硕士学位论文

论纽结的量子模态猜想

ON THE QUANTUM MODULARITY CONJECTURE FOR KNOTS

研　　究　　生：李昀升

指　导　教　师：Stavros Garoufalidis 讲席教授

南方科技大学

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学 位 申 请 人：李昀升
指 导 教 师：Stavros Garoufalidis 讲席教授
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ON THE QUANTUM MODULARITY CONJECTURE FOR KNOTS

Candidate: Li Yunsheng
Supervisor: Chair Prof. Stavros Garoufalidis
Discipline: Mathematics
Date of Defence: May, 2024
Affiliation: Department of Mathematics
Degree-Conferring-Institution: Southern University of Science and Technology

摘 要

量子拓扑学被认为是由 1984 年发现 Jones 多项式开始的, 随后观察到其与物理学的许多联系. 80 年代末, Atiyah, Segal 和 Witten 利用 SU(2) Chern-Simons 理论建立了 Jones 多项式的内在定义, 揭示了 Jones 多项式与物理世界的丰富联系. 围绕 Jones 多项式出现了一系列发现, 其中一个著名的猜想就是本论文的主题, 量子模态猜想.

1995 年, R. Kashaev 利用量子对数函数提出了一个纽结不变量, 并猜想双曲纽结 K 的不变量具有指数增长率, 这一猜想被称为体积猜想 (Volume Conjecture) . 2001 年, H. Murakami 和 J. Murakami 发现 Kashaev 的不变量等于 N 色 Jones 多项式在 N 次单位元根上的取值. 由此, D. Zagier 观察到了在单位元根上的 N 色 Jones 多项式的值之间的模性关系, 并将体积猜想的内容扩展为关于某个函数的模性关系. 这一扩展后的猜想被称为量子模态猜想 (Quantum Modularity Conjecture).

距今更近一些, J. E. Andersen 和 R. Kashaev 基于具有无穷维规范群的 Chern-Simons 理论, 将量子 Teichmüller 理论推广到了一类拓扑量子场论, 引入了 Teichmüller 拓扑量子场论. 在进一步研究 Teichmüller 拓扑量子场论在 4_1 结和 5_2 结的结补空间上的取值时, S. Garoufalidis 和 D. Zagier 发现了一些现象, 表明 Teichmüller 拓扑量子场论的态积分与量子模态猜想之间有着深刻的关系. 此外, 他们的观察还提出了与其他几个主题的丰富联系, 如 Dimofte–Gaiotto–Gukov 指标和量子自旋网络.

本论文将主要介绍 Teichmüller 拓扑量子场论的构造和量子模态猜想的内容, 并通过列举 S. Garoufalidis 和 D. Zagier 的观察结果以及本文作者 (Y. Li), 安妮 (N. An) 和 S. Garoufalidis 共同研究的最新结果和其中一些结果的初等证明来展示它们之间的联系.

关键词：纽结；量子拓扑场论；Teichmüller 量子拓扑场论；全纯量子模形式；量子模态猜想

ABSTRACT

Quantum topology is considered to be initiated by the discovery of the Jones polynomial in 1984, followed with observations of numerous links to physics. In the late '80s, Atiyah, Segal, and Witten established an intrinsic definition of the Jones polynomial using SU(2) Chern-Simons theory, revealing the rich connections of the Jones polynomial with the physical world. Successive findings around the Jones polynomial emerged, including one famous conjecture that is the main topic of this thesis, the Quantum Modularity Conjecture.

In 1995, R. Kashaev introduced a knot invariant using the quantum dilogarithm function, which for a hyperbolic knot K is conjectured to have an exponential growth rate, a conjecture known as the Volume Conjecture. In 2001, H. Murakami and J. Murakami discovered that Kashaev's invariant is equal to the value of the N-colored Jones polynomial at N-th roots of the unity. With this, D. Zagier observed a modular relation between the values of the N-colored Jones polynomial at different roots of the unity and extended the statement of Volume Conjecture to a modular relation of the functions. The extended statement is known as the Quantum Modularity Conjecture (QMC).

More recently, J. E. Andersen and R. Kashaev introduced the Teichmüller TQFT based on Chern-Simons theory with infinite dimensional gauge groups, promoting the quantum Teichmüller theory to a TQFT of categroids. On further investigation into values of the Teichmüller TQFT on knot complements of the 4_1 knot and the 5_2 knot, S. Garoufalidis and D. Zagier discovered phenomena suggesting deep relationships between the state integral from the Teichmüller TQFT and QMC. Furthermore, their observation also suggested rich connections with several other topics, such as the Dimofte–Gaiotto–Gukov index and the quantum spin network.

This thesis will mainly focus on introducing the construction of the Teichmüller TQFT and the contents of QMC, and demonstrate their connections by listing the observations made by S. Garoufalidis and D. Zagier and more recent results along with elementary proofs for some of them from joint work of the author, N. An and S. Garoufalidis.

Keywords: knots; topological quantum field theory; Teichmüller TQFT; holomorphic quantum modular forms; Quantum Modularity Conjecture

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CHAPTER 1 INTRODUCTION

Quantum topology is considered to be initiated by the discovery of the Jones polynomial in 1984, followed with observations of numerous links to physics. In the late '80s, in attempt of establishing an intrinsic and unified definition for the Jones polynomial instead of depending on the knot projections, Atiyah, Segal, and Witten discovered the rich connections of the Jones polynomial with the physical world. In particular, they found that the Jones polynomial could be realized as an invariant computed from the SU(2) Chern-Simons theory.[1] Successive findings around the Jones polynomial emerged, including one famous conjecture that is the main topic of this thesis, the Quantum Modularity Conjecture.

In 1995, R. Kashaev introduced a knot invariant using the quantum dilogarithm function. For each knot K and integer $N \in \mathbb{N}$, Kashaev's knot invariant assigns a complex number $\langle K \rangle_N$. For a hyperbolic knot K, its Kashaev's invariant $\langle K \rangle_N$ is conjectured to have an exponential growth rate of the hyperbolic volume of the knot complement $S^3 \setminus K$ as N tends to the infinity, hence the hyperbolic volume of the knot complement can be recovered from the Kashaev's invariant. This is known as the Volume Conjecture.[2] In 2001, H. Murakami and J. Murakami discovered that $\langle K \rangle_N$ is equal to $J_N^K(\mathrm{e}^{2\pi \mathrm{i}/N})$, the value of the N-colored Jones polynomial at N-th roots of the unity.[3] With this, D. Zagier observed a modular relation between the values of the N-colored Jones polynomial at different roots of the unity and extended the statement of Volume Conjecture to a modular relation of the function on $\mathbb{Q}/\mathbb{Z}$ defined by $\mathbf{J}^K(-a/N) = J_N^K(\mathrm{e}^{2\pi \mathrm{i} a/N})$, where a and N are two coprime integers. The modular relation states that $\mathbf{J}^K(\gamma \cdot x)/\mathbf{J}^K(x)$ admits an exponential growth rate again related to the hyperbolic volume of $S^3 \setminus K$ as $x \to \infty$ in $\mathbb{Q}$, where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $\gamma \cdot x = \frac{ax+b}{cx+d}$.[4] This extended statement is known as the Quantum Modularity Conjecture (QMC).

More recently, J. E. Andersen and R. Kashaev introduced the Teichmüller topological quantum field theory (TQFT) based on Chern-Simons theory with infinite dimensional gauge groups, promoting the quantum Teichmüller theory to a TQFT of categroids. The Teichmüller TQFT assigns to each shaped pseudo 3-manifold, in particular knot complements with certain triangulations, a tempered distribution, which can be represented by a holomorphic function on the cut plane $\mathbb{C}' = \mathbb{C} \setminus (-\infty, 0]$, given by an integral called the state integral. Using the method of residues, the state integral can be factorized into a sum of quadratic products of a q-series and a $\tilde{q}$-series; On the other hand, the asymptotic expansion of state integral as the variable tends to zero along rays gives an asymptotic series – the state integral serves as a bridge connecting these two very different objects. On further investigation into state integrals of the 4_1 knot and the 5_2 knot, S. Garoufalidis and D. Zagier discovered phenomena suggesting deep relationships between the state integral from the Teichmüller TQFT and QMC. For instance, the asymptotic expansions of state integrals behave similarly to those of $\mathbf{J}^K(x)$; in fact, the q-series from state integrals are asymptotically related to the asymptotic expansion of $\mathbf{J}^K(x)$, see observations 1 and 2. Furthermore, their observation also suggested rich connections of state integral with several other topics, such as the Dimofte–Gaiotto–Gukov index and the quantum spin network.

This thesis will mainly focus on the Teichmüller TQFT and the QMC. A primary introduction to the construction of the Teichmüller TQFT and the state integral will be given in chapter 2, outlining the basic components for computation and listing in section 2.4 the specific state integrals in concern of this thesis, namely those of the 4_1 knot, the 5_2 knot and the $(-2, 3, 7)$ pretzel knot. Chapter 3 briefly introduces basic notions of modular forms in section 3.1 and explicitly describes the content of QMC in section 3.2. A summary of the recent results mainly on state integrals of the 4_1 knot and the 5_2 knot observed by S. Garoufalidis and D. Zagier will be presented in chapter 4, followed by results on the $(-2, 3, 7)$ pretzel knot from the joint work of the author, N. An and S. Garoufalidis in section 4.4, along with elementary proofs for some of them.

CHAPTER 2 THE TEICHMÜLLER TQFT

This chapter briefly reviews the Teichmüller TQFT constructed by Andersen and Kashaev[5], explaining the source of the state integral which will be our main concern in chapter 4.

Recall that a Topological Quantum Field Theory (TQFT) in dimension n, as axiomatized by Atiyah[6], is a functor from a category of n-dimensional cobordisms to a category of finite-dimensional vector spaces subject to a sequence of conditions. Instead of a functor between categories, the Teichmüller TQFT is a functor from a sub-categroid of a category of 3-dimensional cobordisms to the categroid of spaces of (complex) tempered distributions, where the definition of categroids is given below.

Definition 2.1 (Categroid): A *categroid* C consists of a family of objects $\mathrm{Obj}(C)$ and for any pair of objects A, B in $\mathrm{Obj}(C)$ a family of morphisms $\mathrm{Mor}_C(A, B)$ such that

- for any three objects A, B, C there is a family of composable morphisms $\mathscr{F}_C(A, B, C) \subset \mathrm{Mor}_C(A, B) \times \mathrm{Mor}_C(B, C)$ and a composition map
$$\circ : \mathscr{F}_C(A, B, C) \to \mathrm{Mor}_C(A, C),$$
such that the composition of composable morphisms is associative;
- for any object A we have an identity morphism $1_A \in \mathrm{Mor}_C(A, A)$ which is composable with any morphism $f \in \mathrm{Mor}_C(A, B)$ or $g \in \mathrm{Mor}_C(B, A)$ and we have
$$1_A \circ f = f, \quad g \circ 1_A = g.$$

Roughly speaking, a categroid is a category where, instead of all, only some of categorically composable morphisms are composable. Functors between categroids are defined similarly as functors between categories.

Therefore, to define the Teichmüller TQFT, we need to define three things: the domain categroid of 3-dimensional cobordisms, the target categroid of tempered distributions, and the TQFT functor. We will define them respectively in sections 2.1 to 2.3.

2.1 The Domain Categroid

To define the domain categroid, let us firstly consider the so called *(triangulated) pseudo 3-manifolds*. The morphisms of our desired categroid of cobordisms will be certain equivalence classes of (triangulated) pseudo 3-manifolds with a series of additional structures.

2.1.1 From pseudo 3-manifolds to gauge equivalence

In this subsection we give a crash course on pseudo 3-manifolds, shape structures and gauge equivalence relations.

Definition 2.2 (Pseudo 3-manifolds): A (triangulated) pseudo 3-manifold is a Δ-complex obtained by gluing finitely many standard 3-simplicies in $\mathbb{R}^3$ (i.e. tetrahedra) with totally ordered vertices along codimension-1 faces with respect to vertex-order-preserving and orientation-reversing simplicial maps such that

- every codimension-1 face belongs to exactly one or two 3-simpilicies;
- for every pair of 3-simplicies T and T', there is a sequence of 3-simplicies

$$T = T_0, T_1, \cdots T_k = T',$$

such that the intersection $T_i \cap T_{i+1}$ is a codimension-1 face for all $i = 0, \cdots, k-1$.

Note that the second condition guarantees that pseudo 3-manifolds are connected in a relatively strong sense, which allows us to define the TQFT functor in a relatively simple way, as we will see in section 2.3.

Let X be a pseudo 3-manifold. For an integer i, we will denote by $\Delta_i(X)$ the set of i-dimensional cells in X. For any $i > j$, we also denote tautologically

$$\Delta_i^j(X) \coloneqq \{(a,b) \mid a \in \Delta_i(X), b \in \Delta_j(a)\},$$

where the cell a, when considered as an Δ-complex and taken the set $\Delta_j(a)$, is thought to be its original standard form without any identification on its boundary (with itself) induced by gluings. We have natural projection maps

$$\phi_{i,j} : \Delta_i^j(X) \to \Delta_i(X), \quad \phi^{i,j} : \Delta_i^j(X) \to \Delta_j(X).$$

Note that two different edges on (and paired with) one tetrahedron might be mapped to the same edge by $\phi^{3,1}$ if they are glued together in X.

Definition 2.3 (Shape structure): A *shape structure* on a pseudo 3-manifold X is a map $\alpha_X : \Delta_3^1(X) \to \mathbb{R}_{>0}$ such that

$$\alpha_X(T,e_1) + \alpha_X(T,e_2) + \alpha_X(T,e_3) = \pi, \tag{2-1}$$

for any $T \in \Delta_3(X)$ and edges $e_1, e_2, e_3 \in \Delta_1(T)$ such that $e_1 \cap e_2 \cap e_3$ is a vertex of T. The values of α_X on edges of tetrahedra are called *dihedral angles*. An oriented pseudo

3-manifold with a shape structure is called a *shaped pseudo* 3-*manifold*. We denote the set of all shape structures on X by $S(X)$.

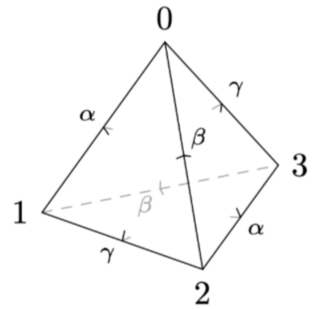

Figure 2-1 A tetrahedron with ordered vertices and dihedral angles.[5]

It is straightforward to see that the dihedral angles at opposite edges of any tetrahedron are the same, see fig. 2-1. The condition in eq. (2-1) allows us to associate to each tetrahedron T the geometric structure of an ideal hyperbolic tetrahedron by entering the complex shape variables at edges $e_1 \in \Delta_1(T)$,

$$\frac{\sin \alpha_X(T, e_2)}{\sin \alpha_X(T, e_3)} \mathrm{e}^{\mathrm{i}\alpha_X(T,e_1)},$$

into Thurston's hyperbolicity equations[7]§4, where e_2 and e_3 are edges such that $e_1 \cap e_2 \cap e_3$ is a vertex of T and e_1, e_2, e_3 corresponds to the counter-clockwise cyclic order of edges around the vertex $e_1 \cap e_2 \cap e_3$ as seen from the outside of T.

Definition 2.4 (Weight function): For a shaped pseudo 3-manifold X, we associate a *weight function*

$$\omega_X : \Delta_1(X) \to \mathbb{R}_{>0},$$

which associates to each edge $e \in \Delta_1(X)$ the sum of dihedral angles around it

$$\omega_X(e) \coloneqq \sum_{(T,e') \in (\phi^{3,1})^{-1}(e)} \alpha_X(T, e').$$

Definition 2.5 (Level): A *leveled shaped pseudo* 3-*manifold* is a pair (X, ℓ_X) consisting of a shaped pseudo 3-manifold X and a real number $\ell_X \in \mathbb{R}$ called the level. We denote by $LS(X)$ the set of all leveled shaped structures on X.

The choice of the level ℓ_X is arbitrary, independent of any other structure on X. The level is introduced as a parameter to participate in the construction of the gauge equivalence relation and the TQFT functor so that the TQFT will be well-defined.

Given an oriented tetrahedron T and a vertex on the tetrahedron, the orientation induces a cyclic order on the three edges meeting at the vertex. To be explicit, embedding T into $\mathbb{R}^3$ in an orientation-preserving manner, the cyclic order of the edges is counter-clockwise as seen from the outside of T. Moreover, this cyclic order is compatible with the pairing of opposite edges, hence the orientation of T determines a cyclic order on the set of pairs of opposite edges of T. Given any two edges e, e' of T, we define a skew-symmetric symbol

$$\varepsilon_{e,e'} \in \{0, \pm 1\}, \quad \varepsilon_{e,e'} = -\varepsilon_{e',e}$$

such that $\varepsilon_{e,e'} = 0$ if e and e' belong to the same pair of opposite edges of T and $\varepsilon_{e,e'} = +1$ if the pair of opposite edges associated with e is the cyclical preceding of that of e'.

