Notes on Chern-Gauss-Bonnet Theorem via Supergeometry

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Content

1. A Brief Introduction to Supergeometry

1.1 A Short Preparation of Superalgebra

Before entering into the geometry part, one should be equipped with a minimal amount of knowledge of superalgebra, the "super" version of algebra, which plays a central role in the "super" part of supergeometry.

Roughly speaking, superalgebra is nothing but algebra in $\mathbb{Z}_2$-graded context. We shall establish the notions following the usual order in abstract algebra, that is, we start from ring and then proceed to algebra and module.

Definition 1.1.1 (Superring)**.** A *superring* R is a $\mathbb{Z}_2$-graded ring, i.e. a (usually non-commutative) unitary ring with a decomposition as abelian groups $R = R_0 \oplus R_1$, where $0, 1 \in \mathbb{Z}_2$, such that $R_i R_j \subset R_{i+j}$ for any $i, j \in \mathbb{Z}_2$.

Remark 1.1.1. Just like in the classical context, one can define the notion of non-unitary superring, non-unitary superalgebra and even non-associative superalgebra. None of these is in our consideration, hence we will simply not ignore them and live only in the world that is associative and unitary. Also, we will never consider the zero ring, consequently every ring homomorphism from a field to a ring is injective by default.

Elements in R_0 are said to be *homogeneous with even parity* 0 and those in R_1 are said to be *homogeneous with odd parity* 1. The assignment of parity gives a function $p : (R_0 \cup R_1) \setminus \{0\} \to \mathbb{Z}_2$. The requirement that $R_i R_j \subset R_{i+j}$ is equivalent to demanding that the parity is additive under multiplication, i.e.

$$p(ab) = p(a) + p(b),$$

for any homogeneous $a, b \in R$. Note that $0 \in R$ can be seen to have both even and odd parities and that the multiplicative unit $1 \in R$ is forced to have even parity since $p(1) = p(1 \cdot 1) = 2p(1) = 0$. Similar convention of the parity applies to things that are $\mathbb{Z}_2$-graded, as one will soon see.

A superring R is *supercommutative* if for any homogeneous $a, b \in R$ there is

$$ab = (-1)^{p(a)p(b)} ba.$$

It follows that in a supercommutative superring odd elements anticommute and are nilpotent, i.e. $ab = -ba$ and $a^2 = 0$ for any odd a and b. Usually we refer to supercommutative superring with one "super" omitted, i.e. by saying *supercommutative ring* or *commutative superring*.

Remark 1.1.2. Note that every ring can be graded trivially by $R = R_0 \oplus \{0\}$ with $R_0 = R$, every ring can be viewed as a superring. It is the same thing to ask for a ring to be commutative as to ask for a trivially graded ring to be supercommutative. In this point of view, notions in superalgebra become generalizations of those in classical abstract algebra, and things that are not "super" are seen to be graded trivially by default.

Remark 1.1.3 (Koszul Sign Rule). Usually we will consider only supercommutative things. The principle of adding a sign up to the parity when switching the position of two adjacent things is known as the *Koszul sign rule*. One will see that it appears everywhere in supergeometry.

For the reason in the preceding remark, one can assume safely that every superring is supercommutative from now on.

By abuse of notation, when it comes to power up -1 by the parity of some elements, one may use the element itself for its parity, i.e. write $(-1)^{p(a)p(b)} = (-1)^{ab}$. As one will see, since the parity is always additive under multiplication (and composition once we define the parity of morphism), $(-1)^{p(ab)}$ can be always written as $(-1)^{a+b}$ and thereby no confusion arises.

On building the category of superrings, it is natural to ask for a forgetful functor from this category to Ring. Due to the requirement of supercommutativity of the multiplicative structure, it is pointless to consider ring homomorphisms that does not preserve the parity. Hence

Definition 1.1.2 (Superring Homomorphism)**.** A superring homomorphism φ from superrings R to R' is a ring homomorphism φ from R to R' that preserves the parity, i.e. $\varphi(R_i) \subset R'_i$ for any $i \in \mathbb{Z}_2$.

Indeed, superring homomorphism is exactly the usual homomorphism between $\mathbb{Z}_2$-graded rings.

