Fundation of Supergeometry and Its Application in Mathematical Physics

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[**ABSTRACT**]: Supergeometry is a natural extension of the theory of differential geometry, which enjoys values on its own right as a purely mathematical object, and also turns out to be useful in physics: it gives a model of spacetime that unifies quantum science and gravity. The first part of this thesis gives a detailed and mathematically strict introduction to supergeometry, rearranged from the lecture notes by Covolo and Poncin[1], the paper of Leites[2] and the notes by Deligne and Morgan[3]. The second part focus on an explicit discussion on the important example $\underline{\text{SMan}}(\mathbb{R}^{0|\delta}, X)$, following Berwick-Evans' work[4].

[**Key words**]: Math; Geometry and Topology; Mathematical Physics; Differential Geometry; Supergeometry

Content

1. An Introduction to Supergeometry

1.1 A Short Introduction of Superalgebra

Before entering into the geometry part, we give a crash course to preliminary superalgebra.

Roughly speaking, superalgebra is $\mathbb{Z}_2$-graded algebra. We shall establish the notions following the usual order in abstract algebra, that is, we start from ring and then proceed to algebra and module.

Definition 1.1.1 (Superring)**.** A *superring* R is a $\mathbb{Z}_2$-graded ring, i.e., a (usually non-commutative) unitary ring with a decomposition as abelian groups $R = R_0 \oplus R_1$, where $0, 1 \in \mathbb{Z}_2$, such that $R_i R_j \subset R_{i+j}$ for any $i, j \in \mathbb{Z}_2$.

Elements in R_0 are said to be *homogeneous with even parity* and those in R_1 are said to be *homogeneous with odd parity*. The assignment of parity gives a function

$$p\colon (R_0 \cup R_1) \setminus \{0\} \to \mathbb{Z}_2.$$

The requirement that $R_i R_j \subset R_{i+j}$ is equivalent to demanding that the parity is additive under multiplication, i.e.,

$$p(ab) = p(a) + p(b),$$

for any homogeneous $a, b \in R$. Note that $0 \in R$ can be seen to have both even and odd parities and that the multiplicative unit $1 \in R$ is forced to have even parity since $p(1) = p(1 \cdot 1) = 2p(1) = 0$. Similar convention of the parity applies to objects that are $\mathbb{Z}_2$-graded, as we will see soon.

A superring R is *supercommutative* if for any homogeneous $a, b \in R$ there is

$$ab = (-1)^{p(a)p(b)} ba. \tag{1.1.1}$$

It follows that, in a supercommutative superring, two odd elements anticommute with each other and are nilpotent, i.e., $ab = -ba$ and $a^2 = 0$ for any odd elements a and b. Usually we refer to supercommutative superring with one "super" omitted, i.e., by saying *supercommutative ring* or *commutative superring*.

Example 1.1.1. Noticing that every ring R can be graded trivially by putting $R_0 := R$, we see that every ring can be viewed as a superring. To ask for a ring to be commutative is the same thing as to ask for a trivially graded ring to be supercommutative. In this point of view, notions in superalgebra are natural generalizations of those in classical abstract algebra, and things that are not "super" are seen to be graded trivially by default.

Example 1.1.2. Given a smooth manifold X, the algebra of smooth forms $\Omega^\bullet(X) = \Omega^\bullet_0(X) \oplus \Omega^\bullet_1(X)$ is a supercommutative ring, where

$$\Omega^\bullet_0(X) \coloneqq \bigoplus_{n \geq 0} \Omega^{2n}(X), \quad \Omega^\bullet_1(X) \coloneqq \bigoplus_{n \geq 0} \Omega^{2n+1}(X).$$

One will see that $\Omega^\bullet(X)$ is in fact a supercommutative $\mathbb{R}$-algebra, once the definition is given.

Remark 1.1.1 (Koszul Sign Rule). Usually we will consider only supercommutative objects. The principle of adding a sign up to the parity when switching the position of two adjacent objects is known as the *Koszul sign rule*; for superrings it is eq. (1.1.1). We will see that it appears everywhere in supergeometry.

For the reason in the above remark, one can assume safely that every superring is supercommutative from now on.

When building the category of superrings, it is natural to ask for a forgetful functor from this category to Ring. Due to the requirement of supercommutativity of the multiplicative structure, it is pointless to consider ring homomorphisms that does not preserve the parity. Hence

Definition 1.1.2 (Superring Homomorphism)**.** Let R and R' be two superrings. A superring homomorphism $\varphi\colon R \to R'$ is a ring homomorphism φ from R to R' that preserves the parity, i.e., $\varphi(R_i) \subset R'_i$ for each $i \in \mathbb{Z}_2$.

We denote the category of supercommutative rings as SRing.

Recall that an algebra over a commutative ring K, K-algebra, is a ring A along with a ring homomorphism $\varphi\colon K \to A$ such that $\varphi(K) \subset Z(A)$ where $Z(A)$ is the multiplicative center of A. Adding the word "super" before each single word, we obtain the notion of *superalgebra*.

Definition 1.1.3 (Superalgebra)**.** A superalgebra over a supercommutative ring R, super R-algebra, is a superring A along with a superring homomorphism $\varphi\colon R \to A$ such that $\varphi(R) \subset Z(A)$ where $Z(A)$ is the supercenter of A, i.e., the sub-superring generated by $\{a \in A_0 \cup A_1 \mid ab = (-1)^{p(a)p(b)}ba,\ \forall b \in A_0 \cup A_1\}$ the set of homogeneous elements that super-commute with all homogeneous elements in A.

Let A and B be two super R-algebras. Morphisms from A to B are superring morphisms from A to B such that the following triangle commutes:

$$\begin{tikzcd} A \arrow[r] & B \\ R \arrow[u] \arrow[ur] & \end{tikzcd}$$

The category of *supercommutative R-algebras*, i.e., super R-algebras that are supercommutative as superrings, is denoted as R-SAlg.

Definition 1.1.4 (Supermodule)**.** A *supermodule* M over a superring $R = R_0 \oplus R_1$, super R-module, is a (left) R-module (R seen as a ring) with a $\mathbb{Z}_2$-graded structure $M = M_0 \oplus M_1$ (direct sum as abelian groups), such that the multiplication by scalars respects the parity, i.e., $R_i M_j \subset M_{i+j}$ for any $i, j \in \mathbb{Z}_2$.

Equivalently, $R_i M_j \subset M_{i+j}$ is the same as that

$$p(rm) = p(r) + p(m)$$

for any homogeneous $r \in R$ and $m \in M$.

Example 1.1.3. A superring is a supermodule over itself.

When it comes to the contexts where supercommutativity is always assumed, the left A-module structure gives rise to a right R-module structure, defined by $mr := (-1)^{p(r)p(m)} rm$.

Remark 1.1.2. Roughly speaking, the induced right R-module structure allows writing the scalars on both sides. This will save a lot of efforts when we put multiple supermodules over a supercommutative ring together via the tensor product.

For the special case where $R = k$ is a (trivially graded) field, we obtain the notion of *super vector space*.

Definition 1.1.5 (Super Vector Space)**.** A super vector space V over k (usually with characteristic 0) is a $\mathbb{Z}_2$-graded k-vector space, i.e., a vector space with a direct sum decomposition (as vector spaces) $V = V_0 \oplus V_1$. If V_0 and V_1 have dimension p and q respectively, then V is said to have dimension $p|q$.

Similar to the classical context, we may consider a supermodule M over a superring A which at the same time is a super R-algebra, then M has a natural structure of a supermodule over R. More specifically, if R is a field, then M has a natural structure of a super vector space over R.

One needs to be careful when talking about morphisms between supermodules. Of course the family of morphisms that preserve the parity is a natural choice.

Definition 1.1.6 (Supermodule Homomorphism)**.** A supermodule homomorphism f from M to N two super R-modules is a R-module homomorphism that preserves the parity, i.e., $f(M_i) \subset N_i$ for any $i \in \mathbb{Z}_2$.

These are the morphisms in the category R - SMod of super R-modules. Without further specification, a morphism always preserves the parity.

However, it makes sense in practice to consider parity-reversing morphisms between supermodules, and their linear combinations with parity-preserving ones. The parity-preserving ones are said to be homogeneous with even parity and the parity-reversing ones are said to be homogeneous with odd parity. According to the Koszul sign rule, instead of the usual R-linearity, we demand the super R-linearity, i.e., for a group homomorphism $f\colon M \to N$ to be a homogeneous morphism of super R-modules, there should be

$$f(rm) = (-1)^{p(f)p(r)} r f(m) \tag{1.1.2}$$

for any $m \in M$ and homogeneous $r \in R$. This is compatible with the induced right R-module structure, i.e., we have $f(mr) = f(m)r$, as one can easily verify. Formally,

Definition 1.1.7 (Homogeneous Morphism of Supermodules)**.** Let M, N be two super R-modules and $f \in \mathrm{Hom}_{\mathrm{Ab}}(M, N)$. The map f is

- an even morphism if $f(M_i) \subset N_i$ for any $i \in \mathbb{Z}_2$ and $f(rm) = rf(m)$ for any $m \in M$ and $r \in R$.
- an odd morphism if $f(M_i) \subset N_{i+1}$ for any $i \in \mathbb{Z}_2$ and $f(rm) = (-1)^{p(r)} rf(m)$ for any $m \in M$ and homogeneous $r \in R$.

The set of all even morphisms from M to N is denoted as $\mathbf{Hom}_0(M, N)$ and the set of all odd ones is denoted as $\mathbf{Hom}_1(M, N)$. The assignment of the parity gives a function $(\mathbf{Hom}_0(M, N) \cup \mathbf{Hom}_1(M, N)) \setminus \{0\} \to \mathbb{Z}_2$, and one sees that the convention of super R-linearity in eq. (1.1.2) fits the definition. Also, note that there is $\mathbf{Hom}_0(M, N) = \mathrm{Hom}_{R\text{-}\mathrm{SMod}}(M, N)$.

The direct sum as abelian groups gives the *internal Hom set*

$$\mathbf{Hom}(M, N) \coloneqq \mathbf{Hom}_0(M, N) \oplus \mathbf{Hom}_1(M, N).$$

When R is supercommutative, $\mathbf{Hom}(M, N)$ has a natural super R-module structure where the addition and scalar multiplication are defined point-wisely.

When R is trivially graded, the super R-linearity is the same as the usual R-linearity, and it is easy to see that $\mathbf{Hom}(M, N) = \mathrm{Hom}_{R\text{-}\mathrm{Mod}}(M, N)$ in this case. In particular, for super k-vector spaces M and N we have $\mathbf{Hom}(M, N) = \mathrm{Hom}_{\mathrm{Vect}_k}(M, N)$.

The notations $\mathbf{End}(M, N)$ and $\mathbf{Aut}(M, N)$ are defined similarly.

1.2 Basics of Supermanifolds

The local model for supermanifolds is the *smooth superdomain*:

Definition 1.2.1 (Smooth Superdomain)**.** A smooth superdomain $\mathcal{U}^{p|q} = (U, C^\infty_{p|q})$ of dimension $p|q$ is an open subset U of $\mathbb{R}^p$ endowed with a sheaf $C^\infty_{p|q}$ defined for each open subset $V \subset U$ by

$$C^\infty_{p|q}(V) \coloneqq C^\infty(V)[\xi^1, \cdots, \xi^q],$$

where $C^\infty(V)[\xi^1, \cdots, \xi^q]$ is the exterior algebra generated by $\xi^1, \cdots, \xi^q$ over $C^\infty(V)$, i.e., the free $C^\infty(V)$-algebra generated by $\xi^1, \cdots, \xi^q$ modulo the relation that ξ's are anticommutative, and the restriction maps are induced by the restriction of functions $C^\infty(V) \to C^\infty(W)$ for any $W \subset V$. $C^\infty_{p|q}(V)$ is a supercommutative ring by setting $\xi^1, \cdots, \xi^q$ to be odd.

Definition 1.2.2 (Supermanifold)**.** A *supermanifold* $\mathcal{M} = (M, \mathcal{O})$ *of dimension* $p|q$ is a ringed space whose underlying space M is a manifold and structural sheaf $\mathcal{O}$ is a sheaf of super $\mathbb{R}$-algebras, such that the pair is locally $\mathbb{R}$-isomorphic to smooth superdomains of dimension $p|q$, i.e., for any point of M there exists a neighborhood W of that point such that there exists a homeomorphism $\varphi\colon W \cong U \subset \mathbb{R}^p$ along with an isomorphism $\varphi^*\colon C^\infty_{p|q} \cong \varphi_*\mathcal{O}|_W$ of sheaves of super $\mathbb{R}$-algebras.

Example 1.2.1. A classical differential manifold of dimension n is a supermanifold of dimension $n|0$ when it is endowed with the sheaf C^∞ of smooth functions.

Example 1.2.2 (The Parity-reversed Tangent Bundle πTX)**.** For an ordinary smooth manifold X, we can consider the supermanifold $\pi TX := (X, \Omega^\bullet)$, the *parity-reversed tangent bundle*, where 1-forms in $\Omega^\bullet(X)$ are set to be odd. Clearly, if X is of dimension n, then πTX is of dimension $n|n$.

With slight abuse of notation, for a supermanifold $\mathcal{M} = (M, \mathcal{O})$ we usually write $C^\infty(\mathcal{M}) := \mathcal{O}(M)$.

In analogy to the classical theory, we call $(W, \mathcal{O}|_W) \cong \mathcal{U}^{p|q} = (U, (x, \xi))$ a *(super) coordinate neighborhood* with super coordinates $(x, \xi) = (x_1, \cdots, x_p, \xi^1, \cdots, \xi^q) \in C^\infty_{p|q}(U) = C^\infty(U)[\xi^1, \cdots, \xi^q]$, where $x_i\colon U \subset \mathbb{R}^p \to \mathbb{R}\colon (a_1, \cdots, a_p) \mapsto a_i$ is the usual coordinate function.

Definition 1.2.3 (Morphism of Supermanifolds)**.** Let $(M, \mathcal{O})$ and $(N, \mathcal{R})$ be two supermanifolds. A morphism $\Psi = (\psi, \psi^*)$ from $(M, \mathcal{O})$ to $(N, \mathcal{R})$ consists of

- a continuous map between the underlying spaces $\psi\colon M \to N$,
- a morphism of sheaves of super $\mathbb{R}$-algebras $\psi^*\colon \mathcal{R} \to \psi_*\mathcal{O}$.

Morphisms of supermanifolds compose in the obvious way, giving rise to the category of supermanifolds, SMan.

Example 1.2.3 (The Superspace $\mathbb{R}^{p|q}$)**.** Given a domain $U \subset \mathbb{R}^p$ along with a diffeomorphism $\varphi\colon U \cong \mathbb{R}^p$, it induces an isomorphism of supermanifolds

$$\mathcal{U}^{p|q} = (U, C^\infty_{p|q}) \cong (\mathbb{R}^p, C^\infty_{p|q}) =: \mathbb{R}^{p|q}$$

via $\varphi^*\colon C^\infty(\mathbb{R}^p) \cong C^\infty(U)$. Since every neighborhood in $\mathbb{R}^p$ contains a neighborhood of coordinate ball which is diffeomorphic to $\mathbb{R}^p$, replacing smooth superdomain with the superspace $\mathbb{R}^{p|q}$ in definition 1.2.2 gives exactly the same definition of supermanifold.

More generally, diffeomorphisms between underlying spaces of smooth superdomains of dimension $p|q$ induce isomorphisms of superdomains.

Example 1.2.4 (The Point pt)**.** It is easy to see that the point $\mathrm{pt} := \mathbb{R}^{0|0}$ is final in the category SMan. This makes the definition of categorical group objects applicable to SMan, giving the definition of *super Lie groups*.

Example 1.2.5 (Open Subsupermanifold)**.** Let $\mathcal{M} = (M, \mathcal{O})$ be a supermanifold. For any open subset $U \subset M$, $\mathcal{U} := (U, \mathcal{O}|_U)$ gives an open sub-supermanifold of $\mathcal{M}$, with the inclusion morphism given by $U \hookrightarrow M$ and the restriction of sheaves $\mathcal{O} \to \mathcal{O}|_U$.

