Notes on Chern-Gauss-Bonnet Theorem via Supergeometry

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1. A Brief Introduction to Supergeometry

1.1 A Short Preparation of Superalgebra

Before entering into the geometry part, one should be equipped with a minimal amount of knowledge of superalgebra, the "super" version of algebra, which plays a central role in the "super" part of supergeometry.

Roughly speaking, superalgebra is nothing but algebra in $\mathbb{Z}_2$-graded context. We shall establish the notions following the usual order in abstract algebra, that is, we start from ring and then proceed to algebra and module.

Definition 1.1.1 (Superring)**.** A *superring* R is a $\mathbb{Z}_2$-graded ring, i.e. a (usually non-commutative) unitary ring with a decomposition as abelian groups $R = R_0 \oplus R_1$, where $0, 1 \in \mathbb{Z}_2$, such that $R_i R_j \subset R_{i+j}$ for any $i, j \in \mathbb{Z}_2$.

Remark 1.1.1. Just like in the classical context, one can define the notion of non-unitary superring, non-unitary superalgebra and even non-associative superalgebra. None of these is in our consideration, hence we will simply not ignore them and live only in the world that is associative and unitary. Also, we will never consider the zero ring, consequently every ring homomorphism from a field to a ring is injective by default.

Elements in R_0 are said to be *homogeneous with even parity* 0 and those in R_1 are said to be *homogeneous with odd parity* 1. The assignment of parity gives a function $p : (R_0 \cup R_1) \setminus \{0\} \to \mathbb{Z}_2$. The requirement that $R_i R_j \subset R_{i+j}$ is equivalent to demanding that the parity is additive under multiplication, i.e.

$$p(ab) = p(a) + p(b),$$

for any homogeneous $a, b \in R$. Note that $0 \in R$ can be seen to have both even and odd parities and that the multiplicative unit $1 \in R$ is forced to have even parity since $p(1) = p(1 \cdot 1) = 2p(1) = 0$. Similar convention of the parity applies to things that are $\mathbb{Z}_2$-graded, as one will soon see.

A superring R is *supercommutative* if for any homogeneous $a, b \in R$ there is

$$ab = (-1)^{p(a)p(b)} ba.$$

It follows that in a supercommutative superring odd elements anticommute and are nilpotent, i.e. $ab = -ba$ and $a^2 = 0$ for any odd a and b. Usually we refer to supercommutative superring with one "super" omitted, i.e. by saying *supercommutative ring* or *commutative superring*.

Remark 1.1.2. Note that every ring can be graded trivially by $R = R_0 \oplus \{0\}$ with $R_0 = R$, every ring can be viewed as a superring. It is the same thing to ask for a ring to be commutative as to ask for a trivially graded ring to be supercommutative. In this point of view, notions in superalgebra become generalizations of those in classical abstract algebra, and things that are not "super" are seen to be graded trivially by default.

Remark 1.1.3 (Koszul Sign Rule). Usually we will consider only supercommutative things. The principle of adding a sign up to the parity when switching the position of two adjacent things is known as the *Koszul sign rule*. One will see that it appears everywhere in supergeometry.

For the reason in the preceding remark, one can assume safely that every superring is supercommutative from now on.

By abuse of notation, when it comes to power up -1 by the parity of some elements, one may use the element itself for its parity, i.e. write $(-1)^{p(a)p(b)} = (-1)^{ab}$. As one will see, since the parity is always additive under multiplication (and composition once we define the parity of morphism), $(-1)^{p(ab)}$ can be always written as $(-1)^{a+b}$ and thereby no confusion arises. Also, since everything is the direct sum of its homogeneous components, we usually assume that everything is homogeneous when using this convention.

On building the category of superrings, it is natural to ask for a forgetful functor from this category to Ring. Due to the requirement of supercommutativity of the multiplicative structure, it is pointless to consider ring homomorphisms that does not preserve the parity. Hence

Definition 1.1.2 (Superring Homomorphism)**.** A superring homomorphism φ from superrings R to R' is a ring homomorphism φ from R to R' that preserves the parity, i.e. $\varphi(R_i) \subset R'_i$ for any $i \in \mathbb{Z}_2$.

Indeed, superring homomorphism is exactly the usual homomorphism between $\mathbb{Z}_2$-graded rings.

The category of superrings is usually denoted as SRing. For our purpose, we will take SRing as the category of supercommutative rings.

Recall that an algebra over a commutative ring K, K-algebra, is a ring A along with a ring homomorphism $\varphi : K \to A$ such that $\varphi(K) \subset Z(A)$ where $Z(A)$ is the multiplicative center of A. Adding the word "super" before each single word, we obtain the notion of *superalgebra*.

Definition 1.1.3 (Superalgebra)**.** A superalgebra over a supercommutative ring R, super R-algebra, is a superring A along with a superring homomorphism $\varphi : R \to A$ such that $\varphi(R) \subset Z(A)$ where $Z(A)$ is the supercenter of A, i.e. $Z(A) := \{a \in A \mid ab = (-1)^{ab}ba,\ \forall b \in A\}$.

Morphisms from A to B two super R-algebras are superring morphisms from A to B such that the triangle
$$\begin{tikzcd} A \arrow[r] & B \\ R \arrow[u] \arrow[ur] & \end{tikzcd}$$
commutes. The category of super R-algebras is denoted

as R-SAlg. For our purpose, we will take R-SAlg as the category of *supercommutative R-algebras*, i.e. super R-algebras that are supercommutative as superrings.

We can now talk about the *supermodule*.

Definition 1.1.4 (Supermodule)**.** A supermodule M over a superring $R = R_0 \oplus R_1$, super R-module, is a (left) R-module (R seen as a ring) with a $\mathbb{Z}_2$-graded structure $M = M_0 \oplus M_1$ (direct sum as abelian groups), such that the multiplication by scalars respects the parity, i.e. $R_i M_j \subset M_{i+j}$ for any $i, j \in \mathbb{Z}_2$.

Equivalently, $R_i M_j \subset M_{i+j}$ is the same as that

$$p(rm) = p(r) + p(m)$$

for any homogeneous $r \in R$ and $m \in M$.

Clearly, a superring is a supermodule over itself.

When it comes to contexts where supercommutativity is always assumed, the left A-module structure gives rise to a right R-module structure, defined by $mr := (-1)^{rm} rm$.

Remark 1.1.4. Roughly speaking, the induced right R-module structure allows writing the scalars on both sides. For instance, this will save a lot of efforts when putting multiple supermodules over a supercommutative ring together via the tensor product.

When the superring is taken as a field, the requirement $R_i M_j \subset M_{i+j}$ can be abandoned and we obtain the notion of *super vector space*.

Definition 1.1.5 (Super Vector Space)**.** A super vector space V over k (usually with characteristic 0) is a $\mathbb{Z}_2$-graded k-vector space, i.e. a vector space with a direct sum decomposition (as vector spaces) $V = V_0 \oplus V_1$. If V_0 and V_1 have dimension p and q respectively, then V is said to have dimension $p|q$.

Similar to the classical context, we may consider a supermodule M over a superring A which is at the same time a super R-algebra, then M has a natural structure of a supermodule over R. More specifically, if R is a field, then M has a natural structure of a super vector space over R.

Cares should be taken when talking about morphisms between supermodules. Of course the family of morphisms that preserve the parity is a natural choice.

Definition 1.1.6 (Supermodule Homomorphism)**.** A supermodule homomorphism f from M to N two super R-modules is a R-module homomorphism that preserves the parity, i.e. $f(M_i) \subset N_i$ for any $i \in \mathbb{Z}_2$.

The category of super R-modules is formed with this choice of morphisms, and is denoted as R-SMod. It is this sense a supermodule morphism refers to. Without further specification, a morphism always preserves the parity.

However, it makes sense in practice to consider between supermodules parity-reversing morphisms and their linear combination with parity-preserving ones. The parity-preserving ones are said to be homogeneous with even parity 0 and the parity-reversing ones are said to be homogeneous with odd parity 1. According to the Koszul sign rule, we demand instead of the usual R-linearity the super R-linearity, i.e. for a homomorphism $f : M \to N$ of abelian groups to be a homogeneous (even or odd) morphism of super R-modules, there should be

$$f(rm) = (-1)^{fr} r f(m) \tag{1.1.1}$$

for any $m \in M$ and homogeneous $r \in R$. This is compatible with the induced right R-module structure, i.e. we have $f(mr) = f(m)r$ as one can easily verify. Formally,

Definition 1.1.7 (Homogeneous Morphism of Supermodules)**.** Let $f \in \mathrm{Hom}_{\mathsf{Ab}}(M, N)$ where M and N are super R-modules. f is

- an even morphism if $f(M_i) \subset N_i$ for any $i \in \mathbb{Z}_2$ and $f(rm) = rf(m)$ for any $r \in R$ and $m \in M$.
- an odd morphism if $f(M_i) \subset N_{i+1}$ for any $i \in \mathbb{Z}_2$ and $f(rm) = (-1)^r r f(m)$ for any $m \in M$ and homogeneous $r \in R$.

The set of all even morphisms from M to N is denoted as $\mathbf{Hom}_0(M, N)$ and the set of all odd ones is denoted as $\mathbf{Hom}_1(M, N)$. The assignment of the parity gives a function $(\mathbf{Hom}_0(M, N) \cup \mathbf{Hom}_1(M, N)) \setminus \{0\} \to \mathbb{Z}_2$, and the convention of super R-linearity in eq. (1.1.1) fits well to the definition. Also, note that there is $\mathbf{Hom}_0(M, N) = \mathrm{Hom}_{R\text{-}\mathsf{SMod}}(M, N)$.

