ALGEBRAIC ASPECTS OF HOLOMORPHIC QUANTUM MODULAR FORMS

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Abstract. Matrix-valued holomorphic quantum modular forms are intricate objects associated to 3-manifolds (in particular to knot complements) that arise in successive refinements of the Volume Conjecture of knots and involve three holomorphic, asymptotic and arithmetic realizations. It is expected that the algebraic properties of these objects can be deduced from the algebraic properties of descendant state integrals, and we illustrate this for the case of the $(-2, 3, 7)$-pretzel knot.

Contents

1. Introduction

The Volume Conjecture of Kashaev links the asymptotics of the Jones polynomial of a hyperbolic knot and its parallels with the hyperbolic geometry of the knot complement [Kas97].

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Explicitly, the conjecture (combined with the results of Murakami–Murakami [MM01]) asserts that for a hyperbolic knot K in 3-space [Thu77], we have

$$\lim_{N\to\infty} \frac{1}{N} \log |J_N^K(e^{2\pi i/N})| = \frac{\mathrm{Vol}(S^3 \setminus K)}{2\pi} \tag{1}$$

where $J_N^K(q) \in \mathbb{Z}[q^{\pm 1}]$ is the Jones polynomial of K, colored with the N-dimensional irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$, and normalized to be 1 for the unknot. The definition of the colored Jones polynomial, that we omit, may be found in [RT90]

The Volume Conjecture is considered one of the main problems of quantum topology. Although it is currently known only for a handful of knots, it can be strengthened in numerous ways to include a statement about asymptotics to all orders in N, and with exponentially small terms included. One of these successive refinements of the quantum modularity conjecture for the Kashaev invariant of a knot lead to the concept of a matrix-valued holomorphic quantum modular forms introduced and studied in [GZ24, GZ23]. The latter are rather intricate objects that involve matrices of

- holomorphic objects, that is q-series with integer coefficients convergent when $|q| \neq 1$.
- asymptotic/analytic objects, that is factorially divergent formal power series that are Borel resummable and whose Stokes phenomenon is explained in terms of the q-series above.
- arithmetic objects, that is collections of functions defined near each complex root of unity that arithmetically determine each other p-adically.

This sounds like a daunting collection of objects (associated for example to knots) that come from different worlds and are somehow stitched together. Despite this, it turns out that matrix-valued holomorphic quantum modular forms have algebraic aspects that can be formulated and proven using the algebraic properties of a variant of the Andersen–Kashaev invariants [AK14], namely the descendant state integrals.

The three simplest hyperbolic knots are the 4_1 knot, the 5_2 knot and the $(-2, 3, 7)$ pretzel knot [Thu77, CDW] shown in Figure 1.

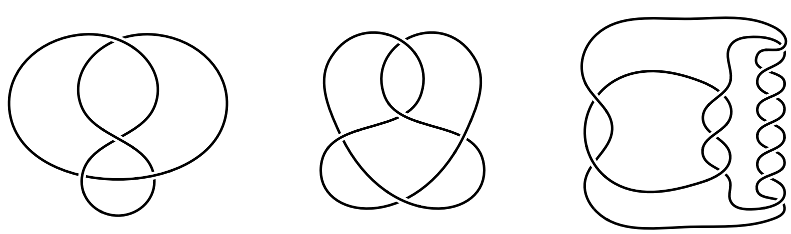

FIGURE 1. The three simplest hyperbolic knots from left to right: 4_1, 5_2 and the $(-2, 3, 7)$-pretzel knot.

The corresponding matrices of the 4_1 and the 5_2 knot are 2×2 and 3×3 due to the fact that all boundary-parabolic $\mathrm{SL}_2(\mathbb{C})$-representations are conjugates of the geometric representation which is defined over $\mathbb{Q}(\sqrt{-3})$ and over the cubic field of discriminant -23. The holomorphic quantum modular forms of the first two were studied in detail in [GZ24, GZ23] as well as in [GGMn21, GGMn23, GGMnW].