To establish a deeper understanding of morphisms of supermanifolds, we need to know what the super version of local ring is.

Definition 1.2.4 (Homogeneous Ideal)**.** A homogeneous ideal I of a superring R is an ideal I of the ring R such that $I = (I \cap R_0) \oplus (I \cap R_1)$, i.e., the homogeneous components of each element of I still live in I.

Since we require morphisms between superrings to preserve the parity, it is clear that the inverse image of a homogeneous ideal under a morphism of superrings is also a homogeneous ideal.

Definition 1.2.5 (Local Superring)**.** A superring $R = R_0 \oplus R_1$ is local if it admits a unique maximal homogeneous ideal, i.e., it has only one homogeneous ideal that is maximal with respect to the inclusion.

A superalgebra is local if it is local as a superring. For a supermanifold $\mathcal{M} = (M, \mathcal{O})$, the stalk $\mathcal{O}_x = C^\infty_{p|q,x}$ of the structural sheaf at a point $x \in U \subset M$ is local, with the unique maximal homogeneous ideal $\mathfrak{m}_x$ consisting of all the non-units. More concretely, for $f \in C^\infty_{p|q}(V)$ we may write it as

$$\begin{aligned} f(x,\xi) = \sum_\alpha f_\alpha(x)\xi^\alpha &= \sum_{l=0}^{q} \sum_{\alpha_1 < \cdots < \alpha_l} f_{\alpha_1\cdots\alpha_l}(x)\xi^{\alpha_1}\cdots\xi^{\alpha_l} \\ &= f_0(x) + \sum_{l=1}^{q} \sum_{\alpha_1 < \cdots < \alpha_l} f_{\alpha_1\cdots\alpha_l}(x)\xi^{\alpha_1}\cdots\xi^{\alpha_l}, \end{aligned}$$

with the coefficients f_α's living in $C^\infty(V)$. A monomial term $f_{\alpha_1\cdots\alpha_l}(x)\xi^{\alpha_1}\cdots\xi^{\alpha_l}$ is said to have cohomological degree l. Since the part with nonzero cohomological degrees

$$\sum_{l=1}^{q} \sum_{\alpha_1 < \cdots < \alpha_l} f_{\alpha_1\cdots\alpha_l}(x)\xi^{\alpha_1}\cdots\xi^{\alpha_l}$$

is nilpotent, for any $x \in V$ the germ $[f]_x$ is a non-unit if and only if $f_0(x) = 0$, consequently the homogeneous components of a non-unit are also non-units. Hence

Theorem 1.2.1 (The Unique Maximal Homogeneous Ideal). *The unique maximal homogeneous ideal of the stalk $C^\infty_{p|q,x}$ is given by*

$$\mathfrak{m}_x = \{[f]_x \mid f_0(x) = 0\}. \tag{1.2.1}$$

□

With the above result, the residue field $\kappa(x) = C^\infty_{p|q,x}/\mathfrak{m}_x$ is identified with $\mathbb{R}$ via the isomorphism $[f]_x \mapsto f_0(x)$. In classical theory, quotient by $\mathfrak{m}_x$ is essentially the same as forgetting the difference in infinitesimal neighborhoods around x, which yields the evaluation at x. Given $f \in \mathcal{O}(V)$ for any open $V \subset M$ and let $x \in V$ varies, this observation induces a real-valued function on V which is locally $(f|_U)_0 \in C^\infty(U)$ in coordinate neighborhoods U. By claiming that these induced real-valued functions are smooth, a smooth structure on M is defined; the compatibility of restriction maps (which are super $\mathbb{R}$-algebra morphisms) ensures that this smooth structure is well-defined. The continuous map ψ in definition 1.2.3 turns out to be smooth under this structure, as we will see later. Consequently, for two supermanifolds to be isomorphic, they must have the same dimension and their underlying spaces endowed with the induced smooth structure must be diffeomorphic.

The above discussion gives a morphism of super $\mathbb{R}$-algebras

$$\begin{aligned} \varepsilon_V\colon \quad \mathcal{O}(V) &\longrightarrow \quad C^\infty(V) \\ f &\longmapsto \varepsilon_V(f)\colon \varepsilon_V(f)(x) := f_0(x) \end{aligned}$$

Let V vary and it is easy to see that ε_V is natural in V, giving a morphism of sheaves

$$\varepsilon\colon \mathcal{O} \to C^\infty.$$

This gives an inclusion morphism $(M, C^\infty) \to \mathcal{M}$.

Note that though for a smooth superdomain $\mathcal{U}^{p|q}$ we have simply $C^\infty(U) \subset C^\infty_{p|q}(U)$, the above ε does not necessarily admit a canonical right inverse $C^\infty \to \mathcal{O}$, due to the complexity of the global nature. A choice of $C^\infty \to \mathcal{O}$ always exists though, giving the existence of restrictions $\mathcal{M} \to (M, C^\infty)$; it is a consequence of the following theorem by Batchelor[5]:

Theorem 1.2.2 (Batchelor's theorem). *Every supermanifold is isomorphic to $(M, \Gamma(\wedge^\bullet(E^\vee)))$ for some ordinary smooth finite-rank vector bundle E over M.* □

It is natural to ask for a morphism between supermanifolds to descend to a morphism between their underlying smooth manifolds. We have seen that the structural sheaf of a supermanifold descends naturally to the structural sheaf of the underlying smooth manifold via ε, so the question is that whether this way of descending is compatible with the morphisms. The answer is yes.

From the above we know that the image of a superfunction $f \in \mathcal{O}(V)$ in the residue field $\kappa(x) = \mathbb{R}$ can be seen as the "evaluation" of f at a point $x \in M$, and ε is induced by this "evaluation". A morphism $\Psi = (\psi, \psi^*)\colon \mathcal{M} = (M, \mathcal{O}) \to \mathcal{N} = (N, \mathcal{R})$ induces a map on stalks $\psi_x^*\colon \mathcal{R}_{\psi(x)} \to \mathcal{O}_x$, and for it to be compatible with the "evaluation", the only requirement is that it induces an isomorphism on the residue fields,

$$\mathbb{R} = \kappa(\psi(x)) = \mathcal{R}_{\psi(x)}/\mathfrak{m}_{\psi(x)} \cong \mathcal{O}_x/\mathfrak{m}_x = \kappa(x) = \mathbb{R},$$

which is equivalent to that ψ_x^* preserves the maximal homogeneous ideal, i.e., $\psi_x^*(\mathfrak{m}_{\psi(x)}) \subset \mathfrak{m}_x$. This is not included by definition 1.2.3, but it is automatically satisfied according to the following proposition.

Proposition 1.2.3. *Let A and B be two local super $\mathbb{R}$-algebras such that $A/\mathfrak{m}_A \cong B/\mathfrak{m}_B \cong \mathbb{R}$. If $f\colon A \to B$ is a morphism of super $\mathbb{R}$-algebras, then $f(\mathfrak{m}_A) \subset \mathfrak{m}_B$.*

Proof. Since f is an $\mathbb{R}$-algebra morphism, the composition

$$\mathbb{R} \to A \xrightarrow{f} B \to B/\mathfrak{m}_B \cong \mathbb{R}$$

is the identity on $\mathbb{R}$ where the arrows are the obvious ones. Hence $g := A \xrightarrow{f} B \to B/\mathfrak{m}_B \cong \mathbb{R}$ is surjective. This tells that $\ker g = \mathfrak{m}_A$. Therefore

$$f(\mathfrak{m}_A) = f(\ker g) \subset \ker(B \to B/\mathfrak{m}_B \cong \mathbb{R}) = \mathfrak{m}_B$$

as desired. □

With the residue field preserved, for any open subset $V \subset N$, $\psi^*\colon \mathcal{R}(V) \to \mathcal{O}(\psi^{-1}(V))$ induces a morphism $\widetilde{\psi^*}\colon C^\infty(V) \to C^\infty(\psi^{-1}(V))$ such that the following diagram is commutative:

$$\begin{tikzcd} \mathcal{R}(V) \arrow[r, "\psi^*"] \arrow[d, "\varepsilon_V"] & \mathcal{O}(\psi^{-1}(V)) \arrow[d, "\varepsilon_{\psi^{-1}(V)}"] \\ C^\infty(V) \arrow[r, "\widetilde{\psi^*}"] & C^\infty(\psi^{-1}(V)) \end{tikzcd}$$

By evaluating the image of functions under $\widetilde{\psi^*}$ at each point $x \in \psi^{-1}(V)$, one sees that $\widetilde{\psi^*}$ coincides with the precomposition by $\psi\colon M \to N$, concluding that ψ is smooth under the induced smooth structure.

In fact, one can verify that ε is equal to the quotient by the *nilpotent ideal*, i.e., the ideal generated by all nilpotent elements, so the existence of $\widetilde{\psi^*}$ is rather trivial. However, one cannot see the fact that $\widetilde{\psi^*}$ is the precomposition by ψ, unless one looks at the germs.

For a classical smooth manifold M, a smooth map $f\colon M \to U \subset \mathbb{R}^r$ is determined by the components $f_i \in C^\infty(M)$ such that $f(p) = (f_1(p), \cdots, f_r(p))$ for any $p \in M$. To give such components is equivalent to determining the pullbacks $y_i \mapsto f_i \in C^\infty(M)$ of the coordinate functions $y_1, \cdots, y_r \in C^\infty(U)$; conversely, every assignment $y_i \mapsto f_i$ such that $\mathrm{Im}(f_1, \cdots, f_r) \subset U$ gives a unique smooth map $f\colon M \to U$. The analogy holds for supermanifolds, which is known as the *Fundamental Theorem of Supermorphisms*.

Before stating and proving the theorem, the technique of *Approximation by Polynomial* should be established. Let $\mathcal{M} = (M, \mathcal{O})$ be a supermanifold of dimension $p|q$ as always. Let $x_0 \in M$ be any point and choose a coordinate neighborhood $\mathcal{U}^{p|q} = (U, C^\infty_{p|q})$ of x_0. By translation, we can assume that $x_0 = 0$ in U for conciseness. For a smooth function $f_0 \in C^\infty(U)$ to vanish at $x_0 = 0$, there must be $f_0(x) \sim O(x) := O(\|x\|)$ near 0 by the Taylor approximation. Hence we can rewrite eq. (1.2.1) as

$$\mathfrak{m}_{x_0} = \left\{ [f]_{x_0=0} \;\middle|\; f(x,\xi) = O(x) + \sum_{l=1}^{q} \sum_{\alpha_1 < \cdots < \alpha_l} f_{\alpha_1 \cdots \alpha_l}(x) \xi^{\alpha_1} \cdots \xi^{\alpha_l} \right\}.$$

It follows that for $k \geq 1$,

$$\begin{aligned} \mathfrak{m}^k_{x_0} = \Bigg\{ [f]_{x_0=0} \;\Bigg|\; f(x,\xi) = &\sum_{l=0}^{k-1} \sum_{\alpha_1 < \cdots < \alpha_l} O(x^{k-l}) \xi^{\alpha_1} \cdots \xi^{\alpha_l} \\ &+ \sum_{l=k}^{q} \sum_{\alpha_1 < \cdots < \alpha_l} f_{\alpha_1 \cdots \alpha_l}(x) \xi^{\alpha_1} \cdots \xi^{\alpha_l} \Bigg\}, \end{aligned}$$

where $O(x^s) := O(\|x\|^s)$ and $\sum_{l=k}^q$ is zero if $k \geq q$. In particular,

$$\mathfrak{m}^{q+1}_{x_0} = \left\{ [f]_{x_0=0} \;\middle|\; f(x,\xi) = O(x^{q+1}) + \sum_i O(x^q)\xi^i + \cdots + O(x)\xi^1 \cdots \xi^q \right\}, \quad (1.2.2)$$

which concludes that

Lemma 1.2.4. *Let $\mathcal{M} = (M, \mathcal{O})$ be a supermanifold of dimension $p|q$, $x_0 \in U \subset M$ and $f \in \mathcal{O}(U)$. If $[f]_{x'} \in \mathfrak{m}^{q+1}_{x'}$ for a dense set of x' in some neighborhood of $x_0 \in M$, then $[f]_{x_0} = 0$.* □

In particular, if $f, g \in \mathcal{O}(V)$ satisfy $[f-g]_x \in \mathfrak{m}^{q+1}_x$ for any $x \in V$, then $f = g$.

Theorem 1.2.5 (Approximation by Polynomial)**.** *Let $(M, \mathcal{O})$ be a supermanifold of dimension $p|q$, $x_0 \in M$ be an arbitrary point and $f \in \mathcal{O}(V)$ be a section of a neighborhood V of x_0. For any fixed degree of approximation $k \in \mathbb{N}^*$, there exists a coordinate neighborhood $(U, C^\infty_{p|q})$ of x_0 with super coordinates (x, ξ) and a polynomial $P = P(x,\xi) \in \mathbb{R}[x_1, \cdots, x_p, \xi^1, \cdots, \xi^q] \subset C^\infty_{p|q}(U)$, such that*

$$[f]_{x_0} - [P]_{x_0} \in \mathfrak{m}^k_{x_0}.$$

Proof. By translation we may assume that $x_0 = 0 \in U$. Restrict f to $C^\infty_{p|q}(V \cap U)$ so that we can write $f = \sum_\alpha f_\alpha(x)\xi^\alpha$. Using the Taylor approximation, we can find a polynomial P_α such that

$$f_\alpha(x) = P_\alpha(x) + O(x^k)$$

for each α. It follows that

$$f = \sum_\alpha f_\alpha \xi^\alpha = \sum_\alpha P_\alpha(x)\xi^\alpha + \sum_\alpha O(x^k)\xi^\alpha.$$

Since $\sum_\alpha O(x^k)\xi^\alpha \in \mathfrak{m}^k_{x_0}$, put $P := \sum_\alpha P_\alpha(x)\xi^\alpha$ and we are done. □

Roughly speaking, theorem 1.2.5 is the super version of Taylor approximation, where $\mathfrak{m}^k_{x_0}$ serves as $O(\|x - x_0\|^k)$. The larger the k grows, the closer the approximation is.

Note that under the requirement that $\psi^*\colon C^\infty_{p|q}(U) \to \mathcal{O}(M)$ is a morphism of super $\mathbb{R}$-algebras, the values of ψ^* on polynomials is determined by its values on the super coordinates $x_1, \cdots, x_p, \xi^1, \cdots, \xi^q \in C^\infty_{p|q}(U)$. Theorem 1.2.5 along with lemma 1.2.4 implies that ψ^* is completely determined by its values on the super coordinates as a consequence of $\psi^*(\mathfrak{m}^{q+1}_{\psi(x)}) = \psi^*(\mathfrak{m}_{\psi(x)})^{q+1} \subset \mathfrak{m}^{q+1}_x$. Conversely, for any parity-preserving assignment of values on super coordinates such that the induced smooth map $M \to U$ is well-defined with image contained by U, there exists a corresponding morphism of supermanifolds $\mathcal{M} \to \mathcal{U}^{p|q}$. Formally,

Theorem 1.2.6 (Fundamental Theorem of Supermorphisms)**.** *Let $\mathcal{M} = (M, \mathcal{O})$ be a supermanifold of dimension $p'|q'$ and $\mathcal{U}^{p|q}$ be a smooth superdomain of dimension $p|q$. If $(s, \sigma) = (s_1, \cdots, s_p, \sigma^1, \cdots, \sigma^q)$ is a $(p+q)$-tuple of superfunctions in $\mathcal{O}(M)$ such that*

- *$s_1, \cdots, s_p$ are homogeneous with even parity and $\sigma^1, \cdots, \sigma^q$ are odd,*
- *$\mathrm{Im}(\varepsilon_M s_1, \cdots, \varepsilon_M s_p) \subset U$,*

then there exists a unique morphism of supermanifolds $\Psi = (\psi, \psi^*)\colon \mathcal{M} \to \mathcal{U}^{p|q}$ such that

$$s_i = \psi^* y_i \quad \textit{and} \quad \sigma^j = \psi^* \eta^j \tag{1.2.3}$$

for each $1 \le i \le p$ and $1 \le j \le q$, where (y, η) are the super coordinates of $\mathcal{U}^{p|q}$.

