Set 2: Released 2023/3/15

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1. Find the Dehn function of the semigroup $\langle\, x, y \mid yx = 1\,\rangle$.

Proof. Since each step of reduction $yx \to 1$ reduces the length of the word by 2, it costs no more than $n/2$ steps to reduce a word of length n. Conversely, for any n, the word $y^{\frac{n}{2}}x^{\frac{n}{2}}$ costs exactly $n/2$ steps to reduce, concluding that the Dehn function is $O(n)$. □

2. Prove that free groups of rank ≥ 2 are not abelian.

Proof. For any free group of rank ≥ 2, let x and y be two distinct elements of its free generators, then the reduced form $xyx^{-1}y^{-1}$ is nontrivial and lives in the commutator subgroup of the free group. Therefore the commutator subgroup is nontrivial, concluding that the free group if nonabelian. □

3. Let $Fr(2)$ be a free group on free generators x, y. Prove that the subgroup of $Fr(2)$ generated by elements $X = \{x^n y x^n, n \in \mathbb{N}\}$ is a free group on the set of free generators X.

Proof. Consider the free group $F(S)$ where $S = \{s_i\}_{i\in\mathbb{N}}$. Define a group homomorphism $\varphi\colon Fr(S) \to Fr(2)$ by sending s_i to $x^i y x^i$. Clearly φ maps $F(S)$ onto the subgroup generated by X, thus it suffices to show that φ is injective. Suppose not, say a reduced form $s_{i_1}^{\varepsilon_1}\cdots s_{i_k}^{\varepsilon_k}$ is sent to the identity by φ, then we have

$$x^{\varepsilon_1 i_1}y^{\varepsilon_1}x^{\varepsilon_1 i_1}\cdots x^{\varepsilon_k i_k}y^{\varepsilon_k}x^{\varepsilon_1 i_k} = 1.$$

However, when placing $x^{\varepsilon i}y^{\varepsilon}x^{\varepsilon i}$ and $x^{\varepsilon' i'}y^{\varepsilon'}x^{\varepsilon' i'}$ together, the only chance that y^{ε} or $y^{\varepsilon'}$ get canceled is that $\varepsilon = -\varepsilon'$ and $i = i'$, hence the above equation contradicts the assumption that $s_{i_1}^{\varepsilon_1}\cdots s_{i_k}^{\varepsilon_k}$ is a reduced form. Therefore φ must be injective. □

4. The element g of a free group $Fr(n)$ in reduced form $g = x_{i_1}^{\varepsilon_1}\cdots x_{i_k}^{\varepsilon_k}$, $\varepsilon_j = \pm 1$, is called *cyclically reduced* if $x_{i_1}^{\varepsilon_1}\cdot x_{i_k}^{\varepsilon_k} \neq 1$. An element g can always be represented as $g = x_{i_1}^{\varepsilon_1}\cdots x_{i_s}^{\varepsilon_s}\sigma(g)x_{i_s}^{-\varepsilon_s}\cdots x_{i_1}^{-\varepsilon_1}$, where $\sigma(g)$ is cyclically reduced.

Prove that two elements $g_1, g_2 \in Fr(n)$ are conjugate if and only if $\sigma(g_1) = vw$, $\sigma(g_2) = wv$ for some elements v, w.

Proof. If $\sigma(g_1) = vw$, $\sigma(g_2) = wv$ for some elements v, w, write $g_1 = x_{i_1}^{\varepsilon_1}\cdots x_{i_s}^{\varepsilon_s}\sigma(g_1)x_{i_s}^{-\varepsilon_s}\cdots x_{i_1}^{-\varepsilon_1}$ and $g_2 = x_{j_1}^{\varepsilon_1'}\cdots x_{j_k}^{\varepsilon_k'}\sigma(g_2)x_{j_k}^{-\varepsilon_k'}\cdots x_{i_1}^{-\varepsilon_1'}$, then

$$g_1 = \left(x_{i_1}^{\varepsilon_1}\cdots x_{i_s}^{\varepsilon_s}w^{-1}x_{j_k}^{-\varepsilon_k'}\cdots x_{i_1}^{-\varepsilon_1'}\right) g_2 \left(x_{i_1}^{\varepsilon_1}\cdots x_{i_s}^{\varepsilon_s}w^{-1}x_{j_k}^{-\varepsilon_k'}\cdots x_{i_1}^{-\varepsilon_1'}\right)^{-1},$$

hence g_1 and g_2 are conjugate.

Conversely, if g_1 and g_2 are conjugate, say $g_1 = hg_2h^{-1}$. Keep the notation that $g_1 = x_{i_1}^{\varepsilon_1}\cdots x_{i_s}^{\varepsilon_s}\sigma(g_1)x_{i_s}^{-\varepsilon_s}\cdots x_{i_1}^{-\varepsilon_1}$ and $g_2 = x_{j_1}^{\varepsilon_1'}\cdots x_{j_k}^{\varepsilon_k'}\sigma(g_2)x_{j_k}^{-\varepsilon_k'}\cdots x_{i_1}^{-\varepsilon_1'}$, then we have

$$\sigma_2(g_2) = \left(x_{j_k}^{-\varepsilon_k'}\cdots x_{i_1}^{-\varepsilon_1'}h^{-1}x_{i_1}^{\varepsilon_1}\cdots x_{i_s}^{\varepsilon_s}\right)\sigma(g_1)\left(x_{j_k}^{-\varepsilon_k'}\cdots x_{i_1}^{-\varepsilon_1'}h^{-1}x_{i_1}^{\varepsilon_1}\cdots x_{i_s}^{\varepsilon_s}\right)^{-1}.$$

Put $w \coloneqq x_{j_k}^{-\varepsilon_k'}\cdots x_{i_1}^{-\varepsilon_1'}h^{-1}x_{i_1}^{\varepsilon_1}\cdots x_{i_s}^{\varepsilon_s}$ and $v \coloneqq \sigma(g_1)w^{-1}$, and the result follows. □

5. Prove that a free group does not contain nonidentical elements of finite orders.

Proof. Since nontrivial free groups always contain infinitely many elements and every subgroup of a free group is free, there cannot be a nonidentical element of finite order in a free group, because if that happens, then neither is the cyclic subgroup generated by that element of finite order trivial nor does it contain infinitely many elements, contradiction. □