On the other hand, it was pointed out in [GZ24, Sec.2.1] that the corresponding matrices associated to the $(-2,3,7)$ pretzel knot are 6×6 due to the fact that the geometric representation is defined over the same number field as for 5_2, but in addition to that there are three additional boundary-parabolic $\mathrm{SL}_2(\mathbb{C})$-representations defined over $\mathbb{Q}(2\cos(2\pi/7))$. In this paper we will study the algebraic aspects of the holomorphic quantum modular forms associated to the $(-2,3,7)$-pretzel knot completing the partial work of [GZ24, GZ23]. Explicitly, we will show that

(a) The factorization of the descendant state integral defines a 6×6 matrix of (deformed) q-hypergeometric series; see Theorem 1.
(b) The matrix is a fundamental solution of a self-dual linear q-difference equation; see Theorem 2.
(c) The corresponding cocycle is a holomorphic function that extends from $\tau\in\mathbb{C}\setminus\mathbb{R}$ to the cut-plane $\mathbb{C}'=\mathbb{C}\setminus(-\infty,0]$; see Theorem 4.
(d) The stationary phase of the descendant state integral determines a 6×6 matrix of asymptotic series; see Theorem 5.

Along with the detailed definitions and proofs, we will give an explanation of why the proofs work from first principles.

2. The descendant state integral

2.1. The state integral. A key role in our paper is the state integral invariant of 3-dimensional manifolds with torus boundary (in particular knot complements) introduced by Andersen–Kashaev [AK14]. This is a multi-dimensional integral whose integrand is a product of the Faddeev quantum dilogarithm $\Phi_b(x)$ [Fad95] times an exponential of a quadratic form, assembled out of a suitable triangulation of the manifold.

In the case of the $(-2,3,7)$-pretzel knot, the Andersen–Kashaev state integral is a four-dimensional state integral that can be reduced to the following one-dimensional state integral as shown in [GK15, Eqn.(58)]:

$$Z_{(-2,3,7)}(\tau) \;=\; \int_{\mathbb{R}+i\frac{c_b}{2}+i\varepsilon} \Phi_{\sqrt{\tau}}(x)^2\Phi_{\sqrt{\tau}}(2x-c_b)e^{-\pi i(2x-c_b)^2}\,\mathrm{d}x \tag{2}$$

where $\tau=b^2$ and $c_b=i(b+b^{-1})/2$. Two key properties of this absolutely convergent state integral are that

(a) it defines a holomorphic function on the cut-plane $\mathbb{C}'=\mathbb{C}\setminus(-\infty,0]$, and
(b) when $\tau\in\mathbb{C}\setminus\mathbb{R}$, it can be factorized as a sum of products of q-series and $\tilde{q}$-series where $q=e^{2\pi i\tau}$ and $\tilde{q}=e^{-2\pi i/\tau}$.

This factorization property is expected to hold for all state integrals that appear naturally in complex Chern–Simons theory of knots and 3-manifolds as explained in [BDP14, GK17] and in the case of the above state integral, it was given in [GZ23, Prop.12].

The main reason behind this factorization is algebraic and follows from the quasi-periodicity of the Faddeev quantum dilogarithm which implies that it is a meromorphic function with poles in the lattice points of a two-dimensional cone and prescribed residues. Upon applying the residue theorem, the sum over the cone (which is coupled by the exponential of a quadratic form with integer coefficients) decouples due to the fact that $e^{2\pi ik} = 1$ for all integers k.

2.2. **The descendant state integral and its q-series.** A descendant version of the state integral (2) obtained by inserting in the integrand of (2) the exponential of a linear form in two integer variables $\lambda, \lambda' \in \mathbb{Z}$, in an analogous way as was done for the 4_1 and 5_2 knots in [GGMn21, Sec.4.3]. The descendant state integral of $(-2,3,7)$ pretzel knot is

$$Z^{(\lambda,\lambda')}_{(-2,3,7)}(\tau) \;=\; \int_{\mathbb{R}+i\frac{c_b}{2}+i\varepsilon} \Phi_{\sqrt{\tau}}(x)^2\Phi_{\sqrt{\tau}}(2x-c_b)e^{-\pi i(2x-c_b)^2+2\pi(\lambda b-\lambda' b^{-1})x}\,\mathrm{d}x\,. \tag{3}$$

This integral can be factorized as a finite sum of products of q-series and $\tilde{q}$-series for the same reason that the integral (2) does. Deforming the contour of integration upwards, applying the residue theorem, collecting the residues and observing the same decoupling as was done in [GK17, GK15], we obtain the factorization into q-series.

