 to line 9. We use $c_i(t)$, $\tau_i(t)$, and $\mathcal{A}(t)$ to denote the quantities of interest at iteration t. Note that weights c_i and set \mathcal{A} may be updated throughout an iteration. Therefore, we use $\mathcal{A}'(t)$ and $c_i'(t)$ to denote the updated quantities at the end of iteration t. Note that $c_i'(t-1) = c_i(t)$ and $\mathcal{A}'(t-1) = \mathcal{A}(t)$.

B.1 Two auxiliary lemmas

We first show that when Algorithm 2 terminates, most of data points in S_0 are remained in A.

Lemma 6. For every iteration $t \ge 1$ in the while-loop of Algorithm 2,

$$\sum_{i \in \mathcal{S}_0 \cap \mathcal{A}(t)} c_i(t)\tau_i(t) \leq \alpha m \tilde{\sigma}^2 \tag{49}$$

$$\sum_{i \in S_0} (1 - c_i(t)) \le \frac{\alpha}{4} \sum_{i=1}^m (1 - c_i(t))$$
(50)

$$|\mathcal{S}_0 \cap \mathcal{A}(t)| \ge \frac{\alpha(2+\alpha)m}{4-\alpha}.\tag{51}$$

Intuitively, Lemma 6 says that in every iteration: (1) the summation of the projected residual error over the non-corrupted data is small; (2) the weights of non-corrupted data points are reduced by a relatively small amount; (3) and more importantly, most non-corrupted data points are not removed.

Proof of Lemma 6. The proof is by induction on (50) and (51). Recall that we use $t = 1, \dots$ to denote the iteration number in the **while**-loop.

Base case: t = 1. Note that $\mathcal{A}(1) = [m]$, and $c_i(1) = 1$ for all $i \in \mathcal{A}(1)$. Therefore, (50) and (51) hold for t = 1 trivially.

Induction Step: Suppose (50) and (51) hold for t, and the **while**— has not terminate at iteration t. We aim to show (50) and (51) hold for t + 1.

We first prove (49) holds for t. Recall that

$$\tau_i(t) = \left(y_i - \sum_{j \in \mathcal{A}(t)} \widehat{y}_j W_{ji}(t) \right)^{\top} U(t) \left(y_i - \sum_{j \in \mathcal{A}(t)} \widehat{y}_j W_{ji}(t) \right),$$

where W(t) is a minimizer to (6) and U(t) is a maximizer to (7) at iteration t, respectively. Since (W(t), U(t)) is a saddle point, it follows from (46) that $W(t) \in \arg\min_{W \in \mathcal{W}} \psi(W, U(t))$. Moreover, this minimization is decoupled over all data points in $\mathcal{A}(t)$ and hence each column of W(t) is optimized independently. Therefore, by letting $W_{*i}(t)$ denote the column of W(t) corresponding to

 $i \in \mathcal{A}(t)$, we have

$$W_{*i}(t) \in \arg\min_{w} \left(y_i - \sum_{j \in \mathcal{A}(t)} \widehat{y}_j w_j \right)^{\top} U(t) \left(y_i - \sum_{j \in \mathcal{A}(t)} \widehat{y}_j w_j \right)$$
s. t.
$$\sum_{j \in \mathcal{A}(t)} w_j = 1$$

$$0 \le w_j \le \frac{4 - \alpha}{\alpha (2 + \alpha) m}.$$
(52)

Let $\widetilde{w} \in \mathbb{R}^{|\mathcal{A}(t)|}$ be the column stochastic vector such that

$$\widetilde{w}_j \triangleq \frac{\mathbf{1}_{\{j \in \mathcal{S}_0 \cap \mathcal{A}(t)\}}}{|\mathcal{S}_0 \cap \mathcal{A}(t)|}, \quad \forall \ j \in \mathcal{A}(t).$$

By the induction hypothesis, \widetilde{w} is feasible to (52). Let $Y_{\mathcal{A}(t)} \in \mathbb{R}^{d \times n}$ be the matrix with \widehat{y}_i with $i \in \mathcal{A}(t)$ as columns. Moreover,

$$Y_{\mathcal{A}(t)}\widetilde{w} = \sum_{j \in \mathcal{A}(t)} \widehat{y}_j \widetilde{w}_j = \frac{1}{|\mathcal{S}_0 \cap \mathcal{A}(t)|} \sum_{j \in \mathcal{S}_0 \cap \mathcal{A}(t)} y_j \triangleq \mu_{\mathcal{S}_0 \cap \mathcal{A}(t)}.$$