The direct sum as abelian groups gives the *internal Hom set* $\mathbf{Hom}(M, N) := \mathbf{Hom}_0(M, N) \oplus \mathbf{Hom}_1(M, N)$. When R is supercommutative, $\mathbf{Hom}(M, N)$ has a natural super R-module structure where the addition and scalar multiplication are defined point-wisely.

When R is trivially graded, the super R-linearity is the same as the usual R-linearity, and it is easy to see that $\mathbf{Hom}(M, N) = \mathrm{Hom}_{R\text{-}\mathsf{Mod}}(M, N)$ in this case. In particular, for super k-vector spaces M and N we have $\mathbf{Hom}(M, N) = \mathrm{Hom}_{\mathsf{Vect}_k}(M, N)$.

The notations $\mathbf{End}(M, N)$ and $\mathbf{Aut}(M, N)$ are defined similarly.

1.2 Basics of Supermanifolds

In short,

Definition 1.2.1 (Supermanifold)**.** A *supermanifold* $\mathcal{M} = (M, \mathcal{O})$ *of dimension* $p|q$ is an underlying differentiable manifold M endowed with a structural sheaf $\mathcal{O} : \mathrm{Open}(M) \to \mathbb{R}\text{-}\mathrm{SAlg}$ of super $\mathbb{R}$-algebras such that the pair is locally $\mathbb{R}$-isomorphic to *smooth superdomains* of dimension $p|q$.

Where a smooth superdomain is

Definition 1.2.2 (Smooth Superdomain)**.** A smooth superdomain $\mathcal{U}^{p|q} = (U, \mathcal{C}^\infty_{p|q})$ of dimension $p|q$ is an open subset U of $\mathbb{R}^p$ endowed with a sheaf $\mathcal{C}^\infty_{p|q}$ defined for each open subset $V \subset U$ by

$$\mathcal{C}^\infty_{p|q}(V) \coloneqq \mathcal{C}^\infty(V)[\xi^1, \cdots, \xi^q]$$

where $\mathcal{C}^\infty(V)[\xi^1, \cdots, \xi^q]$ is the exterior algebra generated by $\xi^1, \cdots, \xi^q$ over $\mathcal{C}^\infty(V)$, i.e. the free $\mathcal{C}^\infty(V)$-algebra generated by $\xi^1, \cdots, \xi^q$ modulo the relation that ξ's are anticommutative, and the restriction maps are induced by the restriction of functions $\mathcal{C}^\infty(V) \to \mathcal{C}^\infty(W)$ if $W \subset V$. $\mathcal{C}^\infty_{p|q}(V)$ is a supercommutative ring by considering $\xi^1, \cdots, \xi^q$ to be odd.

and by locally $\mathbb{R}$-isomorphic we mean that

> For any point of M there exists a neighborhood W of that point such that $(W, \mathcal{O}|_W)$, where $\mathcal{O}|_W$ is the sheaf $\mathcal{O}$ restricted on W, is $\mathbb{R}$-isomorphic to some smooth superdomain $\mathcal{U}^{p|q} = (U, \mathcal{C}^\infty_{p|q})$ in the sense that there exists a diffeomorphism $\varphi : W \to U$ along with a natural isomorphism $\varphi^* : \mathcal{C}^\infty_{p|q} \cong \varphi_*\mathcal{O}|_W$ where $\varphi_*\mathcal{O}|_W : \mathrm{Open}(U) \xrightarrow{\varphi^{-1}} \mathrm{Open}\, O(W) \to \mathbb{R}\text{-}\mathrm{SAlg}$ is the pushforward sheaf of $\mathcal{O}|_W$ by φ.

For instance, a classical differential manifold of dimension n is a supermanifold of dimension $n|0$ if it is endowed with the sheaf $\mathcal{C}^\infty$ of smooth functions, and is a supermanifold of dimension $n|n$ if it is endowed with the sheaf $\Omega*$ of smooth forms. With slight abuse of notation, sometimes for a supermanifold $\mathcal{M} = (M, \mathcal{O})$ we may write $\mathcal{C}^\infty(\mathcal{M}) = \mathcal{O}(M)$.

In analogy to the classical theory, we call $(W, \mathcal{O}|_W) \cong \mathcal{U}^{p|q} = (U, (x, \xi))$ a *super coordinate neighborhood* with super coordinates $(x, \xi) = (x_1, \cdots, x_p, \xi^1, \cdots, \xi^q) \in \mathcal{C}^\infty_{p|q}(U)$, where $x_i : U \to \mathbb{R} : (a_1, \cdots, a_r) \mapsto a_i$.

Remark 1.2.1. In above description, the word "differentiable manifold" can be replaced by "a second countable Hausdorff topological space". Since the smooth structure of the underlying space is readily encoded by the sheaf under the given local condition, the second countable Hausdorff topological space becomes a differentiable manifold automatically, following the same line how a classical differentiable manifold is described in the manner of sheaf theory. We shall not talk much about these to make things brief.

Morphism between supermanifolds is defined, of course, in a similar manner.

Definition 1.2.3 (Morphism of Supermanifolds)**.** A morphism $\Psi = (\psi, \psi^*)$ from $(M, \mathcal{O})$ to $(N, \mathcal{R})$ two supermanifolds consists of

- a smooth map between the underlying spaces $\psi : M \to N$,
- a natural transformation $\psi^* : \mathcal{R} \Rightarrow \psi_*\mathcal{O}$ of functors from $\mathrm{Open}(N)$ to $\mathbb{R}\text{-}\mathrm{SAlg}$.

The morphisms of supermanifolds compose in the obvious way, giving rise to the category of supermanifolds, SMan.

In particular, for two supermanifolds to be isomorphic, they must have a same dimension and their underlying spaces must be diffeomorphic.

Example 1.2.1 (The Superspace $\mathbb{R}^{p|q}$)**.** Given a domain $U \subset \mathbb{R}^p$ along with a diffeomorphism $\varphi : U \cong \mathbb{R}^p$, it induces an isomorphism of supermanifolds

$$\mathcal{U}^{p|q} = (U, \mathcal{C}^\infty_{p|q}) \cong (\mathbb{R}^p, \mathcal{C}^\infty_{p|q}) =: \mathbb{R}^{p|q}$$

via $\varphi^* : \mathcal{C}^\infty(\mathbb{R}^p) \cong \mathcal{C}^\infty(U)$. Since every neighborhood in $\mathbb{R}^p$ contains a neighborhood of coordinate ball which is diffeomorphic to $\mathbb{R}^p$, replacing smooth superdomain with the superspace $\mathbb{R}^{p|q}$ in definition 1.2.1 gives exactly the same definition of supermanifold.

More generally as one can see, diffeomorphisms between underlying spaces of smooth superdomains of dimension $p|q$ induce isomorphisms of supermanifolds.

To establish a deeper understanding of morphisms of supermanifolds, we need to know what the super version of local ring is.

Definition 1.2.4 (Homogeneous Ideal)**.** A homogeneous ideal I of a superring R is an ideal I of the ring R such that $I = (I \cap R_0) \oplus (I \cap R_1)$, i.e. the homogeneous components of each element of I still live in I.

Since we require morphisms between superrings to preserve the parity, it is clear that the inverse image of a homogeneous ideal is also a homogeneous ideal.

Definition 1.2.5 (Local Superring)**.** A superring $R = R_0 \oplus R_1$ is local if it admits a unique maximal homogeneous ideal, i.e. it has only one homogeneous ideal that is maximal with respect to inclusion.

A superalgebra is local if it is local as a superring. The stalk $\mathcal{O}_x = \mathcal{C}^\infty_{p|q,x}$ of the structural sheaf at a point $x \in U \subset M$ is local, with the unique maximal homogeneous ideal $\mathfrak{m}_x$ consisting of all the non-units. More concretely, for $f \in \mathcal{C}^\infty_{p|q}(V)$ we may write it as

$$\begin{aligned} f(x,\xi) = \sum_\alpha f_\alpha(x)\xi^\alpha &= \sum_{l=0}^{q} \sum_{\alpha_1<\cdots<\alpha_l} f_{\alpha_1\cdots\alpha_l}(x)\xi^{\alpha_1}\cdots\xi^{\alpha_l} \\ &= f_0(x) + \sum_{l=1}^{q} \sum_{\alpha_1<\cdots<\alpha_l} f_{\alpha_1\cdots\alpha_l}(x)\xi^{\alpha_1}\cdots\xi^{\alpha_l}, \end{aligned}$$

with the coefficients f_α in $\mathcal{C}^\infty(V)$. A monomial term $f_{\alpha_1\cdots\alpha_l}(x)\xi^{\alpha_1}\cdots\xi^{\alpha_l}$ is said to have cohomological degree l. Since the part with nonzero cohomological degree

$$\sum_{l=1}^{q} \sum_{\alpha_1<\cdots<\alpha_l} f_{\alpha_1\cdots\alpha_l}(x)\xi^{\alpha_1}\cdots\xi^{\alpha_l}$$

is nilpotent, for any $x \in V$ the germ $[f]_x$ is a non-unit if and only if $f_0(x) = 0$, consequently the homogeneous components of a non-unit are also non-units. Hence

