Approaching Chaos In the Case of Logistic Model

Yunqing Li
School of physical Science and Technology, Lanzhou University
Gansu, China
Liyunqing0428@outlook.com

Abstract

In this essay, the property of chaos is explored by using a numerical iteration method. By implementing the classic 1-D logistic model, the way that the behavior of a non-linear system goes from non-chaotic to fully chaotic via a period-cascading process is explained. At last, a yet to be solved problem is noticed and explained.

Keywords

chaos, Logistic Map, period-cascading

Introduction

The chaos theory has been a popular topic since its arrival in 1970s. In short, it means the future behavior of some dynamic systems (systems of differential equations, all of which being non-linear and somewhat complicated), cannot be predicted *at all*. The theory was insightfully summarized by Edward Lorenz as:

When the present determines the future, but the approximate present does not approximately determine the future.

The purpose of this essay is to try to approach chaos using the classic Logistic recurrence relation, and see what happens to the system as the system is getting more complex as the essential parameter *a* increases from 0 to 4. We would see the system starts with totally non-chaotic behavior, then as *a* increases, its behavior becomes more complex and unpredictable. After going through a *period-cascading* process, the system becomes fully unpredictable, thus chaos arrives.

The Logistic model is iterated many steps in order to show its future behavior. The main tool chosen for this task is Numpy and Scipy, aided by a plotting tool Matplotlib, all of which are free, open-source packages based on Python.

The Logistic Map

The one-dimensional logistic model is outlined by the following recurrence relation:

$$. x_{n+1} = a\, x_n (1-x_n)$$

It means each consequent *x* can be deduced from the present *x*. So given the initial value *x* and parameter *a*, the future behavior of the model can be *determined*.

The behavior of the model as parameter *a* increases

As a general note, in this section, the model was iterated 1000 times, long enough to show its future behavior. Also, a constant initial value *x* was chosen to be 0.6 to improve consistency across different *a*.

A was chosen in the spread of (0, 4), with the whole system collapses elsewhere. Finally, we mainly focus on the 950 to 1000 steps of the results, i.e. the future behavior.

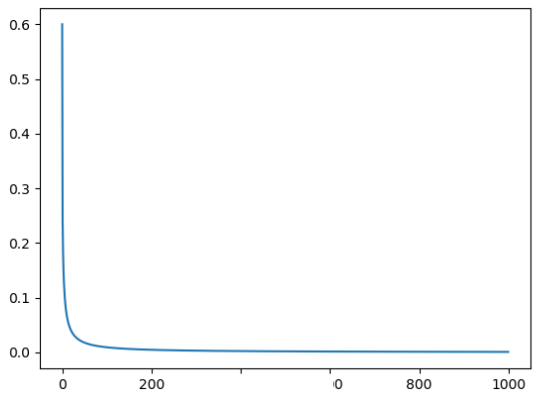


Fig.1

As shown in fig.1, when *a* is 0, the model quickly goes from its initial value, 0.6 to 0 and stays there forever, thus the future of the model is totally predictable: 0.

What if *a* is larger? In fig.2, several as were chosen in increasing order except the last one (for a special reason explained later).