Proof. The uniqueness is easy. Note that for any morphism (ψ, ψ^*) satisfying eq. (1.2.3), there must be $\psi = (\varepsilon_M s_1, \cdots, \varepsilon_M s_p)$ since

$$y_i \circ \psi = \widetilde{\psi^*}(y_i) = \varepsilon_M \psi^* y_i = \varepsilon_M s_i.$$

Hence given any two morphisms (ψ_1, ψ_1^*) and (ψ_2, ψ_2^*) satisfying eq. (1.2.3), there is $\psi_1 = \psi_2 =: \psi$. For any $V \subset U$, $f \in C^\infty_{p|q}(V)$ and $x_0 \in \psi^{-1}(V)$, write $y_0 := \psi(x_0)$ and by theorem 1.2.5 there exists a polynomial $P_{x_0} = P_{x_0}(y, \eta) \in C^\infty_{p|q}(U)$ labeled by x_0 such that $[f]_{y_0} - [P_{x_0}]_{y_0} \in \mathfrak{m}^{q'+1}_{y_0}$. Apply ψ_i^* and we obtain

$$[\psi_i^*(f)]_{x_0} - [\psi_i^*(P_{x_0})]_{x_0} \in \psi_i^*(\mathfrak{m}^{q'+1}_{y_0}) \subset \mathfrak{m}^{q'+1}_{x_0}, \quad i = 1, 2.$$

Since $\psi_1^*(P_{x_0}) = \psi_2^*(P_{x_0})$,

$$[\psi_1^*(f) - \psi_2^*(f)]_{x_0} = ([\psi_1^*(f)]_{x_0} - [\psi_1^*(P_{x_0})]_{x_0}) - ([\psi_2^*(f)]_{x_0} - [\psi_2^*(P_{x_0})]_{x_0}) \in \mathfrak{m}^{q'+1}_{x_0}.$$

Let $x_0 \in \psi^{-1}(V)$ vary and we see by lemma 1.2.4 that $\psi_1^*(f) = \psi_2^*(f)$, concluding $\psi_1^* = \psi_2^*$.

For the existence, we put $\psi := (\varepsilon_M s_1, \cdots, \varepsilon_M s_p) \colon M \to U$ and then construct $\psi^* \colon C^\infty_{p|q}(V) \to \mathcal{O}(\psi^{-1}(V))$ for any open set $V \subset U$.

The superfunction s_i, restricted to $\psi^{-1}(V)$, can be decomposed as

$$s_i = \varepsilon s_i + n^i,$$

where $n^i := s_i - \varepsilon s_i$ is the nilpotent part of s_i. For any superfunction $f = f(y, \eta) = \sum_\alpha f_\alpha(y)\eta^\alpha \in C^\infty_{p|q}(V)$, under eq. (1.2.3) there must be

$$\psi^*(f) = \sum_\alpha \psi^*(f_\alpha(y))(\psi^*\eta)^\alpha = \sum_\alpha \psi^*(f_\alpha(y))\sigma^\alpha,$$

so it remains only to determine $\psi^*(f_\alpha(y))$ for each α. Intuitively, one would like that there is $\psi^*(f_\alpha(y)) = f_\alpha(\psi^* y) = f_\alpha(s) = f_\alpha \circ s$. However, s_i's may not be real-valued. The remedy to this situation is to use the trick of formal Taylor expansion, substituting $s = \varepsilon s + n := (\varepsilon s_1, \cdots, \varepsilon s_p) + (n^1, \cdots, n^p)$ into $T(f; \varepsilon s)$, obtaining

$$\psi^*(f_\alpha(y)) = f_\alpha(s) = f_\alpha(\varepsilon s + n) := \sum_\beta \frac{1}{\beta!}(\partial_y^\beta f_\alpha)(\varepsilon s)n^\beta = \sum_\beta \frac{1}{\beta!}((\partial_y^\beta f_\alpha) \circ \varepsilon s)(x)n^\beta.$$

Note that n^β is well-defined as n^i's, being even, commute with each other. Thus we conclude that the defining formula for ψ^* is

$$\psi^*(f) = \sum_\alpha \sum_\beta \frac{1}{\beta!}(\partial_y^\beta f_\alpha)(\varepsilon s)n^\beta \sigma^\alpha.$$

This defining formula works well. The sums are finite due to nilpotency. Since n^i's are even, ψ^* defined in this way is parity-preserving. $(\partial_y^\beta f_\alpha)(\varepsilon s) = ((\partial_y^\beta f_\alpha) \circ \varepsilon s)(x)$ is smooth in x, hence the coefficients are legal. Since the Taylor expansion commutes with addition and multiplication, ψ^* is indeed a morphism of super $\mathbb{R}$-algebras. The verification that ψ^* does satisfy eq. (1.2.3) is straightforward. Finally, the just-proved uniqueness forces that this definition of ψ^* is compatible with the restriction maps, giving a well-defined morphism of sheaves of super $\mathbb{R}$-algebras and hence a morphism of supermanifolds. $\square$

Note that in the proof of theorem 1.2.6, the way of "evaluating" a smooth function at a "super point" is essentially the same as the way one generalizes a real analytic function to a holomorphic function on the complex plane using Taylor expansion.

Locally, a supermorphism $(\psi, \psi^*)\colon \mathcal{M} \to \mathcal{N}$ is a super $\mathbb{R}$-algebra morphism

$$\psi^*\colon C^\infty(V)[\eta^1, \cdots, \eta^{q'}] \to C^\infty(U)[\xi^1, \cdots, \xi^q]$$

under coordinate neighborhoods $\mathcal{U}^{p|q} = (U, (x, \xi))$ in $\mathcal{M}$ and $\mathcal{V}^{p'|q'} = (V, (y, \eta))$ in $\mathcal{N}$ with $U \subset \psi^{-1}(V)$. Committing the usual abuse of notation in classical theory, we write locally that

$$\begin{aligned} y_i &= \psi^* y_i = s_i(x, \xi) = y_i(x, \xi) \qquad \text{(even)}, \\ \eta^j &= \psi^* \eta^j \sigma^j(x, \xi) = \eta^j(x, \xi) \qquad \text{(odd)}. \end{aligned} \tag{1.2.4}$$

The Fundamental Theorem asserts that the local representations determine the supermorphism completely.

With the convention of "evaluating" smooth functions at "super points", the local behaviour of morphisms between supermanifolds is completely similar to those between ordinary manifolds, i.e., $\psi^* f(y, \eta) = f(y(x, \xi), \eta(x, \xi))$. Intuitively, this tells how the imaginary coordinates (the ξ's and η's) serve as coordinates.

Example 1.2.6. With the above notations, let $(\psi, \psi^*)\colon \mathbb{R}^{1|2} \to \mathbb{R}^{1|1}$ be defined by

$$\begin{aligned} y &= y(x, \xi) = x + \xi^1 \xi^2, \\ \eta &= \eta(x, \xi) = f(x)\xi^1 + g(x)\xi^2, \end{aligned}$$

for some $f, g \in C^\infty(\mathbb{R})$. We have

$$\begin{aligned} \psi^* \sin y &= \sin(x + \xi^1\xi^2) = \sin x + (\cos x)\xi^1\xi^2, \\ \psi^* \cos y &= \cos(x + \xi^1\xi^2) = \cos x - (\sin x)\xi^1\xi^2, \\ \psi^*(\sin y \cos y) &= \sin x \cos x + (\cos^2 x - \sin^2 x)\xi^1\xi^2. \end{aligned}$$

Remark 1.2.1. In the decomposition $s = \varepsilon s + n$, writing $n = \xi + \eta$, it is not hard to see that

$$\sum_\beta \frac{1}{\beta!}(\partial_y^\beta f)(\varepsilon s) \cdot (\xi + \eta)^\beta = \sum_{\gamma, \alpha} \frac{1}{\gamma! \cdot \alpha!}(\partial_y^{\alpha+\gamma} f)(\varepsilon s)\xi^\gamma \eta^\alpha,$$

by an induction from the 1-dimensional case. Since

$$\sum_{\gamma, \alpha} \frac{1}{\gamma! \cdot \alpha!}(\partial_y^{\alpha+\gamma} f)\xi^\gamma \eta^\alpha = \sum_\alpha \frac{1}{\alpha!}(\partial_y^\alpha f)(\varepsilon s + \xi)\eta^\alpha,$$

we see that the trick of formal Taylor expansion commutes with addition of the nilpotent part of the parameter. Hence it is legal to take only part of the nilpotent out of the parameter $f(\varepsilon s + n)$.

More generally, we have the following theorem; a short proof of this theorem admitting the analogy result of ordinary manifolds is given in appendix A.

Theorem 1.2.7 (Supermanifolds are Affine)**.** *For any two supermanifolds $\mathcal{M}$ and $\mathcal{N}$, the functor $C^\infty\colon \mathrm{SMan}^{op} \to \mathbb{R}\text{-}\mathrm{SAlg}$ that brings $\mathcal{M}$ to $C^\infty(\mathcal{M})$ induces a natural bijection*

$$\mathrm{SMan}(\mathcal{M}, \mathcal{N}) \cong \mathbb{R}\text{-}\mathrm{SAlg}(C^\infty(\mathcal{N}), C^\infty(\mathcal{M})). \qquad \square$$

As an application of this theorem along with theorem 1.2.6, we have

Example 1.2.7 (Universal Property of $C^\infty(\mathbb{R}^{p|q})$)**.** Let $x_1, \cdots, x_p, \xi^1, \cdots, \xi^q$ be the coordinates on $\mathbb{R}^{p|q}$, then for any supermanifold S and any parity-preserving assignment

$$\{x_1, \cdots, x_p, \xi^1, \cdots, \xi^q\} \to C^\infty(S),$$

there exists a unique super $\mathbb{R}$-algebra morphism $\varphi\colon C^\infty(\mathbb{R}^{p|q}) \to C^\infty(S)$ such that the following diagram commutes:

$$\begin{tikzcd}
C^\infty(\mathbb{R}^{p|q}) \arrow[r, dashed, "\varphi"] & C^\infty(S) \\
\{x_1, \cdots, x_p, \xi^1, \cdots, \xi^q\} \arrow[u, hook] \arrow[ur] &
\end{tikzcd}$$

Note that theorem 1.2.6 alone does not give the uniqueness.

Example 1.2.8 (Direct Sum of Parity Reversed Vector Bundles)**.** Recall in example 1.2.2 that for an ordinary smooth manifold X, we have a supermanifold πTX; the inclusion $C^\infty(X) \hookrightarrow \Omega^\bullet(X)$ gives the canonical projection $\pi TX \to X$. The direct sum of two copies of πTX, denoted as $\pi TX \oplus \pi TX$, is the supermanifold that satisfies the following universal property diagram:

$$\begin{tikzcd}
S \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \pi TX \oplus \pi TX \arrow[r] \arrow[d] & \pi TX \arrow[d] \\
& \pi TX \arrow[r] & X
\end{tikzcd}$$

where S denotes an arbitrary supermanifold. Theorem 1.2.7 thus indicates that

$$\pi TX \oplus \pi TX \cong (X, \Omega^\bullet \otimes_{C^\infty} \Omega^\bullet).$$

1.3 Differential Calculus on Supermanifolds

In the classical theory, vector bundles of rank n over a smooth manifold M are 1-to-1 with locally free sheaves of C^∞-modules of rank n over M, by sending a vector bundle E to its sheaf of sections $\Gamma(E)$; see page 33 in the book of Ramanan[6]. Given a locally free sheaf $\mathcal{E}$, its fiber of vectors at a point x, being the evaluation of the sections at x, is given by $\mathcal{E}_x/\mathfrak{m}_x\mathcal{E}_x$ where $\mathfrak{m}_x$ is the maximal ideal of C_x^∞. These generalize to the super version immediately.

A *free super module of rank* $p|q$ over superring R is

Definition 1.3.1 (Free Super R-module of rank $p|q$)**.** A free super R-module of rank $p|q$, denoted as $R^{p|q}$, is a super R-module that admits a basis $(e_i)_{1\leq i\leq p+q}$ where e_i is even for $1 \leq i \leq p$ and is odd for $p+1 \leq i \leq p+q$. This means that

$$R^{p|q} = Re_1 \oplus \cdots \oplus Re_{p+q},$$

where the direct sums are direct sums of abelian groups.

Definition 1.3.2 (Super Vector Bundle)**.** Given a supermanifold $\mathcal{M} = (M, \mathcal{O})$. A *super vector bundle of rank* $p|q$ over $\mathcal{M}$ is a locally free sheaf $\mathcal{E}$ of super $\mathcal{O}$-modules of rank $p|q$ over M, i.e., for any $x \in M$ there exists a neighborhood U of x such that $\mathcal{E}(U) \cong \mathcal{O}(U)^{p|q}$.

Similarly, the (super) tangent sheaf over $\mathcal{M} = (M, \mathcal{O})$ is defined to be the sheaf of *superderivations* of $\mathcal{O}(U)$ for each U open in M, where the superderivations are linear combinations of the homogeneous ones:

Definition 1.3.3 (Homogeneous Superderivation)**.** A homogeneous superderivation of parity i of the super $\mathbb{R}$-algebra $\mathcal{O}(U)$ is an $\mathbb{R}$-linear map $D \in \mathbf{End}_i\, \mathcal{O}(U)$ of parity i, satisfying the *graded Leibniz rule*

$$D(st) = D(s)t + (-1)^{ij} s(Dt) \tag{1.3.1}$$

for all $s \in \mathcal{O}_j(U)$ and $t \in \mathcal{O}(U)$.

We denote the set of all superderivations of parity i of $\mathcal{O}(U)$ by $\mathrm{Der}_i\, \mathcal{O}(U)$. Clearly, $\mathrm{Der}_i\, \mathcal{O}(U)$ is an $\mathbb{R}$-vector space, thus the set

$$\mathrm{Der}\, \mathcal{O}(U) := \mathrm{Der}_0\, \mathcal{O}(U) \oplus \mathrm{Der}_1\, \mathcal{O}(U)$$

of all superderivations of $\mathcal{O}(U)$ is a super vector space over $\mathbb{R}$. Also, the point-wisely defined super $\mathcal{O}(U)$-module structure applies to $\mathrm{Der}\, \mathcal{O}(U)$, i.e., we define, for $D \in \mathrm{Der}_i\, \mathcal{O}(U)$ and $s \in \mathcal{O}_j(U)$, $sD \in \mathrm{Der}_{i+j}\, \mathcal{O}(U)$ by

$$(sD)(t) := s \cdot D(t), \text{ for any } t \in \mathcal{O}(U).$$

This makes $\operatorname{Der}\mathcal{O}(U)$ a super $\mathcal{O}(U)$-module.

We have obtained for each open $U \subset M$ a super $\mathcal{O}(U)$-module $\operatorname{Der}\mathcal{O}(U)$. To make these form a sheaf, we need to construct a restriction map $\operatorname{Der}\mathcal{O}(U) \to \operatorname{Der}\mathcal{O}(V)$ whenever $V \subset U$. This requires the local feature of derivation, for which in the classical theory one uses the smooth bump function. Below gives the super version of smooth bump function.

Definition 1.3.4 (Support)**.** The *support* of a superfunction $s \in \mathcal{O}(U)$ is the closed subset $\operatorname{supp} s := U \setminus \Omega$ in U, where

$$\Omega = \{x \in U \mid \exists \text{ a neighborhood } V \subset U \text{of } x \text{ such that } s|_V = 0\}.$$

Equivalently we have

$$\operatorname{supp} s = \{x \in U \mid [s]_x \neq 0 \in \mathcal{O}_x\}.$$

Definition 1.3.5 (Super Bump Function)**.** A *super bump function around* $x \in U \subset M$ *supported in* U is a section $\gamma \in \mathcal{O}_0(M)$ with $\operatorname{supp}\gamma \subset U$ and $\gamma|_V = 1$ for some neighborhood $V \subset U$ of x.