Definition 2.6 (Gauge equivalent): Two leveled shaped pseudo 3-manifolds (X, ℓ_X) and (Y, ℓ_Y) are *gauge equivalent* if there exist an isomorphism $h: X \to Y$ of the underlying cellular structures of X and Y and a function

$$g: \Delta_1(X) \to \mathbb{R}$$

such that

$$\Delta_1(\partial X) \subset g^{-1}(0),$$

$$\alpha_Y(h(T), h(e)) = \alpha_X(T, e) + \pi \sum_{e' \in \Delta_1(T)} \varepsilon_{e,e'} g(\iota_T(e')), \quad \forall (T, e) \in \Delta_3^1(X),$$

where the orientation of T is the one inherited from X and ι_T is the canonical map that maps T into X, and

$$\ell_Y = \ell_X + \sum_{e \in \Delta_1(X)} \left(g(e) \sum_{(T,e') \in (\phi^{3,1})^{-1}(e)} \left(\frac{1}{3} - \frac{\alpha(T, e')}{\pi} \right) \right).$$

It is easy to see that the weights on the edges are gauge invariant in the sense that

$$\omega_X = \omega_Y \circ h.$$

Definition 2.7 (Based gauge equivalent): Two leveled shape structures (α_X, ℓ_X) and (α'_X, ℓ'_X) on an oriented pseudo 3-manifold X are *based gauge equivalent* if (X, α_X, ℓ_X) and (X, α'_X, ℓ'_X) are gauge equivalent in the sense of definition 2.6, where the isomorphism $h: X \to X$ in the equivalence is the identity map.

In fact, since X and Y become topologically indistinguishable after fixing a cellular isomorphism $h: X \to Y$, the only nontrivial part of gauge equivalence is about the leveled shape structures, and it is easy to see that the (based) gauge equivalence is well-defined as an equivalence relation. By forgetting the level we define similarly (based) gauge equivalence relation for shaped pseudo 3-manifolds.

2.1.2 The $3-2$ Pachner moves, admissibility, and the categroid $\mathcal{B}_a$

To define our desired categroid we need a more refined version of gauge equivalence relations called admissible equivalence relations, which rely on the $3-2$ Pachner moves. Before introducing the $3-2$ Pachner moves, we need the following notion.

Definition 2.8 (Balanced): An edge $e \in \Delta_1(X)$ of a shaped pseudo 3-manifold X is *balanced* if it is internal and $\omega_X(e) = 2\pi$. An edge is *unbalanced* if it is not balanced. A shaped pseudo 3-manifold X is *fully balanced* if all edges of X are balanced.

By definition, a shaped pseudo 3-manifold can be fully balanced only if its boundary is empty.

Let X be a shaped pseudo 3-manifold and $e \in \Delta_1(X)$ be a balanced edge of X shared by exactly three distinct tetrahedra T_1, T_2 and T_3. The tetrahedra T_1, T_2, T_3 compose a shaped pseudo 3-submanifold S of X with the only internal and balanced edge e. The $3-2$ *Pachner move* is to replace the triangulation of the topological space underlying S with another triangulation S_e consisting of only two tetrahedra T_4 and T_5 such that the induced triangulation of ∂S stays the same. The triangulation S_e is constructed as shown in fig. 2-2, where the left side stands for S_e and the right side stands for S.

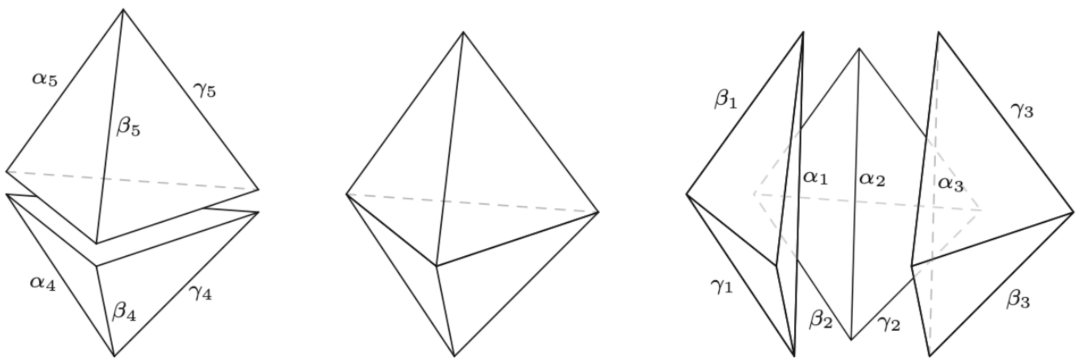

Figure 2-2 The $3-2$ Pachner move[5]

We have $\Delta_1(S_e) = \Delta_1(S) \setminus \{e\}$. Using the labels of the dihedral angles in fig. 2-2, the shape structure on S_e is given by

$$\begin{aligned} \alpha_4 &= \beta_2 + \gamma_1 \quad & \alpha_5 &= \beta_1 + \gamma_2 \\ \beta_4 &= \beta_1 + \gamma_3 \quad & \beta_5 &= \beta_3 + \gamma_1 \\ \gamma_4 &= \beta_3 + \gamma_2 \quad & \gamma_5 &= \beta_2 + \gamma_3. \end{aligned} \tag{2-2}$$

The condition that e is balanced, i.e. $\alpha_1 + \alpha_2 + \alpha_3 = 2\pi$, guarantees that the above equations give a well-defined shape structure on S_e. Conversely, given shaped T_4 and T_5, if we have (positive) solutions for the dihedral angles of any of T_1, T_2 or T_3, the rest dihedral angles of T_1, T_2 and T_3 follow immediately via eq. (2-2) and automatically satisfy the balanced condition. Furthermore, for any shaped T_4 and T_5, the corresponding solutions of the shape structures of T_1, T_2 and T_3 are all gauge equivalent.

Definition 2.9 (Shaped $3-2$ Pachner move): Given two shaped pseudo 3-manifolds X and Y. We say that Y is obtained from X by a *shaped* $3-2$ *Pachner move* along $e \in \Delta_1(X)$ if Y is obtained from X by replacing S by S_e as constructed above. The shaped pseudo 3-manifold obtained from X by a shaped $3-2$ Pachner move along e will be denoted as X_e.

Definition 2.10 (Leveled shaped $3-2$ Pachner move): Given two leveled shaped pseudo 3-manifolds (X, ℓ_X) and (Y, ℓ_Y). We say that (Y, ℓ_Y) is obtained from (X, ℓ_X) by a *leveled shaped* $3-2$ *Pachner move* along $e \in \Delta_1(X)$ if $Y = X_e$ and

$$\ell_Y = \ell_X + \frac{1}{12\pi} \sum_{(T,e_1)\in(\phi^{3,1})^{-1}(e)} \sum_{e_2\in\Delta_1(T)} \varepsilon_{e_1,e_2}\alpha_X(T, e_2).$$

Definition 2.11 (Pachner refinement): Let X and Y be two (leveled) shaped pseudo 3-manifolds. X is a *Pachner refinement* of Y if there exists a finite sequence of (leveled) shaped pseudo 3-manifolds

$$X = X_0, X_2, \cdots, X_n = Y,$$

such that for any $i \in \{0, \cdots, n-1\}$, X_{i+1} is obtained from X_i by a (leveled) shaped $3-2$ Pachner move.

Therefore X is a Pachner refinement of Y means that Y is obtained from X by a finite sequence of $3-2$ Pachner moves. Note that the word “refinement” is in the sense that the triangulation of X contains more edges and tetrahedra than that of Y.

Definition 2.12 (Admissible): An oriented pseudo 3-manifold X is *admissible* if

$$H_2(X - \Delta_0(X), \mathbb{Z}) = 0.$$

Note that since Pachner refinements preserve the set of vertices and the topological structures of original spaces, Pachner refinements of admissible pseudo 3-manifolds are still admissible.

Definition 2.13 (Admissibly equivalent): Two admissible (leveled) shaped pseudo 3-manifolds X and Y are *admissibly equivalent* if there exists Pachner refinements X' and Y' of X and Y respectively such that X' and Y' are gauge equivalent.

We can now define our desired categroid. For this, we need to introduce a canonical way to perceive oriented pseudo 3-manifolds as cobordisms.

Let X be an oriented pseudo 3-manifold, the orientation of X induces an orientation on each tetrahedron $T \in \Delta_3(X)$, according to which we give an orientation-preserving embedding of T into $\mathbb{R}^3$. Recall that we required the vertices of T to be totally ordered, say $T = [v_0, v_1, v_2, v_3]$, the relative position of the ordered vertices of T embedded in $\mathbb{R}^3$ gives a sign for T, defined by

$$\operatorname{sign}(T) = \operatorname{sign}\left(\det(v_1 - v_0, v_2 - v_0, v_3 - v_0)\right).$$

With this sign of T, we define the signs of the faces of T by

$$\operatorname{sign}(\partial_i T) = (-1)^i \operatorname{sign}(T), \quad i \in \{0, \cdots, 3\},$$

where ∂_i is the canonical boundary map

$$\partial_i\left([v_0, \cdots, v_j]\right) = [v_0, \cdots, v_{i-1}, v_{i+1}, \cdots, v_j], \quad 0 \le i \le j.$$

See fig. 2-3 for a graphical demonstration.

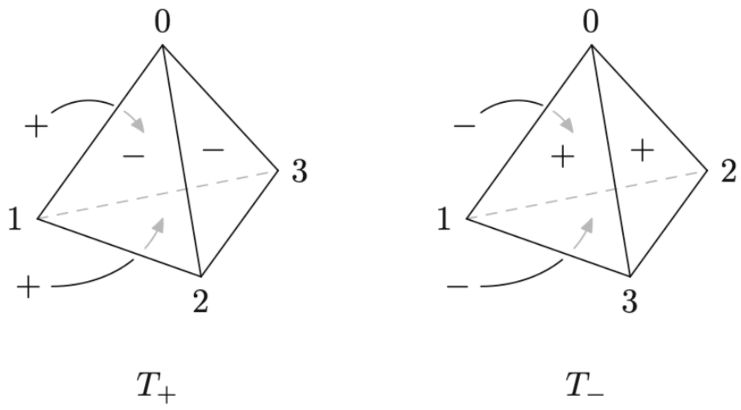

Figure 2-3 The signs of faces of tetrahedra[5]

Since the boundary of X consists of a subset of faces of tetrahedra in $\Delta_3(X)$, the restriction of the sign function $\operatorname{sign}_X : \Delta_2(X) \to \{\pm 1\}$ to $\Delta_2(\partial X)$ divides ∂X into two parts, namely

$$\partial X = \partial_+ X \cup \partial_- X, \quad \partial_\pm X = \bigcup_{A \in \operatorname{sign}_X^{-1}(\pm 1) \cap \partial X} A.$$

In this way, a leveled shaped pseudo 3-manifold X becomes a morphism by

$$X \in \operatorname{Hom}_{\mathcal{B}}(\partial_- X, \partial_+ X),$$

in the cobordism category $\mathcal{B}$ whose objects are pseudo 2-manifolds (defined similarly as in definition 2.2) and composition is gluing with respect to vertex-order-preserving and orientation-reversing simplicial maps with addition of levels and the obvious composition of dihedral angles. Take the sub-categroid of admissible leveled shaped pseudo 3-manifolds and quotient the admissible equivalence relation, we obtain our desired domain categroid $\mathcal{B}_a$.

Definition 2.14 (The domain categroid $\mathcal{B}_a$): The domain categroid $\mathcal{B}_a$ consists of as objects simplicial isomorphism classes of pseudo 2-manifolds and as morphisms admissible equivalence classes of admissible leveled shaped pseudo 3-manifolds, with the composition described above.

2.2 The Target Categroid

The morphisms in the target categroid will be (complex) tempered distributions, which are continuous linear functionals on (complex) Schwartz spaces.

Definition 2.15 (Schwartz space): For $n \in \mathbb{N}$, the *(complex) Schwartz space* $\mathcal{S}(\mathbb{R}^n)$ is the topological vector space whose

- underlying (complex) vector space is

$$\mathcal{S}(\mathbb{R}^n) := \left\{ f \in C^\infty(\mathbb{R}^n, \mathbb{C}) \middle| \sup_{x \in \mathbb{R}^n} \|x^\alpha \partial_\beta f(x)\| < \infty,\ \forall \alpha, \beta \in \mathbb{N}^n \right\};$$

- topology is that induced by semi-norms $p_{\alpha,\beta}(f) := \sup_{x\in\mathbb{R}^n} \|x^\alpha \partial_\beta f(x)\|$ for all $\alpha, \beta \in \mathbb{N}^n$.

The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is a dense subspace of $L^2(\mathbb{R}^n)$ (over $\mathbb{C}$) with the usual inner product

$$\begin{array}{ccc} L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) & \longrightarrow & \mathbb{C} \\ (f, g) & \mapsto & \langle f \mid g \rangle := \int_{\mathbb{R}^n} \bar{f}(x) g(x)\, \mathrm{d}x \end{array}$$

Definition 2.16 (Tempered distribution): For $n \in \mathbb{N}$, the space of *(complex) tempered distribution* $\mathcal{S}'(\mathbb{R}^n)$ is the space of continuous linear functionals on the (complex) Schwartz space $\mathcal{S}(\mathbb{R}^n)$.

Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, the restriction of continuous functionals on $L^2(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ gives an injection $\left(L^2(\mathbb{R}^n)\right)' \to \mathcal{S}'(\mathbb{R}^n)$. The Riesz representation theorem for Hilbert spaces gives a natural isomorphism between $L^2(\mathbb{R}^n)$ and its dual vector space $\left(L^2(\mathbb{R}^n)\right)'$ of continuous linear functionals, $L^2(\mathbb{R}^n) \cong \left(L^2(\mathbb{R}^n)\right)' : f \mapsto \langle f \mid \cdot \rangle$. Therefore $L^2(\mathbb{R}^n)$ can be seen as a subspace of $\mathcal{S}'(\mathbb{R}^n)$ via the following map

$$\begin{array}{ccc} L^2(\mathbb{R}^n) & \longhookrightarrow & \mathcal{S}'(\mathbb{R}^n) \\ f & \mapsto & \left(g \mapsto \int_{\mathbb{R}^n} \bar{f}(x) g(x)\, \mathrm{d}x\right) \end{array}$$

which also restricts to a natural inclusion $\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$. Furthermore, this inspires us to adopt the following notation for an element φ in $\mathcal{S}'(\mathbb{R}^n)$, that for any $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\varphi(f) =: \int_{\mathbb{R}^n} \varphi(\mathbf{x}) f(\mathbf{x})\, \mathrm{d}\mathbf{x}, \tag{2-3}$$

hence blurring the difference between functions and distributions. An advantage of this notation is that the formal integration on the right hand side gives us a way to define change of coordinates for elements in $\mathcal{S}'(\mathbb{R}^n)$.

Definition 2.17 (The target categroid $\mathcal{D}$): The target categroid $\mathcal{D}$ consists of as objects finite sets and as morphisms from a finite set n to m tempered distributions in $\mathcal{S}'(\mathbb{R}^{n \sqcup m})$.

Using the notation introduced in eq. (2-3), the composition in $\mathcal{D}$ can be described as the following,

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(n,m) \times \mathrm{Hom}_{\mathcal{D}}(m,k) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}}(n,k) \\ (\varphi, \psi) & \mapsto & \left(f \mapsto \int_{\mathbb{R}^{n \sqcup m \sqcup k}} \varphi(\mathbf{x},\mathbf{y}) \psi(\mathbf{y},\mathbf{z}) f(\mathbf{x},\mathbf{z})\, \mathrm{d}(\mathbf{x},\mathbf{y},\mathbf{z})\right) \end{array}$$

where $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ denote the standard coordinates of $\mathbb{R}^n$, $\mathbb{R}^m$ and $\mathbb{R}^k$, respectively. Note that the above map is not well-defined for all pairs of (φ, ψ), hence $\mathcal{D}$ is a categroid instead of a category. An explicit description for the subset $\mathscr{F}_{\mathcal{D}}(n,m,k)$ of $\mathrm{Hom}_{\mathcal{D}}(n,m) \times \mathrm{Hom}_{\mathcal{D}}(m,k)$ where the above map is well-defined can be found in Andersen and Kashaev's paper[5]§3.

The nuclear theorem[8]Theorem V.12 provides us with an isomorphism (of vector spaces)

$$\begin{array}{ccc} \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^m)) & \cong & \mathcal{S}'(\mathbb{R}^{n \sqcup m}) \\ \varphi : f \mapsto \begin{pmatrix} g \\ \downarrow \\ \tilde{\varphi}(f \otimes g) \end{pmatrix} & \leftrightsquigarrow & \tilde{\varphi} \end{array}$$

where

$$(f \otimes g)(x_1, \cdots, x_{n+m}) := f(x_1, \cdots, x_n) g(x_{n+1}, \cdots, x_{n+m}).$$

Elements in $\mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^m))$ admit a natural definition of adjoint. For any $\varphi \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^m))$, its adjoint $\varphi^* \in \mathcal{L}(\mathcal{S}(\mathbb{R}^m), \mathcal{S}'(\mathbb{R}^n))$ is uniquely defined by

$$\varphi^*(g)(f) := \overline{\varphi(\bar{f})(\bar{g})}.$$

Definition 2.18 (Adjoint of tempered distributions): The adjoint of a tempered distribution $\varphi \in \mathcal{S}'(\mathbb{R}^n)$ is the element $\varphi^* \in \mathcal{S}'(\mathbb{R}^n)$ defined by

$$\varphi^*(f) := \overline{\varphi(\bar{f})}.$$

2.3 The TQFT Functor

The final missing piece for our TQFT functor is Faddeev's quantum dilogarithm. The following subsection briefly summarizes the definition and some basic properties of Faddeev's quantum dilogarithm.

