EXTERIOR ALGEBRA, HYPERGRAPH AND SOME RELATIVE INEQUALITIES

A summary of the lecture note and the paper Combinatorics in the Exterior Algebra and the Bollobás Two Families Theorem

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1 A brief review and the proof ao an analogous claim

In the chapter on Bollobas two family theorem, we discussed the set intersections in exterior algebra which inspired us to prove the upper bound of the sets'(with fixed number of elements) number in two skew-crossing families via exterior product,and prove the claim of theorem 3.4.5:

Theorem 1.1. Let $A = (A_i)_{i\in[m]}$ and $B = (B_i)_{i\in[m]}$ be two skew cross-intersecting families. . If the elements of A are all r -sets and the elements of B are all s -sets, then $m \leq \begin{pmatrix} r+s \\ s \end{pmatrix}$.

So with the same idea, here is a claim to prove.

Proposition 1.2. Let $X_1, \ldots, X_n$ be n disjoint sets and let $X = \cup_{h\in[n]} X_h$. Let (A_i, B_i), $i \in [m]$, be m pairs of disjoint subsets of X such that $|A_i \cap X_h| \leq r_h$,$|B_i \cap X_h| \leq s_h$ for all (i, h) $\in [m] \times [n]$, and that $A_i \cap B_j \neq \emptyset$ for all i, j with $1 \leq i < j \leq m$. Then $m \leq \prod_{h\in[n]} \begin{pmatrix} r_h + s_h \\ \mathrm{s}_h \end{pmatrix}$

Proof. Denote $A_i \cap X_h = A_{ih}$ and $B_i \cap X_h = B_{ih}$. It's easy to see that $\{A_{ih}\}$,$\{B_{ih}\}$ are disjoint for i, and they are also two cross-intersecting families. Besides, $\cup_{h\in[n]} A_{ih} = A$, $\cup_{h\in[n]} B_{ih} = B$ since A,B are both subsets of X.

for fixed $h \in [n]$,we have

$v_{A_{ih}} \wedge v_{B_{jh}} \neq 0, \quad \text{if } i = j$

$v_{A_{ih}} \wedge v_{B_{jh}} = 0, \quad \text{if } i < j$

Such that $v_{A_{ih}}$ is the exterior product of the elements in A_{ih}

Then $v_{A_{1h}} ... v_{A_{mh}}$ are linear independent in $\wedge^{r_h} V$. So $\mathrm{m}_h \leq \dim_{\mathbb{R}} \wedge^{r_h} V = \begin{pmatrix} r_h + s_h \\ s_h \end{pmatrix}$.

Since for $h \in [n]$, $v_{A_{i1}} \wedge ... \wedge v_{A_{in}} \neq 0$ and $v_{B_{i1}} \wedge ... \wedge v_{B_{in}} \neq 0$ for any i , we can consider the matrix $[v_{A_{i1}} \wedge v_{B_{j1}}, ..., v_{A_{in}} \wedge v_{B_{jn}}]$,

then it's easy to see m $\leq \dim_{\mathbb{R}} \wedge^r V \leq \prod_{h\in[n]} \begin{pmatrix} \mathrm{r}_h + s_h \\ \mathrm{s}_h \end{pmatrix}$.

□

2 Return to the definition of hypergraph and some basic properties

Given an integer n≥1 , we write [n]=$\{1, \ldots, n\}$. For $0 \leq r \leq n$, we write $\begin{pmatrix} [n] \\ r \end{pmatrix}$ =$\{$A $\subseteq [n] : |A| = r\}$ for the collection of r-element subsets of [n]. We call a hypergraph $\mathcal{A}$ r -uniform when $\mathcal{A} \subseteq \begin{pmatrix} [n] \\ r \end{pmatrix}$.

$For F \in$ $\mathrm{GL}_n(\mathbb{R})$, we denote the columns of the matrix as

$$F = (f_1|\dots|f_n) = (f_{ij})$$

Then we will identify F with the ordered basis $\{f_1, \dots, f_n\}$ formed by the columns of its standard matrix. For A $\in \begin{pmatrix} [n] \\ r \end{pmatrix}$ write $f_A = \bigwedge_{a\in A} f_a \in \Lambda^r V$, where the elements of A are listed in increasing order.