If we don't require V to be large, then it is easy to see the existence of super bump functions: For any $x \in M$, let U be an arbitrary neighborhood of x and $\mathcal{W} = (W, C^\infty_{p|q})$ be a super coordinate neighborhood of x contained in U. For any neighborhood V of x with $\overline{V} \subset W$ and open set B with $\overline{V} \subset B \subset \overline{B} \subset W$, there exists by the classical theory a smooth bump function $f \in C^\infty(W) \subset \mathcal{O}_0(W) \subset C^\infty_{p|q}(W)$ with $\operatorname{supp} f \subset B$ and $f|_V = 1$. As $\operatorname{supp} f$ is closed in W and is contained by $\overline{B}$, $\operatorname{supp} f$ is closed in M, hence we can extend f by zeros to the entire M, obtaining a super bump function $\widetilde{f} \in \mathcal{O}_0(M)$ around x supported in U.

For any $s \in \mathcal{O}(U)$, let $\gamma \in \mathcal{O}_0(M)$ be a super bump function supported in U with $\gamma|_V = 1$ for some open V with $\overline{V} \subset U$, then $\operatorname{supp}\gamma|_U s \subset \operatorname{supp}\gamma$ tells that $\gamma|_U s$ can be extended by zeros to M. This gives

Lemma 1.3.1 (Extension by Bump Function)**.** *For any point $x \in U$ and any section $s \in \mathcal{O}(U)$, there exists a global section $S \in \mathcal{O}(M)$ and a neighborhood $V \subset U$ of x such that $S|_V = s|_V$ and* $\operatorname{supp} S \subset \operatorname{supp} s$*. Moreover, if s is homogeneous of parity i, then so does S.*
□

We now state and prove the locality of superderivation.

Proposition 1.3.2 (Local Feature of Superderivations)**.** *Every superderivation $D \in$* $\operatorname{Der}\mathcal{O}(U)$ *is local in the sense that for any $V \subset U$ and $s, t \in \mathcal{O}(U)$, if $s|_V = t|_V$, then $(Ds)|_V = (Dt)|_V$.*

Proof. By linearity it suffices to show that if $s|_V = 0$ then $(Ds)|_V = 0$. For any $x \in V$, there exists a super bump function $\gamma \in \mathcal{O}_0(U)$ supported in V with $\gamma|_W = 1$ for some neighborhood $W \subset V$ of x. Since $\operatorname{supp}(\gamma s) \subset \operatorname{supp} s \cap \operatorname{supp} \gamma \subset (U \setminus V) \cap V = \varnothing$, $\gamma s = 0$, hence

$$0 = (D(\gamma s))|_W = (D\gamma)|_W s|_W + \gamma|_W (Ds)|_W = (Ds)|_W.$$

Let $x \in V$ vary and we conclude that $(Ds)|_V = 0$. □

We can now construct the desired restriction map.

Proposition 1.3.3 (Restriction of Superderivations)**.** *For any $D \in \operatorname{Der} \mathcal{O}(U)$ and open subset $V \subset U$, there exists a unique $D|_V \in \operatorname{Der} \mathcal{O}(V)$ such that*

$$D|_V s|_V = (Ds)|_V \tag{1.3.2}$$

for any $s \in \mathcal{O}(U)$. Moreover, if D is homogeneous of parity i, then so is $D|_V$.

Proof. For the uniqueness, for any $D' \in \operatorname{Der} \mathcal{O}(V)$ satisfying eq. (1.3.2), we have for any $s \in \mathcal{O}(V)$ and $x \in V$, some $S \in \mathcal{O}(U)$ with $S|_W = s|_W$ for some neighborhood $W \subset V$ of x by lemma 1.3.1, which gives

$$(D's)|_W \xlongequal{\text{proposition 1.3.2}} (D'S|_V)|_W \xlongequal{\text{eq. (1.3.2)}} (DS)|_W. \tag{1.3.3}$$

By proposition 1.3.2, $(DS)|_W$ is independent of the choice of S, hence eq. (1.3.3) tells that $D's$ is locally determined by D and s. The identity axiom of sheaf concludes that D' is uniquely determined.

For the existence, we use eq. (1.3.3) as the definition of $D|_V s$ for any $s \in \mathcal{O}(V)$, i.e., $(D|_V s)|_W := (DS)|_W$ where S and W are as the above; proposition 1.3.2 ensures that this local definition glues up to give $D|_V s$. Clearly $D|_V$ satisfies eq. (1.3.2); if D is homogeneous, it is obvious that $D|_V$ has the same parity as D. It remains only to verify that $D|_V$ satisfies the graded Leibniz rule eq. (1.3.1). It suffices to do this locally: for $D \in \operatorname{Der} \mathcal{O}_i(U)$, any $s \in \mathcal{O}_j(V)$, $t \in \mathcal{O}(V)$ and $x \in V$ with $S, T \in \mathcal{O}(U)$ such that $S|_W = s|_W$ and $T|_W = t|_W$ for a neighborhood $W \subset V$ of x, we have

$$\begin{aligned}
(D|_V(st))|_W &= (D(ST))|_W = (DS)|_W T|_W + (-1)^{ij} S|_W (DT)|_W \\
&= (D|_V s)|_W t|_W + (-1)^{ij} s|_W (D|_V t)|_W \\
&= \big((D|_V s)t + (-1)^{ij} s(D|_V t)\big)\Big|_W.
\end{aligned}$$

□

Equation (1.3.2) tells that this restriction map $\operatorname{Der} \mathcal{O}(U) \to \operatorname{Der} \mathcal{O}(V) \colon D \mapsto D|_V$ is compatible with the restriction $\mathcal{O}(U) \to \mathcal{O}(V)$. The gluing of superderivations follows from that of superfunctions. Therefore $\operatorname{Der} \mathcal{O}$ gives a sheaf of super $\mathcal{O}$-modules which we call the *tangent sheaf*, denoted as $T\mathcal{M}$ for a supermanifold $\mathcal{M}$.

For $T\mathcal{M}$ to be a super vector bundle, it remains to check that it is locally free. In fact, its local description is very similar to the tangent sheaf in the classical theory.

Let $\mathcal{M}$ be of dimension $p|q$ and let $(U,(x,\xi))$ be a super coordinate neighborhood with coordinates (x,ξ). We define $p+q$ superderivations

$$\begin{aligned} \partial_{x_i} &\in \mathrm{Der}_0\, \mathcal{O}(U), \quad 1\le i\le p,\\ \partial_{\xi^j} &\in \mathrm{Der}_1\, \mathcal{O}(U), \quad 1\le j\le q, \end{aligned}$$

by putting for any $s=\sum_\alpha s_\alpha(x)\xi^\alpha \in \mathcal{O}(U)$,

$$\begin{aligned} \partial_{x_i} s &:= \sum_\alpha \left(\partial_{x_i} s_\alpha(x)\right)\xi^\alpha \\ \partial_{\xi^j} s &:= \sum_\alpha s_\alpha(x)(\partial_{\xi^j}\xi^\alpha) \end{aligned} \tag{1.3.4}$$

where

$$\partial_{\xi^j}\xi^\alpha := \begin{cases} (-1)^{\#\{\alpha_i<j\}}\xi^{(\alpha_1,\cdots,\hat{j},\cdots,\alpha_k)} & j\in\alpha\\ 0 & j\notin\alpha \end{cases}$$

where $\hat{j}$ implies that j is deleted. This is the same as putting $\partial_{\xi^j}\xi^i := \delta_{ij}$ for each $i=1,\cdots,q$, where δ is the Kronecker delta, and then extending it by the graded Leibniz rule. Also, one may think $\partial_{\xi^j}\xi^\alpha$ as reordering ξ^α such that ξ^j is the first on the left and then killing ξ^j.

Theorem 1.3.4 (Local Description of Tangent Sheaf)**.** *Let $\mathcal{M}$ be of dimension $p|q$ and let $(U,(x,\xi))$ be a super coordinate neighborhood with coordinates (x,ξ), then $(\partial_x,\partial_\xi)$ is a basis of the super $\mathcal{O}(U)$-module $T\mathcal{M}(U)$, i.e., any $X\in T\mathcal{M}(U)$ admits a unique decomposition*

$$X=\sum_{1\le i\le p}\mathcal{X}^i\partial_{x_i}+\sum_{1\le j\le q}\mathscr{X}^j\partial_{\xi^j}, \tag{1.3.5}$$

where $\mathcal{X}^i, \mathscr{X}^j\in\mathcal{O}(U)$.

Proof. The uniqueness of the decomposition is easy, because if X admits decomposition eq. (1.3.5), there must be $\mathcal{X}^i = Xx_i$ and $\mathscr{X}^j = X\xi^j$. For the existence, we put $\mathcal{X}^i = Xx_i$ and $\mathscr{X}^j = X\xi^j$ and measure the difference

$$Y := X-\left(\sum_{1\le i\le p}\mathcal{X}^i\partial_{x_i}+\sum_{1\le j\le q}\mathscr{X}^j\partial_{\xi^j}\right).$$

It suffices to show that $Ys=0$ for any $s\in\mathcal{O}(U)$. Clearly, $Yx_i = Y\xi^j = 0$ for any i,j. By the Leibniz rule we thus have

$$YP=0,$$

for any polynomial $P\in\mathbb{R}[x_1,\cdots,x_p,\xi^1,\cdots,\xi^q]\subset\mathcal{O}(U)$. The remaining follows from the technique of approximation by polynomial, theorem 1.2.5:

Fix any $s \in \mathcal{O}(U)$. For any $x_0 \in U$, there exists a polynomial P such that

$$[s]_{x_0} - [P]_{x_0} \in \mathfrak{m}_{x_0}^{q+1}.$$

By the local feature, superderivations induce maps on stalks, which gives

$$[Ys]_{x_0} = [Ys]_{x_0} - [YP]_{x_0} = Y([s]_{x_0} - [P]_{x_0}) \in Y\mathfrak{m}_{x_0}^{q+1}.$$

By lemma 1.2.4, it suffices to show that $Y\mathfrak{m}_{x_0}^{q+1} \subset \mathfrak{m}_{x_0}^{q+1}$. For any $f \in \mathfrak{m}_{x_0}^{q+1}$, we may apply a translation so that $x_0 = 0$ and then write by eq. (1.2.2),

$$\begin{aligned} f &= O(x^{q+1}) + \sum_{\alpha} O(x^q)\xi^\alpha + \cdots + O(x)\xi^1 \cdots \xi^q \\ &\xlongequal{\textit{Taylor Expansion with Lagrange Remainder}} \sum \varepsilon_I(x) x_{i_1} \cdots x_{i_{q+1-l}} \xi^{\alpha^1} \cdots \xi^{\alpha^l}, \end{aligned}$$

where $I = (i_1, \cdots, i_{q+1-l})$ and ε_I's are the smooth functions given by the Taylor expansion. Since Y vanishes on polynomials, by the Leibniz rule we obtain

$$Yf = \sum (Y\varepsilon_I(x))\, x_{i_1} \cdots x_{i_{q+1-l}} \xi^{\alpha^1} \cdots \xi^{\alpha^l},$$

which tells that $Yf \in \mathfrak{m}_{x_0}^{q+1}$ by eq. (1.2.2). □

Therefore we conclude that $T\mathcal{M}$ is indeed a super vector bundle on $\mathcal{M}$ of the same rank as the dimension of $\mathcal{M}$.

Analogous to the classical theory, we define *super tangent vectors* as

Definition 1.3.6 (Homogeneous Super Tangent Vector)**.** Let $\mathcal{M} = (M, \mathcal{O})$ be a supermanifold and and $x \in M$, a homogeneous super tangent vector of parity i at x of $\mathcal{M}$, is a derivation of parity i at x of $\mathcal{O}_x$, i.e., an $\mathbb{R}$-linear map

$$X_x \colon \mathcal{O}_x \to \mathbb{R}$$

of parity i with $\mathbb{R}$ trivially graded, such that for any $s \in \mathcal{O}_{x,j}$ and any $t \in \mathcal{O}_x$, the graded Leibniz rule

$$X_x(st) = (X_x s)(\varepsilon t)(x) + (-1)^{ij}(\varepsilon s)(x)(X_x t),$$

is satisfied.

The super $\mathbb{R}$-vector space of all super tangent vectors (i.e., the space of the $\mathbb{R}$-linear combinations of the homogeneous super tangent vectors) is denoted as $T_x\mathcal{M}$, called the *super tangent space of $\mathcal{M}$ at x.*

Let $x \in U$ and $[X]_x \in (T\mathcal{M})_x$, $[X]_x$ induces a map $\widetilde{X}\colon \mathcal{O}_x \to \mathcal{O}_x$ by its representative X. It is easy to verify that the composition $X_x := \pi \circ \widetilde{X}\colon \mathcal{O}_x \to \mathcal{O}_x \to \mathcal{O}_x/\mathfrak{m}_x = \mathbb{R}$ gives a super tangent vector at x. If X is homogeneous, then the parity of X_x is the same as X. Intuitively, this means that the "evaluation" at x of a super tangent vector field is a super tangent vector at x. Indeed, the map $T\mathcal{M}(U) \to (T\mathcal{M})_x \to T_x\mathcal{M}\colon X \mapsto X_x$ is surjective, as we have

Theorem 1.3.5 (Local Description of Super Tangent Space)**.** *Let $\mathcal{M}$ be of dimension $p|q$, $x_0 \in M$ be any point and $(U,(x,\xi))$ be a super coordinate neighborhood of x_0. The super tangent space $T_{x_0}\mathcal{M}$ is a super vector space over $\mathbb{R}$ with basis the $p+q$ vectors $\partial_{x_i,x_0} \in T_{x_0,0}\mathcal{M}$, $\partial_{\xi^j,x_0} \in T_{x_0,1}\mathcal{M}$, $1 \leq i \leq p$ and $1 \leq j \leq q$, where $\partial_{x_i} \in (T\mathcal{M})_0(U)$ and $\partial_{\xi^j} \in (T\mathcal{M})_1(U)$ are defined as in eq.* (1.3.4)*.*

Proof. The proof is essentially the same as that of theorem 1.3.4. □

Corollary 1.3.6. *The dimension of super tangent space equals the dimension of the supermanifold,*

$$\dim T_x\mathcal{M} = \dim \mathcal{M}. \qquad \square$$

Since $X_x = 0$ if and only if $\operatorname{Im}\widetilde{X} \subset \mathfrak{m}_x$, one sees immediately that the kernel of $(T\mathcal{M})_x \to T_x\mathcal{M}$ is exactly $\mathfrak{m}_x(T\mathcal{M})_x$. Hence

Corollary 1.3.7. *For any $x \in M$,*

$$T_x\mathcal{M} \cong (T\mathcal{M})_x/\mathfrak{m}_x(T\mathcal{M})_x. \qquad \square$$

Therefore this definition of the super tangent spaces fits the description at the beginning of this section.

According to definition 1.3.6, it is easy to define the super version of tangent map.

Definition 1.3.7 (Tangent Map)**.** Let $\mathcal{M} = (M,\mathcal{O})$ and $\mathcal{N} = (N,\mathcal{R})$ be two supermanifolds. Let $\Psi = (\psi,\psi^*)\colon \mathcal{M} \to \mathcal{N}$ be a morphism of supermanifolds, the tangent map $T_x\Psi$ of Ψ at $x \in M$ is the morphism of super vector spaces

$$\begin{array}{rccc} T_x\Psi\colon & T_x\mathcal{M} & \longrightarrow & T_{\psi(x)}\mathcal{N} \\ & X_x & \longmapsto & X_x \circ \psi^* \end{array}$$

where $\psi^*\colon \mathcal{R}_{\psi(x)} \to \mathcal{O}_x$ is the pullback between stalks.

Since ψ^* preserves the parity, it is easy to verify that $T_x\Psi$ is well-defined and preserves the parity.

Clearly, if Ψ is the identity morphism on $\mathcal{M}$, then $T_x\Psi$ is the identity on $T_x\mathcal{M}$ for any $x \in M$. It is also clear that taking tangent commutes with composition, i.e.,

Proposition 1.3.8. *Let $\Psi = (\psi, \psi^*)\colon \mathcal{M} \to \mathcal{N}$ and $\Phi = (\varphi, \varphi^*)\colon \mathcal{N} \to \mathcal{P}$ be morphisms of supermanifolds, then for any $x \in M$,*

$$T_x(\Phi \circ \Psi) = T_{\psi(x)}\Phi \circ T_x\Psi. \qquad \square$$

Hence we conclude that taking tangent is functorial.