Theorem 1.2.1 (The Unique Maximal Homogeneous Ideal)**.** *The unique maximal homogeneous ideal of the stalk $\mathcal{C}^\infty_{p|q,x}$ is given by*

$$\mathfrak{m}_x = \{[f]_x \mid f_0(x) = 0\}. \tag{1.2.1}$$

From this it is easy to see that the residue field $\kappa(x) = \mathcal{C}^\infty_{p|q,x}/\mathfrak{m}_x$ is exactly $\mathbb{R}$ via the isomorphism $[f]_x \mapsto f_0(x)$. In classical theory, quotient by $\mathfrak{m}_x$ is essentially the same as forgetting the difference in infinitesimal neighborhoods around x, which yields the evaluation at x. Given $f \in \mathcal{O}(V)$ for any open $V \subset M$ and let $x \in V$ varies, this observation induces a real-valued function on V which is locally $(f|_U)_0 \in \mathcal{C}^\infty(U)$. This gives a morphism of super $\mathbb{R}$-algebras

$$\begin{aligned} \varepsilon_V \colon \mathcal{O}(V) &\longrightarrow \mathcal{C}^\infty(V) \\ f &\longmapsto \varepsilon_V(f) : \varepsilon_V(f)(x) := f_0(x) \end{aligned}$$

Let V vary and it is easy to see that ε_V is natural in V, giving rise to a natural transformation $\varepsilon : \mathcal{O} \Rightarrow \mathcal{C}^\infty$. Note that though for a smooth superdomain $\mathcal{U}^{p|q}$ we have simply $\mathcal{C}^\infty(U) \subset \mathcal{C}^\infty_{p|q}(U)$, ε does not necessarily admit a canonical right inverse $\mathcal{C}^\infty \Rightarrow \mathcal{O}$, due to the complexity of the global nature.

Remark 1.2.2. ε can be also seen as the local quotient of nilpotent components, which seems to be algebraically simpler. However, if one knows about the language of algebraic geometry, then one would agree that the above approach via residue fields is the most natural.

What have these to do with morphisms of supermanifolds? It is natural to ask for a morphism between supermanifolds to descend to a morphism between smooth manifolds. We have seen that the structural sheaf of a supermanifold descends naturally to the structural sheaf of the underlying smooth manifold via ε, so the question is that whether this way of descending is compatible with the morphisms. The answer is a nicely yes.

From the above we know that the "evaluation" of a "superfunction" $f \in \mathcal{O}(V)$ at a point $x \in M$ can be seen as its image in the residue field $\kappa(x) = \mathbb{R}$ and ε is induced by this "evaluation". A morphism $\Psi : \mathcal{M} = (M, \mathcal{O}) \to \mathcal{N} = (N, \mathcal{R})$ induces a map on stalk $\psi_x^* : \mathcal{R}_{\psi(x)} \to \mathcal{O}_x$, and for it to be compatible with the "evaluation" the only requirement is that it induces an isomorphism on the residue fields, $\mathbb{R} = \kappa(\psi(x)) = \mathcal{R}_{\psi(x)}/\mathfrak{m}_{\psi(x)} \cong \mathcal{O}_x/\mathfrak{m}_x = \kappa(x) = \mathbb{R}$, which is equivalent to that ψ_x^* preserves the maximal homogeneous ideal, i.e. $\psi_x^*(\mathfrak{m}_{\psi(x)}) \subset \mathfrak{m}_x$. This is not included by definition 1.2.3, but it is automatically satisfied according to the following proposition.

Proposition 1.2.2. *Let $f : A \to B$ be a morphism of local super $\mathbb{R}$-algebras A and B such that $A/\mathfrak{m}_A \cong B/\mathfrak{m}_B = \mathbb{R}$, then $f(\mathfrak{m}_A) \subset \mathfrak{m}_B$.*

Proof. The composition
$$\mathbb{R} \to A \xrightarrow{f} B \to B/\mathfrak{m}_B = \mathbb{R}$$
is the identity on $\mathbb{R}$ where the arrows are the obvious ones. Hence $g := A \xrightarrow{f} B \to B/\mathfrak{m}_B = \mathbb{R}$ is surjective. This tells that $\ker g = \mathfrak{m}_A$. Therefore

$$f(\mathfrak{m}_A) = f(\ker g) \subset \ker(B \to B/\mathfrak{m}_B = \mathbb{R}) = \mathfrak{m}_B$$

as desired. □

Remark 1.2.3. One can verify easily that the followings are equivalent:

- $\psi_x^*(\mathfrak{m}_{\psi(x)}) \subset \mathfrak{m}_x$,
- $\mathfrak{m}_{\psi(x)} \subset (\psi_x^*)^{-1}(\mathfrak{m}_x)$,
- $\mathfrak{m}_{\psi(x)} = (\psi_x^*)^{-1}(\mathfrak{m}_x)$.

However, there is not necessarily $\psi_x^*(\mathfrak{m}_{\psi(x)}) = \mathfrak{m}_x$ for obvious reason.

With the residue field preserved (under the identification $\kappa(x) = \mathbb{R}$), for any open subset $V \subset N$, $\psi^* : \mathcal{R}(V) \to \mathcal{O}(\psi^{-1}(V))$ induces a morphism $\widetilde{\psi^*} : \mathcal{C}^\infty(V) \to \mathcal{C}^\infty(\psi^{-1}(V))$ such that the diagram

$$\begin{tikzcd} \mathcal{R}(V) \arrow[r, "\psi^*"] \arrow[d, "\varepsilon_V"] & \mathcal{O}(\psi^{-1}(V)) \arrow[d, "\varepsilon_{\psi^{-1}(V)}"] \\ \mathcal{C}^\infty(V) \arrow[r, "\widetilde{\psi^*}"] & \mathcal{C}^\infty(\psi^{-1}(V)) \end{tikzcd}$$

is commutative, as one can see by evaluating the image in $\mathcal{C}^\infty(\psi^{-1}(V))$ at each $x \in \psi^{-1}(V)$. Moreover, the induced $\widetilde{\psi^*}$ turns out to coincide with the precomposition by $\psi : M \to N$.

Remark 1.2.4. In above deduction, the smoothness of $\psi : M \to N$ is never used. The fact that the precomposition by $\psi : M \to N$ is well-defined, mapping smooth functions to smooth ones, follows purely algebraic from the nature of $\psi^* : \mathcal{R} \Rightarrow \psi_* \mathcal{O}$. Therefore the requirement of smoothness of ψ in definition 1.2.3 can be replaced by only continuity, and the smoothness would follow automatically by above deduction. This is the philosophy when describing the geometry using sheaf theory: the geometrical information is always encoded in the sheaf, instead of the underlying space.

For classical smooth manifolds, a smooth map $f : M \to U \subset \mathbb{R}^r$ is determined by the components $f_i \in \mathcal{C}^\infty(M)$ such that $f(p) = (f_1(p), \cdots, f_r(p))$ for any $p \in M$. To give a set of such components is equivalent to determining the pullbacks $y_i \mapsto f_i \in \mathcal{C}^\infty(M)$ of the coordinates $y_1, \cdots, y_r \in \mathcal{C}^\infty(U)$, and every assignment of $y_i \mapsto f_i$ such that $\operatorname{Im} f \subset U$ gives a unique smooth map $f : M \to U$. The analogy holds for supermanifolds, known as the *Fundamental Theorem of Supermorphisms*.

Before stating and proving the theorem, the technique of *Approximation by Polynomial* should be established. Let $\mathcal{M} = (M, \mathcal{O})$ be a supermanifold of dimension $p|q$ as always. Let $x_0 \in M$ be any point and choose a neighborhood of superdomain $\mathcal{U}^{p|q} = (U, \mathcal{C}^\infty_{p|q})$. By translation, we can assume that $x_0 = 0$ in U for conciseness. For a smooth function $f_0 \in \mathcal{C}^\infty(U)$ to vanish at $x_0 = 0$, there is $f_0(x) \sim O(x) := O(\|x\|)$ near 0 by Taylor approximation. Hence we can rewrite eq. (1.2.1) as

$$\mathfrak{m}_{x_0} = \left\{ [f]_{x_0=0} \;\middle|\; f(x,\xi) = O(x) + \sum_{l=1}^{q} \sum_{\alpha_1 < \cdots < \alpha_l} f_{\alpha_1 \cdots \alpha_l}(x) \xi^{\alpha_1} \cdots \xi^{\alpha_l} \right\}.$$

It follows that for $k \geq 1$,

$$\begin{aligned} \mathfrak{m}^k_{x_0} = \Bigg\{ [f]_{x_0=0} \;\Bigg|\; f(x,\xi) = &\sum_{l=0}^{k-1} \sum_{\alpha_1 < \cdots < \alpha_l} O(x^{k-l}) \xi^{\alpha_1} \cdots \xi^{\alpha_l} \\ &+ \sum_{l=k}^{q} \sum_{\alpha_1 < \cdots < \alpha_l} f_{\alpha_1 \cdots \alpha_l}(x) \xi^{\alpha_1} \cdots \xi^{\alpha_l} \Bigg\}, \end{aligned}$$

where $O(x^s) := O(\|x\|^s)$ and $\sum_{l=k}^q$ is taken to be zero if $k \geq q$. In particular,

$$\mathfrak{m}^{q+1}_{x_0} = \left\{ [f]_{x_0=0} \;\middle|\; f(x,\xi) = O(x^{q+1}) + \sum_{\alpha} O(x^q) \xi^\alpha + \cdots + O(x) \xi^1 \cdots \xi^q \right\}. \tag{1.2.2}$$

It follows that

Lemma 1.2.3. *Let $\mathcal{M} = (M, \mathcal{O})$ be a supermanifold of dimension $p|q$, $x_0 \in U \subset M$ and $f \in \mathcal{O}(U)$. If $[f]_{x'} \in \mathfrak{m}^{q+1}_{x'}$ for a dense set of x' in some neighborhood of $x_0 \in M$, then $[f]_{x_0} = 0$.*

In particular, if $f, g \in \mathcal{O}(V)$ satisfies $f - g \in \mathfrak{m}^{q+1}_x$ for any $x \in V$, then $f = g$.

