1. Questions 1-2 are about noisy targets.

Consider the bin model for a hypothesis h that makes an error with probability μ in approximating a deterministic target function f (both h and f outputs $\{-1, +1\}$). If we use the same h to approximate a noisy version of f given by

$$P(\mathbf{x}, y) = P(\mathbf{x})P(y|\mathbf{x})$$

$$P(y|\mathbf{x}) = \begin{cases} \lambda & y = f(\mathbf{x}) \\ 1 - \lambda & \text{otherwise} \end{cases}$$

What is the probability of error that h makes in approximating the noisy target y?

Ans:

If $y = f(x)$, the probability of error is $\lambda \cdot \mu$; otherwise, the probability is $(1 - \lambda) \cdot (1 - \mu)$. Hence the answer is $\lambda\mu + (1 - \lambda) \cdot (1 - \mu)$.

2.

Following Question 1, with what value of λ will the performance of h be independent of μ?

Ans:

$\lambda\mu + (1 - \lambda) \cdot (1 - \mu) = 1 - \lambda + (2\lambda - 1)\mu$

The coefficient of μ is zero. → $\lambda = 0.5$.

3.

Questions 3-5 are about generalization error, and getting the feel of the bounds numerically. Please use the simple upper bound $N^{d_{\mathrm{vc}}}$ on the growth function $m_{\mathcal{H}}(N)$, assuming that $N \geq 2$ and $d_{vc} \geq 2$.

For an $\mathcal{H}$ with $d_{\mathrm{vc}} = 10$, if you want 95% confidence that your generalization error is at most 0.05, what is the closest numerical approximation of the sample size that the VC generalization bound predicts?

Ans:

$\varepsilon = 0.05, \text{bound} = 5\%, d_{vc} = 10$

$4 \cdot (2N)^{10} \cdot e^{-\frac{1}{8} \cdot \frac{1}{400} \cdot N} \leq 5\%$ → $N \approx 460{,}000$

4.

There are a number of bounds on the generalization error ϵ, all holding with probability at least $1-\delta$. Fix $d_{\mathrm{vc}}=50$ and $\delta=0.05$ and plot these bounds as a function of N. Which bound is the tightest (smallest) for very large N, say $N=10,000$?

Note that Devroye and Parrondo & Van den Broek are implicit bounds in ϵ.

Ans:

Run hw2_4.py with $N=10000$, the answer is Devroye.

5.

Continuing from Question 4, for small N, say $N=5$, which bound is the tightest (smallest)?

Ans:

Run hw2_4.py with $N=5$, the answer is Parrondo and Van den Broek.

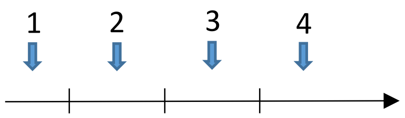
6.

In Questions 6-11, you are asked to play with the growth function or VC-dimension of some hypothesis sets.

What is the growth function $m_{\mathcal{H}}(N)$ of "positive-and-negative intervals on $\mathbb{R}$"? The hypothesis set $\mathcal{H}$ of "positive-and-negative intervals" contains the functions which are $+1$ within an interval $[\ell,r]$ and -1 elsewhere, as well as the functions which are -1 within an interval $[\ell,r]$ and $+1$ elsewhere.

For instance, the hypothesis $h_1(x)=\mathrm{sign}(x(x-4))$ is a negative interval with -1 within $[0,4]$ and $+1$ elsewhere, and hence belongs to $\mathcal{H}$. The hypothesis $h_2(x)=\mathrm{sign}((x+1)(x)(x-1))$ contains two positive intervals in $[-1,0]$ and $[1,\infty)$ and hence does not belong to $\mathcal{H}$.

Ans:



In this example, N=3.

Since a function can be +1 or -1 within an interval, we need to subtract those double-counted functions, for example, (1, 3, +1) gives OOX, while (3, 4, -1) also gives OOX. Note that in a double-counted case, (a, b, y), either a or b lies on edges(1 or 4): if a=1, then b must be 2 or 3, and vice versa. Thus the total number is

$$2\binom{N+1}{2}-2(N-1)=N^2-N+2.$$

7.

Continuing from the previous problem, what is the VC-dimension of the hypothesis set of "positive-and-negative intervals on $\mathbb{R}$"?

Ans:

4 is the minimal break point (e.g. OXOX), so the VC-dimension is 3.

8.

What is the growth function $m_{\mathcal{H}}(N)$ of "positive donuts in $\mathbb{R}^2$"?

The hypothesis set $\mathcal{H}$ of "positive donuts" contains hypotheses formed by two concentric circles centered at the origin. In particular, each hypothesis is $+1$ within a "donut" region of $a^2 \le x_1^2 + x_2^2 \le b^2$ and -1 elsewhere. Without loss of generality, we assume $0 < a < b < \infty$.