2.3.1 Faddeev's quantum dilogarithm

Definition 2.19 (Faddeev's quantum dilogarithm): *Faddeev's quantum dilogarithm* is a function of two complex arguments z and b for $|\operatorname{Im} z| < \frac{1}{2}|b + b^{-1}|$ by the formula

$$\Phi_b(z) := \exp\left(\int_C \frac{e^{2izw}\,dw}{4\sinh(wb)\sinh(w/b)w}\right),$$

where the contour $\boldsymbol{C}$ runs along the real axis, deviating into the upper half plane in the vicinity of the origin and extended by the functional equation

$$\Phi_b(z - ib^{\pm 1}/2) = (1 + e^{2\pi b^{\pm 1} z})\Phi_b(z + ib^{\pm 1}/2)$$

to a meromorphic function in $z \in \mathbb{C}$.

It is clear by definition that

$$\Phi_b(z) = \Phi_{-b}(z) = \Phi_{1/b}(z).$$

We may define equivalently that

$$\Phi_b(z) := \frac{\left(e^{2\pi(z+c_b)b}; q\right)_\infty}{\left(e^{2\pi(z-c_b)b^{-1}}; \tilde{q}\right)_\infty}, \quad \operatorname{Re} b > 0,\ \operatorname{Im} b \geq 0, \quad \forall z \in \mathbb{C}, \tag{2-4}$$

and then extend by the above symmetric properties, where

$$q := e^{2i\pi b^2}, \quad \tilde{q} := e^{-2i\pi b^{-2}}, \quad c_b := i(b + b^{-1})/2, \tag{2-5}$$

and $(x; q)_m$ is the *Pochhammer symbol*

$$(x; q)_m := \prod_{i=0}^{m-1}(1 - q^i x), \quad m \in \mathbb{N} \cup \{\infty\},$$

provided that $|q| < 1$ when $m = \infty$. Note that when b is in the first quadrant, we do have $|q|, |\tilde{q}| < 1$ where q and $\tilde{q}$ are defined as in eq. (2-5).

Using eq. (2-4), it is not hard to see that when b is in the first quadrant, $\Phi_b(z)$ has poles and zeros (in z) as below:

$$\textbf{poles: } c_b + i\mathbb{N}b + i\mathbb{N}b^{-1}, \quad \textbf{zeros: } -c_b - i\mathbb{N}b - i\mathbb{N}b^{-1}. \tag{2-6}$$

The following identities will be useful in the process of expanding the state integral using the method of residues later in chapter 4.

Proposition 2.1: Faddeev's quantum dilogarithm satisfies the following identities:

- (inversion relation) for any $b, z \in \mathbb{C}$

$$\Phi_b(z)\Phi_b(-z) = \mathrm{e}^{\pi \mathrm{i} x^2}\Phi_b(0)^2, \quad \Phi_b(0)^2 = \left(\frac{q}{\tilde{q}}\right)^{\frac{1}{24}}. \tag{2-7}$$

- (pseudo-periodicity[9]Lemma 2.1) for b in the first quadrant and any $n, m \in \mathbb{N}$,

$$\Phi_b(x + c_b + \mathrm{i}mb + \mathrm{i}nb^{-1}) = \frac{1}{1 - \mathrm{e}^{2\pi b^{-1}x}} \frac{1}{(q\mathrm{e}^{2\pi bx}; q)_m} \frac{1}{(\tilde{q}^{-1}\mathrm{e}^{2\pi b^{-1}x}; \tilde{q}^{-1})_n} \frac{(q\mathrm{e}^{2\pi bx}; q)_\infty}{(\tilde{q}\mathrm{e}^{2\pi b^{-1}x}; \tilde{q})_\infty}.$$

- (expansion near poles[9]Lemma 2.1, Proposition 2.2) for b in the first quadrant and any $n, m \in \mathbb{N}$,

$$\Phi_b(x + c_b + \mathrm{i}mb + \mathrm{i}nb^{-1}) = \frac{(q;q)_\infty}{(\tilde{q};\tilde{q})_\infty} \frac{\exp\left(-\sum_{l=1}^\infty \frac{1}{l!}E_l^{(m)}(q)(2\pi bx)^l\right)\exp\left(\sum_{l=1}^\infty \frac{1}{l!}\tilde{E}_l^{(m)}(\tilde{q})(2\pi bx)^l\right)}{(q;q)_m(\tilde{q}^{-1};\tilde{q}^{-1})_n\left(1 - \mathrm{e}^{2\pi b^{-1}x}\right)},$$

where

$$E_l^{(m)}(q) := \sum_{s=1}^\infty \frac{s^{l-1}q^{s(m+1)}}{1 - q^s},$$

$$\tilde{E}_l^{(n)}(\tilde{q}) := \begin{cases} -n + E_1^{(n)}(\tilde{q}) & l = 1 \\ E_l^{(n)}(\tilde{q}) & l \in 2\mathbb{N}^* + 1 \\ 2E_l^{(0)}(\tilde{q}) - E_l^{(n)}(\tilde{q}) & l \in 2\mathbb{N}^* \end{cases} \tag{2-8}$$

2.3.2 Construction of the TQFT functor

Let us now define our TQFT functor. Given $b \in \mathbb{C} \setminus \mathrm{i}\mathbb{R}$, we define a functor $F_b : \mathcal{B}_a \to \mathcal{D}$ such that

- (on objects) for a pseudo 2-manifold $\Sigma \in \mathrm{Obj}(\mathcal{B}_a)$,

$$F_b(\Sigma) := \Delta_2(\Sigma) \in \mathrm{Obj}(\mathcal{D}).$$

- (on morphisms) for a representative (X, α_X, ℓ_X) of a morphism in $\mathcal{B}_a$,

$$F_b(X, \alpha_X, \ell_X) := \mathrm{e}^{\pi \mathrm{i}\frac{\ell_X}{4}(b+b^{-1})^2} Z_b(X, \alpha_X) \in \mathcal{S}'\left(\mathbb{R}^{\Delta_2(\partial X)}\right),$$

where $Z_b(X, \alpha_X)$ is independent of the level ℓ_X.

The value of Z_b on a single tetrahedron T with $\mathrm{sign}(T) = 1$ is an element $Z_b(T, \alpha_T) \in \mathcal{S}(\mathbb{R}^{\Delta_2(T)}) \subset \mathcal{S}'(\mathbb{R}^{\Delta_2(T)})$ given by

$$Z_b(T, \alpha_T)(x_0, x_1, x_2, x_3) = \delta(x_0 - x_1 + x_2)\frac{\mathrm{e}^{2\pi \mathrm{i}(x_3 - x_2)\left(x_0 + \frac{\alpha_3}{2\mathrm{i}}(b+b^{-1})\right) + \pi \mathrm{i}\frac{\varphi_T}{4}(b+b^{-1})^2}}{\Phi_b\left(x_3 - x_2 + \frac{1-\alpha_1}{2\mathrm{i}}(b + b^{-1})\right)},$$

where δ is Dirac's delta-function supported at $0 \in \mathbb{R}$,

$$\varphi_T := \alpha_1\alpha_3 + \frac{\alpha_1 - \alpha_3}{3} - \frac{2(b+b^{-1})^{-2}+1}{6}, \quad \alpha_i := \frac{1}{\pi}\alpha_T(\partial_0\partial_i T), \quad i \in \{1,2,3\},$$

and x_i corresponds to the indeterminant

$$x_i : \partial_i(T) \to \mathbb{R}.$$

For oppositely oriented tetrahedron T^* (i.e. $\mathrm{sign}(T^*) = -1$), we define

$$Z_b(\bar{T}) = Z_b(T)^*,$$

where T is the positively oriented tetrahedron corresponding to T^* (with the same shape structure) and $Z_b(T)^*$ is the adjoint of $Z_b(T)$ as defined in definition 2.18.

The value of Z_b on an arbitrary admissible leveled shaped pseudo 3-manifold (X, α_X) is obtained by composing the values of Z_b on each tetrahedron in $\Delta_2(X)$ accordingly in $\mathcal{D}$. The strong connectedness condition in definition 2.2 guarantees that such composition always exists for us to obtain a morphism from $\partial_- X$ to $\partial_+ X$.

The above procedure gives a unique well-defined functor $F_b : \mathcal{B}_a \to \mathcal{D}$. Moreover, it is a $*$-functor in the following sense.[5]Theorem 4

Definition 2.20: A functor $F : \mathcal{B}_a \to \mathcal{D}$ is a *$*$-functor* if

$$F(X^*) = F(X)^*,$$

where X^* is X with opposite orientation, and $F(X)^*$ is the adjoint of $F(X)$ in the sense of definition 2.18.

In the papers of Andersen and Kashaev, they gave a series of detailed examples of calculation of the precise values of Z_b on spaces of knots complement using the graphical presentation of pseudo 3-manifolds[11]p.12 [5]§11, which we do not duplicate here.

2.4 The State Integral

The *state integral* Z_K of a knot $K \subset S^3$ will be the holomorphic function (in the variable b) on the cut plane $\mathbb{C}' := \mathbb{C} \setminus (-\infty, 0]$ given by, up to some prefactors, the value of Z_b in the TQFT functor on a one-vertex H-triangulation of (S^3, K).

Definition 2.21 (One-vertex H-triangulation): Let (M, K) be a pair of a closed oriented 3-manifold M and a knot $K \subset M$. A *one-vertex H-triangulation* of (M, K) is a Δ-triangulation of M with only one vertex and a distinguished edge representing the knot K.

A detailed description of how to construct a one-vertex H-triangulation (S^3, K) for a knot K is given in the paper of Kashaev, Luo and Vartanov[12]§4.1.

The following lists the state integrals of knots that will appear later in chapter 4, computed by Andersen and Kashaev[5]§11. We will adopt the notation $\tau \coloneqq b^2$ and take the limit $\varepsilon \to 0^+$ without writing out the limit repeatedly.

- (The 4_1 knot)

$$Z_{4_1}(\tau) = \int_{\mathbb{R}+\mathrm{i}\varepsilon} \Phi_{\sqrt{\tau}}(x)^2 \mathrm{e}^{-\pi \mathrm{i} x^2}\, \mathrm{d}x, \quad (\tau \in \mathbb{C}')$$

- (The 5_2 knot)

$$Z_{5_2}(\tau) = \int_{\mathbb{R}+\mathrm{i}\varepsilon} \Phi_{\sqrt{\tau}}(x)^3 \mathrm{e}^{-2\pi \mathrm{i} x^2}\, \mathrm{d}x. \quad (\tau \in \mathbb{C}')$$

- (The $(-2,3,7)$ pretzel knot)[13]eq. (58)

$$Z_{(-2,3,7)}(\tau) = \int_{\mathbb{R}+\mathrm{i}\frac{c_b}{2}+\mathrm{i}\varepsilon} \Phi_{\sqrt{\tau}}(x)^2 \Phi_{\sqrt{\tau}}(2x - c_b) \mathrm{e}^{-\pi \mathrm{i}(2x-c_b)^2}\, \mathrm{d}x. \quad (\tau \in \mathbb{C}')$$

CHAPTER 3 MODULAR FORMS AND QUANTUM MODULAR FORMS

3.1 Basics about Modular Forms

In this section, we briefly review the basic definitions and terminologies of modular forms and introduce a family of examples called the Eisenstein series.

Let $\mathfrak{H} = \{x + \mathrm{i}y \mid x, y \in \mathbb{R}, y > 0\}$ be the upper half plane in $\mathbb{C}$. We define the group action of

$$\mathrm{SL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| ad - bc = 1 \right\},$$

on $\mathfrak{H}$ by, for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and $z \in \mathfrak{H}$,

$$\gamma \cdot z = \frac{az+b}{cz+d}.$$

It is straightforward to see that this action is well-defined. Indeed, we have

$$\mathrm{Im}(\gamma \cdot z) = \frac{\mathrm{Im}(z)}{|cz+d|^2},$$

and the associativity follows from direct computation.

In the following we will mainly focus on the subgroup $\Gamma_1 \coloneqq \mathrm{SL}_2(\mathbb{Z})$ of $\mathrm{SL}_2(\mathbb{R})$.

Definition 3.1 (Modular forms): A *modular form of weight k* on Γ_1 is a holomorphic function $f : \mathfrak{H} \to \mathbb{C}$ such that

$$(f|_k\gamma)(z) \coloneqq (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = f(z), \tag{3-1}$$

for any $z \in \mathfrak{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1$.

Most of the time the weight k will be considered to be an integer, but the definition still makes sense even if we allow k to be rational numbers. The identity in eq. (3-1) is usually referred to as the *modularity transformation property*.

Since $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_1$, modular forms are 1-periodic, hence we may adopt the 1-invariant notation $q \coloneqq \mathrm{e}^{2\pi \mathrm{i} z}$ and expand a modular form f as a q-series,

$$f(z) = \sum_{n=0}^{\infty} a_n q^n.$$

The coefficients a_n's are called *Fourier coefficients*.

Let $M_k(\Gamma_1)$ be the set of modular forms of weight k on Γ_1, then it is a vector space over $\mathbb{C}$. Since the multiplication of a modular form of weight k and a modular form of weight l is a modular form of weight $k+l$, the direct sum $M_*(\Gamma_1) := \bigoplus_{k=0}^{\infty} M_k(\Gamma_1)$ admits a structure of graded $\mathbb{C}$-algebra. Since $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \Gamma_1$, we see from *eq*. (3-1) that $M_k(\Gamma_1) = 0$ when k is odd. It is easily seen that $M_0(\Gamma_1) = \mathbb{C}$. Note also that an estimation of order of growth along with the fact that modular forms are holomorphic would imply that $M_k(\Gamma_1) = 0$ for $k < 0$.[14]§1.1

One of the most famous families of examples of modular forms is the *Eisenstein series*. There are several different approaches to their definition, here we only introduce one of them.[14]§2

Let k be a positive even integer and consider the stabilizer of the infinity, the subgroup Γ_∞ of Γ_1, which is explicitly $\left\{\pm\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) \middle| n \in \mathbb{N}\right\}$. Let Γ_1 act (by right) on the set of holomorphic functions on $\mathfrak{H}$ via $f \mapsto f|_k\gamma$, we see that Γ_∞ preserves constant functions, in particular the constant 1. Hence the summation

$$\sum_{\Gamma_\infty\gamma \in \Gamma_\infty\backslash\Gamma_1} 1|_k\gamma$$

over (right) cosets $\Gamma_\infty\backslash\Gamma_1$ of Γ_∞ in Γ_1 is invariant under Γ_1, provided the absolute convergence. Since left multiplication by $\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$ preserves the bottom row of matrices and acts transitively on any set of matrices in Γ_1 with the same bottom rows, $\Gamma_\infty\backslash\Gamma_1$ is bijectively represented by coprime pairs of integers (c,d) up to signs. Therefore

$$\sum_{\Gamma_\infty\gamma \in \Gamma_\infty\backslash\Gamma_1} 1|_k\gamma = \frac{1}{2}\sum_{\substack{c,d\in\mathbb{Z} \\ (c,d)=1}} \frac{1}{(cz+d)^k}.$$

For $k > 2$ the absolute convergence of the right-hand side is easily seen.

Definition 3.2 (Eisenstein series): For any even integer $k > 2$, the *Eisenstein series of weight* k is the modular form $\mathcal{E}_k(z)$ defined by

$$\mathcal{E}_k(z) = \frac{1}{2}\sum_{\substack{c,d\in\mathbb{Z} \\ (c,d)=1}} \frac{1}{(cz+d)^k}. \tag{3-2}$$

The Fourier expansion of the Eisenstein series for even k with $k > 2$ is

$$1 + \frac{1}{\zeta(k)}\frac{(2\pi \mathrm{i})^k}{(k-1)!}\sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n, \tag{3-3}$$

where $\zeta(k) = \sum_{r\geq 1} 1/r^k$ is the value at k of the Riemann zeta function and $\sigma_{k-1}(n)$ for

$n \in \mathbb{N}$ denotes the sum of the $(k-1)$-th powers of the positive divisors of n, namely

$$\sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n = \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} r^{k-1} q^{mr}. \tag{3-4}$$

For $k = 2$, the right-hand side of eq. (3-2) is not absolutely convergent, but we may define it using eq. (3-3) which is absolutely convergent, hence

$$\mathcal{E}_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n.$$

This corresponds to a certain order of summation on the right-hand side of eq. (3-2). However, since the summation is not interchangeable for $k = 2$, $\mathcal{E}_2(z)$ is no longer a modular form. Nonetheless, it still satisfies a quasi-modularity condition:[14]Proposition 6

Proposition 3.1: For $z \in \mathfrak{H}$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1$, we have

$$\mathcal{E}_2(z) = (cz+d)^2 \mathcal{E}_2(z) + \frac{6}{\pi \mathrm{i}} c(cz+d).$$

Using eq. (3-4), one sees easily that

$$\mathcal{E}_2(z) = 1 - 24 \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2}. \tag{3-5}$$

We will see that Eisenstein series of weight 2 of the form in eq. (3-5) will appear repeatedly in chapter 4, along with the following similar series

$$\mathcal{E}_1(z) := 1 - 4 \sum_{n=1}^{\infty} \frac{q^n}{1-q^n}, \tag{3-6}$$

which we call the Eisenstein series of weight 1.

3.2 Quantum Modular Forms and the Quantum Modularity Conjecture

Generally speaking, quantum modular forms are objects occurring in perturbative quantum field theory that have properties similar to the modularity transformation property. Due to the variety of these objects, it is hard to give a definition that applies to all of them. However, we can give "definitions" that demonstrate what kind of objects they are, attempting to cover as large range of objects as possible.