Theorem 1.2.4 (Approximation by Polynomial)**.** *Let $(M, \mathcal{O})$ be a supermanifold of dimension $p|q$, $x_0 \in M$ an arbitrary point and $f \in \mathcal{O}(V)$ any section for a neighborhood V of x_0. For any fixed degree of approximation $k \in \mathbb{N}^*$, there exists a polynomial $P = P(x,\xi) \in \mathbb{R}[x_1, \cdots, x_p, \xi^1, \cdots, \xi^q] \subset \mathcal{C}^\infty_{p|q}(U)$, where (x,ξ) are super coordinates given by a coordinate neighborhood $(U, \mathcal{C}^\infty_{p|q})$ of x_0, such that*

$$[f]_{x_0} - [P]_{x_0} \in \mathfrak{m}^k_{x_0}.$$

Proof. By translation we may assume $x_0 = 0 \in U$. Bring f into $\mathcal{C}^\infty_{p|q}(V \cap U)$ and use the Taylor approximation to find polynomial $P_\alpha := P_{\alpha_1 \cdots \alpha_l}$ such that

$$f_\alpha(x) := f_{\alpha_1 \cdots \alpha_l}(x) = P_\alpha(x) + O(x^k)$$

for each $\alpha_1 < \cdots < \alpha_l$. It follows that

$$f = \sum_\alpha f_\alpha \xi^\alpha = \sum_\alpha P_\alpha(x)\xi^\alpha + \sum_\alpha O(x^k)\xi^\alpha,$$

where by convention $\xi^\alpha := \xi^{\alpha_1}\cdots\xi^{\alpha_l}$. Since $\sum_\alpha O(x^k)\xi^\alpha \in \mathfrak{m}_{x_0}^k$, put $P := \sum_\alpha P_\alpha(x)\xi^\alpha$ and we are done. $\square$

Roughly speaking, theorem 1.2.4 may be seen as the super version of Taylor approximation, where $\mathfrak{m}_{x_0}^k$ serves as $O(\|x-x_0\|^k)$. The larger the k grows, the closer the approximation is.

What is good about polynomials is that under the requirement of a morphism $\psi^* : \mathcal{C}^\infty_{p|q}(U) \to \mathcal{O}(M)$ being a morphism of super $\mathbb{R}$-algebras, its value on polynomials is determined by its value on the super coordinates $x_1, \cdots, x_p, \xi^1, \cdots, \xi^q$. Theorem 1.2.4 thus asserts that such a morphism is completely determined by its value on the super coordinates as a consequence of $\psi^*(\mathfrak{m}_{\psi(x)}^k) = \psi^*(\mathfrak{m}_{\psi(x)})^k \subset \mathfrak{m}_x^k$. Moreover, for any parity-preserving assignment such that the induced smooth map $M \to U$ is well-defined with image contained by U there is a corresponding morphism of supermanifolds $\mathcal{M} \to \mathcal{U}^{p|q}$. This is exactly what the Fundamental Theorem is about:

Theorem 1.2.5 (Fundamental Theorem of Supermorphisms)**.** *Let $\mathcal{M} = (M, \mathcal{O})$ be a supermanifold of any dimension $p'|q'$ and $\mathcal{U}^{p|q}$ be a smooth superdomain of dimension $p|q$. If $(s, \sigma) = (s_1, \cdots, s_p, \sigma^1, \cdots, \sigma^q)$ is a $(p+q)$-tuple of superfunctions in $\mathcal{O}(M)$ such that*

- *$s_1, \cdots, s_p$ are homogeneous with even parity and $\sigma^1, \cdots, \sigma^q$ are odd,*
- *$\mathrm{Im}(\varepsilon_M s_1, \cdots, \varepsilon_M s_p) \subset U$,*

then there exists a unique morphism of supermanifolds $\Psi = (\psi, \psi^*) : \mathcal{M} \to \mathcal{U}^{p|q}$ such that

$$s_i = \psi^* y_i \quad \textit{and} \quad \sigma^j = \psi^* \eta^j \tag{1.2.3}$$

for each $1 \le i \le p$ and $1 \le j \le q$, where (y, η) are the super coordinates of $\mathcal{U}^{p|q}$.

Proof. The uniqueness is rather easy. Firstly, for any morphism (ψ, ψ^*) satisfying eq. (1.2.3) there must be $\psi = (\varepsilon_M s_1, \cdots, \varepsilon_M s_p)$ since

$$y_i \circ \psi = \widetilde{\psi^*}(y_i) = \varepsilon_M \psi^* y_i = \varepsilon_M s_i.$$

Now given any two morphisms (ψ_1, ψ_1^*) and (ψ_2, ψ_2^*) satisfying eq. (1.2.3). For any $V \subset U$, $f \in \mathcal{C}^\infty_{p|q}(V)$ and $x_0 \in \psi^{-1}(V)$, write $y_0 = \psi(x_0)$ and by theorem 1.2.4 there exists a polynomial $P_{x_0} = P_{x_0}(y, \eta) \in \mathcal{C}^\infty_{p|q}(U)$ labeled by x_0 such that $[f]_{y_0} - [P_{x_0}]_{y_0} \in \mathfrak{m}_{y_0}^{q'+1}$.

Apply ψ_i^* and we obtain

$$[\psi_i^*(f)]_{x_0} - [\psi_i^*(P_{x_0})]_{x_0} \in \mathfrak{m}_{x_0}^{q'+1}, \quad i = 1, 2.$$

Therefore since $\psi_1^*(P_{x_0}) = \psi_2^*(P_{x_0})$,

$$[\psi_1^*(f) - \psi_2^*(f)]_{x_0} = ([\psi_1^*(f)]_{x_0} - [\psi_1^*(P_{x_0})]_{x_0}) - ([\psi_2^*(f)]_{x_0} - [\psi_2^*(P_{x_0})]_{x_0}) \in \mathfrak{m}_{x_0}^{q'+1}.$$

Let $x_0 \in \psi^{-1}(V)$ vary and we see by lemma 1.2.3 that $\psi_1^*(f) = \psi_2^*(f)$, concluding $\psi_1^* = \psi_2^*$.

For the existence, we put the underlying smooth map to be $\psi = (\varepsilon_M s_1, \cdots, \varepsilon_M s_p) : M \to U$ and then construct $\psi^* : \mathcal{C}_{p|q}^\infty(V) \to \mathcal{O}(\psi^{-1}(V))$ for any $V \subset U$. ψ^* can be constructed by passing locally to super coordinate neighborhoods $\mathcal{W}$ of $\psi^{-1}(V)$, defining morphisms of super $\mathbb{R}$-algebras

$$\psi^*|_W : \mathcal{C}_{p|q}^\infty(V) \to \mathcal{C}_{p'|q'}^\infty(W),$$

and then gluing the results together via the restrictions $\mathcal{O}(\psi^{-1}(V)) \to \mathcal{C}_{p'|q'}^\infty(W)$ to obtain the map $\psi^* : \mathcal{C}_{p|q}^\infty(V) \to \mathcal{O}(\psi^{-1}(V))$. Once the local definition is made satisfying eq. (1.2.3), its canonical is permitted and the legality of gluing follows immediately because morphisms of super $\mathbb{R}$-algebras satisfying eq. (1.2.3) must agree in the overlap.

Now it suffices define for any super coordinate neighborhood $\mathcal{W}$ of $\psi^{-1}(V)$ a morphism of super $\mathbb{R}$-algebras $\mathcal{C}_{p|q}^\infty(V) \to \mathcal{C}_{p'|q'}^\infty(W)$ satisfying eq. (1.2.3) with (s, σ) restricted to W.

Let (x, ξ) be the super coordinates given by $\mathcal{W}$. s_i, seen as restricted to W, can be decomposed as

$$s_i = \sum_\beta s_\beta^i(x)\xi^\beta = \varepsilon_W s_i + n^i,$$

where n^i is the nilpotent part. For a superfunction $f = f(y, \eta) = \sum_\alpha f_\alpha(y)\eta^\alpha \in \mathcal{C}_{p|q}^\infty(V)$, there must be

$$\psi^*|_W(f) = \sum_\alpha \psi^*|_W(f_\alpha(y))(\psi^*\eta)^\alpha = \sum_\alpha \psi^*|_W(f_\alpha(y))\sigma^\alpha,$$

so it remains only to determine $\psi^*|_W(f_\alpha(y))$. Intuitively, one would like that there is $\psi^*|_W(f_\alpha(y)) = f_\alpha(\psi^* y) = f_\alpha(s) = f_\alpha \circ s$. However, s_i is not real-valued. The remedy to this situation is to use the trick of formal Taylor expansion, substituting $s = \varepsilon s + n := (\varepsilon s_1, \cdots, \varepsilon s_p) + (n^1, \cdots, n^p)$ into $T(f; \varepsilon s)$, obtaining

$$\psi^*|_W(f_\alpha(y)) = f_\alpha(s) = f_\alpha(\varepsilon s + n) := \sum_\beta \frac{1}{\beta!}(\partial_y^\beta f_\alpha)(\varepsilon s)n^\beta = \sum_\beta \frac{1}{\beta!}((\partial_y^\beta f_\alpha) \circ \varepsilon s)(x)n^\beta.$$

Note that n^β is well-defined as n^i's are even, hence commute with each other. Thus we conclude that the defining formula for $\psi^*|_W$ is

$$\psi^*|_W(f) = \sum_\alpha \sum_\beta \frac{1}{\beta!}(\partial_y^\beta f_\alpha)(\varepsilon s)n^\beta \sigma^\alpha.$$

This defining formula works well. The sums are finite due to nilpotency. Since n^i's are even, $\psi^*|_W$ defined in this way is parity-preserving. $(\partial_y^\beta f_\alpha)(\varepsilon s) = ((\partial_y^\beta f_\alpha) \circ \varepsilon s)(x)$ is smooth in x, hence the coefficients are legal. Since the Taylor expansion commutes algebraically with addition and multiplication, $\psi^*|_W$ is indeed a morphism of super $\mathbb{R}$-algebras. Finally, the verification that $\psi^*|_W$ does satisfy eq. (1.2.3) is straightforward. □

In above proof, the way of evaluating a smooth function at a "super point" is essentially the same as the way one generalizes a smooth function to a holomorphic function on the complex plane using Taylor expansion, e.g. the famous Euler's formula $e^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$.