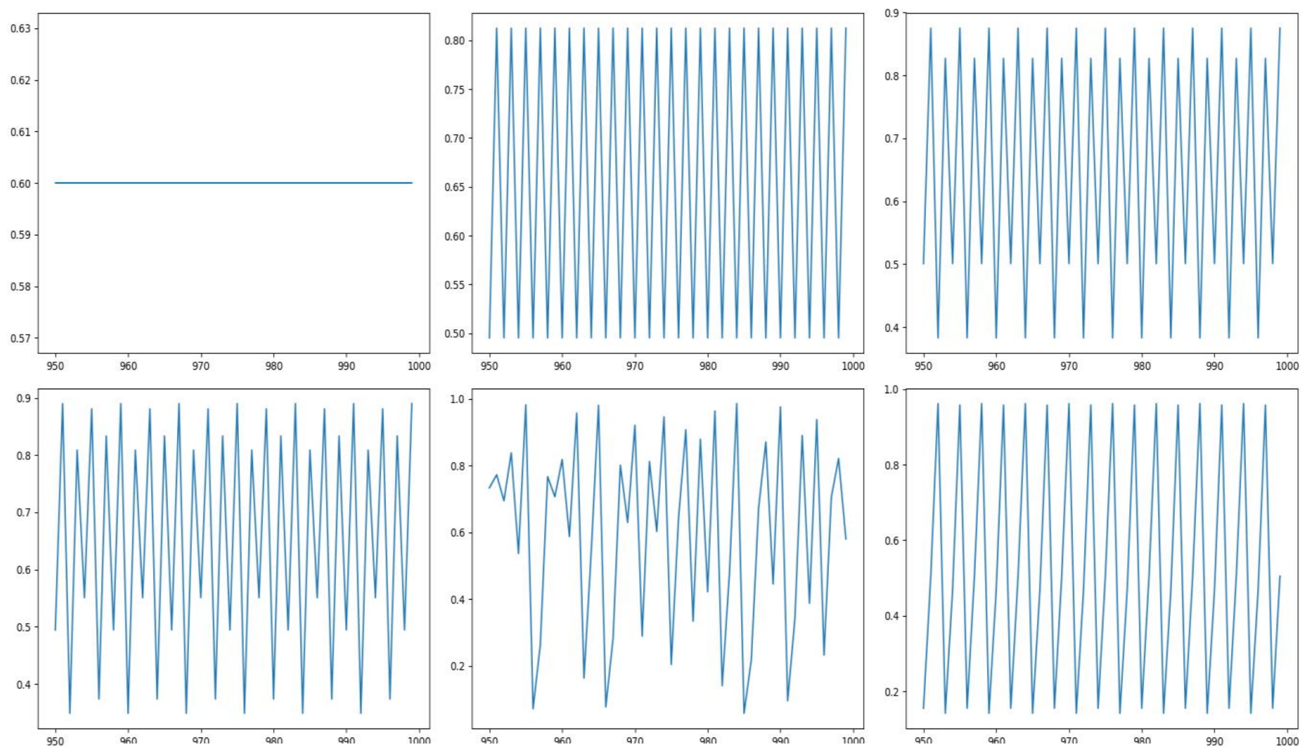


Fig.2, different value of *a* were chosen and the same 950 to 1000 steps were traced.From top left to bottom right, a = 2.5, 3.250, 3.500, 3.560, 3.947,3.847

As can be seen in fig.2, when *a* is 2.5, the future behavior is still non-chaotic, however this time the steady results holds somewhere near 0.6. When *a* is 3.250, the model becomes less predictable: the results cycles between somewhere around 0.50 and 0.82 and show no sign of stop. Mind here that the system is still somewhat predictable: the future value of *x* is *either* 0.50 *or* 0.82. As *a* increases to 3.500, the result cycles in a set of 4 values; and the size of such set becomes 8 when *a* is 3.560.
These shows the model is getting much more fragile, i.e. more sensitive to *a*, as *a* increases. The result cycles in a set with size of 1, then 2, then 4, then 8···. This is a sign of that the model is approaching real chaos *stepwise.* Also note that the step size of each step is getting smaller, explicitly denoted by the Feigenbaum constant, given below,