Also, we have the chain rule:

Proposition 1.3.9. *Let $(\psi, \psi^*)\colon \mathcal{M} = (M, \mathcal{O}) \to (N, \mathcal{R})$ be a supermorphism. If $V \subset N$ is a coordinate neighborhood parametrized by $v = (y, \eta)$ and $\psi^{-1}(V)$ be parametrized by $u = (x, \xi)$, then*

$$\partial_{u^{\mathfrak{a}}} \circ \psi^* = \sum_{\mathfrak{b}} \partial_{u^{\mathfrak{a}}}(\psi^* v^{\mathfrak{b}})\psi^* \circ \partial_{v^{\mathfrak{b}}},$$

where, with $\dim \mathcal{M} = p|q$,

$$u^{\mathfrak{a}} := \begin{cases} x_{\mathfrak{a}} & 1 \leq \mathfrak{a} \leq p, \\ \xi^{\mathfrak{a}-p} & p+1 \leq \mathfrak{a} \leq p+q. \end{cases}$$

and similar convention applies to $v^{\mathfrak{b}}$.

Proof. It is easy to see that the equality holds on evaluation at polynomial sections, hence the similar argument in the proof of theorem 1.3.4 applies. $\square$

Restrict these to the stalk at a point $\psi(x) \in V$ and take the quotient by the maximal ideal (or one can use a similar proof as the above), we obtain

$$T_x\Psi(\partial_{u^{\mathfrak{a}},x}) = \sum_{\mathfrak{b}} \partial_{u^{\mathfrak{a}},x}(\psi^* v^{\mathfrak{b}})\partial_{v^{\mathfrak{b}},\psi(x)}.$$

Hence we have a matrix representation of the tangent map, the Jacobian, similar to that in classical case. However, tedious sign appears if we want to arrange proposition 1.3.9 in the usual manner of matrix multiplication when dealing with composition of supermorphisms. To fix this, the matrix should be modified.

Definition 1.3.8 (The Modified Super Jacobian Matrix)**.** The *modified super Jacobian matrix* of a supermorphism $\Psi\colon \mathcal{M} \to \mathcal{N}$, where $\dim \mathcal{M} = p|q$ and $\dim \mathcal{N} = p'|q'$, under local coordinates $V \subset N$ with (y, η) and $\psi^{-1}(V)$ with (x, ξ), is the $(p'+q') \times (p+q)$ supermatrix, written in the convention $y = y(x, \xi) = \psi^* y$ and $\eta = \eta(x, \xi) = \psi^* \eta$,

$$J\Psi = \begin{pmatrix} \partial_x y & \partial_\xi y \\ -\partial_x \eta & \partial_\xi \eta \end{pmatrix}.$$

On evaluation at a point, we have

$$J_{x_0}\Psi = \begin{pmatrix} \partial_{x,x_0}y & 0 \\ 0 & \partial_{\xi,x_0}\eta \end{pmatrix} = \begin{pmatrix} \partial_{x,x_0}y & \partial_{\xi,x_0}y \\ \partial_{x,x_0}\eta & \partial_{\xi,x_0}\eta \end{pmatrix} = \text{the matrix representation of } T_{x_0}\Psi.$$

With the modified Jacobian matrix, we have by direct computation using proposition 1.3.9,

Proposition 1.3.10. *Let $\Psi\colon \mathcal{M} \to \mathcal{N}$ and $\Phi\colon \mathcal{N} \to \mathcal{P}$ be two supermorphisms, then under any local coordinates representation, we have*

$$J(\Phi \circ \Psi) = J\Phi \cdot J\Psi$$

where entries of $J\Phi$ are considered as their pullback by Ψ when taking the matrix multiplication. □

Remark 1.3.1. The reason why only the lower-left term has a minus sign can be explained by

$$\partial_{x_i} \mapsto \partial_{x_i}(y_j)\partial_{y_j} + \partial_{x_i}(\eta^k)\partial_{\eta^k} = \partial_{y_j} \cdot (\partial_{x_i}(y_j)) - \partial_{\eta^k} \cdot (\partial_{x_i}(\eta^k))$$
$$\partial_{\xi^i} \mapsto \partial_{\xi^i}(y_j)\partial_{y_j} + \partial_{\xi^i}(\eta^k)\partial_{\eta^k} = \partial_{y_j} \cdot (\partial_{\xi^i}(y_j)) + \partial_{\eta^k} \cdot (\partial_{\xi^i}(\eta^k)).$$

That is, to make the left-multiplication of matrix representations of morphisms of supermodules be compatible with the composition of morphisms, the entries should be the scalars written at the right-hand side of the basis, instead of those written at the left-hand side as in the classical case.

Definition 1.3.9 (Cotangent Sheaf)**.** The *cotangent sheaf* of a supermanifold $\mathcal{M} = (M, \mathcal{O})$ is the dual of its tangent sheaf, i.e., it is the sheaf of morphisms of sheaves

$$\Omega^1\mathcal{M} \coloneqq T^*\mathcal{M} \coloneqq \mathbf{Hom}_{\text{Sheaf of } \mathcal{O}\text{ - Mod}}(T\mathcal{M}, \mathcal{O}).$$

The sections of $\Omega^1\mathcal{M}$ are called *super differential* 1*-forms*.

Note that $\Omega^1\mathcal{M}$ is also a sheaf of super $\mathcal{O}$-modules.

Definition 1.3.10 (Differential of Superfunction)**.** For any open subset $U \subset M$ and $i \in \mathbb{Z}_2$, we define the *differential of a superfunction* $f \in \mathcal{O}_i(U)$, for any open $V \subset U$,

$$\mathrm{d}_V f \in \mathbf{Hom}_i(T\mathcal{M}(V), \mathcal{O}(V)),$$

by

$$(\mathrm{d}_V f)(D) = (-1)^{ij} Df|_V \in \mathcal{O}(V),$$

for all $D \in (T\mathcal{M})_j(V) = \mathrm{Der}_j\, \mathcal{O}(V)$. Clearly, $\mathrm{d}_V f$ gives a morphism of sheaves from $T\mathcal{M}|_U$ to $\mathcal{O}|_U$ as V varies, hence $\mathrm{d}f \in \Omega^1\mathcal{M}(U)$. For non-homogeneous f, $\mathrm{d}f$ is defined as the sum of differentials of the homogeneous components of f.

The differentiation d_V preserves the parity and hence gives an $\mathcal{O}(V)$-module morphism. By the definition of restriction of superderivation, we see that d gives a morphism of sheaves from $\mathcal{O}$ to $\Omega^1\mathcal{M}$.

Moreover, it is easy to verify that

Proposition 1.3.11. *For any $f, g \in \mathcal{O}(U)$,*

$$\mathrm{d}(fg) = (\mathrm{d}f)g + f(\mathrm{d}g). \qquad \square$$

Note that everything satisfies the Koszul sign rule.

Being the dual of the locally free $T\mathcal{M}$, $\Omega^1\mathcal{M}$ is also locally free with the dual basis.

Theorem 1.3.12. *The cotangent sheaf $\Omega^1\mathcal{M}$ is locally free with basis $(\mathrm{d}x_1, \cdots, \mathrm{d}\xi^q)$ in a coordinate neighborhood $(U, (x, \xi))$.* $\square$

By definition,

$$\mathrm{d}x_i(\partial_{x_j}) = \delta_{ij}, \quad \mathrm{d}x_i(\partial_{\xi^j}) = 0.$$
$$\mathrm{d}\xi^i(\partial_{x_j}) = 0, \quad \mathrm{d}\xi^i(\partial_{\xi^j}) = -\delta_{ij}.$$

Therefore, any super differential 1-form ω reads locally as

$$\omega = \sum_i \mathrm{d}x_i f_i(x,\xi) + \sum_j \mathrm{d}\xi^j g_j(x,\xi),$$

where the coefficients f_i's are given by $\omega(\partial_{x_i})$ and g_j's are given by $\omega_0(\partial_{\xi^j}) - \omega_1(\partial_{\xi^j})$ where ω_0 and ω_1 are the even and odd homogeneous components of ω, as one can verify. It follows that

$$\mathrm{d}f = \sum_i \mathrm{d}x_i(\partial_{x_i} f) + \sum_j \mathrm{d}\xi^j(\partial_{\xi^j} f),$$

so d reads locally as

$$\mathrm{d} = \sum_i \mathrm{d}x_i \partial_{x_i} + \sum_j \mathrm{d}\xi^j \partial_{\xi^j}.$$

By wedging the 1-forms up, we can talk about k-forms in the super context.

Let A and B be two supermodules over a supercommutative ring R, their tensor product, $A \otimes_R B := R^{A\times B}/\sim$ is the free R-module $R^{A\times B} := \bigoplus_{(a,b)\in A\times B} R_{(a,b)}$ modulo the relations

$$1_{(a+a',b)} =: (a+a', b) = (a,b) + (a',b), \quad (a, b+b') = (a,b) + (a,b'),$$

$$r_{(a,b)} =: r(a,b) = (ra, b), \quad (ar, b) = (a, rb),$$

for any $a, a' \in A$, $b, b' \in B$ and $r \in R$. Writing $a \otimes b$ for the equivalence class of (a,b),

$A \otimes_R B$ is naturally $\mathbb{Z}_2$-graded by

$$A \otimes_R B = \bigoplus_{k \in \mathbb{Z}_2} \bigoplus_{i+j=k} \left\{ \sum a \otimes b \,\middle|\, a \in A_i,\ b \in B_j \right\}.$$

With the induced right module structure on A and B, one sees that the induced right module structure on $A \otimes_R B$ obeys Koszul sign rule. The tensor product of two morphisms of super R-modules $f\colon A \to A'$ and $g\colon B \to B'$ is defined by

$$\begin{array}{rccc} f \otimes_R g\colon & A \otimes_R B & \longrightarrow & A' \otimes_R B' \\ & a \otimes b & \longmapsto & (-1)^{p(g)p(a)} f(a) \otimes g(b) \end{array}$$

for homogeneous g and $a \in A$; the definition for non-homogeneous cases is given by the sum of homogeneous components. Note that this makes $(f \otimes_R g)(a \otimes b) = (-1)^{p(g)p(a)} f(a) \otimes g(b)$ follow the sign rule.

For a single supermodule A over supercommutative ring R, we can consider its tensor with itself. Write $A^{\otimes n} := A \otimes_R \cdots \otimes_R A$ for the tensor product of n copies of A and $A^{\otimes 0} = R$ by convention, the super R-module

$$T^\bullet A = \bigoplus_{n \geq 0} A^{\otimes n}$$

is the *tensor super R-algebra of the supermodule A*, where the direct sum is that of modules, which gives a $\mathbb{Z}_2$-grading on $T^\bullet A$. The super R-module $T^\bullet A$ becomes a superring when equipped with tensor product as the multiplication, and its R-algebra structure follows from the inclusion $R = A^{\otimes 0} \hookrightarrow T^\bullet A$.

Now, to define the wedge product, we use Deligne's formalism. We put the ideal $I_A := (a \otimes a' + (-1)^{p(a)p(a')} a' \otimes a \mid a, a' \in A_0 \cup A_1)$ of $T^\bullet A$. The *exterior super R-algebra of the supermodule A* is the quotient

$$\wedge_D A := T^\bullet A / I_A.$$

We often omit the subscript D and write $a \wedge a'$ as the equivalence class of $a \otimes a'$ in $\wedge A$. Apart from the $\mathbb{Z}_2$-grading, $\wedge A$ is also graded cohomologically by

$$\wedge A = \bigoplus_{n \geq 0} \wedge^n A.$$

Definition 1.3.11 (Super Differential Form)**.** Let $\mathcal{M} = (M, \mathcal{O})$ be a supermanifold. For any open $U \subset M$, the set of *super differential forms* over U is defined by

$$(\Omega\mathcal{M})(U) := \wedge(\Omega^1\mathcal{M})(U) = \bigoplus_{k \geq 0} \wedge^k (\Omega^1\mathcal{M})(U).$$

Elements in $(\Omega^k\mathcal{M})(U) := \wedge^k(\Omega^1\mathcal{M})(U)$ are called *super differential k-forms on U*.

The sheaf structure of $\Omega^1\mathcal{M}$ induces the sheaf structure of $\Omega\mathcal{M}$. Note that, unlike the classical case, there not necessarily no top forms in $\Omega\mathcal{M}$, because the wedge of two odd elements is symmetric. It is easy to check by definition that

$$f(\omega\wedge\omega') = (-1)^{i(p(\omega)+p(\omega'))}(\omega\wedge\omega')f,$$
$$\omega\wedge\omega' = (-1)^{kl+p(\omega)p(\omega')}\omega'\wedge\omega,$$

for any homogeneous $f\in\mathcal{O}_i(U)$, $\omega\in(\Omega^k\mathcal{M})(U)$ and $\omega'\in(\Omega^l\mathcal{M})(U)$.

Any k-form ω reads locally (non-uniquely)

$$\omega|_U = \sum f\,\mathrm{d}f_1\wedge\cdots\wedge\mathrm{d}f_k$$

for some open U and f's in $\mathcal{O}(U)$. The exterior differentiation $\mathrm{d}\colon\mathcal{O}\to\Omega^1\mathcal{M}$ extends uniquely to give $\mathrm{d}\colon\Omega\mathcal{M}\to\Omega\mathcal{M}$ by

$$(\mathrm{d}\omega)|_U := \sum \mathrm{d}f\wedge\mathrm{d}f_1\wedge\cdots\wedge\mathrm{d}f_k.$$

The proof of the well-definedness of d and the following propositions is necessarily identical to the corresponding results in the classical theory.

Proposition 1.3.13. *The operator* d *is a derivation which squares to zero:*

$$\mathrm{d}(\omega\wedge\omega') = \mathrm{d}\omega\wedge\omega' + (-1)^k\omega\wedge\mathrm{d}\omega'$$

for any form ω' and k-form ω, and

$$\mathrm{d}^2 = 0.$$

1.4 Integration on Supermanifolds

The Berezinian is the super version of determinant. Given a $(p+q)\times(p+q)$ supermatrix T, i.e., a matrix whose entries live in a superring R blocked as

$$T = \begin{pmatrix} K & L \\ M & N \end{pmatrix},$$

where K is $p\times p$, N is $q\times q$, K and N have even entries and L and M have odd ones. If T is invertible, then T quotient the nilpotent ideal I of R,

$$\begin{pmatrix} K \bmod I & 0 \\ 0 & N \bmod I \end{pmatrix}$$

is also invertible with inverse the quotient of T^{-1}. Since K and N are even with entries living in the commutative ring R_0, det(K) and det(N) are defined and are even units adding nilpotents, which are again units, telling that K and N are invertible. With this we decompose

$$T = \begin{pmatrix} K & L \\ M & N \end{pmatrix} = \begin{pmatrix} 1 & LN^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} K - LN^{-1}M & 0 \\ M & N \end{pmatrix},$$

in light of which the Berezinian of T is defined by

$$\operatorname{Ber}(T) \coloneqq \det(K - LN^{-1}M)\det(N)^{-1}. \tag{1.4.1}$$

Note that $K - LN^{-1}M$ has even entries, the determinant det$(K - LN^{-1}M)$ does make sense. Also, Ber(T) is even. By a tricky matrix argument in the paper of Leites[2], we have

Proposition 1.4.1. *Given two $(p+q) \times (p+q)$ invertible supermatrices X and Y, then*

$$\operatorname{Ber}(XY) = \operatorname{Ber}(X) \cdot \operatorname{Ber}(Y). \qquad \square$$

To integrate on supermanifolds, we define firstly the integration on superdomains. As usual, functions are integrated via densities:

Definition 1.4.1 (Densities on Superdomains)**.** Let $(t_1, \cdots, t_p, \theta^1, \cdots, \theta^q)$ be the standard coordinates of $\mathbb{R}^{p|q}$, a *density* on a superdomain $\mathcal{U}^{p|q} = (U, C^\infty_{p|q})$ is an $\mathbb{R}$-linear map $C^\infty_{p|q,c}(U) \to \mathbb{R}$, where $C^\infty_{p|q,c}(U)$ is the set of compactly supported superfunctions on U, of the form

$$\begin{array}{ccc} C^\infty_{p|q,c}(U) & \longrightarrow & \mathbb{R} \\ g = \sum_I g_I(t)\theta^I & \longmapsto & \sum_I \int_U g_I(t) f_I(t)\, \mathrm{d}t_1 \cdots \mathrm{d}t_p \end{array}$$

for some $f_I(t) \in C^\infty(U)$, where $I = (i_1, \cdots, i_k)$ with $i_1 < \cdots < i_k$ varies among all nonempty strictly increasing multi-indices of dimension no more than q and $\theta^I \coloneqq \theta^{i_1} \cdots \theta^{i_k}$.