A supermorphism $(\psi, \psi^*) : \mathcal{M} \to \mathcal{N}$ is locally a super $\mathbb{R}$-algebra morphism

$$\psi^* : \mathcal{C}^\infty(V)[\eta^1, \cdots, \eta^{q'}] \to \mathcal{C}^\infty(U)[\xi^1, \cdots, \xi^q]$$

over ψ under coordinate neighborhoods $\mathcal{U}^{p|q} = (U, (x, \xi))$ of $\mathcal{M}$ and $\mathcal{V}^{p'|q'} = (V, (y, \eta))$ of $\mathcal{N}$ such that $U \subset \psi^{-1}(V)$. With the usual abuse of notation in classical theory, we may write locally that

$$\begin{aligned} y_i &= \psi^* y_i = s_i(x, \xi) = y_i(x, \xi) \qquad \text{(even)}, \\ \eta^j &= \psi^* \eta^j \sigma^j(x, \xi) = \eta^j(x, \xi) \qquad \text{(odd)}. \end{aligned} \tag{1.2.4}$$

The Fundamental Theorem asserts that the local representation eq. (1.2.4) determines the supermorphism completely.

With the convention of evaluating smooth functions at "super points", the local behaviour of supermorphisms is completely similar to that of classical ones, and one might see intuitively how the imaginary coordinates (the ξ's and η's) work in this way.

Example 1.2.2. Let $(\psi, \psi^*) : \mathbb{R}^{1|2} \to \mathbb{R}^{1|1}$ be defined by

$$\begin{aligned} y &= y(x, \xi) = x + \xi^1\xi^2, \\ \eta &= \eta(x, \xi) = f(x)\xi^1 + g(x)\xi^2, \end{aligned}$$

where the choice of $f, g \in \mathcal{C}^\infty(\mathbb{R})$ does not really matter. We have

$$\begin{aligned} \psi^* \sin y &= \sin x + (\cos x)\xi^1\xi^2, \\ \psi^* \cos y &= \cos x - (\sin x)\xi^1\xi^2, \\ \psi^*(\sin y \cos y) &= \sin x \cos x + (\cos^2 x - \sin^2 x)\xi^1\xi^2. \end{aligned}$$

1.3 Differential Calculus on Supermanifolds

Tangent Things

In description of the classical theory using sheaves, vector bundles of rank n over a smooth manifold M are 1-to-1 with locally free sheaves of $\mathcal{C}^\infty$-modules of rank n over M by sending a vector bundle E to its sheaf of sections $\Gamma(M, E)$.[1] Given a locally free sheaf $\mathcal{E}$, the set of vectors associated to a point x, being the evaluation at x of the sections, is given by $\mathcal{E}_x/\mathfrak{m}_x\mathcal{E}_x$ where $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{C}^\infty_x$. These generalize to the super version immediately.

Definition 1.3.1 (Super Vector Bundle)**.** A *super vector bundle of rank* $p|q$ over a supermanifold $\mathcal{M} = (M, \mathcal{O})$ is a locally free sheaf of super $\mathcal{O}$-modules of rank $p|q$ over M.

where a *free super module of rank* $p|q$ over superring R is

Definition 1.3.2 (Free Super R-module of rank $p|q$)**.** A free super R-module of rank $p|q$, denoted as $R^{p|q}$, is a super R-module that admits a basis $(e_i)_{1\le i\le p+q}$ where e_i is even for $1 \le i \le p$ and is odd for $p+1 \le i \le p+q$. This means that

$$R^{p|q} = e_1 R \oplus \cdots \oplus e_{p+q} R.$$

Similarly, the (super) tangent sheaf over $\mathcal{M} = (M, \mathcal{O})$ is defined to be the sheaf of *superderivations* on $\mathcal{O}(U)$, U open in M, where

Definition 1.3.3 (Homogeneous Superderivation)**.** A homogeneous superderivation of parity i of the super $\mathbb{R}$-algebra $\mathcal{O}(U)$ is an $\mathbb{R}$-linear map $D \in \mathbf{End}_i\, \mathcal{O}(U)$ of parity i, where $\mathcal{O}(U)$ is regarded as a super $\mathbb{R}$-module, which satisfies the *graded Leibniz rule*

$$D(st) = D(s)t + (-1)^{ij} s(Dt) \tag{1.3.1}$$

for all $s \in \mathcal{O}_j(U)$ and $t \in \mathcal{O}(U)$.

We denote by $\mathrm{Der}_i(U)$ the set of all superderivations of parity i of $\mathcal{O}(U)$. Clearly, $\mathrm{Der}_i(U)$ is an $\mathbb{R}$-vector space. Thus the set

$$\mathrm{Der}\, \mathcal{O}(U) \coloneqq \mathrm{Der}_0\, \mathcal{O}(U) \oplus \mathrm{Der}_1\, \mathcal{O}(U)$$

of all superderivations of $\mathcal{O}(U)$ is a super vector space over $\mathbb{R}$. Also, the point-wisely defined super $\mathcal{O}(U)$-module structure applies to $\mathrm{Der}\, \mathcal{O}(U)$, with parity given by

$$sD \in \mathrm{Der}_{i+j}\, \mathcal{O}(U),$$

for $D \in \mathrm{Der}_i\, \mathcal{O}(U)$ and $s \in \mathcal{O}_j(U)$.

We have obtained for each open $U \subset M$ a super $\mathcal{O}(U)$-module $\operatorname{Der}\mathcal{O}(U)$. To make this a sheaf, we need to construct a restriction map $\operatorname{Der}\mathcal{O}(U) \to \operatorname{Der}\mathcal{O}(V)$ whenever $V \subset U$. This requires the local feature of derivation, the proof of which in the classical theory uses the smooth bump function. Below gives the super version of bump function.

Definition 1.3.4 (Support)**.** The *support* of a superfunction $s \in \mathcal{O}(U)$ is the closed subset $\operatorname{supp} s := U \setminus \Omega$, where

$$\Omega = \{x \in U \mid \exists \text{ a neighborhood } V \subset U \text{of } x \text{ such that } s|_V = 0\}.$$

Equivalently we have

$$\operatorname{supp} s = \{x \in U \mid [s]_x \neq 0 \in \mathcal{O}_x\}.$$

Definition 1.3.5 (Super Bump Function)**.** A *super bump function around $x \in M$ supported in U* is a section $\gamma \in \mathcal{O}_0(M)$ with $\operatorname{supp}\gamma \subset U$ and $\gamma|_V = 1$ for some $V \subset U$ neighborhood of x.

If we don't need V to be large, then it is easy to see the existence of super bump functions. For any $x \in M$, let U be an arbitrary neighborhood of x and $\mathcal{W}$ be a super coordinate neighborhood of x contained in U. For any neighborhood V of x whose closure is contained in W and open set B with $\overline{V} \subset B \subset \overline{B} \subset W$, there exists by classical theory a smooth bump function $f \in \mathcal{C}^\infty(W) \subset \mathcal{O}_0(W) \subset \mathcal{C}^\infty_{p|q}(W)$ with $\operatorname{supp} f \subset B$ and $f|_V = 1$. As $\operatorname{supp} f \subset \overline{B} \subset W$ is closed in M, we can extend f by zero to M using the gluing axiom of sheaf, obtaining the desired super bump function $\widetilde{f} \in \mathcal{O}_0(M)$ around x supported in U.

For any $s \in \mathcal{O}(U)$, let $\gamma \in \mathcal{O}_0(M)$ be a super bump function supported in $B \subset \overline{B} \subset U$ with $\gamma|_V = 1$, then $\gamma|_U s$ is supported in $\overline{B}$, hence can be extended by zero to M. This gives

Lemma 1.3.1 (Extension by Bump)**.** *For any point $x \in U$ and any section $s \in \mathcal{O}(U)$, there exists a global section $S \in \mathcal{O}(M)$ and a neighborhood $V \subset U$ of x such that $S|_V = s|_V$ and* $\operatorname{supp} S \subset \operatorname{supp} s$. *Moreover, if s is homogeneous of parity i, then so does S.* □

We now state and prove the locality of superderivation.