Let us keep the notations $\mathfrak{H}$ and Γ_1 in the previous section. A quantum modular form, instead of a holomorphic function on $\mathfrak{H}$, is an assignment on the orbit of the cusp ∞; *cusps* are points to be added in the compactification. Noticing that an element $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1$ sends ∞ to $\frac{a}{c}$, the orbit of ∞ is exactly $\mathbb{Q} \cup \{\infty\}$, which we may naturally denote as $\mathbb{P}^1(\mathbb{Q})$ and equip it with the discrete topology.

Definition 3.3 (Weak quantum modular form): A *weak quantum modular form* is a function $f : \mathbb{P}^1(\mathbb{Q}) \setminus S \to \mathbb{C}$ for some finite subset S of $\mathbb{P}^1(\mathbb{Q})$ such that for each element $\gamma \in \Gamma_1$, the function $h_\gamma : \mathbb{P}^1(\mathbb{Q}) \setminus \left(S \cup \gamma^{-1}(S)\right) \to \mathbb{C}$ defined by

$$h_\gamma(x) = f(x) - (f|_k\gamma)(x) \tag{3-7}$$

extends to a function on a co-finite subspace of $\mathbb{P}^1(\mathbb{R})$ with properties of continuity or (real) analyticity.

For any two elements $\gamma_1, \gamma_2 \in \Gamma_1$, we have

$$h_{\gamma_1}|_k\gamma_2 = f|_k\gamma_2 - f|_k\gamma_1\gamma_2,$$

hence

$$h_{\gamma_1\gamma_2} = f - f|_k\gamma_1\gamma_2 = (f - f|_k\gamma_1) + (f|_k\gamma_1 - f|_k\gamma_1\gamma_2) = h_{\gamma_1} + h_{\gamma_1}|_k\gamma_2.$$

Therefore to check that f is a quantum modular form it suffices to check it for a set of generators of Γ_1.

Definition 3.4 (Strong quantum modular form): A *strong quantum modular form* is a power-series-valued function $f : \mathbb{P}^1(\mathbb{Q}) \setminus S \to \mathbb{C}[\![\varepsilon]\!] : x \mapsto f(x + \mathrm{i}\varepsilon)$ for some finite subset S of $\mathbb{P}^1(\mathbb{Q})$ such that for each $\gamma \in \Gamma_1$ there exists a real-analytic function h_γ on a neighborhood of a co-finite subspace of $\mathbb{P}^1(\mathbb{R})$ in $\mathbb{P}^1(\mathbb{C})$ whose expansions at rational points agree with the modular equation of f, i.e. for any $x \in \mathbb{Q}$ in the domain of h_γ,

$$h_\gamma(z) = f(z) - \left(f|_k\gamma\right)(z), \quad z \to x.$$

In many examples a strong quantum modular form f admits an extension $(\mathbb{C} \setminus \mathbb{R}) \cup \mathbb{Q} \to \mathbb{C}$ which is analytic on $\mathbb{C} \setminus \mathbb{R}$ and has vertical asymptotic expansions approaching rational points coinciding the power series given by values of f.

More generally speaking, a common property shared and should be concerned with of quantum modular forms is extendibility via certain "modular relations", relations considering the action of Γ_1 on $\mathfrak{H}$. In chapter 4, we will see examples of q-series defined on $\mathbb{C} \setminus \mathbb{R}$ that are extendible to the cut plane $\mathbb{C}' = \mathbb{C} \setminus (-\infty, 0]$. Those objects are known as *holomorphic quantum modular forms*.

The main topic that we are about to delve into, the *Quantum Modularity Conjecture*, however, involves neither a weak quantum modular form nor a strong quantum modular form in the strict sense of our definitions above. Still, we will be able to see their similarity and modular properties.

In 1995, using the quantum dilogarithm function

$$(x;q)_m \coloneqq \prod_{n=0}^{m-1}(1-xq^n), \quad (|q|<1)$$

R. Kashaev introduced a knot invariant related to a positive integer N, which is denoted as $\langle K\rangle_N$ for a knot K[2]. For any knot K and positive integer N, the invariant $\langle K\rangle_N$ is a complex number such that $\langle K\rangle_N \in \mathbb{Z}[e^{\frac{2\pi i}{N}}]$. Kashaev conjectured that, if K is hyperbolic, which means that the complement $S^3 \setminus K$ can be given a hyperbolic structure, then the absolute value of $\langle K\rangle_N$ grows exponentially as N increases. More precisely, the following full asymptotic expansion was conjectured

$$\langle K\rangle_N \sim N^{\frac{3}{2}} e^{\frac{V(K)}{2\pi}N}\Phi^{(K)}\left(\frac{2\pi i}{N}\right), \quad N\to\infty, \tag{3-8}$$

where $V(K)$ is the hyperbolic volume of $S^3 \setminus K$ and $\Phi^{(K)}(\hbar)$ is a divergent power series in $\hbar$[15]. This conjectural expansion is known as the *Volume Conjecture*.[16]

In 2001, H. Murakami and J. Murakami discovered that Kashaev's invariant $\langle K\rangle_N$ is equal to the evaluation of the colored Jones polynomial $J_N^K(q)$ at $q=\eta_N$, where $\eta_N \coloneqq e^{\frac{2\pi i}{N}}$.[3] For the 4_1 knot, the colored Jones polynomial is given by

$$J_n^{4_1}(q) = \sum_{m=0}^{n-1} q^{-mn}\prod_{j=1}^{m}(1-q^{n-j})(1-q^{n+j}).$$

Note that when $m\ge n$, the product $\prod_{j=1}^{m}(1-q^{n-j})(1-q^{n+j})$ vanishes. When q is an N-th root of unity, we have

$$J_N^{4_1}(q) = \sum_{m=0}^{\infty}\left((1-q)(1-q^2)\cdots(1-q^m)\right)^2 = \sum_{m=0}^{\infty}(q;q)_m^2.$$

The absolute value of $J_N^{4_1}(q)$ is a $\bar{\mathbb{Q}}\cap\mathbb{R}$-valued function, where $\bar{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$, and its first few values are as follows:

q	1	-1	$\zeta_3^{\pm1}$	$\pm i$	$\zeta_5^{\pm1}$	$\zeta_5^{\pm2}$	$\zeta_6^{\pm1}$
$\left\lvert J_N^{4_1}(q)\right\rvert$	1	5	13	27	$46+2\sqrt{5}$	$46-2\sqrt{5}$	89

Table 3-1 First few values of $|J_N^{4_1}(q)|$ at roots of the unity[4]

As eq. (3-8) suggests, we have asymptotic expansion

$$\left|J_N^{4_1}\left(e^{\frac{2\pi i}{N}}\right)\right| = \frac{1}{3^{1/4}}N^{3/2}e^{\frac{V(4_1)}{2\pi}N}\left(1+\frac{11}{36\sqrt{3}}\frac{\pi}{N}+\frac{697}{7776}\frac{\pi^2}{N^2}+\frac{724351}{4199040\sqrt{3}}\frac{\pi^3}{N^3}+\cdots\right) \tag{3-9}$$

as $N\to\infty$, where the coefficients are all algebraic numbers. Since we are now dealing with colored Jones polynomials (instead of the Kashaev's invariant), we can further expand it at roots of the unity other than $\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}}$. For instance,

$$\left|J_N^{4_1}\left(-\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}}\right)\right| = \kappa(N)\cdot\frac{3^{1/4}}{2^{3/2}}N^{3/2}\mathrm{e}^{\frac{V(4_1)}{2\pi}\frac{N}{4}}\left(1+\frac{41}{36\sqrt{3}}\frac{\pi}{N}+\frac{12625}{7776}\frac{\pi^2}{N^2}+\cdots\right), \tag{3-10}$$

where

$$\kappa(N) = \begin{cases} 27 & N \equiv 1 \bmod 2, \\ 1 & N \equiv 2 \bmod 4, \\ 5 & N \equiv 0 \bmod 4 \end{cases}$$

Comparing with table 3-1, we see that

$$\kappa(N) = \left|J_N^{4_1}\left(\mathrm{e}^{\frac{\pi\mathrm{i}}{2}(N+2)}\right)\right|.$$

In fact, for rational values of N, eq. (3-10) still holds after replacing $\kappa(N)$ with $\left|J_N^{4_1}\left(\mathrm{e}^{\frac{\pi\mathrm{i}}{2}(N+2)}\right)\right|$. The similar holds for eq. (3-9) after multiplying the right-hand side by $\left|J_N^{4_1}\left(\mathrm{e}^{\pi\mathrm{i}N}\right)\right|$. More generally, for any knot K, if we define

$$\mathbf{J}^K:\ \mathbb{Q}/\mathbb{Z}\to\bar{\mathbb{Q}}\cap\mathbb{R},\quad \mathbf{J}^K(-p/q) := \left|J_q^K\left(\mathrm{e}^{2\pi\mathrm{i}p/q}\right)\right|,$$

where p and q are coprime, this observation then extends to the following conjectural expansion that

$$\frac{\mathbf{J}^K(\gamma\cdot X)}{\mathbf{J}^K(X)} = (cX+d)^{\frac{3}{2}}\mathrm{e}^{\frac{V(K)}{2\pi}\left(X+\frac{d}{c}\right)}\Phi_{a/c}^{(K)}\left(\frac{2\pi\mathrm{i}}{c(cX+d)}\right),\quad X\to\infty \text{ in } \mathbb{Q}, \tag{3-11}$$

for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_1$ with $c > 0$, where $\Phi_\alpha^{(K)}(\hbar)$ is a power series with algebraic coefficients depending on $\alpha\in\mathbb{Q}/\mathbb{Z}$. Therefore eq. (3-11) states a (conjectural) modular property, which is known as the Quantum Modularity Conjecture. Note that the case where $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0\end{smallmatrix}\right)$ and $X = N$ of eq. (3-11) implies eq. (3-8)[4].

CHAPTER 4 THE QUANTUM MODULARITY CONJECTURE AND STATE INTEGRALS

Over the past years the Quantum Modularity Conjecture (QMC), as introduced in section 3.2, has become one of the most outstanding problems in quantum topology, and during the research into it multiple phenomena and consequences have been revealed[17-19]. The phenomena, mostly observed by S. Garoufalidis, R. Kashaev and D. Zagier in their research of the 4_1 knot, the 5_2 knot and the $(-2, 3, 7)$ pretzel knot, indicate a close relationship of the conjecture with the Dimofte–Gaiotto–Gukov index[20] and the Andersen–Kashaev state integral which we have introduced in chapter 2, two knot invariants that were introduced in 2011[5,18,21]. The invariants also turned out to be related to the quantum spin network. Most of these relations are given in terms of the corresponding q-series rising from the conjecture, invariants and spin network[18].

A family of evidence for the QMC has been presented by Garoufalidis and Zagier.[18,22] Although a proof for the 4_1 knot is easy, currently for very few knots a rigorous proof of the Quantum Modularity Conjecture has been given.

In this chapter, we will mainly focus on introducing the remarkable phenomena observed in Andersen–Kashaev state integrals for the 4_1 knot, the 5_2 knot and the $(-2, 3, 7)$ pretzel knot. A reason for the existence of relations between the state integrals and the QMC is that they both come from Chern–Simons theory, of infinite dimensional (the Teichmüller TQFT) and finite dimensional ($\mathrm{SL}_2(\mathbb{C})$ Chern–Simons theory) respectively. We will focus more on the $(-2, 3, 7)$ pretzel knot in section 4.4 while giving brief summaries of the discoveries for the 4_1 knot and the 5_2 knot in sections 4.1 to 4.3, as there are newly obtained results for the $(-2, 3, 7)$ pretzel knot by the author, N. An and S. Garoufalidis.

This chapter will be more likely a list of observations and results, with proofs for some of them in section 4.4 and a minimum amount of comments. For the affluent stories and reasons behind these results, the reader is kindly referred to their original papers.[9,13,18,23]

Throughout this chapter, the following notations will be used consistently:

$$\tau \in \mathbb{C}' = \mathbb{C} \setminus (-\infty, 0],$$

$$b = \sqrt{\tau}, \quad \hbar = 2\pi \mathrm{i}\tau,$$

$$q = \mathrm{e}^{2\pi\mathrm{i}\tau}, \quad \tilde{q} = \mathrm{e}^{-2\pi\mathrm{i}\tau^{-1}}, \quad c_b = \frac{\mathrm{i}(b+b^{-1})}{2}.$$

4.1 The 4_1 Knot

Recall from section 2.4 that the state integral of the 4_1 knot is a holomorphic function on $\mathbb{C}' := \mathbb{C} \setminus (-\infty, 0]$, defined by

$$Z_{4_1}(\tau) = \int_{\mathbb{R}+\mathrm{i}\varepsilon} \Phi_{\sqrt{\tau}}(x)^2 \mathrm{e}^{-\pi\mathrm{i}x^2}\,\mathrm{d}x, \quad (\tau \in \mathbb{C}')$$

where $\Phi_b(x)$ is Faddeev's quantum dilogarithm as we have discussed in section 2.3.1.

When $\operatorname{Im}\tau > 0$ (so that $|q| < 1$ in the following), the state integral can be expanded into a combination of q-series $G_0(q)$ and $G_1(q)$,

$$2\mathrm{i}\left(\frac{\tilde{q}}{q}\right)^{\frac{1}{24}} Z_{4_1}(\tau) = \tau^{\frac{1}{2}} G_1(q)G_0(\tilde{q}) - \tau^{-\frac{1}{2}} G_0(q)G_1(\tilde{q}), \tag{4-1}$$

where $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$, $\tilde{q} = \mathrm{e}^{-2\pi\mathrm{i}\tau^{-1}}$. Explicitly, the q-series are given by

$$G_0(q) = \sum_{n=0}^{\infty}(-1)^n \frac{q^{\frac{n(n+1)}{2}}}{(q;q)_n^2}, \quad G_1(q) = \sum_{n=0}^{\infty}\left(1 + 2n - 4\sum_{s=1}^{\infty}\frac{q^{s(n+1)}}{1-q^s}\right)(-1)^n \frac{q^{\frac{n(n+1)}{2}}}{(q;q)_n^2},$$

recalling that $(x;q)_n$ is the Pochhammer symbol

$$(x;q)_n := \prod_{i=0}^{n-1}(1 - xq^i), \quad n \in \mathbb{N} \cup \{\infty\}.$$

The computation using the method of residues, along with that of the 5_2 knot and 1-dimensional state integrals in general, has been given in detail by Garoufalidis and Kashaev[9,13]. The symmetry $\Phi_b(x) = \Phi_{b^{-1}}(x)$ implies that $Z_{4_1}(\tau) = Z_{4_1}(\tau^{-1})$ whenever $\tau \in \mathbb{C} \setminus \mathbb{R}$, hence we can extend $G_0(q)$ and $G_1(q)$ to $|q| > 1$ by

$$G_0(q) = G_0(q^{-1}), \quad G_1(q) = -G_1(q^{-1}), \quad (q \in \mathbb{C},\ |q| \neq 1)$$

so that the factorization eq. (4-1) holds for all $\tau \in \mathbb{C} \setminus \mathbb{R}$[18].

In Garoufalidis and Zagier's recent paper[18], the following observations were made:

Let $\widehat{\Phi}_{4_1}(\hbar)$ be defined by

$$\widehat{\Phi}_{4_1}(\hbar) = \mathrm{e}^{\frac{\mathrm{i}V(4_1)}{\hbar}} \Phi^{(4_1)}(\hbar),$$

where $\Phi^{(4_1)}(\hbar)$ is given by eq. (3-8) for $K = 4_1$, then

Observation 1: When τ tends to 0 along any ray in the interior of the upper half-plane,

$$G_0(\mathrm{e}^{2\pi\mathrm{i}\tau}) \sim \sqrt{\tau}\left(\widehat{\Phi}_{4_1}(2\pi\mathrm{i}\tau) - \mathrm{i}\widehat{\Phi}_{4_1}(-2\pi\mathrm{i}\tau)\right)$$

to all orders in τ.

Observation 2: When τ tends to 0 in a cone in the interior of the upper half-plane

$$G_1(\mathrm{e}^{2\pi\mathrm{i}\tau}) \sim \frac{1}{\sqrt{\tau}}\left(\widehat{\Phi}_{4_1}(2\pi\mathrm{i}\tau) + \mathrm{i}\widehat{\Phi}_{4_1}(-2\pi\mathrm{i}\tau)\right)$$

to all orders in τ.

Observation 3: For $|q| < 1$, we have

$$G_0(q) = (q;q)_\infty \sum_{n=0}^{\infty}(-1)^n \frac{q^{\frac{n(3n+1)}{2}}}{(q;q)_n^3} = \frac{1}{(q;q)_\infty}\sum_{n,m=0}^{\infty}(-1)^{n+m}\frac{q^{\frac{(n+m)(n+m+1)}{2}}}{(q;q)_n(q;q)_m},$$

and

$$G_1(q) = \sum_{n=0}^{\infty}(1+6n)(-1)^n \frac{q^{\frac{n(n+1)}{2}}}{(q;q)_n^2}.$$

The series $\sum_{n=0}^{\infty}(-1)^n \frac{q^{\frac{n(3n+1)}{2}}}{(q;q)_n^3}$ occurred in Garoufalidis' work on the stability of the coefficients of the evaluation of the regular quantum spin network[24].