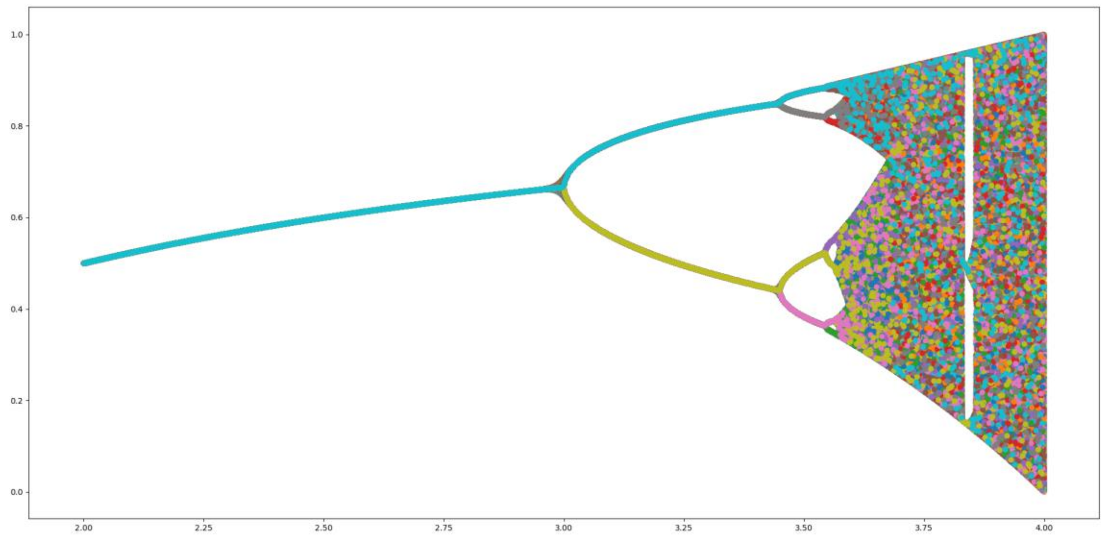
$$\delta = \lim_{n\to\infty} \frac{a_{n-1} - a_{n-2}}{a_n - a_{n-1}} = 4.669\,201\,609\ \ldots,$$

which shows the step size is shrinking at an almost constant speed.

With that said, we can calculate a critical value *a*, after which the *x* cycles in a set of size infinity, i.e. the behavior is now truly unpredictable, or in another word, chaotic.
Let *a* = 2.998 as the start point of the bifurcation process, we can calculate the critical value would be about 3.596.

Choose *a* to be 3.947, which is larger than the critical value, it can be seen in the 5th subplot of fig.2 that the behavior shows no certain pattern, a sign of chaos.

In order to show how the system approaches chaos more clearly, fig.3 was given. In such figure *a* was the control variable, and, as above, 1000 steps was iterated for each *a*, yet this time the 900 to 1000 steps of each *a* were used in the plot in hope of showing more information in the chaotic area. Fig.3 clearly shows that the set of possible results soon exploded after a process of period-cascading (In order to improve clarity, the Scatter plot scheme was chosen).



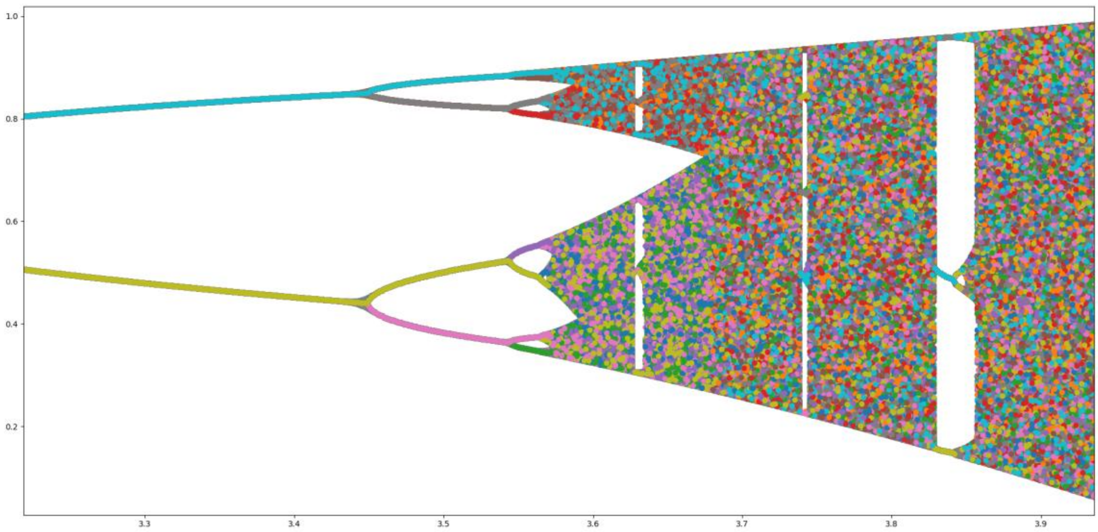


Fig.3

To show the system is extremely sensitive to initial value if it is chaotic, hold a = 3.947 as a constant, then perturb the initial value x by an insignificant amount, 0.00000001, to 0.60000001 and trace the same 1 to 50 steps as when we did with the initial value 0.6 (shown in fig.4).