Theorem 1. *For all $\tau \in \mathbb{C}\setminus\mathbb{R}$, we have*

$$\begin{aligned} &2e^{\frac{\pi i}{4}}q^{-\frac{\lambda}{2}}\tilde{q}^{-\frac{\lambda'}{2}}Z^{(\lambda,\lambda')}_{(-2,3,7)}(\tau)\\ &= -\frac{1}{2\tau}h_{\lambda,0}(\tau)h_{\lambda',2}(\tau^{-1}) + h_{\lambda,1}(\tau)h_{\lambda',1}(\tau^{-1}) - \frac{\tau}{2}h_{\lambda,2}(\tau)h_{\lambda',0}(\tau^{-1})\\ &- i\Big(\frac{1}{2}h_{\lambda,3}(\tau)h_{\lambda',4}(\tau^{-1}) - \frac{1}{2}h_{\lambda,4}(\tau)h_{\lambda',3}(\tau^{-1}) + h_{\lambda,5}(\tau)h_{\lambda',5}(\tau^{-1})\Big)\,. \end{aligned} \tag{4}$$

In the above theorem

$$h_{\lambda,j}(\tau) \coloneqq H_{\lambda,j}(e^{2\pi i\tau}), \quad H_{\lambda,j}(q) \;=\; \begin{cases} H^{+}_{\lambda,j}(q) & \text{if } |q|<1\\ (-1)^{\delta_j}H^{-}_{-\lambda,j}(q^{-1}) & \text{if } |q|>1 \end{cases} \tag{5}$$

where $H_{\lambda,j}(q)$ are q-series defined in Section 3.2 for $|q| \neq 1$ and $\delta = (0,1,2,0,0,0)$ is a weight vector. $H^{\pm}_{\lambda,j}(q)$ are power series of $q^{1/8}$ whose first few terms are given by

$$\begin{aligned} H^{+}_{0,0}(q) &= 1+q^3+3q^4+7q^5+13q^6+\cdots & H^{-}_{0,0}(q) &= 1+q^2+3q^3+7q^4+13q^5+\cdots\\ H^{+}_{0,1}(q) &= 1-4q-8q^2-3q^3+3q^4+\cdots & H^{-}_{0,1}(q) &= 1-4q-5q^2+q^3+7q^4+\cdots\\ H^{+}_{0,2}(q) &= \frac{2}{3}-6q+6q^2+\frac{242}{3}q^3+200q^4+\cdots & H^{-}_{0,2}(q) &= \frac{5}{6}-10q+\frac{17}{6}q^2+\frac{141}{2}q^3+\frac{971}{6}q^4+\cdots\\ H^{+}_{0,3}(q) &= q^{1/8}(q+2q^{3/2}+4q^2+6q^{5/2}+\cdots) & H^{-}_{0,3}(q) &= q^{-1/8}(q+2q^{3/2}+4q^2+6q^{5/2}+\cdots)\\ H^{+}_{0,4}(q) &= 1+q^3-q^4+3q^5-3q^6+\cdots & H^{-}_{0,4}(q) &= 1+q^2-q^3+3q^4-3q^5+\cdots\\ H^{+}_{0,5}(q) &= q^{1/8}(q-2q^{3/2}+4q^2-6q^{5/2}+\cdots) & H^{-}_{0,5}(q) &= q^{-1/8}(q-2q^{3/2}+4q^2-6q^{5/2}+\cdots) \end{aligned} \tag{6}$$