Usually we write a density μ as $g \mapsto \int \mu g$. Clearly, a density on $\mathcal{U}^{p|q}$ restricts to a density on any open sub-supermanifold of $\mathcal{U}^{p|q}$. Also, the gluing of smooth functions gives the gluing of densities, telling that the set of densities form a sheaf of super vector spaces on $\mathbb{R}^{p|q}$, with parity defined by their parity as super linear maps. Moreover, the set of densities on $\mathcal{U}^{p|q}$ has a natural structure of $C^\infty_{p|q}$-module, given by

$$\int (\mu u) g \coloneqq \int \mu(ug)$$

for any $u \in C^\infty_{p|q}(U)$ and $g \in C^\infty_{p|q,c}(U)$.

It is easy to see that the density $[\mathrm{d}t_1 \cdots \mathrm{d}t_p\, \mathrm{d}\theta^1 \cdots \mathrm{d}\theta^q]$ on $\mathcal{U}^{p|q}$ defined by

$$g = g_0(t) + \cdots + h(t)\theta^q \cdots \theta^1 \mapsto \int_U \mathrm{d}t_1 \cdots \mathrm{d}t_p h(t)$$

is a basis of the $C^\infty_{p|q}$-module $\mathcal{D}(\mathcal{U}^{p|q})$ of densities on $\mathcal{U}^{p|q}$. Hence

Proposition 1.4.2. *The $C^\infty_{p|q}$-module $\mathcal{D}(\mathcal{U}^{p|q})$ is free of rank $1|0$ if q is even, of rank $0|1$ if q is odd.* □

We have the change of variables formula, which is proposition 3.10.2 in the notes of Deligne & Morgan[3]

Proposition 1.4.3 (Change of Variables). *Let $\Phi = (\varphi, \varphi^*) \colon \mathcal{U}^{p|q} \to \mathcal{V}^{p|q}$ be an isomorphism, then for any $g \in C^\infty_c(\mathcal{V}^{p|q}) := C^\infty_{p|q,c}(V)$,*

$$\int_V [\mathrm{d}t_1 \cdots \mathrm{d}t_p\, \mathrm{d}\theta^1 \cdots \mathrm{d}\theta^q] g = \pm \int_U [\mathrm{d}t_1 \cdots \mathrm{d}t_p\, \mathrm{d}\theta^1 \cdots \mathrm{d}\theta^q]\, \mathrm{Ber}(J\Phi)\varphi^*(g),$$

where the sign is positive if φ is orientation-preserving and is negative if it is orientation-reversing. □

In particular a density precomposed with a change of parametrization is still a density.

Note that a compactly supported function on an open submanifold $\mathcal{U}$ of $\mathcal{M}$ is extendible by zeros, giving a inclusion $C^\infty_c(\mathcal{U}) \hookrightarrow C^\infty_c(\mathcal{M})$. A *density on supermanifold* $\mathcal{M}$ is thus defined to be an assignment $C^\infty_c(\mathcal{M}) \to \mathbb{R}$ that is locally a density on superdomains, i.e., the composition $C^\infty_c(\mathcal{U}) \hookrightarrow C^\infty_c(\mathcal{M}) \to \mathbb{R}$ is a density for any coordinate neighborhood $\mathcal{U}$ in $\mathcal{M}$.

One sees immediately that this is a natural generalization of the integration on ordinary manifolds.

Example 1.4.1. For an ordinary orientable manifold M, a top form ω along with an orientation of M gives a density on M by $g \mapsto \int_M g\omega$.

1.5 Generalized Supermanifolds and $\mathbb{R}^{1|1}$

Since sheaves are generally difficult to work with, one often thinks of supermanifolds in term of their S-points, i.e., instead of $\mathcal{M}$ itself one considers the sets $\mathrm{SMan}(S, \mathcal{M})$ as S varies among all supermanifolds[7]; theorem 1.2.7 makes it easy to describe the set $\mathrm{SMan}(S, \mathcal{M})$. The fully faithful Yoneda embedding $\mathrm{SMan} \to \mathrm{Hom}_{\mathrm{Cat}}(\mathrm{SMan}^{op}, \mathrm{Set}) \colon \mathcal{M} \mapsto \mathrm{SMan}(-, \mathcal{M})$ gives an equivalence of categories between SMan and the subcategory of representable functors in $\mathrm{Hom}_{\mathrm{Cat}}(\mathrm{SMan}^{op}, \mathrm{Set})$. These motivate that

Definition 1.5.1 (Generalized Supermanifolds)**.** A *generalized supermanifold* is a functor $\mathcal{F}\colon\ \mathrm{SMan}^{op} \to \mathrm{Set}$.

What are smooth (super)functions on generalized supermanifolds? In the ordinary theory, we have $C^\infty(M) = \mathrm{Man}(M, \mathbb{R})$. For a supermanifold $\mathcal{M}$, by theorem 1.2.6 we have as sets

$$\mathrm{SMan}(\mathcal{M}, \mathbb{R}^{1|1}) \cong \{\{x\} \to C_0^\infty(\mathcal{M})\} \times \{\{\theta\} \to C_1^\infty(\mathcal{M})\} \cong C_0^\infty(\mathcal{M}) \oplus C_1^\infty(\mathcal{M}) = C^\infty(\mathcal{M}).$$

Moreover, noticing by the universal property that $\mathbb{R}^{1|1} \times \mathbb{R}^{1|1} \cong \mathbb{R}^{2|2}$, $\mathbb{R}^{1|1}$ is equipped with an algebra structure

- Addition:

$$\begin{aligned} +\colon \mathbb{R}^{1|1} \times \mathbb{R}^{1|1} \to \mathbb{R}^{1|1}\colon \quad C^\infty(\mathbb{R}^{1|1}) &\longrightarrow C^\infty(\mathbb{R}^{1|1} \times \mathbb{R}^{1|1}) \\ x &\longmapsto x_1 + x_2 \\ \theta &\longmapsto \theta_1 + \theta_2 \end{aligned}$$

- Multiplication:

$$\begin{aligned} \cdot\colon \mathbb{R}^{1|1} \times \mathbb{R}^{1|1} \to \mathbb{R}^{1|1}\colon \quad C^\infty(\mathbb{R}^{1|1}) &\longrightarrow C^\infty(\mathbb{R}^{1|1} \times \mathbb{R}^{1|1}) \\ x &\longmapsto x_1 x_2 + \theta_1 \theta_2 \\ \theta &\longmapsto x_1 \theta_2 + x_2 \theta_1 \end{aligned}$$

Strictly speaking, $\mathbb{R}^{1|1}$ is not an algebra with these; but $\mathrm{SMan}(\mathcal{M}, \mathbb{R}^{1|1})$ becomes a super $\mathbb{R}$-algebra such that the above set-bijection is an isomorphism of super algebras, with addition and multiplication graphically

$$\begin{tikzcd}
 & \mathcal{M} \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, dashed, "{(f,g)}"] \arrow[dd, bend right=60, "f\cdot g"'] \arrow[dd, bend left=60, "f+g"] & \\
\mathbb{R}^{1|1} & \mathbb{R}^{1|1} \times \mathbb{R}^{1|1} \arrow[l] \arrow[r] \arrow[d, shift right, "\cdot"'] \arrow[d, shift left, "+"] & \mathbb{R}^{1|1} \\
 & \mathbb{R}^{1|1} &
\end{tikzcd}$$

and the grading inherited from $C^\infty(\mathcal{M})$.

Remark 1.5.1. The algebra structure on $\mathrm{SMan}(\mathcal{M}, \mathbb{R}^{1|1})$ can also be seen as induced from $C^\infty(\mathcal{M})$ via the set-bijection, which spelled out is exactly the above.

Remark 1.5.2. It follows that $\mathbb{R}^{1|1}$ represents the functor $C^\infty\colon \text{SMan}^{op} \to \mathbb{R}\text{-SAlg}\colon \mathcal{M} \mapsto C^\infty(\mathcal{M})$.

By the Yoneda Lemma,

$$C^\infty(\mathcal{M}) \cong \text{SMan}(\mathcal{M}, \mathbb{R}^{1|1}) \cong \text{Hom}(\text{SMan}(-, \mathcal{M}), \text{SMan}(-, \mathbb{R}^{1|1})). \tag{1.5.1}$$

Moreover, the super algebra structure on $\text{SMan}(S, \mathbb{R}^{1|1})$ induces a super algebra structure on $\text{Hom}(\text{SMan}(-, \mathcal{M}), \text{SMan}(-, \mathbb{R}^{1|1}))$ which makes the above composition an isomorphism of super algebras. Therefore smooth functions on $\mathcal{M}$ can be defined as natural transformations from the functor $\text{SMan}(-, \mathcal{M})$ to $\text{SMan}(-, \mathbb{R}^{1|1}) \cong C^\infty$. This generalizes to give smooth functions on generalized supermanifolds, i.e., for a generalized supermanifold $\mathcal{F}\colon \text{SMan}^{op} \to \text{Set}$, we put

$$C^\infty(\mathcal{F}) \coloneqq \text{Hom}(\mathcal{F}, \text{SMan}(-, \mathbb{R}^{1|1})) \cong \text{Hom}(\mathcal{F}, C^\infty).$$

In terms of the S-points, the algebra structure of $C^\infty(\mathcal{F})$ is described by

$$\begin{array}{rccc} f+g\colon & \mathcal{F}(S) & \longrightarrow & C^\infty(S) \\ & x & \longmapsto & f(x)+g(x) \end{array}$$

$$\begin{array}{rccc} f\cdot g\colon & \mathcal{F}(S) & \longrightarrow & C^\infty(S) \\ & x & \longmapsto & f(x)\cdot g(x) \end{array}$$

for any $f, g \in C^\infty(\mathcal{F})$.

2. A Discussion on $\underline{\text{SMan}}(\mathbb{R}^{0|\delta}, X)$

Let δ be a non-negative integer and X be an ordinary manifold of dimension n. In the following we consider the generalized supermanifold $\underline{\text{SMan}}(\mathbb{R}^{0|\delta}, X)$ defined by

$$\begin{array}{rccc} \underline{\text{SMan}}(\mathbb{R}^{0|\delta}, X)\colon & \text{SMan}^{op} & \longrightarrow & \text{Set} \\ & S & \longmapsto & \text{SMan}(S \times \mathbb{R}^{0|\delta}, X) \end{array}$$

whose action on morphisms is obvious. It is a basic but important object in the study of supersymmetric field theories, for example see the paper of Stolz and Teichner[8]. Also, a proof of the Chern-Gauss-Bonnet theorem using $\underline{\text{SMan}}(\mathbb{R}^{0|2}, X)$ is given by Berwick-Evans[4], whose outline is described in section 2.3.

2.1 General Properties of $\underline{\mathrm{SMan}}(\mathbb{R}^{0|\delta}, X)$

Let $\theta^1, \cdots, \theta^\delta$ be the (odd) coordinates on $\mathbb{R}^{0|\delta}$, the universal property of product tells that $S \times \mathbb{R}^{0|\delta}$ is in fact the supermanifold $(|S| \times \mathrm{pt}, C^\infty(S)[\theta^1, \cdots, \theta^\delta])$, where $|S|$ denotes the base manifold of S.

According to theorem 1.2.7, an S-point $\Phi \in \underline{\mathrm{SMan}}(\mathbb{R}^{0|\delta}, X)(S)$ is determined by its morphism of super $\mathbb{R}$-algebras $\Phi^*\colon C^\infty(X) \to C^\infty(S \times \mathbb{R}^{0|\delta}) = C^\infty(S)[\theta^1, \cdots, \theta^\delta]$. The direct sum decomposition of $C^\infty(S)[\theta^1, \cdots, \theta^\delta]$ via θ-coordinates gives

$$\Phi^* = f + \sum_I \phi_I \theta^I,$$

where $I = (i_1, \cdots, i_k)$ with $i_1 < \cdots < i_k$ varies among all nonempty strictly increasing multi-indices of dimension no more than q, $\theta^I := \theta^{i_1} \cdots \theta^{i_k}$ and $f, \phi_I\colon C^\infty(X) \to C^\infty(S)$ are linear maps. The requirement that Φ^* is a super $\mathbb{R}$-algebra morphism gives further restrictions to f and ϕ_I; for instance, f must be a super $\mathbb{R}$-algebra morphism, hence it induces a map of supermanifolds $(S =)S \times \mathrm{pt} \to X$.

Given a super $\mathbb{R}$-algebra homomorphism $\varphi\colon A \to B$, an $\mathbb{R}$-linear map $\psi\colon A \to B$ is an *even derivation with respect to* φ if $\psi(A_i) \subset B_i$ for each $i \in \mathbb{Z}_2$ and

$$\psi(ab) = \psi(a)\varphi(b) + \varphi(a)\psi(b), \quad \forall a, b \in A;$$

is an *odd derivation with respect to* φ if $\psi(A_i) \subset B_{i+1}$ for each $i \in \mathbb{Z}_2$ and

$$\psi(ab) = \psi(a)\varphi(b) + (-1)^{p(a)}\varphi(a)\psi(b), \quad \forall a \in A_0 \cup A_1,\ b \in A.$$

Derivations with respect to φ are the linear combinations of even and odd ones. The set of all derivations with respect to φ is denoted as $\mathrm{Der}_\varphi(A, B)$.

It is easy to verify the following lemma:

Lemma 2.1.1. *Let $\mathcal{M}$ and $\mathcal{N}$ be two supermanifolds and $f\colon \mathcal{N} \to \mathcal{M}$ be a morphism of supermanifolds. The map*

$$\begin{array}{ccc} (\mathrm{Der}\, C^\infty(\mathcal{M})) \otimes_f C^\infty(\mathcal{N}) & \longrightarrow & \mathrm{Der}_f(C^\infty(\mathcal{M}), C^\infty(\mathcal{N})) \\ D \otimes_f g & \longmapsto & g \cdot (f^* \circ D) \end{array}$$

is an isomorphism of $C^\infty(\mathcal{N})$-modules, whose inverse is expressed locally in coordinates $u = (x, \xi)$ on $\mathcal{M}$, by

$$\begin{array}{ccc} \mathrm{Der}_f(C^\infty(\mathcal{M}), C^\infty(\mathcal{N})) & \longrightarrow & (\mathrm{Der}\, C^\infty(\mathcal{M})) \otimes_f C^\infty(\mathcal{N}) \\ V & \longmapsto & \sum \partial_{u^{\mathfrak{a}}} \otimes_f V(u^{\mathfrak{a}}) \end{array}$$

$\square$

Example 2.1.1 ($\underline{\mathrm{SMan}}(\mathbb{R}^{0|1}, X)$)**.** The S-points of $\underline{\mathrm{SMan}}(\mathbb{R}^{0|1}, X)$ are super $\mathbb{R}$-algebra homomorphisms

$$\Phi^* = f + \phi\theta.$$

The Φ^* is an algebra homomorphism if and only if f is a super $\mathbb{R}$-algebra homomorphism and $\phi\colon C^\infty(X) \to C^\infty(S)$ is an odd derivation with respect to f, i.e.,

$$\phi(ab) = \phi(a)f(b) + (-1)^{p(a)} f(a)\phi(b), \quad \forall a, b \in C^\infty(X),$$

where $(-1)^{p(a)}$ is in fact always 1 since a is always even.