Thus, we have

$$\sum_{i \in \mathcal{S}_0 \cap \mathcal{A}(t)} c_i(t) \tau_i(t) \overset{(a)}{\leq} \sum_{i \in \mathcal{S}_0 \cap \mathcal{A}(t)} c_i(t) (y_i - \mu_{\mathcal{S}_0 \cap \mathcal{A}(t)})^\top U(t) \left(y_i - \mu_{\mathcal{S}_0 \cap \mathcal{A}(t)} \right) \\
\overset{(b)}{\leq} \sum_{i \in \mathcal{S}_0 \cap \mathcal{A}(t)} (y_i - \mu_{\mathcal{S}_0 \cap \mathcal{A}(t)})^\top U(t) \left(y_i - \mu_{\mathcal{S}_0 \cap \mathcal{A}(t)} \right) \\
\overset{(c)}{\leq} \sum_{i \in \mathcal{S}_0 \cap \mathcal{A}(t)} (y_i - \mu_{\mathcal{S}})^\top U(t) \left(y_i - \mu_{\mathcal{S}} \right) \\
&\leq \sum_{i \in \mathcal{S}_0} (y_i - \mu_{\mathcal{S}})^\top U(t) \left(y_i - \mu_{\mathcal{S}} \right) \\
&\stackrel{(d)}{\leq} \operatorname{Tr} \left(U(t) \right) \left\| \sum_{i \in \mathcal{S}_0} (y_i - \mu_{\mathcal{S}}) (y_i - \mu_{\mathcal{S}})^\top \right\| \\
&\stackrel{(e)}{\leq} \alpha m \widetilde{\sigma}^2,$$

where (a) holds by the optimality of $W_{*i}(t)$ to (52); (b) holds because $c_i(t) \leq 1$ and $U(t) \succeq 0$; (c) holds because $\mu_{\mathcal{S}_0 \cap \mathcal{A}(t)} = \frac{1}{|\mathcal{S}_0 \cap \mathcal{A}(t)|} \sum_{i \in \mathcal{S}_0 \cap \mathcal{A}(t)} y_i$ is a minimizer of the quadratic form $\sum_{i \in \mathcal{S}_0 \cap \mathcal{A}(t)} (y_i - u)^{\top} U(t) (y_i - u)$, as a function of u; (d) holds because $|\langle A, B \rangle| \leq ||A|| \, ||B||_*$, where $||B||_*$ is the sum of singlular values of B and $||B||_* = \mathsf{Tr}(B)$ when $B \succeq 0$; (e) follows by (48) and the facts that $|\mathcal{S}_0| \leq \alpha m$ and $\mathsf{Tr}(U(t)) = 1$.

Next we prove (50) and (51). Since by induction hypothesis the **while**— has not terminate at iteration t, it follows that

$$\sum_{i \in \mathcal{A}(t)} c_i(t)\tau_i(t) > 4m\tilde{\sigma}^2.$$
 (53)

Note that the weights of the data points that do not lie in $\mathcal{A}(t)$ are not updated in iteration t, i.e., $c'_i(t) = c_i(t)$ for $i \notin \mathcal{A}(t)$. As a consequence, we have

$$\sum_{i \in \mathcal{S}_0} \left(1 - c_i'(t) \right) = \sum_{i \in \mathcal{S}_0} \left(1 - c_i(t) \right) + \sum_{i \in \mathcal{S}_0 \cap \mathcal{A}(t)} \left(c_i(t) - c_i'(t) \right)$$

$$\leq \frac{\alpha}{4} \sum_{i=1}^m \left(1 - c_i(t) \right) + \frac{1}{\tau_{\max}(t)} \sum_{i \in \mathcal{S}_0 \cap \mathcal{A}(t)} \tau_i(t) c_i(t), \tag{54}$$

where the last inequality follows from induction hypothesis. Furthermore, we have

$$\frac{1}{\tau_{\max}(t)} \sum_{i \in \mathcal{S}_0 \cap \mathcal{A}(t)} \tau_i(t) c_i(t) \stackrel{(a)}{\leq} \frac{1}{\tau_{\max}(t)} \alpha m \widetilde{\sigma}^2 \stackrel{(b)}{<} \frac{\alpha}{4\tau_{\max}(t)} \sum_{i \in \mathcal{A}(t)} \tau_i(t) c_i(t),$$

where (a) holds because we have shown that (49) holds for t; (b) holds because we have assumed without loss of generality that $\sum_{i \in \mathcal{A}(t)} c_i(t) \tau_i^*(t) > 4m\widetilde{\sigma}^2$.