Proposition 1.3.2 (Local Feature of Superderivations)**.** *Every superderivation $D \in \operatorname{Der}\mathcal{O}(U)$ is local in the sense that for any $V \subset U$ and $s, t \in \mathcal{O}(U)$, if $s|_V = t|_V$, then $(Ds)|_V = (Dt)|_V$.*

Proof. By linearity it suffices to show that if $s|_V = 0$ then $(Ds)|_V = 0$. For any $x \in V$, there exists a super bump function $\gamma \in \mathcal{O}_0(U)$ supported in V with $\gamma|_W = 1$ for a neighborhood $W \subset V$ of x. Since $\operatorname{supp}(\gamma s) \subset \operatorname{supp} s \cap \operatorname{supp}\gamma \subset (U \setminus V) \cap V = \varnothing$, $\gamma s = 0$, hence

$$0 = (D(\gamma s))|_W = (D\gamma)|_W s|_W + \gamma|_W (Ds)|_W = (Ds)|_W.$$

Let $x \in V$ vary and we conclude that $(Ds)|_V = 0$. □

The following gives the desired restriction map.

Proposition 1.3.3 (Restriction of Superderivations)**.** *For any $D \in \operatorname{Der} \mathcal{O}(U)$ and open subset $V \subset U$, there exists a unique $D|_V \in \operatorname{Der} \mathcal{O}(V)$ such that*

$$D|_V s|_V = (Ds)|_V \tag{1.3.2}$$

for any $s \in \mathcal{O}(U)$. Moreover, if D is homogeneous of parity i, then so is $D|_V$.

Proof. For the uniqueness, for any $D' \in \operatorname{Der} \mathcal{O}(V)$ satisfying eq. (1.3.2), we have for any $s \in \mathcal{O}(V)$ and $x \in V$, by lemma 1.3.1, some $S \in \mathcal{O}(U)$ with $S|_W = s|_W$ for some $W \subset V$ neighborhood of x, which gives

$$(D's)|_W \xlongequal{\text{proposition 1.3.2}} (D'S|_V)|_W \xlongequal{\text{eq. (1.3.2)}} (DS)|_W. \tag{1.3.3}$$

By proposition 1.3.2, $(DS)|_W$ is independent of the choice of S, hence eq. (1.3.3) tells that $D's$ is locally determined by the information of D and s. The identity axiom of sheaf hence concludes that D' is uniquely determined.

For the existence, we use eq. (1.3.3) as the definition of the value $D|_V s$. By the locality, $(DS)|_W$'s coincide in the overlaps, hence the gluing and identity axioms apply to define $D|_V s$. Substituting S by $s \in \mathcal{O}(U)$, s by $s|_V$ and W by V into eq. (1.3.3) and one obtain eq. (1.3.2). Since the restriction of sections preserves the parity, one can see that $D|_V$ has the same parity to D by substituting homogeneous D and s into eq. (1.3.3) and track the parity of $(DS)|_W$. It remains only to verify that $D|_V$ satisfy the graded Leibniz rule eq. (1.3.1). By the identity axiom of sheaf, it suffices to verify this locally: for $D \in \operatorname{Der} \mathcal{O}_i(U)$, any $s \in \mathcal{O}_j(V)$, $t \in \mathcal{O}(V)$, $x \in V$ with $S, T \in \mathcal{O}(U)$ such that $S|_W = s|_W$, $T|_W = t|_W$ for $W \subset V$ a neighborhood of x, we have

$$\begin{aligned}
(D|_V(st))|_W &= (D(ST))|_W = (DS)|_W T|_W + (-1)^{ij} S|_W (DT)|_W \\
&= (D|_V s)|_W t|_W + (-1)^{ij} s|_W (D|_V t)|_W \\
&= \left((D|_V s)t + (-1)^{ij} s(D|_V t)\right)\Big|_W.
\end{aligned}$$

□

It is immediate by eq. (1.3.2) that the given restriction map $\operatorname{Der} \mathcal{O}(U) \to \operatorname{Der} \mathcal{O}(V) : D \mapsto D|_V$ is compatible with the restriction $\mathcal{O}(U) \to \mathcal{O}(V)$. The gluing of superderivations follows from that of superfunctions. Therefore $\operatorname{Der} \mathcal{O}$ gives a sheaf of super $\mathcal{O}$-modules which we name to be the *tangent sheaf*, denoted as $T\mathcal{M}$ for supermanifold $\mathcal{M}$.

As is mentioned before in definition 1.3.1, for $T\mathcal{M}$ to be a super vector bundle we need to check that it is locally free. Indeed, its local description is very similar to that of the classical theory.

Let $\mathcal{M}$ be of dimension $p|q$ and let $(U,(x,\xi))$ be a super coordinate neighborhood with coordinates (x,ξ). We define $p+q$ superderivations

$$\begin{aligned}\partial_{x_i} &\in \operatorname{Der}_0 \mathcal{O}(U), \quad 1\le i\le p,\\ \partial_{\xi^j} &\in \operatorname{Der}_1 \mathcal{O}(U), \quad 1\le j\le q,\end{aligned}$$

by putting for any $s\in\sum_\alpha s_\alpha(x)\xi^\alpha\in\mathcal{O}(U)$,

$$\begin{aligned}\partial_{x_i} s &:= \sum_\alpha \left(\partial_{x_i} s_\alpha(x)\right)\xi^\alpha\\ \partial_{\xi^j} s &:= \sum_\alpha s_\alpha(x)(\partial_{\xi^j}\xi^\alpha)\end{aligned}$$

where

$$\partial_{\xi^j}\xi^\alpha := \begin{cases}(-1)^{\#\{\alpha_i<j\}}\xi^{(\alpha_1,\cdots,\hat{j},\cdots,\alpha_k)} & j\in\alpha\\ 0 & j\notin\alpha\end{cases}$$

where $\hat{j}$ implies that j is deleted. This is the same as putting $\partial_{\xi^j}\xi^i := \delta_{ij}$ for each i, where δ is the Kronecker delta, and extending it by the Leibniz rule. Or intuitively, one may think $\partial_{\xi^j}\xi^\alpha$ as to reorder ξ^α such that ξ^j is the first on the left and then kill ξ^j.

Theorem 1.3.4 (Local Description of Tangent Sheaf)**.** *Let $\mathcal{M}$ be of dimension $p|q$ and let $(U,(x,\xi))$ be a super coordinate neighborhood with coordinates (x,ξ), then $(\partial_x,\partial_\xi)$ is a basis of the super $\mathcal{O}(U)$-module $T\mathcal{M}(U)$, i.e. any $X\in T\mathcal{M}(U)$ admits a unique decomposition*

$$X=\sum_{1\le i\le p}\mathcal{X}^i\partial_{x_i}+\sum_{1\le j\le q}\mathscr{X}^j\partial_{\xi^j}, \tag{1.3.4}$$

where $\mathcal{X}^i,\mathscr{X}^j\in\mathcal{O}(U)$.

Proof. The uniqueness of the decomposition is easy, since if X admits decomposition eq. (1.3.4), then there must be $\mathcal{X}^i=Xx_i$ and $\mathscr{X}^j=X\xi^j$. For the existence, we put $\mathcal{X}^i=Xx_i$ and $\mathscr{X}^j=X\xi^j$ and measure the difference

$$Y:=X-\left(\sum_{1\le i\le p}\mathcal{X}^i\partial_{x_i}+\sum_{1\le j\le q}\mathscr{X}^j\partial_{\xi^j}\right).$$

It suffices to show that $Ys=0$ for any $s\in\mathcal{O}(U)$. Clearly, $Yx_i=Y\xi^j=0$ for any i,j. By the Leibniz rule we thus have

$$YP=0,$$

for any polynomial $P\in\mathbb{R}[x_1,\cdots,x_p,\xi^1,\cdots,\xi^q]\subset\mathcal{O}(U)$. The key is to use the technique of approximation by polynomial, theorem 1.2.4.

Fix $s \in \mathcal{O}(U)$. For any $x_0 \in U$, there exists a polynomial P such that

$$[s]_{x_0} - [P]_{x_0} \in \mathfrak{m}_{x_0}^{q+1}.$$

By the local feature, superderivations induce maps on stalks, which gives

$$[Ys]_{x_0} = [Ys]_{x_0} - [YP]_{x_0} = Y([s]_{x_0} - [P]_{x_0}) \in Y\mathfrak{m}_{x_0}^{q+1}.$$

By lemma 1.2.3, it suffices to show that $Y\mathfrak{m}_{x_0}^{q+1} \subset \mathfrak{m}_{x_0}^{q+1}$. For any $f \in \mathfrak{m}_{x_0}^{q+1}$, we may apply a translation so that $x_0 = 0$ and then write by eq. (1.2.2),

$$\begin{aligned} f &= O(x^{q+1}) + \sum_{\alpha} O(x^q)\xi^\alpha + \cdots + O(x)\xi^1 \cdots \xi^q \\ &\xlongequal{\text{Taylor Expansion with Lagrange Remainder}} \sum \varepsilon_I(x) x_{i_1} \cdots x_{i_{q+1-l}} \xi^{\alpha^1} \cdots \xi^{\alpha^l}, \end{aligned}$$

where $I = (i_1, \cdots, i_{q+1-l})$ and ε_I is the smooth function given by the Taylor expansion. Since Y vanishes on polynomials, by the Leibniz rule we obtain

$$Yf = \sum \left(Y\varepsilon_I(x)\right) x_{i_1} \cdots x_{i_{q+1-l}} \xi^{\alpha^1} \cdots \xi^{\alpha^l},$$

which tells that $Yf \in \mathfrak{m}_{x_0}^{q+1}$ by eq. (1.2.2). □

We therefore conclude that $T\mathcal{M}$ is indeed a super vector bundle on $\mathcal{M}$ of the same rank to the dimension of $\mathcal{M}$.