The category of superrings is usually denoted as SRing. For our purpose, we will take SRing as the category of supercommutative rings.

Recall that an algebra over a commutative ring K, K-algebra, is a ring A along with a ring homomorphism $\varphi : K \to A$ such that $\varphi(K) \subset Z(A)$ where $Z(A)$ is the multiplicative center of A. Adding the word "super" before each single word, we obtain the notion of *superalgebra*.

Definition 1.1.3 (Superalgebra)**.** A superalgebra over a supercommutative ring R, super R-algebra, is a superring A along with a superring homomorphism $\varphi : R \to A$ such that $\varphi(R) \subset Z(A)$ where $Z(A)$ is the supercenter of A, i.e. $Z(A) := \{a \in A \mid ab = (-1)^{ab}ba,\ \forall b \in A\}$.

Morphisms from A to B two super R-algebras are superring morphisms from A to B such that the triangle $\begin{tikzcd} A \arrow[r] & B \\ R \arrow[u] \arrow[ur] & \end{tikzcd}$ commutes. The category of super R-algebras is denoted as R-SAlg. For our purpose, we will take R-SAlg as the category of *supercommutative R-algebras*, i.e. super R-algebras that are supercommutative as superrings.

We can now talk about the *supermodule*.

Definition 1.1.4 (Supermodule)**.** A supermodule M over a superring $R = R_0 \oplus R_1$, super R-module, is a (left) R-module (R seen as a ring) with a $\mathbb{Z}_2$-graded structure $M = M_0 \oplus M_1$ (direct sum as abelian groups), such that the multiplication by scalars respects the parity, i.e. $R_i M_j \subset M_{i+j}$ for any $i, j \in \mathbb{Z}_2$.

Equivalently, $R_i M_j \subset M_{i+j}$ is the same as that

$$p(rm) = p(r) + p(m)$$

for any homogeneous $r \in R$ and $m \in M$.

Clearly, a superring is a supermodule over itself.

When it comes to contexts where supercommutativity is always assumed, the left A-module structure gives rise to a right R-module structure, defined by $mr := (-1)^{rm} rm$.

Remark 1.1.4. Roughly speaking, the induced right R-module structure allows writing the scalars on both sides. For instance, this will save a lot of efforts when putting multiple supermodules over a supercommutative ring together via the tensor product.

When the superring is taken as a field, the requirement $R_i M_j \subset M_{i+j}$ can be abandoned and we obtain the notion of *super vector space*.

Definition 1.1.5 (Super Vector Space)**.** A super vector space V over k (usually with characteristic 0) is a $\mathbb{Z}_2$-graded k-vector space, i.e. a vector space with a direct sum decomposition (as vector spaces) $V = V_0 \oplus V_1$. If V_0 and V_1 have dimension p and q respectively, then V is said to have dimension $p|q$.

Similar to the classical context, we may consider a supermodule M over a superring A which is at the same time a super R-algebra, then M has a natural structure of a supermodule over R. More specifically, if R is a field, then M has a natural structure of a super vector space over R.

Cares should be taken when talking about morphisms between supermodules. Of course the family of morphisms that preserve the parity is a natural choice.

Definition 1.1.6 (Supermodule Homomorphism)**.** A supermodule homomorphism f from M to N two super R-modules is a R-module homomorphism that preserves the parity, i.e. $f(M_i) \subset N_i$ for any $i \in \mathbb{Z}_2$.

The category of super R-modules is formed with this choice of morphisms, and is denoted as R-SMod. It is this sense a supermodule morphism refers to. Without further specification, a morphism always preserves the parity.

However, it makes sense in practice to consider between supermodules parity-reversing morphisms and their linear combination with parity-preserving ones. The parity-preserving ones are said to be homogeneous with even parity 0 and the parity-reversing ones are said to be homogeneous with odd parity 1. According to the Koszul sign rule, we demand instead of the usual R-linearity the super R-linearity, i.e. for a homomorphism $f : M \to N$ of abelian groups to be a homogeneous (even or odd) morphism of super R-modules, there should be

$$f(rm) = (-1)^{fr} r f(m) \tag{1.1.1}$$

for any $m \in M$ and homogeneous $r \in R$. This is compatible with the induced right R-module structure, i.e. we have $f(mr) = f(m)r$ as one can easily verify. Formally,