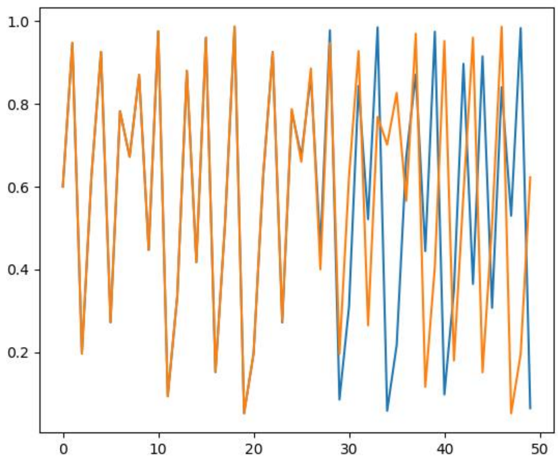


Fig.4

In fig.4 it shows the two lines start off so close that the first 25 steps are almost indistinguishable. But after that they go to totally different paths. That means they are different because the system is so sensitive to initial value that however small the perturbation is, the future behavior is totally unpredictable.

Beyond These

However, fig.3 also shows something unexpected before. After *a* increased pass the critical value, the behavior thereafter should be fully chaotic, yet there is some vacuity in the chaotic region as shown in the local enlargements of fig.3. This means the system returns to non-chaotic behavior at some a, and then, after a short time, explodes again into chaos.
For instance, when *a* is chosen as 3.847, the result set has size of 6. The last subplot shows this behavior. For more clarity, a super local enlargement of fig.3 was given in fig.5. It shows the 6 possible values clearly. Also note "6" is not a number appeared in the period-cascading process. Similar behavior are also spotted elsewhere in Fig.3, with a typical another shown in fig.6, where *a* is around 3.630.

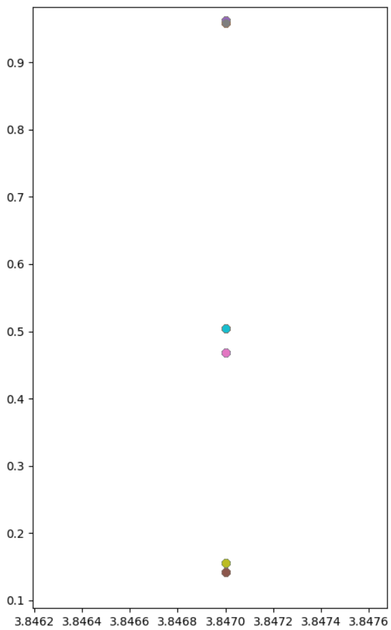


Fig.5

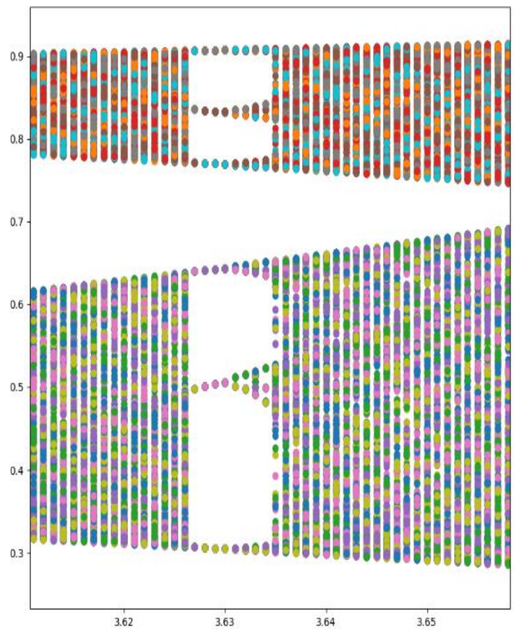


Fig.6

Why this happens is still to be discovered, however at least it can be safely said that the system starts off non-chaotic, then goes into a period-cascading process, after which becomes chaotic. And in some areas in the chaotic region, the system becomes less chaotic, but these cases are rare.

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