Thus, (54) can be further bounded as

$$\sum_{i \in S_0} (1 - c_i'(t)) \leq \frac{\alpha}{4} \sum_{i=1}^m (1 - c_i(t)) + \frac{\alpha}{4\tau_{\max}(t)} \sum_{i \in A(t)} \tau_i(t) c_i(t)
= \frac{\alpha}{4} \left(\sum_{i \notin A(t)} (1 - c_i(t)) + \sum_{i \in A(t)} (1 - c_i(t)) + \frac{1}{\tau_{\max}(t)} \sum_{i \in A(t)} \tau_i^*(t) c_i(t) \right)
= \frac{\alpha}{4} \left(\sum_{i \notin A(t)} (1 - c_i'(t)) + \sum_{i \in A(t)} \left(1 - \left(1 - \frac{\tau_i^*(t)}{\tau_{\max}(t)} \right) c_i(t) \right) \right)
= \frac{\alpha}{4} \sum_{i=1}^m (1 - c_i'(t)),$$

proving (50) for t+1. We rewrite (50) for t+1 as

$$\sum_{i \in \mathcal{S}_0} \left(1 - c_i'(t) \right) \le \frac{\alpha}{4 - \alpha} \sum_{i \notin \mathcal{S}_0} \left(1 - c_i'(t) \right).$$

One the one hand, we have

$$\sum_{i \notin \mathcal{S}_0} (1 - c_i'(t)) \le |\mathcal{S}_0^c| \le (1 - \alpha)m.$$

On the other hand,

$$\sum_{i \in \mathcal{S}_0} \left(1 - c_i'(t) \right) \ge \sum_{i \in \mathcal{S}_0 \setminus \mathcal{A}'(t)} \left(1 - c_i'(t) \right) \ge \frac{1}{2} \left| \mathcal{S}_0 \setminus \mathcal{A}'(t) \right|,$$

where the last inequality holds from the fact that $c'_i(t) \leq 1/2$ for all $i \notin \mathcal{A}'(t)$ – by the data removal criterion in Algorithm 2. Combining the last three displayed equations, we get that

$$\left| \mathcal{S}_0 \setminus \mathcal{A}'(t) \right| \leq \frac{2\alpha(1-\alpha)}{4-\alpha} m,$$

proving (50) for t+1. The proof of Lemma 6 is complete.

Let W be the minimizer of (6) when the **while**-loop terminates. Let W_1 be the result of zeroing out all singular values of W that are greater than 0.9.

Lemma 7. The matrix $W_0 = (W - W_1)(I - W_1)^{-1}$ is a column stochastic matrix, and the rank of the weight matrix W_0 is one.

Remark 8. Let $X_{\mathcal{A}} \subseteq \mathbb{R}^{d \times |\mathcal{A}|}$ be the data matrix with columns being the data points in \mathcal{A} . Let $Z = X_{\mathcal{A}}W_0$. Since W_0 is rank one, all the $|\mathcal{A}|$ columns in the matrix Z are identical. Denote

$$Z = [\widetilde{\mu}, \cdots, \widetilde{\mu}]. \tag{55}$$

Then $\widetilde{\mu}$ is a weighted average of the points in \mathcal{A} .

Proof. We first show that W_0 is a column stochastic matrix:

$$\mathbf{1}^{\top} W_0 = \mathbf{1}^{\top} (W - W_1) (I - W_1)^{-1} \stackrel{(a)}{=} (\mathbf{1}^{\top} - \mathbf{1}^{\top} W_1) (I - W_1)^{-1}$$
$$= \mathbf{1}^{\top} (I - W_1) (I - W_1)^{-1} = \mathbf{1}^{\top},$$

where (a) follows because W is column stochastic.

Next we show that rank of W_0 is one. From (6), we know that $\|W\|_F^2 \leq \frac{4-\alpha}{\alpha(2+\alpha)}$. To see this,

$$||W||_{\mathrm{F}}^2 = \sum_{i,j \in \mathcal{A}} W_{ji}^2 \le \sum_{i,j \in \mathcal{A}} \left(W_{ji} \cdot \max_{i,j \in \mathcal{A}} W_{ji} \right) \le \left(\sum_{i,j \in \mathcal{A}} W_{ji} \right) \frac{4 - \alpha}{\alpha (2 + \alpha) m} \le \frac{4 - \alpha}{\alpha (2 + \alpha)}.$$

When $\alpha \geq \frac{3}{4}$,

$$\frac{4-\alpha}{\alpha(2+\alpha)} \le \frac{52}{33} < 2 \times 0.9^2.$$

Hence, at most one singular value of W can be greater than 0.9. Moreover, since W is column stochastic, its largest singular value is at least 1. Thus, $W - W_1$ is of rank one. As a consequence, W_0 is of rank one.