Ans:

Change Cartesian coordinates (x, y) to polar coordinates (r, θ). Since (r, θ_1) has the exact same value as (r, θ_2), this is just as the 1-dimension case, where the growth function has value $\binom{N+1}{2} + 1$.

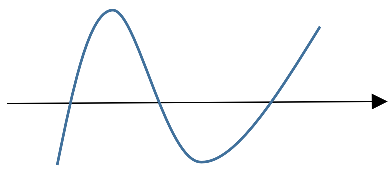
9.

Consider the "polynomial discriminant" hypothesis set of degree D on $\mathbb{R}$, which is given by

$$\mathcal{H} = \left\{ h_{\mathbf{c}} \,\middle|\, h_{\mathbf{c}}(x) = \operatorname{sign}\left(\sum_{i=0}^{D} c_i x^i \right) \right\}$$

What is the VC-dimension of such an $\mathcal{H}$?

Ans:



As the graph shows, a polynomial of degree D can split the real line into at most D+1 intervals. Thus a hypothesis set of degree D can shatter D+1 points.

10.

Consider the "simplified decision trees" hypothesis set on $\mathbb{R}^d$, which is given by

$$\mathcal{H} = \{h_{\mathbf{t},\mathbf{S}} \mid h_{\mathbf{t},\mathbf{S}}(\mathbf{x}) = 2[[\mathbf{v} \in S]] - 1, \text{ where } v_i = [[x_i > t_i]], \\ \mathbf{S} \text{ a collection of vectors in } \{0,1\}^d, \mathbf{t} \in \mathbb{R}^d \quad \}$$

That is, each hypothesis makes a prediction by first using the d thresholds t_i to locate $\mathbf{x}$ to be within one of the 2^d hyper-rectangular regions, and looking up $\mathbf{S}$ to decide whether the region should be $+1$ or -1.

What is the VC-dimension of the "simplified decision trees" hypothesis set?

Ans:

2^d.

11.

Consider the "triangle waves" hypothesis set on $\mathbb{R}$, which is given by

$$\mathcal{H} = \{h_\alpha \mid h_\alpha(x) = \text{sign}(|(\alpha x) \bmod 4 - 2| - 1), \alpha \in \mathbb{R}\}$$

Here $(z \bmod 4)$ is a number $z - 4k$ for some integer k such that $z - 4k \in [0,4)$. For instance, $(11.26 \bmod 4)$ is 3.26, and $(-11.26 \bmod 4)$ is 0.74. What is the VC-dimension of such an $\mathcal{H}$?

Ans:

The wavelengths can be modified by α. For any $N \in \mathbb{R}$, there exists a set of N points which can be shattered by the hypothesis set.

12.

In Questions 12-15, you are asked to verify some properties or bounds on the growth function and VC-dimension.

Which of the following is an upper bounds of the growth function $m_{\mathcal{H}}(N)$ for $N \geq d_{vc} \geq 2$?

Ans:

$m_H(N) \leq 2m_H(N-1) \leq 2^2 m_H(N-2) \leq \cdots \leq 2^i m_H(N-i)$, thus we choose

$$\min_{1 \leq i \leq N-1} 2^i m_{\mathcal{H}}(N-i)$$

13.

Which of the following is not a possible growth functions $m_{\mathcal{H}}(N)$ for some hypothesis set?

Ans:

The answer is $2^{\sqrt{N}}$. To see this, first note that the VC dimension of such an hypothesis set is 1. Thus, $m_H(N) \leq B(N,2) = N+1$, which leads to a contradiction for $N \geq 25$.

14.

For hypothesis sets $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_K$ with finite, positive VC-dimensions $d_{vc}(\mathcal{H}_k)$, some of the following bounds are correct and some are not.

Which among the correct ones is the tightest bound on $d_{vc}(\bigcap_{k=1}^{K}\mathcal{H}_k)$, the VC-dimension of the **intersection** of the sets?

(The VC-dimension of an empty set or a singleton set is taken as zero.)

Ans:

$$0 \leq d_{vc}(\bigcap_{k=1}^{K}\mathcal{H}_k) \leq \min\{d_{vc}(\mathcal{H}_k)\}_{k=1}^{K}$$

15.

For hypothesis sets $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_K$ with finite, positive VC-dimensions $d_{vc}(\mathcal{H}_k)$, some of the following bounds are correct and some are not.

Which among the correct ones is the tightest bound on $d_{vc}(\bigcup_{k=1}^{K}\mathcal{H}_k)$, the VC-dimension of the **union** of the sets?