Let $\mathrm{Ind}_{4_1}(q)$ denote the Dimofte-Gaiotto-Gukov index of the 4_1 knot, which is also a q-series, then

Observation 4:

$$\mathrm{Ind}_{4_1}(q) = G_0(q)G_1(q).$$

4.2 The 5_2 Knot

Recall that the state integral of the 5_2 knot is

$$Z_{5_2}(\tau) = \int_{\mathbb{R}+\mathrm{i}\varepsilon} \Phi_{\sqrt{\tau}}(x)^3 \mathrm{e}^{-2\pi\mathrm{i}x^2}\,\mathrm{d}x. \quad (\tau \in \mathbb{C}')$$

Using the method of residues and extending by symmetry, it factorizes into the following form[18]

$$2\mathrm{e}^{\frac{3i\pi}{4}}\left(\frac{\tilde{q}}{q}\right)^{\frac{1}{8}} Z_{5_2}(\tau) = \tau h_2(\tau)h_0(\tau^{-1}) + 2h_1(\tau)h_1(\tau^{-1}) + \frac{1}{\tau}h_0(\tau)h_2(\tau^{-1}),$$

for $\tau \in \mathbb{C} \setminus \mathbb{R}$, where

$$h_j(\tau) = (\pm 1)^j H_j^{\pm}(\mathrm{e}^{\pm 2\pi\mathrm{i}\tau}), \quad \text{for } \pm\operatorname{Im}(\tau) > 0,$$

with q-series $H_j^{\pm}(q)$ given by

$$H_j^+(q) = \sum_{m=0}^{\infty} t_m(q)p_m^{(j)}(q), \quad H_j^-(q) = \sum_{m=0}^{\infty} T_m(q)P_m^{(j)}(q), \quad (j = 0,1,2)$$

where

$$t_m(q) = \frac{q^{m(m+1)}}{(q;q)_m^3}, \quad T_m(q) = \frac{(-1)^m q^{m(m+1)/2}}{(q;q)_m^3},$$

and

$$p_m^{(0)}(q) = 1,\ p_m^{(1)}(q) = \frac{1+3\mathcal{E}_1(q)}{4} + \sum_{j=1}^{m} \frac{2+q^j}{1-q^j},\ p_m^{(2)}(q) = p_m^{(1)}(q)^2 - \frac{3+\mathcal{E}_2(q)}{24} + \sum_{j=1}^{m} \frac{3q^j}{(1-q^j)^2},$$

$$P_m^{(0)}(q) = 1,\ P_m^{(1)}(q) = \frac{3\mathcal{E}_1(q)-1}{4} + \sum_{j=1}^{m} \frac{1+2q^j}{1-q^j},\ P_m^{(2)}(q) = P_m^{(1)}(q)^2 - \frac{\mathcal{E}_2(q)-3}{24} + \sum_{j=1}^{m} \frac{3q^j}{(1-q^j)^2}.$$

Here $\mathcal{E}_1(q)$ and $\mathcal{E}_2(q)$ are the weight 1 and weight 2 Eisenstein series as introduced in section 3.1.

Parallel to the 4_1 knot, the following observations were made[18]:

Let $\widehat{\Phi}_{5_2}$ be the following vector of series

$$\widehat{\Phi}_{5_2} := \begin{pmatrix} \widehat{\Phi}^{(5_2,\sigma_1)} \\ \widehat{\Phi}^{(5_2,\sigma_3)} \\ \widehat{\Phi}^{(5_2,\sigma_2)} \end{pmatrix},$$

where $\widehat{\Phi}^{(5_2,\sigma_1)}$ is the series for the 5_2 knot in eq. (3-8), $\widehat{\Phi}^{(5_2,\sigma_2)}$ and $\widehat{\Phi}^{(5_2,\sigma_3)}$ are two other series indexed by $\sigma_j \in \mathcal{P}_{5_2}$ where $\mathcal{P}_{5_2}$ coincides with the set of boundary parabolic $\mathrm{SL}_2(\mathbb{C})$-representations of $\pi_1(S^3 \setminus 5_2)$. A definition of $\widehat{\Phi}^{(K,\sigma_j)}$ for a knot K was given by T. Dimofte and Garoufalidis[25-26]. Let $h = \begin{pmatrix} \tau^{-1}h_0 \\ h_1 \\ \tau h_2 \end{pmatrix}$, then

Observation 5:

$$h(\tau) \sim \begin{cases} N_+\widehat{\Phi}(2\pi \mathrm{i}\tau) & \text{when } \arg(\tau) \in (0, 0.19) \\ N_-\widehat{\Phi}(2\pi \mathrm{i}\tau) & \text{when } \arg(\tau) \in \left(-\frac{\pi}{2}, 0\right) \end{cases}$$

where

$$N_+ = \begin{pmatrix} 1/2 & 1/2 & 1 \\ 0 & 1/2 & 1/2 \\ -1/12 & 5/12 & -2/3 \end{pmatrix}, \quad N_- = \begin{pmatrix} -1/2 & -1/2 & 1/2 \\ 3/4 & -1/4 & -1/4 \\ -13/12 & -1/12 & 1/12 \end{pmatrix}.$$

For the index, there is

Observation 6:

$$\mathrm{Ind}_{5_2}(q) = 2H_1^+(q)H_1^-(q).$$

Furthermore, the following quadratic relation for the q-series $H_j^{\pm}$'s was also observed

Observation 7:

$$H_0^+(q)H_2^-(q) - 2H_1^+(q)H_1^-(q) + H_2^+(q)H_0^-(q) = 0.$$

For the 4_1 knot, this could not be seen since it is trivially

$$G_0(q)G_1(q) - G_1(q)G_0(q) = 0,$$

as a consequence of that the 4_1 knot is amphichiral.

4.3 The Descendant State Integral

By adding a factor $\mathrm{e}^{2\pi\left(\lambda\tau^{1/2}-\mu\tau^{-1/2}\right)x}$ to the integrand we obtain the descendant state integral[19]. For example, the descendant state integral of the 4_1 knot is

$$Z_{4_1}^{(\lambda,\mu)}(\tau) = \int_{\mathbb{R}+\mathrm{i}\varepsilon} \Phi_{\sqrt{\tau}}(x)^2 \mathrm{e}^{-\pi \mathrm{i}x^2+2\pi\left(\lambda\tau^{1/2}-\mu\tau^{-1/2}\right)x}\,\mathrm{d}x. \quad (\lambda,\mu\in\mathbb{Z})$$

By the method of residues and the symmetry, it factorizes as the following,

$$Z_{4_1}^{(\lambda,\mu)}(\tau) = (-1)^{\lambda-\mu+1}\frac{\mathrm{i}}{2}q^{\frac{m}{2}+\frac{1}{24}}\tilde{q}^{\frac{\mu}{2}-\frac{1}{24}}\left(\sqrt{\tau}G_0^{(\mu)}(\tilde{q})G_1^{(\lambda)}(q) - \frac{1}{\sqrt{\tau}}G_1^{(\mu)}(\tilde{q})G_0^{(\lambda)}(q)\right), \tag{4-2}$$

where $G_0^{(k)}$ and $G_1^{(k)}$ are defined by

$$G_0^{(k)}(q) = \sum_{n=0}^{\infty}(-1)^n\frac{q^{\frac{n(n+1)}{2}+kn}}{(q;q)_n^2}, \quad G_1^{(k)}(q) = \left(1+2k+2n-4\sum_{s=1}^{\infty}\frac{q^{s(n+1)}}{1-q^s}\right)\sum_{n=0}^{\infty}(-1)^n\frac{q^{\frac{n(n+1)}{2}+kn}}{(q;q)_n^2},$$

for $|q|<1$ and extended to $|q|>1$ by $G_j^{(k)}(q^{-1}) = (-1)^j G_j^{(k)}(q)$. The matrix of these series,

$$w_k(q) = \begin{pmatrix} G_0^{(k)}(q) & G_1^{(k)}(q) \\ G_0^{(k+1)}(q) & G_1^{(k+1)}(q) \end{pmatrix}, \quad (|q|\neq 1)$$

satisfies the following linear q-difference equation[27]:

Theorem 4.1: The matrix $w_k(q)$ is a fundamental solution of the linear q-difference equation

$$y_{k+1}(q) - (2-q^k)y_k(q) + y_{k-1}(q) = 0 \quad (k\in\mathbb{Z}). \tag{4-3}$$

It has constant determinant

$$\det(w_k(q)) = 2, \tag{4-4}$$

and satisfies the symmetry and orthogonality properties

$$w_k(q^{-1}) = w_{-k}(q)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$
$$\frac{1}{2}w_k(q)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}w_k(q^{-1})^T = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},$$

for all integers k and for $|q| \neq 1$.

The factorization eq. (4-2), along with the orthogonal relation above, implies, since the left-hand-side is a holomorphic function on $\tau \in \mathbb{C}'$, that the matrix-valued function

$$W_{\lambda,\mu}(\tau) = (w_\mu(\tilde{q})^T)^{-1}\begin{pmatrix}1/\tau & 0\\ 0 & 1\end{pmatrix}w_\lambda(q)^T, \quad \left(q = \mathrm{e}^{2\pi\mathrm{i}\tau},\ \tilde{q} = \mathrm{e}^{-2\pi\mathrm{i}\tau^{-1}}\right) \tag{4-5}$$

which is originally defined only for $\tau \in \mathbb{C}\setminus\mathbb{R}$, extends holomorphically to $\tau \in \mathbb{C}'$ for all integers λ and μ. We remark here that this example along with theorem 4.5 for the $(-2,3,7)$ pretzel knot demonstrates a remarkable phenomenon that the descendant state integrals naturally give valuable examples of matrix-valued holomorphic quantum modular forms.

A similar story of descendants for the 5_2 knot can be found in Garoufalidis and Zagier's recent paper[18]Sec. 4.3.

In the study of the refined quantum modularity conjecture for the 4_1 knot, the following 2-by-2 matrix of asymptotic series was found by Garoufalidis and Zagier[22]

$$\widehat{\mathbf{\Phi}}_{4_1}(\hbar) = \begin{pmatrix}\widehat{\Phi}_{4_1}(\hbar) & \widehat{\Psi}_{4_1}(\hbar)\\ \mathrm{i}\widehat{\Phi}_{4_1}(\hbar) & -\mathrm{i}\widehat{\Psi}_{4_1}(\hbar)\end{pmatrix},$$

where $\widehat{\Psi}_{4_1}(\hbar) = \mathrm{e}^{\frac{\mathrm{i}V(4_1)}{\hbar}}\Psi^{(4_1)}(\hbar)$ and $\Psi^{(4_1)}(\hbar)$ is a power series in $\hbar$ with similar properties as $\Phi^{(4_1)}(\hbar)$. Let $Q(\tau)$ be the following matrix of linear combinations of $G_j^{(k)}$'s,

$$Q(\tau) = w_0(q)^T\begin{pmatrix}1 & -\frac{1}{2}\\ 0 & 1\end{pmatrix} \overset{eq.\ (4\text{-}3)}{=\!=\!=} \begin{pmatrix}G_0^{(0)}(q) & \frac{1}{2}\left(G_0^{(1)}(q) - G_0^{(-1)}(q)\right)\\ G_1^{(0)}(q) & \frac{1}{2}\left(G_1^{(1)}(q) - G_1^{(-1)}(q)\right)\end{pmatrix},$$

t hen

Observation 8: As $\tau \to 0$ in the upper half-plane, we have:

$$\begin{pmatrix}1/\sqrt{\tau} & 0\\ 0 & \sqrt{\tau}\end{pmatrix}Q(\tau) \sim \begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\widehat{\mathbf{\Phi}}_{4_1}(2\pi\mathrm{i}\tau).$$

As a consequence, by eq. (4-4) that $\det(Q(\tau)) = 2$ for all τ, it follows that

$$\det(\widehat{\mathbf{\Phi}}_{4_1}(\hbar)) = 1,$$

and[18]

$$\widehat{\mathbf{\Phi}}_{4_1}(-\hbar)\widehat{\mathbf{\Phi}}_{4_1}(\hbar)^T = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}.$$

4.4 The $(-2,3,7)$ Pretzel Knot

For the $(-2,3,7)$ pretzel knot, the factorization of the state integral involves 6 pairs of q-series, and some of them are power series in integer powers of $q^{1/2}$, which is different from the case of the 4_1 knot and the 5_2 knot. This new phenomenon is formulated by Garoufalidis and Zagier as the level of knots, and $(-2,3,7)$ is said to have level $N = 2$. Writing the 6 pairs of q-series as $H_j^{\pm}(q)$ for $j = 0,1,\cdots,5$, Garoufalidis and Zagier found the following[18]:

Observation 9: The relation with the index is given by

$$\mathrm{Ind}_{(-2,3,7)}(q) = H_1^+(q)H_1^-(q),$$

and the following quadratic relation holds:

$$\frac{1}{2}H_0^+(q)H_2^-(q) - H_1^+(q)H_1^-(q) + \frac{1}{2}H_2^+(q)H_0^-(q) - H_3^+(q)H_3^-(q) + H_4^+(q)H_4^-(q) - H_5^+(q)H_5^-(q) = 0.$$

Since the $(-2,3,7)$ pretzel knot has 6 boundary parabolic $\mathrm{SL}_2(\mathbb{C})$ representations, there are 6 series $\{\widehat{\Phi}_\alpha^{(\sigma_i)}(\hbar)\}_{j=1}^6$. Similar to the case of the 4_1 knot and the 5_2 knot, consider the vector of asymptotic series corresponding to the $(-2,3,7)$ pretzel knot $\widehat{\Phi}_\alpha(\hbar) \coloneqq \left(\widehat{\Phi}_\alpha^{(\sigma_i)}(\hbar)\right)_{j=1}^6$ and the vector of holomorphic functions $h(\tau) \coloneqq (h_j(\tau))_{j=1}^6$ with weight $(-1,0,1,-1,-1,-1)$, where $h_j(\tau) = (\pm 1)^{\delta_j} H_j^{\pm}(\mathrm{e}^{\pm 2\pi\mathrm{i}\tau})$ for $\pm\,\mathrm{Im}(\tau) > 0$ respectively with $\delta = (0,1,2,0,0,0)$, then

Observation 10: For any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, as $X \in \mathbb{C} \setminus \mathbb{R}$ in a sector near the positive real axis and $X \to \infty$, we have:

$$h|_\gamma(X) \sim \rho(\gamma) \begin{pmatrix} 0 & 1 & -1 & 0 & -1 & -1/2 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2/3 & -2/3 & 0 & 4/3 & 1/6 \\ 0 & -1 & 1 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & -1/2 & -1 & 0 \\ 2 & 0 & 0 & -1/2 & -1 & 0 \end{pmatrix} \widehat{\Phi}_{a/c}\left(\frac{2\pi\mathrm{i}}{cX+d}\right)$$

to all orders in $1/X$, where ρ is a complex representation of $\mathrm{SL}_2(\mathbb{Z})$.

Note that since some of $H_j^{\pm}(q)$ are power series in $q^{1/2}$, here $h_j(\tau)$ are 2-periodic, instead of 1-periodic as in the case of the 4_1 knot and the 5_2 knot.

The following two subsections present the newly obtained results on the descendant state integral of the $(-2, 3, 7)$ pretzel knot, with an emphasis on the algebraic nature of these objects, involving the following aspects:

(a) parallel to eq. (4-2), the factorization of the descendant state integral defines a 6×6 matrix of (deformed) q-hypergeometric series; see theorem 4.2.

(b) parallel to theorem 4.1, the matrix is a fundamental solution of a self-dual linear q-difference equation; see theorems 4.3 and 4.4.

(c) parallel to eq. (4-5), the corresponding cocycle is a holomorphic function that extends from $\tau \in \mathbb{C} \setminus \mathbb{R}$ the cut-plane $\mathbb{C}' = \mathbb{C} \setminus (-\infty, 0]$; see theorem 4.5.

(d) parallel to observation 8, the stationary phase of the descendant state integral determines a 6×6 matrix of asymptotic series, which is related to the q-series given by the factorization; see theorem 4.5 and eq. (4-60).

Moreover, we will present elementary proofs for theorems 4.3 and 4.4 and outline the computation for the stationary phase in section 4.4.2. These are joint work of the author with N. An and S. Garoufalidis.

4.4.1 Factorization of the descendant state integral

The descendant state integral of $(-2, 3, 7)$ pretzel knot is

$$Z^{(\lambda,\lambda')}_{(-2,3,7)}(\tau) = \int_{\mathbb{R}+\mathrm{i}\frac{c_b}{2}+\mathrm{i}\varepsilon} \Phi_{\sqrt{\tau}}(x)^2 \Phi_{\sqrt{\tau}}(2x - c_b) \mathrm{e}^{-\pi\mathrm{i}(2x-c_b)^2+2\pi(\lambda b-\lambda' b^{-1})x}\, \mathrm{d}x, \tag{4-6}$$

where $\lambda, \lambda' \in \mathbb{Z}$, $\tau = b^2$, $\sqrt{\tau} = b$ and $c_b = \mathrm{i}(b + b^{-1})/2$.