B.2 Proof of Lemma 1

Recall that our goal is to show

$$\|\mu_{\mathcal{S}} - \widehat{\mu}\| = O(\sigma\sqrt{1-\alpha}),$$

where $\widehat{\mu} = \frac{1}{|\mathcal{A}|} \sum_{i \in \mathcal{A}} \widehat{y}_i$ is the algorithm output. Recall $Y_{\mathcal{A}} \subseteq \mathbb{R}^{d \times |\mathcal{A}|}$ is the data matrix with columns being the data points in \mathcal{A} . In view of Remark 8, columns of $Z = Y_{\mathcal{A}}W_0$ are identical and denoted by $\widetilde{\mu}$. Our proof is divided into two steps:

• We first show that points in \mathcal{A} are clustered around the center $\widetilde{\mu}$. In addition, by (51) in Lemma 6, the set \mathcal{A} mainly consists of uncorrupted data. As a consequence, we are able to show that

$$\widehat{\mu} = \frac{1}{|\mathcal{A}|} \sum_{i \in \mathcal{A}} \widehat{y}_i \approx \frac{1}{|\mathcal{S}_0 \cap \mathcal{A}|} \sum_{i \in \mathcal{S}_0 \cap \mathcal{A}} \widehat{y}_i = \frac{1}{|\mathcal{S}_0 \cap \mathcal{A}|} \sum_{i \in \mathcal{S}_0 \cap \mathcal{A}} y_i.$$
 (56)

• By (48), points in S_0 are clustered around the center μ_S . In addition, by (51) in Lemma 6, most of the points in S_0 have been preserved. Thus we are able to show that

$$\mu_{\mathcal{S}} = \frac{1}{m} \sum_{i=1}^{m} y_i \approx \frac{1}{|\mathcal{S}_0 \cap \mathcal{A}|} \sum_{i \in \mathcal{S}_0 \cap \mathcal{A}} y_i.$$
 (57)

Putting these two pieces together, the proof of Lemma 1 is complete.

Step 1: We show (56).

When the **while**-loop terminates, in view of (45), we have

$$\left\| Y_{\mathcal{A}}(I-W)\operatorname{diag}\left\{ (c_{\mathcal{A}})^{\frac{1}{2}}\right\} \right\| \leq 2\sqrt{m}\widetilde{\sigma}.,$$
 (58)

where diag $\{(c_{\mathcal{A}})^{\frac{1}{2}}\}$ is the diagonal matrix with diagonal entries given by $\{c_i^{1/2}\}_{i\in\mathcal{A}}$. We will show that $\widehat{y}_i \approx \widetilde{\mu}$ for all $i \in \mathcal{A}$. For this purpose, it is enough to show $||Y_{\mathcal{A}} - Z||$ is small:

$$\begin{split} \left\| Y_{\mathcal{A}} - \widetilde{\mu} \mathbf{1}^T \right\| &= \| Y_{\mathcal{A}} - Z \| = \| Y_{\mathcal{A}} - Y_{\mathcal{A}} W_0 \| \\ &= \left\| Y_{\mathcal{A}} (I - W_1) (I - W_1)^{-1} - Y_{\mathcal{A}} (W - W_1) (I - W_1)^{-1} \right\| \\ &= \left\| Y_{\mathcal{A}} (I - W) (I - W_1)^{-1} \right\| \\ &\leq \left\| Y_{\mathcal{A}} (I - W) \right\| \left\| (I - W_1)^{-1} \right\| \\ &\stackrel{(a)}{\leq} \left\| Y_{\mathcal{A}} (I - W) \right\| \times 10 \\ &\stackrel{(b)}{\leq} 10 \sqrt{2} \left\| Y_{\mathcal{A}} (I - W) \mathrm{diag} \left\{ (c_{\mathcal{A}})^{\frac{1}{2}} \right\} \right\| \\ &\stackrel{(c)}{<} 20 \sqrt{2m} \widetilde{\sigma}, \end{split}$$

where (a) holds because the largest singular value of W_1 is at most 0.9; (b) holds because $c_i \geq \frac{1}{2}$ for all $i \in \mathcal{A}$; (c) follows from (58).