Analogously to the classical theory, we can define *super tangent vectors* as

Definition 1.3.6 (Homogeneous Super Tangent Vector)**.** Let $\mathcal{M} = (M, \mathcal{O})$ be a supermanifold and and $x \in M$, a homogeneous super tangent vector of parity i at x to $\mathcal{M}$, is a derivation of parity i at x of $\mathcal{O}_x$, i.e. an $\mathbb{R}$-linear map

$$X_x : \mathcal{O}_x \to \mathbb{R}$$

of parity i with $\mathbb{R}$ is trivially graded, such that for any $s \in \mathcal{O}_{x,j}$ and any $t \in \mathcal{O}_x$, the graded Leibniz rule

$$X_x(st) = (X_x s)(\varepsilon t)(x) + (-1)^{ij}(\varepsilon s)(x)(X_x t),$$

is satisfied, where ε is defined as after eq. (1.2.1).

The super $\mathbb{R}$-vector space of all super tangent vectors (i.e. the space of the $\mathbb{R}$-linear combinations of the homogeneous ones) is denoted as $T_x\mathcal{M}$, called the *super tangent space of $\mathcal{M}$ at x*.

Let $x \in U$ and $[X]_x \in (T\mathcal{M})_x$, $[X]_x$ induces a map $\widetilde{X} : \mathcal{O}_x \to \mathcal{O}_x$ by its representative X. It is easy to verify that the composition $X_x := \pi \circ \widetilde{X} : \mathcal{O}_x \to \mathcal{O}_x \to \mathcal{O}_x/\mathfrak{m}_x = \mathbb{R}$

gives a super tangent vector at x. If X is homogeneous, then the parity of X_x is the same as X. Intuitively, this means that the evaluation at x of a super tangent vector field is a super tangent vector at x. The assignment $T\mathcal{M}(U) \to (T\mathcal{M})_x \to T_x\mathcal{M} : X \mapsto X_x$ is in fact surjective, as we have

Theorem 1.3.5 (Local Description of Super Tangent Space)**.** *Let $\mathcal{M}$ be of dimension $p|q$, $x_0 \in M$ be any point and $(U, (x, \xi))$ be a super coordinate neighborhood of x_0. The super tangent space $T_{x_0}\mathcal{M}$ is a super vector space over $\mathbb{R}$ with basis $\partial_{x_i, x_0} \in T_{x_0,0}\mathcal{M}$, $\partial_{\xi^j, x_0} \in T_{x_0,1}\mathcal{M}$, $1 \le i \le p$ and $1 \le j \le q$, induced by $\partial_{x_i} \in (T\mathcal{M})_0(U)$ and $\partial_{\xi^j} \in (T\mathcal{M})_1(U)$.*

Proof. The proof is essentially the same as that of theorem 1.3.4. □

In particular,

Corollary 1.3.6.

$$\dim T_x\mathcal{M} = \dim \mathcal{M}. \qquad \square$$

Since $X_x = 0$ if and only if $\operatorname{Im} \widetilde{X} \subset \mathfrak{m}_x$, one sees immediately that the kernel of $(T\mathcal{M})_x \to T_x\mathcal{M}$ is exactly $\mathfrak{m}_x(T\mathcal{M})_x$. Hence

Corollary 1.3.7. *For any $x \in M$,*

$$T_x\mathcal{M} \cong (T\mathcal{M})_x / \mathfrak{m}_x(T\mathcal{M})_x. \qquad \square$$

So our definition of the super tangent space fits well to the description at the beginning of this section.

According to definition 1.3.6, it is easy to define the super version of tangent map.

Definition 1.3.7 (Tangent Map)**.** Let $\Psi = (\psi, \psi^*) : \mathcal{M} = (M, \mathcal{O}) \to \mathcal{N} = (N, \mathcal{R})$ be a morphism of supermanifolds, the tangent map $T_x\Psi$ of Ψ at $x \in M$, is the super vector space morphism defined by

$$\begin{aligned} T_x\Psi \colon T_x\mathcal{M} &\longrightarrow T_{\psi(x)}\mathcal{N} \\ X_x &\longmapsto X_x \circ \psi^* \end{aligned}$$

where $\psi^* : \mathcal{R}_{\psi(x)} \to \mathcal{O}_x$ is the pullback morphism between stalks.

Since ψ^* preserves the parity, it is easy to verify that $T_x\Psi$ is well-defined and preserves the parity.

Clearly, if Ψ is the identity morphism on $\mathcal{M}$, then $T_x\Psi$ is the identity on $T_x\mathcal{M}$ for any $x \in M$. It is also clear that taking tangent commutes with composition, i.e.

Proposition 1.3.8. *Let $\Psi = (\psi, \psi^*) : \mathcal{M} \to \mathcal{N}$ and $\Phi = (\varphi, \varphi^*) : \mathcal{N} \to \mathcal{P}$ be morphisms of supermanifolds, then for any $x \in M$,*

$$T_x(\Phi \circ \Psi) = T_{\psi(x)}\Phi \circ T_x\Psi. \qquad \square$$

Therefore we conclude that taking tangent is functorial.

Also, we have the chain rule:

Proposition 1.3.9. *Let* $(\psi, \psi^*) : \mathcal{M} = (M, \mathcal{O}) \to (N, \mathcal{R})$ *be a supermorphism. If* $V \subset N$ *is a coordinate neighborhood parametrized by* $v = (y, \eta)$ *and* $\psi^{-1}(V)$ *be parametrized by* $u = (x, \xi)$*, then*

$$\partial_{u^{\mathfrak{a}}} \circ \psi^* = \sum_{\mathfrak{b}} \partial_{u^{\mathfrak{a}}}(\psi^* v^{\mathfrak{b}}) \psi^* \circ \partial_{v^{\mathfrak{b}}},$$

where, with $\dim \mathcal{M} = p|q$*,*

$$\partial_{u^{\mathfrak{a}}} := \begin{cases} \partial_{x_{\mathfrak{a}}} & 1 \leq \mathfrak{a} \leq p, \\ \partial_{\xi^{\mathfrak{a}-p}} & p+1 \leq \mathfrak{a} \leq p+q. \end{cases}$$

Proof. It is easy to see that the equality holds on evaluation at polynomial sections, hence the similar argument in the proof of theorem 1.3.4 applies. □

Restrict these to the stalk at a point $\psi(x) \in V$ and quotient by the maximal ideal (or one can use the same proof as above), we obtain

$$T_x\Psi(\partial_{u^{\mathfrak{a}},x}) = \sum_{\mathfrak{b}} \partial_{u^{\mathfrak{a}},x}(\psi^* v^{\mathfrak{b}}) \partial_{v^{\mathfrak{b}},\psi(x)}.$$

Hence we have matrix representation of the tangent map, the Jacobian, similar to that in classical case. However, tedious sign appears if we want to arrange proposition 1.3.9 in the usual manner of matrix multiplication with the pullback omitted when dealing with composition of supermorphisms. To fix this, the matrix should be modified.

Definition 1.3.8 (The Modified Super Jacobian Matrix)**.** The *modified super Jacobian matrix* of a supermorphism $\Psi : \mathcal{M} \to \mathcal{N}$, where $\dim \mathcal{M} = p|q$ and $\dim \mathcal{N} = p'|q'$, under local coordinates $V \subset N$ with (y, η) and $\psi^{-1}(V)$ with (x, ξ), is the $(p'+q') \times (p+q)$ supermatrix, given by the convention $y = y(x, \xi)$ and $\eta = \eta(x, \xi)$,

$$J\Psi = \begin{pmatrix} \partial_x y & -\partial_\xi y \\ \partial_x \eta & \partial_\xi \eta \end{pmatrix}.$$

On evaluation at a point, we have since $\mathbb{R}$ is trivially graded

$$J_{x_0}\Psi = \begin{pmatrix} \partial_{x,x_0} y & 0 \\ 0 & \partial_{\xi,x_0}\eta \end{pmatrix} = \begin{pmatrix} \partial_{x,x_0} y & \partial_{\xi,x_0} y \\ \partial_{x,x_0}\eta & \partial_{\xi,x_0}\eta \end{pmatrix} = \text{ matrix representation of } T_{x_0}\Psi.$$

With the modified Jacobian matrix, we have

Proposition 1.3.10. *Let $\Psi : \mathcal{M} \to \mathcal{N}$ and $\Phi : \mathcal{N} \to \mathcal{P}$ be two supermorphisms, then under any local coordinates representation, we have*

$$J(\Phi \circ \Psi) = J\Phi \cdot J\Psi$$

where entries of $J\Phi$ are considered as their pullback by Ψ when taking the matrix multiplication.

Proof. This is nothing but direct computation using proposition 1.3.9. □

Remark 1.3.1. Under the abuse of notation that $(x, \xi) \mapsto (y, \eta)$, the reason why only the upper-right term has a minus sign can be explained by

$$\begin{aligned}\partial_{x_i} &\mapsto \partial_{x_i}(y_j)\partial_{y_j} + \partial_{x_i}(\eta^k)\partial_{\eta^k} = \partial_{y_j} \cdot (\partial_{x_i}(y_j)) - \partial_{\eta^k} \cdot (\partial_{x_i}(\eta^k)) \\ \partial_{\xi^i} &\mapsto \partial_{\xi^i}(y_j)\partial_{y_j} + \partial_{\xi^i}(\eta^k)\partial_{\eta^k} = \partial_{y_j} \cdot (\partial_{\xi^i}(y_j)) + \partial_{\eta^k} \cdot (\partial_{\xi^i}(\eta^k)).\end{aligned}$$

The thing is, that to make the left-multiplication of matrix representations of morphisms of supermodules be compatible with the composition of morphisms, the entries should be taken to be the scalars written at the right-hand side of the basis, instead at the left-hand side as one is used to in the ordinary case.