The set F_r= {f$_A$ $: A \in \begin{pmatrix} [n] \\ r \end{pmatrix}$ is a basis for $\Lambda^r V$ and $\dim \Lambda^r V$= $\begin{pmatrix} n \\ r \end{pmatrix}$. We write F_{full} $= \bigcup_{r=0}^{n} F_r$, so that F$_{\text{full}}$ is a basis for $\wedge V$, and $\dim \wedge V = 2^n$. For a hypergraph $\mathcal{A}$, write $F(\mathcal{A}) = \operatorname{span}\{f_A : A \in \mathcal{A}\}$.

Given a non-zero $w \in \Lambda V$, define its initial set $\operatorname{ins}_F(w) \in \mathcal{P}(n)$ with respect to F as follows: expand w in the basis F_{full} as $w = \sum_{A\in\mathcal{P}(n)} m_A f_A$.
Then $\operatorname{ins}(w) = \max\{A \in \mathcal{P}(n) : m_A \neq 0\}$(We already gave the basis an order)

Then We can define the initial hypergraph We define the initial hypergraph $\mathcal{H}_F(W) \subseteq \mathcal{P}(n)$ with respect to F of a subspace $W \subseteq \wedge V$ by

$$\mathcal{H}_F(W) = \{\operatorname{ins}(w) : w \in W, w \neq 0\}$$

.

The following are some basic properties

Proposition 2.1. $V = \mathbb{R}^n$ and $F \in \mathrm{GL}_n(\mathbb{R})$. Then
(i) $\dim W = |\mathcal{H}_F(W)|$ for any subspace W $\subseteq \wedge V$.
(ii) $\mathcal{H}_F(F(\mathcal{A})) = \mathcal{A}$ for any hypergraph $\mathcal{A}$.

3 The inequalities

In this section, I summarize some inequalities that were mentioned for or relative to the two family problems with proofs ungiven in the course note. The inequalities will all be proved with the help of exterior algebra.

3.1 self-annihilating subspace and mutually annihilating pairs of subspaces

Theorem 3.1. Let $\mathcal{A} \subseteq 2^{[n]}$ be an intersecting hypergraph. Then $|\mathcal{A}| \leq 2^{n-1}$. Furthermore, if $\mathcal{A}$ is r-uniform, where $r \leq n/2$, then

$$|\mathcal{A}| \leq \binom{n-1}{r-1}.$$

Proof. Firstly, we illustrate some concepts regarding the ”intersecting hypergraph”. Hyperedge is an edge that can contain multiple vertices. Each element A in the hypergraph $\mathcal{A}$ can be viewed as a hyperedge.In intersecting hypergraph any two hyperedges share at least one common vertex.

Note that $\mathcal{A}$ is an intersecting hypergraph, so for any $A, B \in \mathcal{A}$, we have $A \cap B \neq \emptyset$.

When $\mathcal{A} \subseteq 2^{[n]}$, we can find an element $x \in [n]$ that belongs to every hyperedgea in the hypergraph $\mathcal{A}$. Then $\mathcal{A}$ can be written as the binomial coefficients on $x \cup ([n]-x)$, so $|\mathcal{A}| \leq 2^{n-1}$.

When $\mathcal{A}$ is r-uniform, let $\mathcal{A}_x = \{A{-}x \mid A{\in}\mathcal{A}, x{\in}A\}$ Then $\mathcal{A}_x \subseteq 2^{[n]-x}$ is an intersecting hypergraph. Similarly, we can find an element $y \in [n]-x$ that belongs to every hyperedge in $\mathcal{A}_x$. (Since $\mathcal{A}$ is intersecting, every hyperedge A contains element x. By ignoring x, the remaining elements of A form a subset. This subset can be counted using binomial coefficients,as this corresponds to choosing a subset of size r-1 from the remaining elements in [n]-x). Then $\mathcal{A}_x$ can be written as the binomial coefficients on $y \cup ([n]-x-y)$, so $|\mathcal{A}_x| \leq 2^{n-2}$. By $\mathcal{A}_x \leftrightarrow \mathcal{A}$, we obtain $|\mathcal{A}| = \binom{n-1}{r-1}$. □