Theorem 4.2: We have:

$$\begin{aligned}
&2\mathrm{e}^{\frac{\pi\mathrm{i}}{4}} \left(q^{\frac{\lambda}{2}}\tilde{q}^{\frac{\lambda'}{2}}\right)^{-1} Z^{(\lambda,\lambda')}_{(-2,3,7)}(\tau)\\
&\qquad = -\frac{1}{2\tau}h_0(\lambda, \tau)h_2(\lambda', \tau^{-1}) + h_1(\lambda, \tau)h_1(\lambda', \tau^{-1}) - \frac{\tau}{2}h_2(\lambda, \tau)h_0(\lambda', \tau^{-1})\\
&\qquad\quad - \mathrm{i}\left(\frac{1}{2}h_3(\lambda, \tau)h_4(\lambda', \tau^{-1}) - \frac{1}{2}h_4(\lambda, \tau)h_3(\lambda', \tau^{-1}) + h_5(\lambda, \tau)h_5(\mu, \tau^{-1})\right).
\end{aligned}$$

In the above theorem

$$h_{\lambda,j}(\tau) \coloneqq H_{\lambda,j}(\mathrm{e}^{2\pi\mathrm{i}\tau}), \quad H_{\lambda,j}(q) = \begin{cases} H^+_{\lambda,j}(q) & \text{if } |q| < 1\\ (-1)^{\delta_j} H^-_{-\lambda,j}(q^{-1}) & \text{if } |q| > 1 \end{cases} \tag{4-7}$$

where $H_{\lambda,j}(q)$ are q-series defined as the following: Recall that $\mathcal{E}_1(q)$ and $\mathcal{E}_2(q)$ denote the Eisenstein series of weights 1 and 2

$$\mathcal{E}_1(q) = 1 - 4\sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)}, \qquad \mathcal{E}_2(q) = 1 - 24\sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} \tag{4-8}$$

and

$$E_l^{(m)}(q) = \sum_{s=1}^{\infty} \frac{s^{l-1} q^{s(m+1)}}{1-q^s} \tag{4-9}$$

are some series that appear in the factorization of one-dimensional state integrals[9].

$$H^+_{\lambda,j}(q) \;=\; (-1)^{\lambda} \sum_{m=0}^{\infty} t_{\lambda,m}(q) p_{\lambda,j,m}(q), \quad H^-_{\lambda',j}(q) \;=\; (-1)^{\lambda'} \sum_{n=0}^{\infty} T_{\lambda',n} P_{\lambda',j,n}(q), \tag{4-10}$$

with

$$t_{\lambda,m}(q) = \frac{q^{m(2m+1)+\lambda m}}{(q;q)_m^2 (q;q)_{2m}}, \quad T_{\lambda',n}(q) = \frac{q^{n(n+1)+\lambda' n}}{(q;q)_n^2 (q;q)_{2n}}, \tag{4-11}$$

and

$$\begin{aligned}
&p_{\lambda,0,m}(q) = 1, \quad p_{\lambda,1,m}(q) = 4m + \lambda + 1 - 2E_1^{(m)}(q) - 2E_1^{(2m)}(q),\\
&p_{\lambda,2,m}(q) = p_{\lambda,1,m}(q)^2 - 2E_2^{(m)}(q) - 4E_2^{(2m)}(q) - \frac{1}{3}\mathcal{E}_2(q),\\
&P_{\lambda',0,n}(q) = 1, \quad P_{\lambda',1,n}(q) = 2n + \lambda' + 1 - 2E_1^{(n)}(q) - 2E_1^{(2n)}(q),\\
&P_{\lambda',2,n}(q) = P_{\lambda',1,n}(q)^2 + 12E_2^{(0)}(q) - \frac{1}{2} - 2E_2^{(n)}(q) - 4E_2^{(2n)}(q) + \frac{1}{3}\mathcal{E}_2(q),
\end{aligned} \tag{4-12}$$

and for $j = 3, 4, 5$ by:

$$\begin{aligned}
H^+_{\lambda,3}(q) &= \frac{(-1)^{\lambda} q^{1/8}}{(1-q^{1/2})^2} \sum_{m=0}^{\infty} \frac{q^{(2m+1)(m+1)+\lambda(m+1/2)}}{(q^{3/2};q)_m^2 (q;q)_{2m+1}}, & H^-_{\lambda',4}(q) &= \sum_{n=0}^{\infty} \frac{q^{n(n+1)+\lambda' n}}{(-q;q)_n^2 (q;q)_{2n}},\\
H^+_{\lambda,4}(q) &= \sum_{m=0}^{\infty} \frac{q^{(2m+1)m+\lambda m}}{(-q;q)_m^2 (q;q)_{2m}}, & H^-_{\lambda',3}(q) &= \frac{(-1)^{\lambda'} q^{-1/8}}{(1-q^{-1/2})^2} \sum_{n=0}^{\infty} \frac{q^{n(n+2)+\lambda'(n+1/2)}}{(q^{3/2};q)_n^2 (q;q)_{2n+1}},\\
H^+_{\lambda,5}(q) &= \frac{q^{1/8}}{(1+q^{1/2})^2} \sum_{m=0}^{\infty} \frac{q^{(2m+1)(m+1)+\lambda(m+1/2)}}{(-q^{3/2};q)_m^2 (q;q)_{2m+1}}, & H^-_{\lambda',5}(q) &= \frac{q^{-1/8}}{(1+q^{-1/2})^2} \sum_{n=0}^{\infty} \frac{q^{n(n+2)+\lambda'(n+1/2)}}{(-q^{3/2};q)_n^2 (q;q)_{2n+1}}.
\end{aligned} \tag{4-13}$$

When $(\lambda, \mu) = (0, 0)$, this factorization can be connected to that in Garoufalidis and

Zagier's[18]eq. (50) using the following identities:[23]Appx. A

$$\begin{aligned}\frac{(q^{3/2};q)_\infty^2}{(q;q)_\infty^2}\frac{(\tilde{q};\tilde{q})_\infty^2}{(-1;\tilde{q})_\infty^2} &= \frac{\mathrm{e}^{-\frac{\pi\mathrm{i}}{2}}q^{1/8}}{2(1-q^{1/2})^2}\tau,\\ \frac{(-q;q)_\infty^2}{(q;q)_\infty^2}\frac{(\tilde{q};\tilde{q})_\infty^2}{(-\tilde{q}^{-1/2};\tilde{q})_\infty^2} &= \frac{\mathrm{e}^{-\frac{\pi\mathrm{i}}{2}}\tilde{q}^{-1/8}}{2(1-\tilde{q}^{-1/2})^2}\tau,\\ \frac{(-q^{3/2};q)_\infty^2}{(q;q)_\infty^2}\frac{(\tilde{q};\tilde{q})_\infty^2}{(-\tilde{q}^{-1/2};\tilde{q})_\infty^2} &= \frac{\mathrm{e}^{-\frac{\pi\mathrm{i}}{2}}q^{1/8}\tilde{q}^{-1/8}}{(1+q^{1/2})^2(1+\tilde{q}^{-1/2})^2}\tau.\end{aligned} \tag{4-14}$$

The first few terms of $H^{\pm}_{\lambda,j}(q)$ are given by

$$\begin{aligned}
H^+_{0,0}(q) &= 1+q^3+3q^4+7q^5+13q^6+\cdots & H^-_{0,0}(q) &= 1+q^2+3q^3+7q^4+13q^5+\cdots\\
H^+_{0,1}(q) &= 1-4q-8q^2-3q^3+3q^4+\cdots & H^-_{0,1}(q) &= 1-4q-5q^2+q^3+7q^4+\cdots\\
H^+_{0,2}(q) &= \frac{2}{3}-6q+6q^2+\frac{242}{3}q^3+200q^4+\cdots & H^-_{0,2}(q) &= \frac{5}{6}-10q+\frac{17}{6}q^2+\frac{141}{2}q^3+\frac{971}{6}q^4+\cdots\\
H^+_{0,3}(q) &= q^{1/8}(q+2q^{3/2}+4q^2+6q^{5/2}+\cdots) & H^-_{0,3}(q) &= q^{-1/8}(q+2q^{3/2}+4q^2+6q^{5/2}+\cdots)\\
H^+_{0,4}(q) &= 1+q^3-q^4+3q^5-3q^6+\cdots & H^-_{0,4}(q) &= 1+q^2-q^3+3q^4-3q^5+\cdots\\
H^+_{0,5}(q) &= q^{1/8}(q-2q^{3/2}+4q^2-6q^{5/2}+\cdots) & H^-_{0,5}(q) &= q^{-1/8}(q-2q^{3/2}+4q^2-6q^{5/2}+\cdots)
\end{aligned} \tag{4-15}$$

Theorem 4.3: For each $j = 0,\ldots,5$, the sequence $H_{\lambda,j}(q)$ for $|q| \neq 1$ and $\lambda \in \mathbb{Z}$ satisfies the linear q-difference equation

$$\begin{aligned}y_{\lambda+6}(q)+2\,y_{\lambda+5}(q)-(q+q^{\lambda+4})\,y_{\lambda+4}(q)-2(q+1)\,y_{\lambda+3}(q)\\ -\,y_{\lambda+2}(q)+2q\,y_{\lambda+1}(q)+q\,y_\lambda(q)=0\,.\end{aligned} \tag{4-16}$$

Proof: We begin with the case $j = 0$ and $|q| < 1$, hence

$$H_{\lambda,0}(q) = H^+_{\lambda,0}(q) = (-1)^\lambda\sum_{m=0}^{\infty}\frac{q^{m(2m+1)+\lambda m}}{(q;q)_m^2(q;q)_{2m}}\,.$$

Since $(q;q)_m = \prod_{i=1}^m(1-q^i)$, we have

$$\begin{aligned}q^\lambda\sum_{m=0}^{\infty}\frac{q^{m(2m+1)+\lambda m}}{(q;q)_m^2(q;q)_{2m}} &= \sum_{m=0}^{\infty}\frac{q^{m(2m+1)+\lambda(m+1)}}{(q;q)_m^2(q;q)_{2m}} = \sum_{m=1}^{\infty}\frac{q^{(m-1)(2m-1)+\lambda m}}{(q;q)_{m-1}^2(q;q)_{2m-2}}\\ &= \sum_{m=1}^{\infty}\frac{q^{m(2m+1)+\lambda m}}{(q;q)_m^2(q;q)_{2m}}\frac{(1-q^m)^2(1-q^{2m-1})(1-q^{2m})}{q^{4m-1}}\,.\end{aligned}$$

Since $1-q^m = 0$ when $m = 0$, we can replace the summation in the above equation from $m = 0$ to $m = \infty$. Since

$$\frac{(1-q^m)^2(1-q^{2m-1})(1-q^{2m})}{q^{4m-1}} = q^{1-4m}-2q^{1-3m}-q^{-2m}+2q^{1-m}+2q^{-m}-q-2q^m+q^{2m}, \tag{4-17}$$

we obtain

$$\begin{aligned} q^{\lambda}H^{+}_{\lambda,0}(q) =&(-1)^{\lambda}\sum_{m=0}^{\infty}\frac{q^{m(2m+1)+\lambda m}}{(q;q)_m^2(q;q)_{2m}}(q^{1-4m}-2q^{1-3m}-q^{-2m}+2q^{1-m}+2q^{-m}-q-2q^m+q^{2m})\\ =&qH^{+}_{\lambda-4,0}(q)+2qH^{+}_{\lambda-3,0}(q)-H^{+}_{\lambda-2,0}(q)-(2+2q)H^{+}_{\lambda-1,0}(q)-qH^{+}_{\lambda,0}(q)+2H^{+}_{\lambda+1,0}(q)+H^{+}_{\lambda+2,0}(q)\,. \end{aligned}$$

This gives the q-difference equation for $H_{\lambda,j}(q)$ when $j=0$ and $|q|<1$. Similarly one proves the q-difference equation for the cases $j=0,3,4,5$ and whenever $|q|\neq 1$.

For $j=1$ and $|q|<1$, we have

$$H_{\lambda,1}(q)=H^{+}_{\lambda,1}(q)=(-1)^{\lambda}\sum_{m=0}^{\infty}\frac{q^{m(2m+1)+\lambda m}}{(q;q)_m^2(q;q)_{2m}}p^{(1)}_{\lambda,m}(q)\,,$$

where

$$p_{\lambda,1,m}(q)=4m+\lambda+1-2E_1^{(m)}(q)-2E_1^{(2m)}(q).$$

Hence

$$\begin{aligned} qH^{+}_{\lambda-4,1}(q)+2qH^{+}_{\lambda-3,1}(q)-H^{+}_{\lambda-2,1}(q)-(2+2q)H^{+}_{\lambda-1,1}(q)-qH^{+}_{\lambda,1}(q)+2H^{+}_{\lambda+1,1}(q)+H^{+}_{\lambda+2,1}(q)\\ =(-1)^{\lambda}\sum_{m=0}^{\infty}\frac{q^{m(2m+1)+\lambda m}}{(q;q)_m^2(q;q)_{2m}}g_{\lambda,m}(q), \end{aligned}$$

where

$$\begin{aligned} g_{\lambda,m}(q) =&q^{1-4m}p_{\lambda-4,1,m}(q)-2q^{1-3m}p_{\lambda-3,1,m}(q)-q^{-2m}p_{\lambda-2,1,m}(q)\\ &+(2+2q)q^{-m}p_{\lambda-1,1,m}(q)-qp_{\lambda,1,m}(q)-2q^{m}p_{\lambda+1,1,m}(q)+q^{2m}p_{\lambda+2,1,m}(q). \end{aligned} \tag{4-18}$$

We are going to show that $(-1)^{\lambda}\sum_{m=0}^{\infty}\frac{q^{m(2m+1)+\lambda m}}{(q;q)_m^2(q;q)_{2m}}g_{\lambda,m}(q) = q^{\lambda}H^{+}_{\lambda,1}(q)$. Noticing the recursive relation that

$$E_1^{(m)}(q)-E_1^{(m-1)}(q)=\sum_{s=1}^{\infty}\left(\frac{q^{s(m+1)}}{1-q^s}-\frac{q^{sm}}{1-q^s}\right)=\sum_{s=1}^{\infty}-q^{sm}=-\frac{q^m}{1-q^m}, \tag{4-19}$$

we convert $p_{\lambda,1,m}(q)$ into the following form

$$\begin{aligned} p_{\lambda,1,m}(q) =&4m+\lambda+1-2E_1^{(m-1)}(q)-2E_1^{(2m-2)}(q)+\frac{2q^m}{1-q^m}+\frac{2q^{2m-1}}{1-q^{2m-1}}+\frac{2q^{2m}}{1-q^{2m}}\\ =&4(m-1)+\lambda+1-2E_1^{(m-1)}(q)-2E_1^{(2m-2)}(q)+f_{1,m}(q)+4\\ =&p_{\lambda,1,m-1}(q)+f_{1,m}(q)+4, \end{aligned} \tag{4-20}$$

where

$$f_{1,m}(q) \coloneqq \frac{2q^m}{1-q^m}+\frac{2q^{2m-1}}{1-q^{2m-1}}+\frac{2q^{2m}}{1-q^{2m}}. \tag{4-21}$$

Substituting the (4-20) into (4-18), combining the common factors $p_{\lambda,1,m-1}(q)+f_{1,m}(q)$

and applying the identity (4-17), we see that

$$\begin{aligned} g_{\lambda,m}(q) =& \frac{(1-q^m)^2(1-q^{2m-1})(1-q^{2m})}{q^{4m-1}} \left(p_{\lambda,1,m-1}(q) + f_{1,m}(q)\right) \\ & - \left(2q^{1-3m} + 2q^{-2m} - 6(q+1)q^{-m} + 4q + 10q^m - 6q^{2m}\right). \end{aligned}$$

Since

$$\frac{(1-q^m)^2(1-q^{2m-1})(1-q^{2m})}{q^{4m-1}} f_{1,m}(q) = 2q^{1-3m} + 2q^{-2m} - 6(q+1)q^{-m} + 4q + 10q^m - 6q^{2m},$$

we conclude that

$$g_{\lambda,m}(q) = \frac{(1-q^m)^2(1-q^{2m-1})(1-q^{2m})}{q^{4m-1}} p_{\lambda,1,m-1}(q).$$

Therefore

$$\begin{aligned} (-1)^\lambda \sum_{m=0}^\infty \frac{q^{m(2m+1)+\lambda m}}{(q;q)_m^2 (q;q)_{2m}} g_{\lambda,m}(q) =& (-1)^\lambda \sum_{m=1}^\infty \frac{q^{(m-1)(2m-1)+\lambda m}}{(q;q)_{m-1}^2 (q;q)_{2m-2}} p_{\lambda,1,m-1}(q) \\ =& (-1)^\lambda q^\lambda \sum_{m=0}^\infty \frac{q^{m(2m+1)+\lambda m}}{(q;q)_m^2 (q;q)_{2m}} p_{\lambda,1,m}(q) = q^\lambda H^+_{\lambda,1}(q), \end{aligned}$$

as desired. Similarly one proves the q-difference equation for $j = 1, 2$ and $|q| \neq 1$, using the recursive relation (4-19) and

$$E_2^{(m)}(q) - E_2^{(m-1)}(q) = \sum_{s=1}^\infty \left(\frac{sq^{s(m+1)}}{1-q^s} - \frac{sq^{sm}}{1-q^s} \right) = \sum_{s=1}^\infty -sq^{sm} = -\frac{q^m}{(1-q^m)^2}. \tag{4-22}$$

This completes the proof. ∎

Consider the Wronskian

$$W_\lambda(q) = \left(H_{\lambda+i,j}(q)\right)_{0\le i,j\le 5} \qquad |q| \neq 1 \tag{4-23}$$

of the six solutions to the q-difference equation (4-16). We next give an orthogonality property of the Wronskian, which implies that the six sequences of q-series form a fundamental solution set of (4-16) and satisfy quadratic relations.

