Sparse Recovery Illustration

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1 Understanding:

- Initial transmitting signal: $S(t) = c(t)e^{jw_0 t}$
- Received signal:

$$y(t) = \sum_{k=1}^{K} \alpha_k S(t-\tau_k) + n(t) = \sum_{k=1}^{K} \alpha_k c(t-\tau_k) e^{jw_0(t-\tau_k)} + n(t)$$

  since w_0 can't be estimated precisely, let

$$r(t) = y(t)e^{-j\tilde{w}_0 t} = \sum_{k=1}^{K} \alpha_k c(t-\tau_k) e^{-jw_0\tau_k} e^{j(w_0-\tilde{w}_0)t} + n(t)e^{-j\tilde{w}_0 t}$$

- Sampling:

$$\begin{aligned}
&t = m\Delta t, \quad m = 0, ..., N, \\
&r_m \triangleq r(m\Delta t), \\
&w \triangleq w_0 - \tilde{w}_0, \\
&\tilde{\alpha}_k \triangleq \alpha_k e^{-jw_0\tau_k}, \\
&c_m \triangleq c(m\Delta t),
\end{aligned}$$

- Recovery of $\tilde{\alpha}_k$, $k = 1, ..., K$

$$f(x) = \begin{cases} min_{\vec{x}} \|\vec{x}\|_1 \\ s.t. \|diag\{W\}C\vec{x} - \vec{r}\| \leq \rho \\ w \in [-\Omega, \Omega] \end{cases}$$

- Define:

C$\triangleq \begin{bmatrix} c_0 & c_N & c_{N-1} & ... & c_1 \\ c_1 & c_0 & c_N & ... & c_2 \\ c_2 & c_1 & c_0 & ... & c_3 \\ \vdots & \vdots & \vdots & & \vdots \\ c_N & c_{N-1} & c_{N-2} & ... & c_0 \end{bmatrix}$

W$\triangleq \begin{bmatrix} 1 \\ e^{jw\Delta t} \\ e^{jw2\Delta t} \\ \vdots \\ e^{jwN\Delta t} \end{bmatrix}$

$\overrightarrow{r} \triangleq \begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ \vdots \\ r_N \end{bmatrix}$

$\overrightarrow{c} \triangleq eig(C), \Lambda_k$ in $\overrightarrow{c}$ is also given by $\Lambda_k = \sum_{j=0}^{N-1} c_j e^{\frac{2\pi ijk}{N}}, k = 0, ..., N-1.$

w$^{(m)} = e^{\frac{m\times 2\pi j}{N}}$

F$= \begin{bmatrix} 1 & 1 & 1 & ... & 1 \\ 1 & w^{(1)} & w^{(2)} & ... & w^{(N-1)} \\ 1 & w^{(2)} & w^{(4)} & ... & w^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & w^{(N-1)} & w^{2(N-1)} & ... & w^{(N-1)^2} \end{bmatrix}$

Circulant matrix C can be decomposed as C$=$ F$^{-1}diag\{\vec{c}\}F,\ where\ F^{-1} = \frac{1}{N}F^*$

2 Complex Domain:

In the complex form, for matrix multiplication $\mathbf{y} = \mathbf{Hx}$, we can think it as:

$$\mathbf{y} = \begin{bmatrix} \mathbf{real(y)} \\ \mathbf{imag(y)} \end{bmatrix}, \quad \mathbf{Hx} = \begin{bmatrix} \mathbf{real(H)} & -\mathbf{imag(H)} \\ \mathbf{imag(H)} & \mathbf{real(H)} \end{bmatrix} \begin{bmatrix} \mathbf{real(x)} \\ \mathbf{imag(x)} \end{bmatrix}$$

$$\mathbf{H^H H} = \begin{bmatrix} \mathbf{real^T(H)} & \mathbf{imag^T(H)} \\ -\mathbf{imag^T(H)} & \mathbf{real^T(H)} \end{bmatrix} \begin{bmatrix} \mathbf{real(H)} \\ \mathbf{imag(H)} \end{bmatrix}$$