2.3. **A self-dual linear q-difference equation.** As we saw in the previous section, the factorization of the state integral (3) produced six sequences $H_{\lambda,j}(q)$ of q-hypergeometric series for $j = 0,\dots,5$ indexed by $\lambda \in \mathbb{Z}$ and defined for $|q| \neq 1$. We now show that these six sequences are solutions of a common sixth order linear q-difference equation. The algebraic reason for this is that the integrand of the descendant state integral is a q-holonomic function of three variables x, λ and λ', as follows from the quasi-periodicity of the Faddeev quantum dilogarithm (see Equation (17) below). Zeilberger theory implies that the integral is a q-holonomic function of λ and λ' [WZ92]. Due to the factorization of the integral, it follows that its λ-dependent part is a q-holonomic function. An alternative explanation for the linear q-difference equation would be to use the explicit q-hypergeometric formulas for the six q-series. In fact, in this case we can do the corresponding algebraic calculation by elementary telescoping methods and obtain the following.

Theorem 2. *For each $j = 0, \dots, 5$, the sequence $H_{\lambda,j}(q)$ for $|q| \neq 1$ and $\lambda \in \mathbb{Z}$ satisfies the linear q-difference equation*

$$\begin{aligned} y_{\lambda+6}(q) + 2\, y_{\lambda+5}(q) - (q + q^{\lambda+4})\, y_{\lambda+4}(q) - 2(q+1)\, y_{\lambda+3}(q) \\ - y_{\lambda+2}(q) + 2q\, y_{\lambda+1}(q) + q\, y_\lambda(q) = 0\,. \end{aligned} \tag{7}$$

Consider the truncated Wronskian

$$W_\lambda(q) = (H_{\lambda+i,j}(q))_{0 \leq i,j \leq 5} \qquad |q| \neq 1 \tag{8}$$

of the six solutions to the q-difference equation (7). Technically, the Wronskian is a $\mathbb{Z} \times 6$ matrix whose block indexed by the raws corresponding to $i = 0, \dots, 5$ is the above matrix W_λ. We next give an orthogonality property of the truncated Wronskian, which implies that the six sequences of q-series form a fundamental solution set of (7) and satisfy quadratic relations.

Theorem 3. *The determinant of the truncated Wronskian is given by*

$$\det(W_\lambda(q)) = 32 q^{\lambda + \frac{11}{4}}\,. \tag{9}$$

The truncated Wronskian satisfies the orthogonality property

$$W_\lambda(q) \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} W_{-\lambda-5}(q^{-1})^T = \begin{pmatrix} -12 & 8 & -4 & 2 & 0 & 0 \\ 8 & -4 & 2 & 0 & 0 & 0 \\ -4 & 2 & 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 & 2 & -4 \\ 0 & 0 & 0 & 2 & -4 & 8 + 2q^{\lambda+2} \\ 0 & 0 & 2 & -4 & 8 + 2q^{\lambda+3} & -12 - 4q^{\lambda+2} - 4q^{\lambda+3} \end{pmatrix}. \tag{10}$$

Equations (9) and (10) were first guessed by explicit computations of the q-series. But once they were guessed, they were proven algebraically, i.e., by reducing them to identities among rational functions in $\mathbb{Q}(q, q^\lambda)$. This concludes our last algebraic aspect of matrix-valued holomorphic quantum modular forms, discussed in detail in Section 3.4 below.

A consequence of Equation (10) (in fact, of its $(1,6)$-entry) is that the collection of q-series $H^\pm_{\lambda,j}(q)$ satisfies the quadratic relation

$$\begin{aligned} \frac{1}{2} H^+_{\lambda,0}(q) H^-_{\lambda,2}(q) - H^+_{\lambda,1}(q) H^-_{\lambda,1}(q) + \frac{1}{2} H^+_{\lambda,2}(q) H^-_{\lambda,0}(q) \\ - H^+_{\lambda,3}(q) H^-_{\lambda,3}(q) + \frac{1}{4} H^+_{\lambda,4}(q) H^-_{\lambda,4}(q) - H^+_{\lambda,5}(q) H^-_{\lambda,5}(q) = 0\,. \end{aligned} \tag{11}$$

A second consequence of Equation (10) (and in fact, an equivalent statement to it) is that the q-holonomic module associated with the linear q-difference equation (7) is self-dual. For a detailed definition of self-dual q-holonomic modules, we refer the reader to [GW, Sec 2.5].