Theorem 4.4: The determinant of the Wronskian is given by

$$\det(W_\lambda(q)) = 32q^{\lambda+\frac{11}{4}}. \tag{4-24}$$

The Wronskian satisfies the orthogonality property

$$W_\lambda(q)\begin{pmatrix}0&0&\frac{1}{2}&0&0&0\\0&1&0&0&0&0\\\frac{1}{2}&0&0&0&0&0\\0&0&0&-1&0&0\\0&0&0&0&\frac{1}{4}&0\\0&0&0&0&0&-1\end{pmatrix}W_{-\lambda-5}(q^{-1})^T=\begin{pmatrix}-12&8&-4&2&0&0\\8&-4&2&0&0&0\\-4&2&0&0&0&2\\2&0&0&0&2&-4\\0&0&0&2&-4&8+2q^{\lambda+2}\\0&0&2&-4&8+2q^{\lambda+3}&-12-4q^{\lambda+2}-4q^{\lambda+3}\end{pmatrix}. \tag{4-25}$$

A consequence of eq. (4-25) (in fact, of its $(1,6)$-entry) is that the collection of q-series $H^{\pm}_{\lambda,j}(q)$ satisfies the quadratic relation

$$\begin{aligned}&\frac{1}{2}H^+_{\lambda,0}(q)H^-_{\lambda,2}(q)-H^+_{\lambda,1}(q)H^-_{\lambda,1}(q)+\frac{1}{2}H^+_{\lambda,2}(q)H^-_{\lambda,0}(q)\\&\qquad-H^+_{\lambda,3}(q)H^-_{\lambda,3}(q)+\frac{1}{4}H^+_{\lambda,4}(q)H^-_{\lambda,4}(q)-H^+_{\lambda,5}(q)H^-_{\lambda,5}(q)=0\,.\end{aligned} \tag{4-26}$$

Proof: We assume $|q|<1$ and give the proof for this case only; the proof for $|q|>1$ is similar and is omitted. The method can be used to give a systematic proof of the self-duality properties of the q-holonomic modules that appear in the refined quantum modularity conjecture of knot complements or of closed 3-manifolds.

We first compute the determinant of the Wronskian $W_\lambda(q)$. It is well-known that it satisfies the first order linear q-difference equation[28]Lemma 4.7

$$\det(W_{\lambda+1}(q))-q\det(W_\lambda(q))=0\,.$$

It follows that $\det(W_\lambda(q))=q^\lambda c(q)$ for some q-series $c(q)$ independent of λ. We claim that

$$\det(W_\lambda(q))=32q^{\lambda+11/4}+O(q^{3\lambda/2}), \tag{4-27}$$

for all sufficiently large natural numbers λ, which implies that $c(q)=32q^{11/4}$. To show eq. (4-27), recall that $W_\lambda(q)=\left(H^+_{\lambda+i,j}(q)\right)_{0\le i,j\le 5}$ when $|q|<1$. The definition of $H^+_{\lambda,j}(q)$ implies that

$$H^{\pm}_{\lambda,j}(q)=R^{\pm}_{\lambda,j}(q)+O(q^{3\lambda/2}), \tag{4-28}$$

where $R^+_{\lambda,j}(q)$ and $R^-_{\lambda,j}(q)$ are given by

$$
\begin{aligned}
R^+_{\lambda,j}(q) &= (-1)^\lambda \left(p_{\lambda,j,0}(q) + p_{\lambda,j,1} \frac{q^{\lambda+3}}{(1-q)^4(1+q)} \right), \quad j = 0,1,2, \\
R^+_{\lambda,3}(q) &= (-1)^\lambda \frac{q^{1/8}}{(1-q^{1/2})^2} \frac{q^{1+\lambda/2}}{1-q}, \\
R^+_{\lambda,4}(q) &= 1 + \frac{q^{\lambda+3}}{(1+q)^3(1-q)^2}, \\
R^+_{\lambda,5}(q) &= \frac{q^{1/8}}{(1+q^{1/2})^2} \frac{q^{1+\lambda/2}}{1-q},
\end{aligned}
\tag{4-29}
$$

and

$$
\begin{aligned}
R^-_{\lambda,j}(q) &= (-1)^\lambda \left(P_{\lambda,j,0}(q) + P_{\lambda,j,1}(q) \frac{q^{\lambda+2}}{(1-q)^4(1+q)} \right), \quad j = 0,1,2, \\
R^-_{\lambda,3}(q) &= (-1)^\lambda \frac{q^{-1/8}}{(1-q^{-1/2})^2} \frac{q^{\lambda/2}}{1-q}, \\
R^-_{\lambda,4}(q) &= 1 + \frac{q^{\lambda+2}}{(1+q)^3(1-q)^2}, \\
R^-_{\lambda,5}(q) &= \frac{q^{-1/8}}{(1+q^{-1/2})^2} \frac{q^{\lambda/2}}{1-q}.
\end{aligned}
\tag{4-30}
$$

Thus,

$$W_\lambda(q) = R_\lambda(q) + O(q^{3\lambda/2}), \tag{4-31}$$

where $R_\lambda(q) = (R_{\lambda+i,j}(q))_{0\le i,j\le 5}$. Since

$$\det(W_\lambda(q)) + O(q^{3\lambda/2}) = \det(R_\lambda(q)) + O(q^{3\lambda/2}) = 32q^{\lambda+11/4} + O(q^{3\lambda/2}), \tag{4-32}$$

eq. (4-27) follows. It is noteworthy that the Eisenstein series $\mathcal{E}_2(q)$ which appear in the entries of $R_\lambda(q)$ cancel upon taking the determinant. The same happens in the entries of the matrix (4-40) below.

This concludes the proof of (4-24). We next prove the orthogonality property (4-25) following the method of[29]Sec 2.5. By the q-difference equation (4-26), we have

$$W_{\lambda+1}(q) = A(\lambda,q)W_\lambda(q), \quad W_{-\lambda-1}(q^{-1}) = \tilde{A}(\lambda,q)W_{-\lambda}(q^{-1}), \tag{4-33}$$

where

$$A(\lambda, q) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -q & -2q & 1 & 2(1+q) & q+q^{\lambda+4} & -2 \end{pmatrix} \tag{4-34}$$

and

$$\tilde{A}(\lambda, q) = A(-\lambda-1, q^{-1})^{-1} = \begin{pmatrix} -2 & q & 2(1+q) & 1+q^{\lambda-2} & -2q & -q \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Consider

$$Q(\lambda, q) := \begin{pmatrix} -12 & 8 & -4 & 2 & 0 & 0 \\ 8 & -4 & 2 & 0 & 0 & 0 \\ -4 & 2 & 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 & 2 & -4 \\ 0 & 0 & 0 & 2 & -4 & 8+2q^{\lambda+2} \\ 0 & 0 & 2 & -4 & 8+2q^{\lambda+3} & -12-4q^{\lambda+2}-4q^{\lambda+3} \end{pmatrix}.$$

It is easy to see that the matrices A, Q and $\tilde{A}$ (all with entries in the polynomial ring $\mathbb{Q}[q^{\pm 1}, q^{\pm\lambda}]$) satisfy

$$A(\lambda, q)Q(\lambda, q)\tilde{A}(\lambda+5, q) = Q(\lambda+1, q). \tag{4-35}$$

Note that all matrices above are invertible, with determinants

$$\det(A(\lambda, q) = q, \qquad \det(\tilde{A}(\lambda, q) = q, \qquad \det(Q(\lambda, q) = -64q^{5+2\lambda}. \tag{4-36}$$

Using (4-33) and (4-35), we see that

$$W_{\lambda+1}(q)^{-1}Q(\lambda+1, q)\left(W_{-\lambda-6}(q^{-1})^{-1}\right)^T = W_{\lambda}(q)^{-1}Q(\lambda, q)\left(W_{-\lambda-5}(q^{-1})^{-1}\right)^T,$$

hence $W_\lambda(q)^{-1}Q(\lambda,q)\left(W_{-\lambda-5}(q^{-1})^{-1}\right)^T$ is independent of λ. The claim is that we have

$$W_\lambda(q)^{-1}Q(\lambda,q)\left(W_{-\lambda-5}(q^{-1})^{-1}\right)^T = D := \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{4-37}$$

Since we have seen that the left-hand side of (4-37) is independent of λ, it suffices to show that

$$W_\lambda(q)^{-1}Q(\lambda,q)\left(W_{-\lambda-5}(q^{-1})^{-1}\right)^T = D + O(q^{\lambda/2}), \tag{4-38}$$

for any sufficiently large $\lambda \in \mathbb{N}$. Equation (4-31), together with (4-24) gives that

$$W_\lambda(q)^{-1}Q(\lambda,q)\left(W_{-\lambda-5}(q^{-1})^{-1}\right)^T + O(q^{\lambda/2}) = R_\lambda(q)^{-1}Q(\lambda,q)\left(R_{-\lambda-5}(q^{-1})^{-1}\right)^T + O(q^{\lambda/2}) \tag{4-39}$$

and an explicit calculation shows that

$$R_\lambda(q)^{-1}Q(\lambda,q)\left(R_{-\lambda-5}(q^{-1})^{-1}\right)^T + O(q^{\lambda/2}) = D + O(q^{\lambda/2}) \tag{4-40}$$

where $R_{-\lambda-5}(q^{-1})^{-1} + O(q^{\lambda/2})$ can be computed by multiplying the adjugate of $R_{-\lambda-5}(q^{-1}) + O(q^{\lambda/2})$ with the inverse of its determinant (4-31). Equation (4-38) follows. ∎

The following theorem follows directly from theorem 4.2 and eq. (4-25), but states a highly non-trivial result for matrix-valued holomorphic quantum modular form.

Theorem 4.5: (a) The matrix-valued function

$$F_{\lambda,\lambda'}(\tau) := W_{-\lambda'-5}(\tilde{q}^{-1})\begin{pmatrix} 0 & 0 & -\frac{\tau}{2} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2\tau} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\mathrm{i}}{2} & 0 \\ 0 & 0 & 0 & -\frac{\mathrm{i}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mathrm{i} \end{pmatrix} W_\lambda(q)^T \tag{4-41}$$

defined for $\tau = b^2 \in \mathbb{C}\backslash\mathbb{R}$, has entries given by the descendant state integrals up to a prefactor given by theorem 4.2, and therefore extends to a holomorphic function on the cut plane $\mathbb{C}'$.

(b) The matrix-valued function

$$W_{\lambda,\lambda'}(\tau) := \left(W_{\lambda'}(\tilde{q})^T\right)^{-1} \begin{pmatrix} -\frac{1}{\tau} & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\tau & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\mathrm{i}}{2} & 0 \\ 0 & 0 & 0 & -2\mathrm{i} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathrm{i} \end{pmatrix} W_{\lambda}(q)^T \tag{4-42}$$

extends to a holomorphic function of $\tau \in \mathbb{C}'$.

4.4.2 Stationary phase of the descendant state integral

The stationary phase is a well-known method of asymptotic analysis that can be found in many classic books[30-31]. For convenience, we define a renormalized version of the descendant state integral eq. (4-6) given by

$$\hat{Z}^{(\lambda,\lambda')}_{(-2,3,7)}(\tau) \;=\; (\tilde{q}/q)^{\frac{1}{24}} Z^{(\lambda,\lambda')}_{(-2,3,7)}(\tau)\,. \tag{4-43}$$

We will determine the asymptotic expansion of $\hat{Z}^{(\lambda,\lambda')}_{(-2,3,7)}(\tau)$ as $\hbar := 2\pi\mathrm{i}\tau \to 0$ along rays in the upper half plane (i.e., $\arg(\tau) = \theta \in (-\pi, \pi)$ is fixed) in this subsection.

It turns out that there are 6 critical points α

$$(\alpha^3 - \alpha - 1)(\alpha^3 + 2\alpha^2 - \alpha - 1) = 0 \tag{4-44}$$

in two Galois orbits of the cubic number fields with discriminants -23 and 49, respectively. After a change of parametrization of these number fields (to match with the conventions of[22], these critical points are given by

$$\alpha = -\xi + \xi^2, \qquad \xi^3 - \xi^2 + 1 = 0 \tag{4-45a}$$

$$\alpha = -1 - \eta, \qquad \eta^3 + \eta^2 - 2\eta - 1 = 0\,. \tag{4-45b}$$

The next theorem computes the stationary phase expansion of $\hat{Z}^{(\lambda,\lambda')}_{(-2,3,7)}(\tau)$ at each critical point.

Theorem 4.6: The stationary phase of $\hat{Z}^{(\lambda,\lambda')}_{(-2,3,7)}(\tau)$ is given by $\mathrm{e}^{\frac{2\pi\mathrm{i}\lambda'\log\alpha}{\hbar}}\hat{\Phi}^{(\alpha)}(\lambda,\hbar)$, where

$$\hat{\Phi}^{(\alpha)}(\lambda,\hbar) \;=\; \mathrm{e}^{\frac{V_{0,0}(\alpha)}{\hbar}}\Phi^{(\alpha)}(\lambda,\hbar), \qquad \Phi^{(\alpha)}(\lambda,\hbar) \;=\; \frac{\alpha^\lambda}{\sqrt{\mathrm{i}\Delta(\alpha)}}\sum_{k=0}^{\infty} c_k(\alpha,\lambda)\hbar^k \tag{4-46}$$

and

$$\begin{aligned} V_{0,0}(\alpha) &= 2\mathrm{Li}_2(-\alpha) - \mathrm{Li}_2(\alpha^{-2}), \\ \Delta(\alpha) &= -2\alpha^5 + 12\alpha^3 - 2\alpha^2 - 16\alpha - 10\,, \end{aligned} \tag{4-47}$$

and $c_k(\alpha, \lambda) \in \mathbb{Q}(\alpha)[\lambda]$ are polynomials in λ of degree $2k$ with coefficients in $\mathbb{Q}(\alpha)$ with $c_0(\alpha, \lambda) = 1$ given explicitly by a formal Gaussian integration.

The first few terms of the asymptotic series are given below. Since there are two number fields involved, we present the asymptotic series $\hat{\Phi}^{(\sigma)}(\lambda, \hbar)$ separately for each field. For α as in (4-45a), we have

$$\begin{aligned} \hat{\Phi}^{(\sigma)}(\lambda,\hbar) = \frac{\alpha^\lambda e^{\frac{V_{0,0}}{\hbar}}}{\sqrt{\mathrm{i}(-6\xi^2+10\xi-4)}} \Bigg(1+\bigg(\Big(-\frac{1}{46}\xi^2 - \frac{7}{92}\xi + \frac{3}{92}\Big)\lambda^2 + \Big(\frac{3}{46}\xi^2 - \frac{11}{92}\xi + \frac{17}{46}\Big)\lambda \\ + \frac{293}{8464}\xi^2 + \frac{127}{2116}\xi - \frac{681}{8464}\bigg)\hbar + O(\hbar^2)\Bigg), \end{aligned} \tag{4-48}$$

and for α as in (4-45b), we have

$$\begin{aligned} \hat{\Phi}^{(\sigma)}(\lambda,\hbar) = \frac{\alpha^\lambda e^{\frac{V_{0,0}}{\hbar}}}{\sqrt{\mathrm{i}(-4\eta^2+2\eta-2)}} \Bigg(1+\bigg(\Big(\frac{1}{28}\eta^2 + \frac{1}{14}\eta - \frac{1}{28}\Big)\lambda^2 + \Big(\frac{1}{28}\eta^2 - \frac{1}{14}\eta + \frac{3}{14}\Big)\lambda \\ + \frac{1}{16}\eta^2 + \frac{1}{16}\eta - \frac{17}{168}\bigg)\hbar + O(\hbar^2)\Bigg). \end{aligned} \tag{4-49}$$

We can give more terms when $\lambda = 0$. For α as in (4-45a), we have

$$\begin{aligned} \hat{\Phi}^{(\sigma)}(0,\hbar) &= \frac{e^{\frac{V_{0,0}}{\hbar}}}{\sqrt{\mathrm{i}(-6\xi^2+10\xi-4)}} \Bigg(1 + \Big(\frac{293}{8464}\xi^2 + \frac{127}{2116}\xi - \frac{681}{8464}\Big)\hbar \\ &\quad + \Big(\frac{65537}{6229504}\xi^2 - \frac{50607}{6229504}\xi + \frac{2535}{778688}\Big)\hbar^2 + O(\hbar^3)\Bigg), \end{aligned} \tag{4-50}$$

and for α as in (4-45b), we have

$$\begin{aligned} \hat{\Phi}^{(\sigma)}(0,\hbar) &= \frac{e^{\frac{V_{0,0}}{\hbar}}}{\sqrt{\mathrm{i}(-4\eta^2+2\eta-2)}} \Bigg(1 + \Big(\frac{293}{8464}\xi^2 + \frac{127}{2116}\xi - \frac{681}{8464}\Big)\hbar \\ &\quad + \Big(\frac{65537}{6229504}\xi^2 - \frac{50607}{6229504}\xi + \frac{2535}{778688}\Big)\hbar^2 + O(\hbar^3)\Bigg). \end{aligned} \tag{4-51}$$

Proof (Proof of Theorem 4.6): Using the identity eq. (2-7) we convert the descendant state integral into the following form,

$$\hat{Z}^{(\lambda,\lambda')}_{(-2,3,7)}(\hbar) = \left(\frac{q}{\tilde{q}}\right)^{-\frac{1}{24}} \int_{\mathbb{R}+\mathrm{i}\frac{c_b}{2}+\mathrm{i}\varepsilon} \Phi_b(x)^2 \Phi_b(2x - c_b) e^{-\pi \mathrm{i}(2x-c_b)^2 + 2\pi(\lambda b - \lambda' b^{-1})x}\, dx \tag{4-52}$$

$$= \int_{\mathbb{R}+\mathrm{i}\frac{c_b}{2}+\mathrm{i}\varepsilon} \frac{\Phi_b(x)^2}{\Phi_b(-2x+c_b)} e^{2\pi(\lambda b - \lambda' b^{-1})x}\, dx, \tag{4-53}$$

and then apply the approximation[5]eq. (65)

$$\Phi_b\left(\frac{z}{2\pi b}\right) = \exp\left(\sum_{n=0}^{\infty} \hbar^{2n-1}\frac{B_{2n}(1/2)}{(2n)!}\mathrm{Li}_{2-2n}(-\mathrm{e}^{z})\right).$$