By abuse of notation, the isomorphism in lemma 2.1.1 identifies ϕ with an element $\phi \in (\mathrm{Der}\, C^\infty(X)) \otimes_f C^\infty(S)$. This makes ϕ a global section of the *pullback sheaf* (also called the *inverse image functor*) $f^*\pi TX$ on S, where πTX, again by abuse of notation, is the parity-reversed tangent sheaf on X.

Recall in example 1.2.2 that $C^\infty(\pi TX) = \Omega^\bullet(X)$; global sections in $f^*\pi TX$ are in fact one-to-one to S-points in $\underline{\pi TX}(S) \coloneqq \underline{\mathrm{SMan}}(S, \pi TX)$, via the map

$$\begin{array}{ccccc} (\mathrm{Der}\, C^\infty(X)) \otimes_f C^\infty(S) & \longrightarrow & \mathbb{R}\text{-}\mathrm{SAlg}(\Omega^\bullet(X), C^\infty(S)) & \cong & \mathrm{SMan}(S, \pi TX) \\ D \otimes_f g & \longmapsto & (\omega \mapsto g \cdot f^*\omega(D)) & & \end{array}$$

The inverse to this map sends a morphism $\varphi^*\colon \Omega^\bullet(X) \to C^\infty(S)$ to the composition $C^\infty(X) \xrightarrow{\mathrm{d}} \Omega^\bullet(X) \xrightarrow{\varphi^*} C^\infty(S)$, which is a derivation with respect to $f\colon C^\infty(X) \hookrightarrow \Omega^\bullet(X) \to C^\infty(S)$.

These conclude that

$$\underline{\mathrm{SMan}}(\mathbb{R}^{0|1}, X) \cong \underline{\pi TX},$$

which means that $\underline{\mathrm{SMan}}(\mathbb{R}^{0|1}, X)$ is represented by πTX.

More generally, one sees that $\phi_i\colon C^\infty(X) \to C^\infty(S)$ is an odd derivation with respect to f for each $i = 1, \cdots, \delta$.

Recall that in section 1.5, $C^\infty(\underline{\mathrm{SMan}}(\mathbb{R}^{0|\delta}, X))$ is identified with the set of natural transformations $\underline{\mathrm{SMan}}(\mathbb{R}^{0|\delta}, X) \Rightarrow C^\infty$. Given a function $a \in C^\infty(X)$, it induces maps

$$\begin{array}{rccc} a\colon & \underline{\mathrm{SMan}}(\mathbb{R}^{0|\delta}, X)(S) & \longrightarrow & C^\infty(S) \\ & \Phi^* = f + \sum_I \phi_I \theta^I & \longmapsto & f(a) \end{array}$$

with obvious naturality in S, giving a superfunction on $\underline{\mathrm{SMan}}(\mathbb{R}^{0|\delta}, X)$. Since f is an algebra homomorphism, this gives an inclusion of algebras $C^\infty(X) \hookrightarrow C^\infty(\underline{\mathrm{SMan}}(\mathbb{R}^{0|\delta}, X))$.

Similarly, we can define for each multi-index I,

$$\begin{array}{rccc} \mathrm{d}_I a\colon & \underline{\mathrm{SMan}}(\mathbb{R}^{0|\delta}, X)(S) & \longrightarrow & C^\infty(S) \\ & \Phi^* = f + \sum_I \phi_I \theta^I & \longmapsto & \phi_I(a) \end{array}$$

whose naturality is also clear. The following suggests that we may write $\mathrm{d}_{i_k} \cdots \mathrm{d}_{i_1} := \mathrm{d}_I$ for more natural symbolic computation.

Let $x = (x_i)$ be local coordinates on X, then for any $g \in C^\infty(X)$, we have locally

$$\begin{aligned} f(a) + \sum_I \phi_I(a)\theta^I = \Phi^*(a) &= a\left(f(x) + \sum_I \phi_I(x)\theta^I\right) \\ &= a(f(x)) + \sum_\beta \frac{1}{\beta!}(\partial_x^\beta a)\,(f(x))\left(\sum_I \phi_I(x)\theta^I\right)^\beta. \end{aligned}$$

For $\delta = 2$, we compute

$$\begin{aligned} &f(a) + \phi_1(a)\theta^1 + \phi_2(a)\theta^2 + \phi_{(1,2)}(a)\theta^1\theta^2 \\ &\qquad = a(f(x)) + \frac{\partial a}{\partial x_k}(f(x))(\phi_1(x_k)\theta^1 + \phi_2(x_k)\theta^2) \\ &\qquad + \left(\frac{\partial^2 a}{\partial x_k \partial x_l}(f(x))\phi_2(x_l)\phi_1(x_k + \frac{\partial a}{\partial x_k}(f(x))\phi_{(1,2)}(x_k)\right)\theta^1\theta^2. \end{aligned}$$

Hence we conclude

$$\begin{aligned} \mathrm{d}_i a &= \frac{\partial a}{\partial x_k}\,\mathrm{d}_i x_k, \quad i = 1, 2, \\ \mathrm{d}_2\,\mathrm{d}_1 a &= \frac{\partial^2 a}{\partial x_k \partial x_l}\,\mathrm{d}_2 x_l\,\mathrm{d}_1 x_k + \frac{\partial a}{\partial x_k}\,\mathrm{d}_2\,\mathrm{d}_1 x_k. \end{aligned}$$

These can be generalized to any $\delta \geq 0$.

Remark 2.1.1. For d_i's this turns out to be natural: as $\phi_i \in \mathrm{Der}_f(C^\infty(X), C^\infty(S))$, we see that $\mathrm{d}_i g$ is in fact the composition

$$\begin{array}{cccccc} C^\infty(S \times \mathbb{R}^{0|\delta}) & \longrightarrow & \mathrm{Der}_f(C^\infty(X), C^\infty(S)) & \xrightarrow{f^*\,\mathrm{d}g} & C^\infty(S) \\ \Phi^* = f + \sum_I \phi_I \theta^I & \longmapsto & \phi_i & \mapsto & f^*\,\mathrm{d}g(\phi_i) = \phi_i(g). \end{array}$$

In particular, via the Yoneda lemma, one sees that for $\delta = 1$, $\mathrm{d}g \in C^\infty(\underline{\mathrm{SMan}}(\mathbb{R}^{0|1}, X)) \cong \Omega^\bullet(X)$ is exactly the 1-form of the exterior differentiation of g.

2.2 Details about $\underline{\mathrm{SMan}}(\mathbb{R}^{0|2}, X)$

For $\Phi \in \underline{\mathrm{SMan}}(\mathbb{R}^{0|2}, X)(S)$, we write

$$\Phi^* = f + \phi_1\theta^1 + \phi_2\theta^2 + E\theta^1\theta^2,$$

where $\phi_i \colon C^\infty(X) \to C_1^\infty(S)$ and $f, E \colon C^\infty(X) \to C_0^\infty(S)$. A computation shows that Φ^* is an algebra homomorphism if and only if

$$\begin{aligned}
f(ab) &= f(a)f(b), \\
\phi_i(ab) &= \phi_i(a)f(b) + f(a)\phi_i(b), \quad i = 1, 2, \\
E(ab) &= E(a)f(b) + f(a)E(b) - \phi_1(a)\phi_2(b) + \phi_2(a)\phi_1(b),
\end{aligned}$$

for any $a, b \in C^\infty(X)$. Thus S-points of $\underline{\mathrm{SMan}}(\mathbb{R}^{0|2}, X)$ can be identified with quadruples of linear maps (f, ϕ_1, ϕ_2, E) satisfying the above equations.

Recall that given a connection ∇ on X, the associated Hessian is defined by

$$\begin{array}{rccc}
\mathrm{Hess}\colon & \pi TX \otimes \pi TX \cong TX \otimes TX & \longrightarrow & \mathrm{Diff}^{\le 2}(X) \\
& V \otimes W & \longmapsto & VW - \nabla_V W
\end{array}$$

where TX stands for $\mathrm{Der}(C^\infty(X))$ in fact. For an S-point $f\colon C^\infty(X) \to C^\infty(S)$, as $f^*\pi TX = (\mathrm{Der}\, C^\infty(X)) \otimes_f C^\infty(S)$, we can define the pullback of Hessian by f

$$\begin{array}{rccc}
f^*\,\mathrm{Hess}\colon & f^*\pi TX \otimes f^*\pi TX & \longrightarrow & f^*\,\mathrm{Diff}^{\le 2}(X) \\
& (V \otimes_f s) \otimes (W \otimes_f t) & \longmapsto & \mathrm{Hess}(V, W) \otimes_f (st)
\end{array}$$

Since Hess is linear in both V and W, the above $f^*\,\mathrm{Hess}$ is well-defined.

Similarly to example 2.1.1, we see that $\phi_1, \phi_2 \in f^*\pi TX$. Thus we have $f^*\,\mathrm{Hess}(\phi_1, \phi_2) \in f^*\,\mathrm{Diff}^{\le 2}(X)$. For any $a, b \in C^\infty(X)$, since for any $V, W \in \pi TX$,

$$\begin{aligned}
\mathrm{Hess}(V, W)(ab) &= (VW - \nabla_V W)(ab) = V(Wa \cdot b + aWb) - \nabla_V Wa \cdot b - a\nabla_V Wb \\
&= b(VWa - \nabla_V Wa) + Wa \cdot Vb + Va \cdot Wb + a(VWb - \nabla_V Wb) \\
&= b \cdot \mathrm{Hess}(V, W)(a) + Vb \cdot Wa + Va \cdot Wb + a \cdot \mathrm{Hess}(V, W)(b),
\end{aligned}$$

we have

$$f^*\,\mathrm{Hess}(\phi_1, \phi_2)(ab) = f(a) \cdot \mathrm{Hess}(\phi_1, \phi_2)(b) + f(b) \cdot \mathrm{Hess}(\phi_1, \phi_2)(a) + \phi_1(a)\phi_2(b) - \phi_2(a)\phi_1(b).$$

Thus if we put $F := E + f^* \operatorname{Hess}(\phi_1, \phi_2)$, then

$$\begin{aligned} F(ab) &= E(ab) + f^* \operatorname{Hess}(\phi_1, \phi_2)(ab) \\ &= (E + f^* \operatorname{Hess}(\phi_1, \phi_2))(a) f(b) + f(a)(E + f^* \operatorname{Hess}(\phi_1, \phi_2))(b) \\ &= F(a) f(b) + f(a) F(b). \end{aligned}$$

Noticing that $f^* \operatorname{Hess}(\phi_1, \phi_2)$ is even, $F\colon C^\infty(X) \to C_0^\infty(S)$ is an even derivation with respect to f.

Since adding or abstracting $f^* \operatorname{Hess}(\phi_1, \phi_2)$ is invertible, we see that S-points $\Phi \in \underline{\mathrm{SMan}}(\mathbb{R}^{0|2}, X)$ can be identified with quadruples (f, F, ϕ_1, ϕ_2) satisfying the properties described above.

Recall example 1.2.8; let $p\colon TX \to X$ be the usual projection, then $p^*(\pi TX \oplus \pi TX)$ is the supermanifold $(TX, p^*(\Omega^\bullet \otimes_{C^\infty} \Omega^\bullet))$. Since

$$p^*(\Omega^\bullet(X) \otimes_{C^\infty(X)} \Omega^\bullet(X)) = (\Omega^\bullet(X) \otimes_{C^\infty(X)} \Omega^\bullet(X)) \otimes_p C^\infty(TX),$$

S-points of $p^*(\pi TX \oplus \pi TX)$ are one-to-one to pairs

$$(\varphi, \psi) \in \mathbb{R}\text{-}\mathrm{SAlg}(C^\infty(TX), C^\infty(S)) \times \mathbb{R}\text{-}\mathrm{SAlg}(\Omega^\bullet(X) \otimes_{C^\infty(X)} \Omega^\bullet(X), C^\infty(S))$$

such that the following diagram commutes

$$\begin{tikzcd} C^\infty(S) & \Omega^\bullet(X) \otimes_{C^\infty(X)} \Omega^\bullet(X) \arrow[l, "\psi"'] \\ C^\infty(TX) \arrow[u, "\varphi"] & C^\infty(X) \arrow[l, "p^*"] \arrow[u, hook] \end{tikzcd}$$

The universal property of $C^\infty(TX)$ asserts that φ can be identified with (f, F), where $f = \varphi \circ p^*$, see appendix B. The universal property of tensor product along with the same argument in example 2.1.1 tells that ψ can be identified with (ϕ_1, ϕ_2). Therefore S-points of $p^*(\pi TX \oplus \pi TX)$ are one-to-one to quadruples (f, F, ϕ_1, ϕ_2), which is one-to-one to S-points of $\underline{\mathrm{SMan}}(\mathbb{R}^{0|2}, X)$. As all identifications are natural in S, we conclude

Lemma 2.2.1. *For an ordinary manifold X equipped with a connection, we have*

$$\underline{\mathrm{SMan}}(\mathbb{R}^{0|2}, X) \cong \underline{p^*(\pi TX \oplus \pi TX)},$$

i.e., $\underline{\mathrm{SMan}}(\mathbb{R}^{0|2}, X)$ is represented by $p^*(\pi TX \oplus \pi TX)$. $\square$

The connection ∇ on X splits $T(TX)$ into vertical and horizontal bundles $T(TX) \cong V(TX) \oplus H(TX)$, and $V(TX) \cong H(TX) \cong p^*(TX)$. Precomposition of $T(TX) \cong V(TX) \oplus H(TX) \cong p^*(TX \oplus TX)$ gives $\Omega^\bullet(TX) \cong p^*(\Omega^\bullet(X) \otimes_{C^\infty(X)} \Omega^\bullet(X))$, thus

$\pi T(TX) \cong p^*(\pi TX \oplus \pi TX)$. Therefore with lemma 2.2.1, we have

$$\begin{aligned} C^\infty(\underline{\mathrm{SMan}}(\mathbb{R}^{0|2}, X)) &= \mathrm{Hom}(\underline{\mathrm{SMan}}(\mathbb{R}^{0|2}, X), C^\infty) \\ &\cong \mathrm{Hom}(\underline{p^*(\pi TX \oplus \pi TX)}, C^\infty) \\ &\cong \mathrm{Hom}(\underline{\pi T(TX)}, C^\infty) \\ &\cong C^\infty(\pi T(TX)) = \Omega^\bullet(TX). \end{aligned} \tag{2.2.1}$$

Since TX is canonically oriented by local parametrizations $(x_1, \cdots, x_n, \mathrm{d}x_1, \cdots, \mathrm{d}x_n)$, the integration of top forms on TX induces the integration on $\underline{\mathrm{SMan}}(\mathbb{R}^{0|2}, X)$, via

$$C^\infty(\underline{\mathrm{SMan}}(\mathbb{R}^{0|2}, X)) \cong \Omega^\bullet(TX) \xrightarrow{\text{project}} \Omega^{2n}(TX) \xrightarrow{\int} \mathbb{R}, \tag{2.2.2}$$

if only the integration of the obtained top form could be defined.

Example 2.2.1. Recall that given a function $a \in C^\infty(X)$, we have superfunctions

$$a, \mathrm{d}_1 a, \mathrm{d}_2 a, \mathrm{d}_2\, \mathrm{d}_1 a \in C^\infty(\underline{\mathrm{SMan}}(\mathbb{R}^{0|2}, X)).$$

The identification in eq. (2.2.1) sends

$$\begin{array}{ccc} C^\infty(\underline{\mathrm{SMan}}(\mathbb{R}^{0|2}, X)) & \longrightarrow & \Omega^\bullet(TX) \\ fa & \mapsto & p^*a \in C^\infty(TX) \subset \Omega^\bullet(TX) \\ \mathrm{d}_1 a & \mapsto & D_1 a \colon T(TX) \twoheadrightarrow V(TX) \cong p^*TX \xrightarrow{p^*(-)a} C^\infty(TX) \\ \mathrm{d}_2 a & \mapsto & D_2 a \colon T(TX) \twoheadrightarrow H(TX) \cong p^*TX \xrightarrow{p^*(-)a} C^\infty(TX) \\ \mathrm{d}_2\, \mathrm{d}_1 a & \mapsto & i(\mathrm{d}a) - p^*\, \mathrm{Hess}(D_1, D_2)(a) \in C^\infty(TX) \subset \Omega^\bullet(TX) \end{array}$$

where $i \colon \Omega^1(X) \hookrightarrow C^\infty(TX)$ is defined in the beginning of appendix B, and D_1 and D_2 are the compositions D_1 and D_2 are the compositions

$$D_1 \colon C^\infty(X) \xrightarrow{\mathrm{d}} \Omega^\bullet(X) \otimes_{C^\infty(X)} 1 \hookrightarrow \Omega^\bullet(X) \otimes_{C^\infty(X)} \Omega^\bullet(X) \to p^*(\Omega^\bullet(X) \otimes_{C^\infty(X)} \Omega^\bullet(X)) \cong \Omega^\bullet(TX),$$

$$D_2 \colon C^\infty(X) \xrightarrow{\mathrm{d}} 1 \otimes_{C^\infty(X)} \Omega^\bullet(X) \hookrightarrow \Omega^\bullet(X) \otimes_{C^\infty(X)} \Omega^\bullet(X) \to p^*(\Omega^\bullet(X) \otimes_{C^\infty(X)} \Omega^\bullet(X)) \cong \Omega^\bullet(TX).$$

An iterating argument based on lemma 2.2.1 gives the following general result, which is the proposition 3.3 on the paper of Berwick-Evans[4].