Theorem 3.2. Let $V = \mathbb{R}^n$and let W be a self-annihilating subspace of $\wedge V$. Then $\dim W \leq 2^{n-1}$. Furthermore, if W$\subseteq \wedge^r$ V , where $r \leq n/2$, then
$$\dim W \leq \begin{pmatrix} n-1 \\ r-1 \end{pmatrix}$$

Proof. Fix F $\in \mathrm{GL}_n(\mathbb{R})$. By the basic properties in 2.1 and theorem 3.1 , we can consider$\mathcal{H}_F(W)$ as an intersecting hypergraph, as $\dim(W) = |\mathcal{H}_F(W)|$ We need to prove that it is intersecting . Assume that it is not the case, then for some non zero u, w $\in$ W we have A $\cap B = \emptyset$, where $A = \mathrm{ins}_F(u)$ and $B = \mathrm{ins}_F(w)$. Since $u \wedge w = 0$, there must be other sets A',B' in the supports of u, w respectively with $A' \cap B' = \emptyset$ and $A' \cup B' = A \cup B$. It must be true that $|A'| = |A|$ and $|B'| = |B|$, since A and B are both initial sets(which takes the maximum from all the hypergraphs) (so $|A| \geq |A'|$ and $|B| \geq |B'|$) and $|A'| + |B'| = |A| + |B|$.

Let $A_0 = A \cap A'$, $B_0 = B \cap B'$, $X = A \cap B'$, and $Y = B \cap A'$. This gives disjoint decompositions

A=$\mathrm{A}_0 \cup X$, B=$\mathrm{B}_0 \cup Y$,

A$' = A_0 \cup Y$, B$' = B_0 \cup X$,

$A > A' \Longleftrightarrow X > Y \Longleftrightarrow B' > B$

contradicting either $A = \mathrm{ins}_F(u)$ or $B = \mathrm{ins}_F(w)$. □

Theorem 3.3. Let $V = \mathbb{R}^n$ and $1 \leq r,\ s \leq n/2$. Suppose that $U \subseteq \Lambda^r V$ and $W \subseteq \wedge^s V$, and $u \wedge w = 0$ whenever $u \in U$ and $w \in W$. Then
$$\dim U \dim W \leq \begin{pmatrix} n-1 \\ r-1 \end{pmatrix} \begin{pmatrix} n-1 \\ s-1 \end{pmatrix}.$$

Proof. This follows similar lines to the proof of previous theorem .we consider the hypergraphs $\mathcal{A} = \mathcal{H}_F(U)$ and $\mathcal{B} = \mathcal{H}_F(W)$. Then $\mathcal{A}$ is r -uniform, $\mathcal{B}$ is s -uniform, and (arguing as before) we have A $\cap$ B nonempty for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. This means that $\mathcal{A}$ and $\mathcal{B}$ are cross-intersecting systems,then choose any hyperedge A $\in \mathcal{A}$. Due to $\mathcal{A}$

and $\mathcal{B}$ being cross-intersecting hypergraphs, A must intersect every hyperedge in $\mathcal{B}$.This means the r elements in A cannot form any other hyperedge in $\mathcal{A}$. The remaining n-r elements can at most form $\binom{n-r}{s-1}$ hyperedges of size s in $\mathcal{B}$.Repeating this process for each of the at most $\binom{n}{r-1}$ ways to choose A in $\mathcal{A}$, we have: $|\mathcal{A}| \leq \binom{n}{r-1}$ $|\mathcal{B}| \leq \binom{n-r}{s-1}$

Taking the product and summing over all choices of A yields the result.