Definition 1.1.7 (Homogeneous Morphism of Supermodules)**.** Let $f \in \mathrm{Hom}_{\mathsf{Ab}}(M, N)$ where M and N are super R-modules. f is

- an even morphism if $f(M_i) \subset N_i$ for any $i \in \mathbb{Z}_2$ and $f(rm) = rf(m)$ for any $r \in R$ and $m \in M$.
- an odd morphism if $f(M_i) \subset N_{i+1}$ for any $i \in \mathbb{Z}_2$ and $f(rm) = (-1)^r rf(m)$ for any $m \in M$ and homogeneous $r \in R$.

The set of all even morphisms from M to N is denoted as $\mathbf{Hom}_0(M, N)$ and the set of all odd ones is denoted as $\mathbf{Hom}_1(M, N)$. The assignment of the parity gives a function $(\mathbf{Hom}_0(M, N) \cup \mathbf{Hom}_1(M, N)) \setminus \{0\} \to \mathbb{Z}_2$, and the convention of super R-linearity in eq. (1.1.1) fits well to the definition. Also, note that there is $\mathbf{Hom}_0(M, N) = \mathrm{Hom}_{R\text{-}\mathsf{SMod}}(M, N)$.

The direct sum as abelian groups gives the *internal Hom set* $\mathbf{Hom}(M, N) := \mathbf{Hom}_0(M, N) \oplus \mathbf{Hom}_1(M, N)$. When R is supercommutative, $\mathbf{Hom}(M, N)$ has a natural super R-module structure where the addition and scalar multiplication are defined point-wisely.

When R is trivially graded, the super R-linearity is the same as the usual R-linearity, and it is easy to see that $\mathbf{Hom}(M, N) = \mathrm{Hom}_{R\text{-}\mathsf{Mod}}(M, N)$ in this case. In particular, for super k-vector spaces M and N we have $\mathbf{Hom}(M, N) = \mathrm{Hom}_{\mathsf{Vect}_k}(M, N)$.

We have been equipped with a complete view of the most fundamental notions in superalgebra since now and are ready to enter into the geometry part.

1.2 Supermanifolds

In short, a *supermanifold* $\mathcal{M} = (M, \mathcal{O})$ *of dimension* $p|q$ is an underlying differentiable manifold M endowed with a structural sheaf $\mathcal{O} : \mathrm{Open}(M) \to \mathbb{R}\text{-}\mathsf{SAlg}$ of super $\mathbb{R}$-algebras such that the pair is locally $\mathbb{R}$-isomorphic to *smooth superdomains* of dimension $p|q$, where a smooth superdomain is

Definition 1.2.1 (Smooth Superdomain)**.** A smooth superdomain $\mathcal{U}^{p|q}$ of dimension $p|q$ is an open subset U of $\mathbb{R}^p$ endowed with a sheaf $\mathcal{C}^\infty_{p|q}$ defined for each open subset $V \subset U$ by

$$\mathcal{C}^\infty_{p|q}(V) \coloneqq \mathcal{C}^\infty(V)[\xi^1, \cdots, \xi^q]$$

where $\mathcal{C}^\infty(V)[\xi^1, \cdots, \xi^q]$ is the exterior algebra generated by $\xi^1, \cdots, \xi^q$ over $\mathcal{C}^\infty(V)$, i.e. the free $\mathcal{C}^\infty(V)$-algebra generated by $\xi^1, \cdots, \xi^q$ modulo the relation that ξ's are anticommutative, and the restriction maps are induced by the restriction of functions $\mathcal{C}^\infty(V) \to \mathcal{C}^\infty(W)$ if $W \subset V$.

and by locally $\mathbb{R}$-isomorphic we mean that

> For any point of M there exists a neighborhood W of that point such that $(W, \mathcal{O}|_W)$, where $\mathcal{O}|_W$ is the sheaf $\mathcal{O}$ restricted on W, is $\mathbb{R}$-isomorphic to some smooth superdomain $\mathcal{U}^{p|q} = (U, \mathcal{C}^\infty_{p|q})$ in the sense that there exists a diffeomorphism $\varphi : W \to U$ along with a natural isomorphism $\varphi^* : \mathcal{C}^\infty_{p|q} \cong \varphi_*\mathcal{O}|_W$ where $\varphi_*\mathcal{O}|_W : \mathrm{Open}(U) \xrightarrow{\varphi^{-1}} \mathrm{Open}\, O(W) \to \mathbb{R}\text{-}\mathrm{SAlg}$ is the pushforward sheaf of $\mathcal{O}|_W$ by φ.