Cotangent Things

Definition 1.3.9 (Cotangent Sheaf)**.** The *cotangent sheaf* of a supermanifold $\mathcal{M} = (M, \mathcal{O})$ is the dual of its tangent sheaf, i.e. it is the sheaf of morphisms of sheaves

$$\Omega^1\mathcal{M} := T^*\mathcal{M} := \mathbf{Hom}_{\text{Sheaf of } \mathcal{O}\text{ - Mod}}(T\mathcal{M}, \mathcal{O}).$$

The sections of $\Omega^1\mathcal{M}$ are called *super differential* 1*-forms*.

Note that $\Omega^1\mathcal{M}$ is again a sheaf of super $\mathcal{O}$-modules.

Definition 1.3.10 (Differential of Superfunction)**.** For any open subset $U \subset M$ and $i \in \mathbb{Z}_2$, we define the *differential of a superfunction* $f \in \mathcal{O}_i(U)$, for any open $V \subset U$,

$$\mathrm{d}_V f \in \mathbf{Hom}_i(T\mathcal{M}(V), \mathcal{O}(V)),$$

by

$$(\mathrm{d}_V f)(X) = (-1)^{ij} X f|_V \in \mathcal{O}(V),$$

for all $X \in (T\mathcal{M})_j(V) = \mathrm{Der}_j\, \mathcal{O}(V)$. Clearly, $\mathrm{d}_V f$ gives a morphism from $T\mathcal{M}|_U$ to $\mathcal{O}|_U$ as V varies, hence $\mathrm{d}f \in \Omega^1\mathcal{M}(U)$.

For non-homogeneous f, $\mathrm{d}f$ is defined as the sum of differentials of the homogeneous components of f. d_V preserves the parity and hence gives an $\mathcal{O}(V)$-module morphism by above definition. By the definition of restriction of superderivation, we see that d gives a natural transformation from $\mathcal{O}$ to $\Omega^1\mathcal{M}$.

Moreover, it is easy to verify that

Proposition 1.3.11. *For any $f, g \in \mathcal{O}(U)$,*

$$\mathrm{d}(fg) = (\mathrm{d}f)g + f(\mathrm{d}g). \qquad \square$$

This makes d an abstract (even) superderivation.

Remark 1.3.2. One can see that everything defined satisfies the Koszul sign rule. In $(\mathrm{d}_U f)(X) = (-1)^{ij} X f$, f and X swap the position and hence gives a sign up to their parity; in $\mathrm{d}(fg) = (\mathrm{d}f)g + f(\mathrm{d}g)$, d is even, hence no sign arises when f and d_U swap.

Being the dual of locally free $T\mathcal{M}$, $\Omega^1\mathcal{M}$ is also locally free with the dual basis.

Theorem 1.3.12. *The cotangent sheaf $\Omega^1\mathcal{M}$ is locally free with basis $(\mathrm{d}x_1, \cdots, \mathrm{d}\xi^q)$ in a coordinate neighborhood $(U, (x, \xi))$.* $\square$

Example 1.3.1. By definition,

$$\mathrm{d}x_i(\partial_{x_j}) = \delta_{ij}, \quad \mathrm{d}x_i(\partial_{\xi^j}) = 0.$$
$$\mathrm{d}\xi^i(\partial_{x_j}) = 0, \quad \mathrm{d}\xi^i(\partial_{\xi^j}) = -\delta_{ij}.$$

Hence any super differential 1-form ω reads locally (under Einstein summation) as

$$\omega = \mathrm{d}x_i f_i(x, \xi) + \mathrm{d}\xi^j g_j(x, \xi),$$

where the coefficients f_i are given by $\omega(\partial_{x_i})$ and g_j are given by $(-1)^{\omega}\omega(\partial_{\xi^j})$ (or sum of these substituted homogeneous components of ω), as one can verify. It follows that

$$\mathrm{d}_U f = \mathrm{d}x_i(\partial_{x_i} f) + \mathrm{d}\xi^j(\partial_{\xi^j} f),$$

so d reads locally as

$$\mathrm{d} = \mathrm{d}x_i \partial_{x_i} + \mathrm{d}\xi^j \partial_{\xi^j}.$$

Still, we can talk about super k-forms. But we need to know how to wedge things up before that.

Let A and B be two supermodules over a supercommutative ring R, their tensor product, $A \otimes_R B = R^{A\times B}/\sim$ is the free R-module $R^{A\times B}$ modulo the relations

$$1_{(a+a',b)} =: (a + a', b) = (a, b) + (a', b), \quad (a, b + b') = (a, b) + (a, b'),$$

$$r_{(a,b)} =: r(a,b) = (ra, b), \quad (ar, b) = (a, rb),$$

for any $a, a' \in A$, $b, b' \in B$ and $r \in R$. Write $a \otimes b$ as the equivalent class of (a, b), and $A \otimes_R B$ is naturally $\mathbb{Z}_2$-graded by

$$A \otimes_R B = \bigoplus_{k \in \mathbb{Z}_2} \bigoplus_{i+j=k} \left\{ \sum a \otimes b \,\middle|\, a \in A_i,\ b \in B_j \right\}.$$

With the induced right module structure on A and B, one sees that the induced right module structure on $A \otimes_R B$ obeys Koszul sign rule. The tensor product of two morphisms of super R-modules $f : A \to A'$ and $g : B \to B'$ is defined by

$$\begin{aligned} f \otimes_R g \colon A \otimes_R B &\longrightarrow A' \otimes_R B' \\ a \otimes b &\longmapsto (-1)^{ga} f(a) \otimes g(b) \end{aligned}$$

This definition makes writing $(f \otimes_R g)(a \otimes b) = (-1)^{ga} f(a) \otimes g(b)$ follow the sign rule.

For a single supermodule A over supercommutative ring R, we can consider its tensor with itself. Write $A^{\otimes n} := A \otimes_R \cdots \otimes_R A$ the tensor product of n copies of A and $A^{\otimes 0} = R$ by convention, the super R-module

$$T^\bullet A = \bigoplus_{n \geq 0} A^{\otimes n}$$

is the *tensor super R-algebra of the supermodule A*, where the direct product is the usual one of modules which induces an obvious $\mathbb{Z}_2$-grading. $T^\bullet A$ is made a superring by multiplication by tensor, and the R-algebra structure follows from the inclusion $R = A^{\otimes 0} \subset T^\bullet A$.

Now, to define the wedge product, or say the exterior product, for our case we use Deligne's formalism, we put the ideal $I_A := (a \otimes a' + (-1)^{aa'} a' \otimes a \mid a, a' \in A)$ of $T^\bullet A$. The *super exterior R-algebra oof the supermodule A* is the quotient algebra

$$\wedge_D A := T^\bullet A / I_A.$$

We often omit the subscript D and write $a \wedge a'$ as the equivalent class of $a \otimes a'$. Apart from the $\mathbb{Z}_2$-grading, $\wedge A$ is also graded cohomologically (and so is $T^\bullet A$) by

$$\wedge A = \bigoplus_{n \geq 0} \wedge^n A.$$

Now we can define

Definition 1.3.11 (Super Differential Form)**.** Let $\mathcal{M} = (M, \mathcal{O})$ be a supermanifold. For any open $U \subset M$, the set of *super differential forms* over U is defined by

$$(\Omega\mathcal{M})(U) := \wedge(\Omega^1\mathcal{M})(U) = \bigoplus_{k\geq 0} \wedge^k(\Omega^1\mathcal{M})(U) =: \bigoplus_{k\geq 0}(\Omega^k\mathcal{M})(U).$$

Elements in $(\Omega^k\mathcal{M})(U)$ are called *super differential k-forms*. The sheaf structure of $\Omega^1\mathcal{M}$ induces the sheaf structure of $\Omega\mathcal{M}$ immediately. Note that, unlike the ordinary case, there is no top form in $\Omega\mathcal{M}$, because the wedge of two odd elements is symmetric by definition. It is easy to check by definition (or one can take the sign rule for granted) that

$$f(\omega \wedge \omega') = (-1)^{i(\omega+\omega')}(\omega \wedge \omega')f,$$
$$\omega \wedge \omega' = (-1)^{kl+\omega\omega'}\omega' \wedge \omega,$$

for any homogeneous $f \in \mathcal{O}_i(U)$, $\omega \in (\Omega^k\mathcal{M})(U)$ and $\omega' \in (\Omega^l\mathcal{M})(U)$.

Any k-form ω reads locally (non-uniquely)

$$\omega|_U = \sum f\,\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k$$

for some open U and f's in $\mathcal{O}(U)$. The exterior differentiation $\mathrm{d} : \mathcal{O} \Rightarrow \Omega^1\mathcal{M}$ extends uniquely to give $\mathrm{d} : \Omega\mathcal{M} \Rightarrow \Omega\mathcal{M}$ by

$$(\mathrm{d}\omega)|_U := \sum \mathrm{d}f \wedge \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k.$$

The proof of the well-definedness of d and that

Proposition 1.3.13.

$$\mathrm{d}^2 = 0.$$

Proposition 1.3.14.

$$\mathrm{d}(\omega \wedge \omega') = \mathrm{d}\omega \wedge \omega' + (-1)^k\omega \wedge \mathrm{d}\omega'$$

for any form ω' and k-form ω.

is necessarily identical to the corresponding results in the classical theory.

Remark 1.3.3. Since d increases the cohomological degree by 1 and is $\mathbb{R}$-linear, the super version of the de Rham complex arises.

1.4 Integration on Supermanifolds

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