2.4. **A cocyle.** The last topic of our paper concerns the analytic continuation of a cocycle, defined in two forms below (Equations (12) and (13)), from a holomorphic function on $\mathbb{C}\setminus\mathbb{R}$ to one on $\mathbb{C}'=\mathbb{C}\setminus(-\infty,0]$. This remarkable statement, which concerns the holomorphic aspects of matrix-valued holomorphic quantum modular forms, follows immediately from the factorization of the descendant state integrals (Theorem 1), the self-duality property (Theorem 3) and the fact that state integrals are holomorphic functions in the cut-plane $\mathbb{C}'$.

Theorem 4. (a) The matrix-valued function

$$F_{\lambda,\lambda'}(\tau) \coloneqq W_{-\lambda'-5}(\tilde{q}^{-1}) \begin{pmatrix} 0 & 0 & -\frac{\tau}{2} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2\tau} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{i}{2} & 0 \\ 0 & 0 & 0 & -\frac{i}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i \end{pmatrix} W_\lambda(q)^T \tag{12}$$

defined for $\tau=b^2\in\mathbb{C}\backslash\mathbb{R}$, has entries given by the descendant state integrals up to a prefactor given by (4), and therefore extends to a holomorphic function on the cut plane $\mathbb{C}'$.
(b) The matrix-valued function

$$W_{\lambda,\lambda'}(\tau) \coloneqq \big(W_{\lambda'}(\tilde{q})^T\big)^{-1} \begin{pmatrix} -\frac{1}{\tau} & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\tau & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{i}{2} & 0 \\ 0 & 0 & 0 & -2i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i \end{pmatrix} W_\lambda(q)^T \tag{13}$$

extends to a holomorphic function of $\tau\in\mathbb{C}'$.

Note that Equation (13) follows from (12) and (10).

To make contact with the results of the paper [GW], Equation (13) becomes the cocycle $U(-1/\tau)^{-1}D(\tau)U(\tau)$ where $U_\lambda(\tau)=(W_\lambda(e^{2\pi i\tau})^T)^{-1}$ and $D(\tau)$ is the automorphy factor in the middle matrix of the right hand side of (13).

3. The q-series

3.1. **Algebraic properties of the Faddeev quantum dilogarithm.** As stated in the introduction, all of our theorems follow from algebraic properties of state integrals, which in turn follow from some well-known properties of the Faddeev quantum dilogarithm function; see [Fad95] as well as [AK14, App.A]. In this section we review these properties briefly and highlight their algebraic aspects.

To begin with, the Faddeev quantum dilogarithm function is defined by

$$\Phi_b(x) \;=\; \exp\Big(\frac{1}{4}\int_{\mathbb{R}+i\epsilon}\frac{e^{-2ixw}}{\sinh(bw)\sinh(b^{-1}w)}\frac{dw}{w}\Big) \tag{14}$$

When $\mathrm{Im}(b^2)>0$, the Faddeev quantum dilogarithm is given by a ratio of two infinite Pochhammer symbols as follows

$$\Phi_b(x) = \frac{(e^{2\pi b(x+c_b)};q)_\infty}{(e^{2\pi b^{-1}(x-c_b)};\tilde{q})_\infty}\,, \tag{15}$$

where

$$q = e^{2\pi i b^2}, \qquad \tilde{q} = e^{-2\pi i b^{-2}}, \qquad c_b = \frac{i}{2}(b+b^{-1}), \qquad \mathrm{Im}(b^2) > 0\,. \tag{16}$$

Remarkably, the ratio (15) admits an extension to all values of b with $b^2 \in \mathbb{C} \setminus (-\infty, 0]$. $\Phi_b(x)$ is a meromorphic function of x with poles in the set $c_b + i\mathbb{N}b + i\mathbb{N}b^{-1}$. In other words, the poles are given by $x_{m,n} = i\big(m+\frac{1}{2}\big)b + i\big(n+\frac{1}{2}\big)b^{-1}$ with m and n nonnegative integers.