Ans:

$$\max\{d_{vc}(\mathcal{H}_k)\}_{k=1}^{K} \leq d_{vc}(\bigcup_{k=1}^{K}\mathcal{H}_k) \leq K-1+\sum_{k=1}^{K} d_{vc}(\mathcal{H}_k)$$

Let $f(x) = +1\ \forall x \in \mathbb{R}$, and $g(x) = -1\ \forall x \in \mathbb{R}$. Now let $H_1 = \{f\}, H_2 = \{g\}$, both hypothesis sets have VC dimension 0, while $H_1 \cup H_2$ has VC dimension 1.

16.

For Questions 16-20, you will play with the decision stump algorithm.

In class, we taught about the learning model of "positive and negative rays" (which is simply one-dimensional perceptron) for one-dimensional data. The model contains hypotheses of the form:

$$h_{s,\theta}(x) = s \cdot \mathrm{sign}(x - \theta).$$

The model is frequently named the "decision stump" model and is one of the simplest learning models. As shown in class, for one-dimensional data, the VC dimension of the decision stump model is 2.

In fact, the decision stump model is one of the few models that we could easily minimize E_{in} efficiently by enumerating all possible thresholds. In particular, for N examples, there are at most $2N$ dichotomies (see page 22 of lecture 5 slides), and thus at most $2N$ different E_{in} values. We can then easily choose the dichotomy that leads to the lowest E_{in}, where ties an be broken by randomly choosing among the lowest E_{in} ones. The chosen dichotomy stands for a combination of some "spot" (range of θ) and s, and commonly the median of the range is chosen as the θ that realizes the dichotomy.

In this problem, you are asked to implement such and algorithm and run your program on an artificial data set. First of all, start by generating a one-dimensional data by the procedure below:

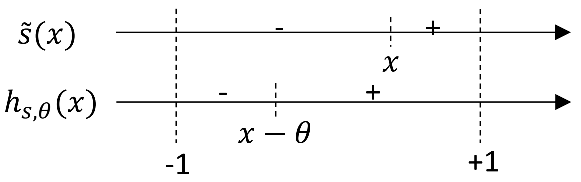
(a) Generate x by a uniform distribution in $[-1, 1]$.

(b) Generate y by $f(x) = \tilde{s}(x)$ + noise where $\tilde{s}(x) = \mathrm{sign}(x)$ and the noise flips the result with 20% probability.

For any decision stump $h_{s,\theta}$ with $\theta \in [-1, 1]$, express $E_{out}(h_{s,\theta})$ as a function of θ and s.

Ans:

From problem 1, we know the probability of error is $\lambda\mu + (1-\lambda)\cdot(1-\mu)$, where $\lambda = 0.8$. To find μ, we need to check case by case.



In this case $(s = 1, \theta > 0)$, the probability of error is $\frac{\theta}{2}$.

After checking three other cases, we conclude that $\mu = \left|\frac{\theta}{2}\right|$ for $s = 1$, and

$\mu = 1 - \left|\frac{\theta}{2}\right|$ for $s = -1$. Thus the answer is $0.5 + 0.3s(|\theta| - 1)$.

17.

Generate a data set of size 20 by the procedure above and run the one-dimensional decision stump algorithm on the data set. Record E_{in} and compute E_{out} with the formula above. Repeat the experiment (including data generation, running the decision stump algorithm, and computing E_{in} and E_{out}) $5,000$ times. What is the average E_{in}? Please choose the closest option.

18.

Continuing from the previous question, what is the average E_{out}? Please choose the closest option.

Ans:

Run hw2_17-18.py and get $\text{average}(E_{in}) \approx 0.17$, $\text{average}(E_{out}) \approx 0.25$.

19.

Decision stumps can also work for multi-dimensional data. In particular, each decision stump now deals with a specific dimension i, as shown below.

$$h_{s,i,\theta}(\mathbf{x}) = s \cdot \text{sign}(x_i - \theta).$$

Implement the following decision stump algorithm for multi-dimensional data:

a) for each dimension $i = 1, 2, \cdots, d$, find the best decision stump $h_{s,i,\theta}$ using the one-dimensional decision stump algorithm that you have just implemented.

b) return the "best of best"' decision stump in terms of E_{in}. If there is a tie , please randomly choose among the lowest-E_{in} ones

The training data $\mathcal{D}_{train}$ is available at:

https://www.csie.ntu.edu.tw/~htlin/mooc/datasets/mlfound_math/hw2_train.dat

The testing data $\mathcal{D}_{test}$ is available at:

https://www.csie.ntu.edu.tw/~htlin/mooc/datasets/mlfound_math/hw2_test.dat

Run the algorithm on the $\mathcal{D}_{train}$. Report the E_{in} of the optimal decision stump returned by your program. Choose the closest option.

20.

Use the returned decision stump to predict the label of each example within $\mathcal{D}_{test}$. Report an estimate of E_{out} by E_{test}. Please choose the closest option.

Ans:

Run hw2_19-20.py and get $E_{in} \approx 0.25$, $E_{out} \approx 0.355$.