Fix any $0 < \epsilon' < 1/2$. Let $\mathcal{T} \subseteq \mathcal{A}$ such that $|\mathcal{T}| \ge (1 - \epsilon')|\mathcal{A}|$. We have

$$\left\| \frac{1}{|\mathcal{T}|} \sum_{i \in \mathcal{T}} \widehat{y}_{i} - \widehat{\mu} \right\| = \left\| \frac{1}{|\mathcal{T}|} \sum_{i \in \mathcal{T}} \widehat{y}_{i} - \frac{1}{|\mathcal{A}|} \sum_{i \in \mathcal{A}} \widehat{y}_{i} \right\| = \left\| \frac{1}{|\mathcal{T}|} \sum_{i \in \mathcal{T}} (\widehat{y}_{i} - \widetilde{\mu}) - \frac{1}{|\mathcal{A}|} \sum_{i \in \mathcal{A}} (\widehat{y}_{i} - \widetilde{\mu}) \right\|$$

$$= \left\| \left(\frac{1}{|\mathcal{T}|} - \frac{1}{|\mathcal{A}|} \right) \sum_{i \in \mathcal{T}} (\widehat{y}_{i} - \widetilde{\mu}) - \frac{1}{|\mathcal{A}|} \sum_{i \in \mathcal{A}/\mathcal{T}} (\widehat{y}_{i} - \widetilde{\mu}) \right\|$$

$$\stackrel{(a)}{\leq} \frac{|\mathcal{A}| - |\mathcal{T}|}{|\mathcal{T}||\mathcal{A}|} \| [Y_{\mathcal{A}} - Z]_{\mathcal{T}} \mathbf{1} \| + \frac{1}{|\mathcal{A}|} \| [Y_{\mathcal{A}} - Z]_{\mathcal{A}/\mathcal{T}} \mathbf{1} \|$$

$$= \left(\frac{|\mathcal{A}| - |\mathcal{T}|}{\sqrt{|\mathcal{T}|}|\mathcal{A}|} + \frac{\sqrt{|\mathcal{A}/\mathcal{T}|}}{|\mathcal{A}|} \right) \|Y_{\mathcal{A}} - Z \|$$

$$\leq 80\sqrt{2}\widetilde{\sigma}\sqrt{\epsilon'}, \tag{59}$$

where $[Y_{\mathcal{A}} - Z]_{\mathcal{T}}$ denotes the submatrix of $Y_{\mathcal{A}} - Z$ – restricting to columns in \mathcal{T} , and $\mathbf{1} \in \mathbb{R}^{|\mathcal{T}|}$; the last inequality holds because $\epsilon' < 1/2$ and

$$|\mathcal{A}| \ge |\mathcal{A} \cap \mathcal{S}_0| \ge \frac{\alpha(2+\alpha)}{4-\alpha}m.$$

Note that

$$\frac{\alpha(2+\alpha)}{4-\alpha} \ge 1 - \frac{5}{3}(1-\alpha) \Leftrightarrow (\alpha-1)^2 \ge 0.$$

Thus, $|\mathcal{A} - \mathcal{A} \cap \mathcal{S}_0| \leq \frac{5}{3}(1 - \alpha)m$. Choosing $\mathcal{T} = \mathcal{A} \cap \mathcal{S}_0$, we obtain

$$\|\mu_{\mathcal{S}_0 \cap \mathcal{A}} - \widehat{\mu}\| \le 80\sqrt{2}\widetilde{\sigma}\sqrt{5(1-\alpha)/3} \le 160\widetilde{\sigma}\sqrt{1-\alpha} = O(\widetilde{\sigma}\sqrt{1-\alpha}). \tag{60}$$

Step 2: We show (57). The proof of (57) is similar to that of (56). Recall that $\mu_{\mathcal{S}} = \frac{1}{m} \sum_{i=1}^{m} y_i$, and that $\mu_{\mathcal{S}_0 \cap \mathcal{A}} = \frac{1}{|\mathcal{S}_0 \cap \mathcal{A}|} \sum_{i \in \mathcal{S}_0 \cap \mathcal{A}} y_i$. We have

$$\|\mu_{\mathcal{S}} - \mu_{\mathcal{S}_0 \cap \mathcal{A}}\| = \left\| \mu_{\mathcal{S}} - \frac{1}{|\mathcal{A} \cap \mathcal{S}_0|} \sum_{i \in \mathcal{S}_0 \cap \mathcal{A}} y_i \right\|$$