Proposition 2.2.2. *Let δ be a non-negative integer. Given a choice of connection on TX, there is an isomorphism*

$$\underline{\mathrm{SMan}}(\mathbb{R}^{0|\delta}, X) \cong \underline{\pi T(T^{\delta-1}X)}$$

as supermanifolds. For $\delta > 2$ this isomorphism requires a framing of $\mathbb{R}^{0|\delta}$. ∎

2.3 The Outline of a Proof of the Chern-Gauss-Bonnet Theorem

As an application of our results of $\underline{\text{SMan}}(\mathbb{R}^{0|2}, X)$, a proof of the Chern-Gauss-Bonnet theorem is given in the paper of Berwick-Evans[4], for which we give a short description here.

Let (X, g) denote an ordinary closed Riemannian manifold of dimension n, and the connection ∇ on X be the Levi-Civita connection. The *partition function* Z_X of $0|2$ dimensional *supersymmetric sigma models* takes the form

$$Z_X(g,h) := \int_{\underline{\text{SMan}}(\mathbb{R}^{0|2},X)} \frac{\exp(-\mathcal{S}_h)}{N},$$

where the integration on $\underline{\text{SMan}}(\mathbb{R}^{0|2}, X)$ is defined by eq. (2.2.2), $\exp(-\mathcal{S}_h)/N$ is the integrand, $\mathcal{S}_h \in C^\infty(\underline{\text{SMan}}(\mathbb{R}^{0|2}, X))$ is the *action functional* in the model and N is the *normalization constant* for which we take $N = (2\pi)^{\frac{n}{2}}$. It turns out that for any S-point $\Phi \in \underline{\text{SMan}}(\mathbb{R}^{0|2}, X)(S)$,

$$\mathcal{S}_h(\Phi) = \frac{1}{2}\langle F, F \rangle + \frac{1}{2}R(\phi_1, \phi_2, \phi_1, \phi_2) - \langle F, \nabla h \rangle - \text{Hess}(\phi_1, \phi_2)h,$$

where R denotes the Riemann curvature tensor associated to the Levi-Civita connection on X.

As a consequence of the general structure of quantization for $0|\delta$-dimensional Euclidean field theories, we have

Corollary 2.3.1. *The number $Z_X(g, h)$ is independent of the metric g and the function h.* □

This is the corollary 1.3, which is proved after proposition 1.39, in the paper of Berwick-Evans[4].

Calculation of $Z_X(g, 0)$ and $\lim\limits_{\lambda\to\infty} Z_X(g, \lambda h)$ gives the following form of Chern-Gauss-Bonnet theorem:

Theorem 2.3.2 (Chern-Gauss-Bonnet). *Let R denote the Riemannian curvature tensor associated to the Levi-Civita connection on a closed Riemannian manifold X, and let* $\text{Pf}(R)$ *be the Pfaffian density of the curvature R. Then*

$$\frac{1}{(2\pi)^{\frac{n}{2}}} \int_X \text{Pf}(R) = Z_X(g,0) \overset{corollary\ 2.3.1}{=\!=\!=\!=} \lim_{\lambda\to\infty} Z_X(g,\lambda h) = \text{Index}(\nabla h),$$

where h is any Morse function on X and $\text{Index}(\nabla h)$ *is the Hopf index of the gradient vector field ∇h of h.* □

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Appendix

Appendix A Supermanifolds are Affine

Since the pullbacks of superfunctions are determined locally, it is clear that the functor $C^\infty\colon \mathrm{SMan}^{op} \to \mathbb{R}\text{-}\mathrm{SAlg}$ is faithful. It remains only to show that C^∞ is full, i.e., let $\mathcal{M} = (M, \mathcal{O})$ and $\mathcal{N} = (N, \mathcal{R})$ be two supermanifolds, we show that for any super $\mathbb{R}$-algebra homomorphism $\varphi\colon \mathcal{R}(N) \to \mathcal{O}(M)$, there exists a supermorphism $\Psi = (\psi, \psi^*)\colon \mathcal{M} \to \mathcal{N}$ such that $\psi^* = \varphi$. As the projection $\varepsilon\colon \mathcal{O}(M) \to C^\infty(M)$ is equal to the quotient of the nilpotent ideal, φ induces an $\mathbb{R}$-algebra morphism $\widetilde{\varphi}\colon C^\infty(N) \to C^\infty(M)$ such that the following diagram commutes

$$\begin{tikzcd}
\mathcal{R}(N) \arrow[r, "\varphi"] \arrow[d, "\varepsilon"'] & \mathcal{O}(M) \arrow[d, "\varepsilon"] \\
C^\infty(N) \arrow[r, dashed, "\widetilde{\varphi}"'] & C^\infty(M)
\end{tikzcd}$$

The following theorem tells that $\widetilde{\varphi}$ is the pullback of smooth functions by a smooth map $\psi\colon M \to N$.

Theorem A.1 (Manifolds are Affine)**.** *Let M and N be two ordinary smooth manifolds. The functor $C^\infty\colon$ $\mathrm{Man}^{op} \to \mathbb{R}\text{-}\mathrm{Alg}$ induces a natural bijection*

$$\mathrm{Man}(M, N) \cong \mathbb{R}\text{-}\mathrm{Alg}(C^\infty(N), C^\infty(M)).$$

Proof. This is a consequence of theorem 3.8 and theorem 7.2 in the book of Nestruev[9]. $\square$

Now we use $\psi\colon M \to N$ to define the restrictions of φ to open subsets $U \subset N$,

$$\varphi|_U\colon \mathcal{R}(U) \to \mathcal{O}(\psi^{-1}(U)),$$

in a way that is compatible to the restrictions, so that φ is extended to a sheaf morphism. This relies on the following lemma

Lemma A.2. *For any $h \in \mathcal{R}(N)$ such that $h|_U = 0$ for some open subset $U \subset N$, $\varphi(h)|_{\psi^{-1}(U)} = 0$.*

Proof. For any $x \in U$, by the argument after definition 1.3.5, there exists a bump function $\rho \in \mathcal{R}(N)$ supported in U with $\rho|_V \equiv 1$ in some neighborhood V of x. Clearly, $\rho h = 0$, hence

$$0 = \varphi(\rho h) = \varphi(\rho)\varphi(h).$$

Restricting the equation to $\psi^{-1}(V)$, we obtain

$$0 = \varphi(\rho)|_{\psi^{-1}(V)}\varphi(h)|_{\psi^{-1}(V)}.$$

Since $\widetilde{\varphi}$ and ε both commute with the restriction,

$$\varepsilon \circ \varphi(\rho)|_{\psi^{-1}(V)} = \widetilde{\varphi} \circ \epsilon(\rho)|_{\psi^{-1}(V)} = \widetilde{\varphi} \circ \epsilon(\rho|_V) = 1.$$

Hence $\varphi(\rho)|_{\psi^{-1}(V)}$ is a unit added a nilpotent, which is also a unit. Thus

$$0 = \varphi(h)|_{\psi^{-1}(V)}.$$

As x varies in U, we conclude that $\varphi(h)|_{\psi^{-1}(U)} = 0$ as desired. $\square$

With this lemma, $\varphi|_U \colon \mathcal{R}(U) \to \mathcal{O}(\psi^{-1}(U))$ is defined as the unique map such that

$$\varphi|_U(g)|_{\psi^{-1}(V)} = \varphi(\widetilde{g})|_{\psi^{-1}(V)},$$

for any $g \in \mathcal{R}(U)$ and $\widetilde{g} \in \mathcal{R}(N)$ with $\widetilde{g}|_V = g|_V$ for some open set $V \subset U$. One verifies easily that this extends φ to a morphism of sheaves, finishing the proof.

Appendix B Universal Property of $C^\infty(TX)$

Considering the $C^\infty(X)$-module $\Omega^1(X)$, we have a $C^\infty(X)$-linear map $i\colon \Omega^1(X) \hookrightarrow C^\infty(TX)$, defined by putting, for any $\omega \in \Omega^1(X)$, $i(\omega) \in C^\infty(TX)$ to be the function that sends a point $(x, v_x) \in TX$ to $\omega(V)(x) \in \mathbb{R}$, where $V \in \Gamma(TX)$ is any global vector field such that $V_x = v_x$. As $\omega\colon \Gamma(TX) \to C^\infty(X)$ is $C^\infty(X)$-linear, $i(\omega)$ is independent of the choice of V. It is easy to see that i is injective.

The universal property of $C^\infty(TX)$ is stated as:

Theorem B.1. *For any S-point of X ($S = (|S|, \mathcal{O})$ is some supermanifold) $\tau\colon S \to X$, $\tau^*\colon C^\infty(X) \to C^\infty(S)$ gives $C^\infty(S)$ a structure of $C^\infty(X)$-algebra. Under this structure, for any $C^\infty(X)$-linear map $\Psi\colon \Omega^1(X) \to C^\infty_0(S) \subset C^\infty(S)$, there exists a unique super $\mathbb{R}$-algebra morphism $\varphi\colon C^\infty(TX) \to C^\infty(S)$ such that $\Psi = \varphi \circ i$, i.e., the following diagram commutes:*

$$\begin{tikzcd} C^\infty(TX) \arrow[r, dashed, "\varphi"] & C^\infty(S) \\ \Omega^1(X) \arrow[u, hook, "i"] \arrow[ur, "\Psi"'] & \end{tikzcd}$$

Proof. If X has a global parametrization $x_1, \cdots, x_n$ so that TX is isomorphic to the product bundle $X \times \mathbb{R}^n$, then

$$C^\infty(TX) = C^\infty(X \times \mathbb{R}^n) \cong \overline{C^\infty(X) \otimes_{\mathbb{R}} C^\infty(\mathbb{R}^n)},$$

where $\overline{C^\infty(X) \otimes_{\mathbb{R}} C^\infty(\mathbb{R}^n)}$ is the smooth envelope of the tensor product, and the result follows from the universal property of the smooth envelope (see definition 3.36 and lemma 4.30 in the book of Nestruev[9]) and the universal property of $C^\infty(\mathbb{R}^n)$ in example 1.2.7 tensoring $C^\infty(X)$:

$$\begin{tikzcd}
\overline{C^\infty(X) \otimes_{\mathbb{R}} C^\infty(\mathbb{R}^n)} \arrow[r, dashed, "\varphi"] & C^\infty(S) \\
\Omega^1(X) = C^\infty(X)[\mathrm{d}x_1, \cdots, \mathrm{d}x_n] \arrow[u, hook, "i"] \arrow[ur, "\Psi"'] &
\end{tikzcd}$$

For a general manifold X, if we could apply some restriction morphisms to bring the statement into local coordinate charts, then the above case would give the desired φ locally. The uniqueness in that case's statement ensures that we can glue the local terms up, obtaining a unique φ such that the following diagram commutes for any coordinate chart U:

$$\begin{tikzcd}
C^\infty(TX) \arrow[r, dashed, "\varphi"] \arrow[d] & C^\infty(S) \arrow[d] \\
C^\infty(U \times \mathbb{R}^n) \arrow[r, "\varphi|_{U \times \mathbb{R}^n}"'] & \mathcal{O}(\tau^{-1}(U))
\end{tikzcd}$$

As the restrictions for C^∞, Ω^1 and i are obvious, we need only construct the restrictions of Ψ. Since Ψ is $C^\infty(X)$-linear, the usual trick of multiplying a bump function tells that for any open subset $U \subset X$, if $\omega \in \Omega^1(X)$ satisfies $\omega|_U = 0$, then $\Psi(\omega)|_{\tau^{-1}(U)} = 0$. With this we can define the restriction of Ψ to U, $\Psi|_U \colon \Omega^1(U) \to \mathcal{O}(\tau^{-1}(U))$, as the unique assignment such that

$$\Psi|_U(\eta)|_{\tau^{-1}(V)} = \Psi(\widetilde{\eta})|_{\tau^{-1}(V)},$$

for any $\eta \in \Omega^1(U)$ and $\widetilde{\eta} \in \Omega^1(X)$ with $\widetilde{\eta}|_V = \eta|_V$ for some open set $V \subset U$. Such $\Psi|_U$ is $C^\infty(U)$-linear and is a reasonable restriction since the following diagram commutes

$$\begin{tikzcd}
\Omega^1(X) \arrow[r, "\Psi"] \arrow[d] & C^\infty(S) \arrow[d] \\
\Omega^1(U) \arrow[r, "\Psi|_U"'] & \mathcal{O}(\tau^{-1}(U))
\end{tikzcd}$$

The fact that φ does make the desired diagram commute follows from that of the restrictions, therefore we have finished the proof. $\square$

Recall that $p \colon TX \to X$ is the usual projection. Given a super $\mathbb{R}$-algebra morphism $\varphi \colon C^\infty(TX) \to C^\infty(S)$, its composition with $p^* \colon C^\infty(X) \to C^\infty(TX)$ and $C^\infty(X) \xrightarrow{\mathrm{d}}$

$\Omega^1(X) \xrightarrow{i} C^\infty(TX)$ respectively gives a super $\mathbb{R}$-algebra morphism $f\colon C^\infty(X) \to C^\infty(S)$ and an even derivation $F\colon C^\infty(X) \to C_0^\infty(S) \subset C^\infty(S)$ with respect to f.

Under the identification $\mathrm{Der}_f(C^\infty(X), C^\infty(S)) \cong (\mathrm{Der}\, C^\infty(X)) \otimes_f C^\infty(S)$, the universal property of $C^\infty(TX)$ gives the following map, which is the inverse of the above assignment that $\varphi \mapsto (f, F)$.

$$\begin{array}{ccccc} (\mathrm{Der}\, C^\infty(X)) \otimes_f C^\infty(S) & \longrightarrow & C^\infty(X)\text{-}\mathrm{SMod}(\Omega^1(X), C^\infty(S)) & \cong & \mathrm{SMan}(S, TX) \\ D \otimes_f g & \longmapsto & (\omega \mapsto g \cdot f^*\omega(D)) =: \Psi & \mapsto & \varphi \end{array}$$

Therefore we obtain

$$\begin{aligned} \mathrm{SMan}(S, TX) &\cong \mathbb{R}\text{-}\mathrm{SAlg}(C^\infty(TX), C^\infty(S)) \\ &\cong C^\infty(X)\text{-}\mathrm{SMod}(\Omega^1(X), C^\infty(S)) \\ &\cong \bigcup_{f \in \mathbb{R}\text{-}\mathrm{SAlg}(C^\infty(TX), C^\infty(S))} \{f\} \times \mathrm{Der}_f(C^\infty(X), C^\infty(S)), \end{aligned}$$

i.e., S-points of TX can be identified with the pairs (f, F).

Acknowledgement

I'd like to thank firstly my advisor Qin Li, who guided me all along the way I wrote this thesis. I'd like to also thank Prof. Dmitri Pavlov, who kindly helped me on MathOverflow with several of my questions that I came up with when writing this thesis. Finally, I'd like to thank every person who treated me kindly during my undergraduate years.