□

3.2 The exterior local LYM inequality

Theorem 3.4. Let $V = \mathbb{R}^n$ be an n -dimensional real vector space, and let $I \subseteq \wedge V$ be a graded ideal. Let $A = \{a_1, \ldots, a_m\}$ be a minimal set of homogeneous generators for I , where $a_i \in \Lambda^{r_i} V$. Then
$\sum_{i=1}^{m} \frac{1}{\begin{pmatrix} n \\ r_i \end{pmatrix}} \leq 1$
Equality occurs only when span $A = \Lambda^r V$ for some r

Proof. Without loss of generality, we may assume that $r_1 \leq r_2 \leq \cdots \leq r_m$. Note that for each r , the elements $\{a_i \mid r_i = r\} \subseteq \Lambda^r V$ are linearly independent by minimality of A . Now define linear subspaces $Z_i \subseteq \Lambda^{r_i} V$ recursively by

$Z_1 = \operatorname{span}\{a_1\} \quad \text{and} \quad Z_{i+1} = \operatorname{span}\left\{\left(\bigwedge^{r_{i+1}-r_i} V\right) \wedge Z_i, a_{i+1}\right\}$.

First, we claim that $a_{i+1} \notin (\wedge^{r_{i+1}-r_i} V) \wedge Z_i$. Because the elements of $(\wedge^{r_{i+1}-r_i} V) \wedge Z_i$ are of the form

$\sum_{j=1}^{i} w_j \wedge a_j$

where $w_j \in \Lambda^{r_{i+1}-r_j} V$, and if a_{i+1} were of this form, then A would not be a minimal generating set of the ideal I. Then each i

$\frac{\dim\left(\bigwedge^{r_{i+1}-r_i} V \wedge Z_i\right)}{\begin{pmatrix} n \\ r_{i+1} \end{pmatrix}} \geq \frac{\dim Z_i}{\begin{pmatrix} n \\ r_i \end{pmatrix}}$

and thus

$\frac{\dim Z_{i+1}}{\begin{pmatrix} n \\ r_{i+1} \end{pmatrix}} = \frac{1+\dim\left(\Lambda^{r_{i+1}-r_i} V \wedge Z_i\right)}{\begin{pmatrix} n \\ r_{i+1} \end{pmatrix}} \geq \frac{1}{\begin{pmatrix} n \\ r_{i+1} \end{pmatrix}} + \frac{\dim Z_i}{\begin{pmatrix} n \\ r \end{pmatrix}}$

We can now proceed recursively down from $1 \geq \frac{\dim Z_m}{\begin{pmatrix} n \\ r_m \end{pmatrix}}$. If equality occurs, that is only possible when $r_{i+1} = r_i$ or $Z_i = \Lambda^{r_i} V$. Hence $\{a_1, \ldots, a_m\}$forms a basis for some $\Lambda^r V$. □

3.3 A word about t-self annihilating subspace

For t>0 , a hypergraph $\mathcal{A}$ is called t -intersecting when $|A \cap B| \geq t$for all $A, B \in \mathcal{A}$. Just as we did for intersecting hypergraphs, we would like to define an analogous notion in the exterior algebra: we will call these subspaces t -self-annihilating. We will use interior products to do so. Note that a hypergraph $\mathcal{A}$ is t -intersecting exactly when the hypergraph $\{A \backslash C : A \in \mathcal{A}\}$ is intersecting for all sets C having at most t-1 elements. In parallel, we define a subspace $W \subseteq \wedge V$ to be t -self-annihilating when

$(u \cdot y^*) \wedge (w \cdot y^*) = 0$

for all $u, v \in W$ and all $y^* \in \Lambda^{<t} V^*$. Note that 1-self-annihilating coincides with self-annihilating as defined above, since $\Lambda^0 V^*$ is a copy of the field $\mathbb{R}$ of scalars: $f_\emptyset^* = 1$.

Then we have the following proposition.

Proposition 3.5. If $W \subseteq \wedge V$ is a t-self-annihilating subspace, then $\mathcal{H}_F(W)$ is t -intersecting.

4 Some plans for the final project

Due to the very limited time, this midterm project is not satisfactory for me as it's not that complete and I think it's not even logically coherent enough. Also it's not appropriate to add this section here. Although i may be busier with all the coursework and research after the midterm, I'll try my best to make a better final project. I'm interested in Shannon's original paper about information theory so I may write an essay about its review and influence. Or I'll do something about graph neural networks which is closer to my personal research interest. Anyway, thank you for reading this project.