We begin with a change of variables $x \mapsto \frac{z}{2\pi b}$, so that

$$\Phi_b(x)^2 = \Phi_b\left(\frac{z}{2\pi b}\right)^2 \sim \exp\left(\sum_{n=0}^{\infty} \hbar^{2n-1}\frac{B_{2n}(1/2)}{(2n)!}2\mathrm{Li}_{2-2n}(-\mathrm{e}^{z})\right)$$

and

$$\begin{aligned}\Phi_b(-2x+c_b) = \Phi_b\left(\frac{-2z+2\pi b c_b}{2\pi b}\right) &\sim \exp\left(\sum_{n=0}^{\infty} \hbar^{2n-1}\frac{B_{2n}(1/2)}{(2n)!}\mathrm{Li}_{2-2n}(-\mathrm{e}^{-2z+2\pi b c_b})\right)\\ &= \exp\left(\sum_{n=0}^{\infty} \hbar^{2n-1}\frac{B_{2n}(1/2)}{(2n)!}\mathrm{Li}_{2-2n}(\mathrm{e}^{-2z+\frac{\hbar}{2}})\right).\end{aligned}$$

Using the identity

$$\mathrm{Li}_{2-2n}(\mathrm{e}^{-2z+s}) = \sum_{k=0}^{\infty} \frac{\mathrm{Li}_{2-2n-k}(\mathrm{e}^{-2z})}{k!}s^k,$$

we have

$$\mathrm{Li}_{2-2n}(-\mathrm{e}^{-2z+\frac{\hbar}{2}}) = \sum_{k=0}^{\infty} \frac{\mathrm{Li}_{2-2n-k}(\mathrm{e}^{-2z})}{k!}\left(\frac{\hbar}{2}\right)^k.$$

Collecting the above equalities up, we obtain

$$\hat{Z}^{(\lambda,\lambda')}_{(-2,3,7)}(\hbar) \sim \frac{\mathrm{i}}{\sqrt{2\pi \mathrm{i}\hbar}}\int \exp\left(\lambda z + \frac{2\pi\mathrm{i}\lambda'}{\hbar}z + V(z,\hbar)\right)\mathrm{d}z,$$

where

$$\begin{aligned}V(z,\hbar) &= \sum_{n=0}^{\infty} \hbar^{2n-1}\frac{B_{2n}(1/2)}{(2n)!}2\mathrm{Li}_{2-2n}(-\mathrm{e}^{z}) - \sum_{n,k\geq 0} \hbar^{2n+k-1}\frac{B_{2n}(1/2)}{(2n)!}\frac{\mathrm{Li}_{2-2n-k}(\mathrm{e}^{-2z})}{k!}\frac{1}{2^k}\\ &= \sum_{n=0}^{\infty} \hbar^{2n-1}\frac{B_{2n}(1/2)}{(2n)!}2\mathrm{Li}_{2-2n}(-\mathrm{e}^{z}) - \Bigg(\sum_{n,k\geq 0} \hbar^{2n+2k-1}\frac{B_{2n}(1/2)}{(2n)!}\frac{\mathrm{Li}_{2-2n-2k}(\mathrm{e}^{-2z})}{(2k)!}\frac{1}{2^{2k}}\\ &\quad + \sum_{n,k\geq 0} \hbar^{2n+2k}\frac{B_{2n}(1/2)}{(2n)!}\frac{\mathrm{Li}_{1-2n-2k}(\mathrm{e}^{-2z})}{(2k+1)!}\frac{1}{2^{2k+1}}\Bigg)\\ &= \sum_{n=0}^{\infty} \hbar^{2n-1}\left(\frac{B_{2n}(1/2)}{(2n)!}2\mathrm{Li}_{2-2n}(-\mathrm{e}^{z}) - \sum_{k=0}^{n} \frac{B_{2n-2k}(1/2)}{(2n-2k)!(2k)!}\frac{\mathrm{Li}_{2-2n}(\mathrm{e}^{-2z})}{2^{2k}}\right)\\ &\quad + \sum_{n=0}^{\infty} \hbar^{2n}\left(-\sum_{k=0}^{n} \frac{B_{2n-2k}(1/2)}{(2n-2k)!(2k+1)!}\frac{\mathrm{Li}_{1-2n}(\mathrm{e}^{-2z})}{2^{2k+1}}\right).\end{aligned}$$

Therefore, if we define

$$\begin{aligned} V_{2n+1}(z) &= -\sum_{k=0}^{\infty} \frac{B_{2n-2k}(1/2)}{(2n-2k)!(2k+1)!} \frac{\mathrm{Li}_{1-2n}(\mathrm{e}^{-2z})}{2^{2k+1}}, \\ V_{2n}(z) &= \frac{B_{2n}(1/2)}{(2n)!} 2\mathrm{Li}_{2-2n}(-\mathrm{e}^{z}) - \sum_{k=0}^{n} \frac{B_{2n-2k}(1/2)}{(2n-2k)!(2k)!} \frac{\mathrm{Li}_{2-2n}(\mathrm{e}^{-2z})}{2^{2k}}, \end{aligned} \tag{4-54}$$

then $V(z,\hbar) = \sum_{n=0}^{\infty} \hbar^{n-1} V_n(z)$, hence

$$\hat{Z}^{(\lambda,\lambda')}_{(-2,3,7)}(\hbar) \sim \frac{\mathrm{i}}{\sqrt{2\pi\mathrm{i}\hbar}} \int \exp\left(\lambda z + \frac{2\pi\mathrm{i}\lambda'}{\hbar} z + \sum_{n=0}^{\infty} \hbar^{n-1} V_n(z)\right) \mathrm{d}z.$$

Solving $\frac{\mathrm{d}}{\mathrm{d}z}\left(2\pi\mathrm{i}\lambda' z + V_0(z)\right) = 0$, we find that the critical point equation is

$$(\alpha^3 - \alpha - 1)(\alpha^3 + 2\alpha^2 - \alpha - 1) = 0, \quad (\alpha = \mathrm{e}^{z}). \tag{4-55}$$

The expansion $V_n(z) = \sum_{m=0}^{\infty} (z - \log\alpha)^m V_{n,m}(\log\alpha)$ at a critical point $z = \log\alpha$ thus gives

$$\begin{aligned} \hat{Z}^{(\lambda,\lambda')}_{(-2,3,7)}(\hbar) &\sim \frac{\mathrm{i}\alpha^{\lambda} \mathrm{e}^{\frac{V_{0,0}+2\pi\mathrm{i}\lambda' \log\alpha}{\hbar}}}{\sqrt{2\pi\mathrm{i}}} \int \mathrm{d}y \mathrm{e}^{V_{0,2}y^2} \exp\left(\lambda\hbar^{\frac{1}{2}} y + \sum_{m\geq 3} \hbar^{\frac{m}{2}-1} y^m V_{0,m} + \sum_{n\geq 1, m\geq 0} \hbar^{n-1+\frac{m}{2}} y^m V_{n,m}\right) \\ &=: \mathrm{e}^{\frac{2\pi\mathrm{i}\lambda' \log\alpha}{\hbar}} \hat{\Phi}(\lambda,\hbar). \end{aligned} \tag{4-56}$$

where the change of variables $z \mapsto \log\alpha + \hbar^{\frac{1}{2}} y$ is applied, and

$$\begin{aligned} V_{0,0} &= 2\mathrm{Li}_2(-\alpha) - \mathrm{Li}_2(\alpha^{-2}), \\ V_{0,1} &= -2\pi\mathrm{i}\lambda', \\ V_{1,0} &= -\frac{1}{2}\mathrm{Li}_1(\alpha^{-2}) = \frac{1}{2}\log(1-\alpha^{-2}), \\ V_{0,2} &= \mathrm{Li}_0(-\alpha) - 2\mathrm{Li}_0(\alpha^{-2}) = -\frac{\alpha^2 - \alpha + 2}{(\alpha-1)(\alpha+1)} = \alpha^5 - \alpha^4 - 7\alpha^3 + \alpha^2 + 4\alpha + 5, \\ V_{2n,m} &= \frac{1}{m!}\left(\frac{B_{2n}(1/2)}{(2n)!} 2\mathrm{Li}_{2-2n-m}(-\alpha) - (-2)^m \mathrm{Li}_{2-2n-m}(\alpha^{-2}) \sum_{k=0}^{n} \frac{B_{2n-2k}(1/2)}{(2n-2k)!(2k)!2^{2k}}\right), \\ V_{2n+1,m} &= -\frac{(-2)^m}{m!} \mathrm{Li}_{1-2n-m}(\alpha^{-2}) \sum_{k=0}^{n} \frac{B_{2n-2k}(1/2)}{(2n-2k)!(2k+1)!2^{2k+1}}. \end{aligned} \tag{4-57}$$

Note that for $n = 1$ and $m = 0$, we have $\hbar^{n-1+\frac{m}{2}} y^m V_{n,m} = V_{1,0}$. Expand the exponential in the integrand, collect $\hbar$'s and use the formal Gaussian integrals, we obtain

$$\hat{\Phi}(\lambda,\hbar) = \frac{\alpha^{\lambda} \mathrm{e}^{\frac{V_{0,0}}{\hbar}} \mathrm{e}^{V_{1,0}}}{\sqrt{2\mathrm{i}V_{0,2}}} (1 + O(\hbar)) = \frac{\alpha^{\lambda} \mathrm{e}^{\frac{V_{0,0}}{\hbar}}}{\sqrt{\mathrm{i}\Delta}} (1 + O(\hbar))$$

where

$$\Delta := \frac{2V_{0,2}}{\mathrm{e}^{2V_{1,0}}} = \frac{-2\alpha^2(\alpha^2 - \alpha + 2)}{(\alpha-1)^2(\alpha+1)^2} = -2\alpha^5 + 12\alpha^3 - 2\alpha^2 - 16\alpha - 10\,.$$

This finishes the proof. ∎

When α satisfies (4-45a), we have

$$V_{1,0} = \frac{1}{2}\log(1-\xi^2), \qquad V_{0,2} = -3\xi^2 + 2\xi, \qquad \Delta = -6\xi^2 + 10\xi - 4 \tag{4-58}$$

whereas when α satisfies (4-45b), we have

$$V_{1,0} = \frac{1}{2}\log(\eta^2+\eta-2), \qquad V_{0,2} = -\eta^2 - 3\eta + 3, \qquad \Delta = -4\eta^2 + 2\eta - 2\,. \tag{4-59}$$

Computing out the formal Gaussian integrals in (4-56), we obtain (4-49) and (4-51).

Naturally, due to the factorization theorem 4.2, one expects the stationary phase to be related to the q-series $H^{\pm}_{\lambda,j}(q)$. Using Richardson and Zagier's extrapolation methods[17,22], we can extrapolate numerically to obtain the coefficients of the asymptotic expansions of the q-series $H^{\pm}_{\lambda,j}(q)$. We find that the coefficients ought to match one of the $\hat{\Phi}^{(\sigma)}(\hbar)$ series, up to some elementary factors, for some value of σ, which of course depends on the ray. For instance, when $\arg(\tau) = \pi/5$, we find numerically that

$$\begin{aligned}
H^+_{0,0}(q) &= \left(\frac{q}{\tilde q}\right)^{1/24} \tau e^{\frac{\pi i}{4}}\hat{\Phi}^{(\sigma_1)}(\hbar), & H^-_{0,0}(q) &= \left(\frac{q}{\tilde q}\right)^{-1/24} \tau e^{\frac{\pi i}{4}}\hat{\Phi}^{(\sigma_2)}(-\hbar),\\
H^+_{0,1}(q) &= \left(\frac{q}{\tilde q}\right)^{1/24} e^{\frac{\pi i}{4}}\hat{\Phi}^{(\sigma_1)}(\hbar), & H^-_{0,1}(q) &= \left(\frac{q}{\tilde q}\right)^{-1/24} e^{\frac{\pi i}{4}}\hat{\Phi}^{(\sigma_2)}(-\hbar),\\
H^+_{0,2}(q) &= \left(\frac{q}{\tilde q}\right)^{1/24} \frac{2}{3\tau} e^{\frac{\pi i}{4}}\hat{\Phi}^{(\sigma_1)}(\hbar), & H^-_{0,2}(q) &= \left(\frac{q}{\tilde q}\right)^{-1/24} \frac{5}{6\tau} e^{\frac{\pi i}{4}}\hat{\Phi}^{(\sigma_2)}(-\hbar),\\
H^+_{0,3}(q) &= \left(\frac{q}{\tilde q}\right)^{1/24} \frac{1}{2} e^{-\frac{\pi i}{4}}\hat{\Phi}^{(\sigma_1)}(\hbar), & H^-_{0,3}(q) &= \left(\frac{q}{\tilde q}\right)^{-1/24} \frac{1}{2} e^{-\frac{\pi i}{4}}\hat{\Phi}^{(\sigma_2)}(-\hbar),\\
H^+_{0,4}(q) &= \tilde q^{-\frac{7}{8}}\left(\frac{q}{\tilde q}\right)^{1/24} 2e^{-\frac{\pi i}{4}}\hat{\Phi}^{(\sigma_6)}(\hbar), & H^-_{0,4}(q) &= \tilde q^{\frac{7}{8}}\left(\frac{q}{\tilde q}\right)^{-1/24} 2e^{-\frac{\pi i}{4}}\hat{\Phi}^{(\sigma_3)}(-\hbar),\\
H^+_{0,5}(q) &= \tilde q^{-\frac{7}{8}}\left(\frac{q}{\tilde q}\right)^{1/24} e^{-\frac{\pi i}{4}}\hat{\Phi}^{(\sigma_6)}(\hbar), & H^-_{0,5}(q) &= \tilde q^{\frac{7}{8}}\left(\frac{q}{\tilde q}\right)^{-1/24} e^{-\frac{\pi i}{4}}\hat{\Phi}^{(\sigma_3)}(-\hbar)\,.
\end{aligned} \tag{4-60}$$

Here $\hat{\Phi}^{(\sigma)}(\hbar) := \hat{\Phi}^{(\sigma)}(0,\hbar)$ and σ_j for $j = 1,\ldots,6$ are the six roots of the polynomial (4-44) with the numerical values

$$\sigma_1 = -0.662 - 0.562i, \qquad \sigma_2 = -0.662 + 0.562i, \qquad \sigma_3 = 1.325, \tag{4-61}$$

corresponding to the field (4-45a) and

$$\sigma_4 = -2.247, \qquad \sigma_5 = -0.555, \qquad \sigma_6 = 0.802\,, \tag{4-62}$$

corresponding to the field (4-45b), respectively. Similar phenomena can be observed for other values of arguments, with slight variations on the factors (usually only by a sign).

The correspondence between the q-series $H^{\pm}_{0,j}(q)$ and the roots σ_j's is presented in table 4-1.

$H^{+}_{0,j}(q)$		
j	arg ($\times\frac{\pi}{100}$)	σ
0	$[1,23]$	σ_1
	$[33,67]$	σ_5
	$[77,99]$	σ_2
1	$[4,36]$	σ_1
	$[47,53]$	σ_5
	$[64,96]$	σ_2
2	$[4,24]$	σ_1
	$[33,67]$	σ_5
	$[76,96]$	σ_2
3	$[1,23]$	σ_1
	$[33,67]$	σ_5
	$[77,99]$	σ_2
4	$[15,85]$	σ_6
5	$[15,85]$	σ_6

$H^{-}_{0,j}(q)$		
j	arg ($\times\frac{\pi}{100}$)	σ
0	$[1,20]$	σ_2
	$[25,75]$	σ_4
	$[80,99]$	σ_1
1	$[4,36]$	σ_2
	$[47,53]$	σ_4
	$[82,97]$	σ_1
2	$[4,20]$	σ_1
	$[26,74]$	σ_5
	$[81,96]$	σ_2
3	$[1,20]$	σ_2
	$[25,75]$	σ_4
	$[80,99]$	σ_1
4	$[15,85]$	σ_3
5	$[15,85]$	σ_3

Table 4-1 Correspondence between $H^{\pm}_{0,j}(q)$'s and σ

Note that inserting the asymptotics (4-60) to the quadratic relation (4-26), one simply obtains that $0 = 0$.

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RESUME AND ACADEMIC ACHIEVEMENTS

Resume

Born in 1999, in Deyang, Sichuan, China.

Admitted to the Southern University of Science and Technology (SUSTech) in September 2018. Obtained a bachelor's degree in Mathematics and Applied Mathematics from the Department of Mathematics, SUSTech, in June 2022.

Started to pursue a master's degree of science in Mathematics and Technology in the Department of Mathematics, SUSTech, since September 2022.

Academic Achievements during the Study for an Academic Degree

Academic Articles

[1] AN N, GAROUFALIDIS S, LI S Y. Algebraic Aspects of Holomorphic Quantum Modular Forms [A]. 2024. arXiv: 2403.02880 (corresponding to sections 4.4.1 and 4.4.2 in graduation thesis)