The Faddeev quantum dilogarithm satisfies the quasi-periodicity

$$\frac{\Phi_b(x+c_b+ib)}{\Phi_b(x+c_b)} = \frac{1}{1-qe^{2\pi bx}}, \qquad \frac{\Phi_b(x+c_b+ib^{-1})}{\Phi_b(x+c_b)} = \frac{1}{1-\tilde{q}^{-1}e^{2\pi b^{-1}x}}\,. \tag{17}$$

Quasi-periodicity among other things, explains the structure of the set of poles, and that the residue of $\Phi_b(x)$ at the pole $x_{m,n}$ is given by [GK17, Lem.2.1]

$$\mathrm{Res}_{x=x_{m,n}}\Phi_b(x) = -\frac{b}{2\pi}\frac{(q;q)_\infty}{(\tilde{q};\tilde{q})_\infty}\frac{1}{(q;q)_m}\frac{1}{(\tilde{q}^{-1};\tilde{q}^{-1})_n}\,. \tag{18}$$

Notice that the poles are parametrized by a pair (m,n) of natural numbers and the residue decouples, i.e., it is the product of a function of m times a function of n.

Although it will not play in our paper, we mention that the Faddeev quantum dilogarithm satisfies the functional equation

$$\Phi_b(x)\Phi_b(-x) = e^{\pi i x^2}\Phi_b(0)^2, \qquad \Phi_b(0)^2 = q^{\frac{1}{24}}\tilde{q}^{-\frac{1}{24}} \tag{19}$$

which implies that its set of zeros is the negative of its set of poles, and also allows us to move $\Phi_b(x)$ from the denominator to the numerator of the integrand of a state-integral.

3.2. **Factorization.** To prove Theorem 1, we observe that the integrand is a meromorphic function of x. We then deform the contour of integration upwards, apply the residue theorem and collect residues. The poles of the Faddeev quantum dilogarithms were discussed in the previous section, and they are parametrized by the lattice points $x_{m,n}$ in a two dimensional cone, and the residues are decoupled functions of m (involving q) and n (involving $\tilde{q}$), see Equation (18). This decoupling persists when we evaluate the exponential function $e^{-\pi i(2x-c_b)^2+2\pi(\lambda b-\lambda' b^{-1})x}$ at $x_{m,n}$. Thus, the sum over the lattice points (m,n) of a two-dimensional lattice becomes a product over the lattice points m of one-dimensional lattice times a product over the lattice points n of the other one-dimensional lattice. Taking into account that the integrand of (3) is the product of three Faddeev quantum dilogarithms, this gives the proof of Theorem 1.

The details of the factorization of the original state integral (2) were given in [GZ23, section A.6]. Following the same proof mutatis mutandis and inserting the integers λ and λ', produces the definition of the q-series given in Equations (20) and (23) below and concludes the proof of Theorem 1 $\square$

The corresponding series are given as follows. We define $H^{\pm}_{\lambda,j}(q)$ for $|q|<1$ and $j=0,1,2$ by:

$$H^+_{\lambda,j}(q) \;=\; (-1)^{\lambda}\sum_{m=0}^{\infty} t_{\lambda,m}(q)p_{\lambda,j,m}(q), \quad H^-_{\lambda',j}(q) \;=\; (-1)^{\lambda'}\sum_{n=0}^{\infty} T_{\lambda',n}P_{\lambda',j,n}(q), \tag{20}$$

with

$$t_{\lambda,m}(q)=\frac{q^{m(2m+1)+\lambda m}}{(q;q)_m^2(q;q)_{2m}}, \quad T_{\lambda',n}(q)=\frac{q^{n(n+1)+\lambda' n}}{(q;q)_n^2(q;q)_{2n}}, \tag{21}$$