$$= \left\| \frac{1}{|\mathcal{A} \cap \mathcal{S}_0|} \sum_{i \in \mathcal{A} \cap \mathcal{S}_0} (y_i - \mu_{\mathcal{S}}) \right\|$$

$$= \frac{1}{|\mathcal{A} \cap \mathcal{S}_0|} \|[Y_{\mathcal{A} \cap \mathcal{S}_0} - \mu_{\mathcal{S}}] \mathbf{1}\|$$

$$\leq \frac{\sqrt{|\mathcal{S}_0|}}{\sqrt{|\mathcal{A} \cap \mathcal{S}_0|}} \tilde{\sigma}$$

$$\leq \sqrt{\frac{4 - \alpha}{\alpha(2 + \alpha)}} \sqrt{1 - \alpha} \tilde{\sigma} \leq \sqrt{2(1 - \alpha)} \tilde{\sigma}.$$

B.3 Modification of Algorithm 2

Suppose the modified Algorithm 2 terminates at iteration t^* . By the modified code we know $|\mathcal{A}(t^*)| \geq \frac{\alpha(2+\alpha)m}{4-\alpha}$; otherwise, the algorithm terminates earlier than t^* . By the termination condition, we also know that

$$\left| \mathcal{A}(t^*) - \left\{ i : \left(1 - \frac{\tau_i}{\tau_{\text{max}}} \right) c_i \le \frac{1}{2} \right\} \right| < \frac{\alpha(2+\alpha)m}{4-\alpha}. \tag{61}$$

Claim 1. There exists an iteration $t' \leq t^*$ such that $\sum_{i \in \mathcal{A}(t')} c_i(t') \tau_i(t') \leq 8m\sigma^2$.

Proof. We prove by contradiction. Suppose

$$\sum_{i \in \mathcal{A}(t)} c_i(t)\tau_i(t) > 8m\sigma^2, \quad \forall t \le t^*.$$
(62)

Note that the modified Algorithm 2 and the original Algorithm 2 differ only in their termination conditions. Recall that the original termination condition is only used in the proof of Lemma 6 to conclude that (53) holds when the **while**-loop does not terminate. Thus, under the hypothesis (given in the last displayed equation), Lemma 6 still holds. It follows that

$$\left| \mathcal{A}(t^*) - \left\{ i : \left(1 - \frac{\tau_i}{\tau_{\max}} \right) c_i \le \frac{1}{2} \right\} \right| \ge \left| \mathcal{S}_0 \cap \left(\mathcal{A}(t^*) - \left\{ i : \left(1 - \frac{\tau_i}{\tau_{\max}} \right) c_i \le \frac{1}{2} \right\} \right) \right|$$

$$\ge \frac{\alpha(2 + \alpha)m}{4 - \alpha},$$

which leads to a contradiction.

Since $\mathcal{A}(t)$ is monotone decreasing, it follows that $\mathcal{A}(t^*) \subseteq \mathcal{A}(t')$. Moreover,

$$|\mathcal{A}(t^*)| \ge \frac{\alpha(2+\alpha)m}{4-\alpha} \ge \frac{\alpha(2+\alpha)}{4-\alpha} \left| \mathcal{A}(t') \right| \ge \left(1 - \frac{5}{3}(1-\alpha)\right) \left| \mathcal{A}(t') \right|.$$

By (59), we know

$$\left\| \frac{1}{|\mathcal{A}(t^*)|} \sum_{i \in \mathcal{A}(t^*)} \widehat{y}_i - \frac{1}{|\mathcal{A}(t')|} \sum_{i \in \mathcal{A}(t')} \widehat{y}_i \right\| \le 80\sqrt{2}\widetilde{\sigma}\sqrt{\frac{5}{3}(1-\alpha)} = O(\sigma\sqrt{1-\alpha}).$$

From Lemma 1, we know

$$\left\| \frac{1}{|\mathcal{A}(t')|} \sum_{i \in \mathcal{A}(t')} \widehat{y}_i - \mu_{\mathcal{S}} \right\| = O(\sigma \sqrt{1 - \alpha}).$$

Combining the last two displayed equations, we have

$$\left\| \frac{1}{|\mathcal{A}(t^*)|} \sum_{i \in \mathcal{A}(t^*)} \widehat{y}_i - \mu_{\mathcal{S}} \right\| = O(\sigma \sqrt{1 - \alpha}).$$