Remark 1.2.1. In above description, the word "differentiable manifold" can be replaced by "a second countable Hausdorff topological space". Since the smooth structure of the underlying space is readily encoded by the sheaf under the given local condition, the second countable Hausdorff topological space becomes a differentiable manifold automatically, following the same line how a classical differentiable manifold is described in the manner of sheaf theory.[1] We shall not talk much about these to make things brief.

Morphism between supermanifolds is defined, of course, in a similar manner, but before giving its definition there are several things to be said. Firstly we need to know what the super version of local ring is.

Definition 1.2.2 (Homogeneous Ideal)**.** A homogeneous ideal I of a superring R is an ideal I of the ring R such that $I = (I \cap R_0) \oplus (I \cap R_1)$, i.e. the homogeneous components of each element of I still live in I.

Definition 1.2.3 (Local Superring)**.** A superring $R = R_0 \oplus R_1$ is local if it admits a unique maximal homogeneous ideal, i.e. it has only one homogeneous ideal that is maximal with respect to inclusion.

A superalgebra is local if it is local as a superring. The stalk $\mathcal{O}_x = \mathcal{C}^\infty_{p|q,x}$ of the structural sheaf at a point $x \in U \subset M$ is local, with the unique maximal homogeneous ideal $\mathfrak{m}_x$

consisting of all the non-units. More concretely, for $f \in \mathcal{C}^\infty_{p|q}(V)$ we may write it as

$$
\begin{aligned}
f(x,\xi) = \sum_\alpha f_\alpha(x)\xi^\alpha &= \sum_{k=0}^{q} \sum_{\alpha_1<\cdots<\alpha_k} f_{\alpha_1\cdots\alpha_k}(x)\xi^{\alpha_1}\cdots\xi^{\alpha_k} \\
&= f_0(x) + \sum_{k=1}^{q} \sum_{\alpha_1<\cdots<\alpha_k} f_{\alpha_1\cdots\alpha_k}(x)\xi^{\alpha_1}\cdots\xi^{\alpha_k},
\end{aligned}
$$

with the coefficients f_α in $\mathcal{C}^\infty(V)$. A monomial term $f_{\alpha_1\cdots\alpha_k}(x)\xi^{\alpha_1}\cdots\xi^{\alpha_k}$ is said to have cohomological degree k. Since the part with nonzero cohomological degree

$$
\sum_{k=1}^{q} \sum_{\alpha_1<\cdots<\alpha_k} f_{\alpha_1\cdots\alpha_k}(x)\xi^{\alpha_1}\cdots\xi^{\alpha_k}
$$

is nilpotent, for any $x \in V$ the germ $[f]_x$ is a non-unit if and only if $f_0(x) = 0$, consequently the homogeneous components of a non-unit are also non-units. Hence

Theorem 1.2.1 (The Unique Maximal Homogeneous Ideal)**.** *The unique maximal homogeneous ideal of the stalk $\mathcal{C}^\infty_{p|q,x}$ is given by*

$$
\mathfrak{m}_x = \{[f]_x \mid f_0(x) = 0\}.
$$

Bibliography

[1] BARTOCCI C, BRUZZO U, HERNÁNDEZ-RUIPÉREZ D. The Goemetry of Supermanifolds[M]. [S.l.]: Kluwer Academic Publishers, 1991.

[2] COVOLO T, PONCIN N. Lectures on Supergeometry[C/OL]//. [S.l. : s.n.], 2012. https://orbilu.uni.lu/bitstream/10993/14295/1/LecturesSupergeometryFinalRevised.pdf.

[3] DELIGNE P, MORGAN J W. Notes on Supersymmetry (following Joseph Bernstein)[M]//DELIGNE P, KAZHDAN D, ETINGOF P, et al. Quantum Fields and Strings: A Course for Mathematicians. [S.l.]: American Mathematical Society, 1999: 41-97.