and

$$\begin{aligned}
&p_{\lambda,0,m}(q)=1, \quad p_{\lambda,1,m}(q)=4m+\lambda+1-2E_1^{(m)}(q)-2E_1^{(2m)}(q),\\
&p_{\lambda,2,m}(q)=p_{\lambda,1,m}(q)^2-2E_2^{(m)}(q)-4E_2^{(2m)}(q)-\frac{1}{3}\mathcal{E}_2(q),\\
&P_{\lambda',0,n}(q)=1, \quad P_{\lambda',1,n}(q)=2n+\lambda'+1-2E_1^{(n)}(q)-2E_1^{(2n)}(q),\\
&P_{\lambda',2,n}(q)=P_{\lambda',1,n}(q)^2+12E_2^{(0)}(q)-\frac{1}{2}-2E_2^{(n)}(q)-4E_2^{(2n)}(q)+\frac{1}{3}\mathcal{E}_2(q),
\end{aligned} \tag{22}$$

and for $j=3,4,5$ by:

$$\begin{aligned}
&H^+_{\lambda,3}(q)=\frac{(-1)^{\lambda}q^{1/8}}{(1-q^{1/2})^2}\sum_{m=0}^{\infty}\frac{q^{(2m+1)(m+1)+\lambda(m+1/2)}}{(q^{3/2};q)_m^2(q;q)_{2m+1}} \quad && H^-_{\lambda',4}(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)+\lambda' n}}{(-q;q)_n^2(q;q)_{2n}}\\
&H^+_{\lambda,4}(q)=\sum_{m=0}^{\infty}\frac{q^{(2m+1)m+\lambda m}}{(-q;q)_m^2(q;q)_{2m}} && H^-_{\lambda',3}(q)=\frac{(-1)^{\lambda'}q^{-1/8}}{(1-q^{-1/2})^2}\sum_{n=0}^{\infty}\frac{q^{n(n+2)+\lambda'(n+1/2)}}{(q^{3/2};q)_n^2(q;q)_{2n+1}}\\
&H^+_{\lambda,5}(q)=\frac{q^{1/8}}{(1+q^{1/2})^2}\sum_{m=0}^{\infty}\frac{q^{(2m+1)(m+1)+\lambda(m+1/2)}}{(-q^{3/2};q)_m^2(q;q)_{2m+1}} && H^-_{\lambda',5}(q)=\frac{q^{-1/8}}{(1+q^{-1/2})^2}\sum_{n=0}^{\infty}\frac{q^{n(n+2)+\lambda'(n+1/2)}}{(-q^{3/2};q)_n^2(q;q)_{2n+1}}\,.
\end{aligned} \tag{23}$$

Here, $\mathcal{E}_1(q)$ and $\mathcal{E}_2(q)$ denote the Eisenstein series of weights 1 and 2

$$\mathcal{E}_1(q)=1-4\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)}, \qquad \mathcal{E}_2(q)=1-24\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2} \tag{24}$$

and

$$E_l^{(m)}(q)=\sum_{s=1}^{\infty}\frac{s^{l-1}q^{s(m+1)}}{1-q^s} \tag{25}$$

are some series that appear in the factorization of one-dimensional state integrals [GK17].

When $(\lambda,\lambda')=(0,0)$, this factorization can be connected to that in [GK15, eq. (52)] (see Appendix A below).

3.3. **The linear q-difference equation.**

In this section we prove Theorem 2. We will use elementary telescoping summation than the more elaborate methods of [WZ92].

We begin with the case $j=0$ and $|q|<1$, hence

$$H_{\lambda,0}(q)=H^+_{\lambda,0}(q)=(-1)^{\lambda}\sum_{m=0}^{\infty}\frac{q^{m(2m+1)+\lambda m}}{(q;q)_m^2(q;q)_{2m}}\,.$$

Since $(q;q)_m = \prod_{i=1}^m (1-q^i)$, we have

$$\begin{aligned} q^\lambda \sum_{m=0}^\infty \frac{q^{m(2m+1)+\lambda m}}{(q;q)_m^2 (q;q)_{2m}} &= \sum_{m=0}^\infty \frac{q^{m(2m+1)+\lambda(m+1)}}{(q;q)_m^2 (q;q)_{2m}} = \sum_{m=1}^\infty \frac{q^{(m-1)(2m-1)+\lambda m}}{(q;q)_{m-1}^2 (q;q)_{2m-2}} \\ &= \sum_{m=1}^\infty \frac{q^{m(2m+1)+\lambda m}}{(q;q)_m^2 (q;q)_{2m}} \frac{(1-q^m)^2(1-q^{2m-1})(1-q^{2m})}{q^{4m-1}} . \end{aligned}$$

Since $1-q^m=0$ when $m=0$, we can replace the summation in the above equation from $m=0$ to $m=\infty$. Since

$$\frac{(1-q^m)^2(1-q^{2m-1})(1-q^{2m})}{q^{4m-1}} = q^{1-4m} - 2q^{1-3m} - q^{-2m} + 2q^{1-m} + 2q^{-m} - q - 2q^m + q^{2m}, \quad (26)$$

we obtain

$$\begin{aligned} q^\lambda H^+_{\lambda,0}(q) =& (-1)^\lambda \sum_{m=0}^\infty \frac{q^{m(2m+1)+\lambda m}}{(q;q)_m^2 (q;q)_{2m}} (q^{1-4m} - 2q^{1-3m} - q^{-2m} + 2q^{1-m} + 2q^{-m} - q - 2q^m + q^{2m}) \\ =& qH^+_{\lambda-4,0}(q) + 2qH^+_{\lambda-3,0}(q) - H^+_{\lambda-2,0}(q) - (2+2q)H^+_{\lambda-1,0}(q) - qH^+_{\lambda,0}(q) + 2H^+_{\lambda+1,0}(q) + H^+_{\lambda+2,0}(q) . \end{aligned}$$

This gives the q-difference equation for $H_{\lambda,j}(q)$ when $j=0$ and $|q|<1$. Similarly one proves the q-difference equation for the cases $j=0,3,4,5$ and whenever $|q|\neq 1$.

For $j=1$ and $|q|<1$, we have

$$H_{\lambda,1}(q) = H^+_{\lambda,1}(q) = (-1)^\lambda \sum_{m=0}^\infty \frac{q^{m(2m+1)+\lambda m}}{(q;q)_m^2 (q;q)_{2m}} p^{(1)}_{\lambda,m}(q) ,$$

where

$$p_{\lambda,1,m}(q) = 4m + \lambda + 1 - 2E_1^{(m)}(q) - 2E_1^{(2m)}(q).$$

Hence

$$\begin{aligned} qH^+_{\lambda-4,1}(q) + 2qH^+_{\lambda-3,1}(q) - H^+_{\lambda-2,1}(q) - (2+2q)H^+_{\lambda-1,1}(q) - qH^+_{\lambda,1}(q) + 2H^+_{\lambda+1,1}(q) + H^+_{\lambda+2,1}(q) \\ = (-1)^\lambda \sum_{m=0}^\infty \frac{q^{m(2m+1)+\lambda m}}{(q;q)_m^2 (q;q)_{2m}} g_{\lambda,m}(q), \end{aligned}$$

where

$$\begin{aligned} g_{\lambda,m}(q) =& q^{1-4m} p_{\lambda-4,1,m}(q) - 2q^{1-3m} p_{\lambda-3,1,m}(q) - q^{-2m} p_{\lambda-2,1,m}(q) \\ &+ (2+2q)q^{-m} p_{\lambda-1,1,m}(q) - qp_{\lambda,1,m}(q) - 2q^m p_{\lambda+1,1,m}(q) + q^{2m} p_{\lambda+2,1,m}(q). \end{aligned} \quad (27)$$

We are going to show that $(-1)^\lambda \sum_{m=0}^\infty \frac{q^{m(2m+1)+\lambda m}}{(q;q)_m^2 (q;q)_{2m}} g_{\lambda,m}(q) = q^\lambda H^+_{\lambda,1}(q)$. Noticing the recursive relation that

$$E_1^{(m)}(q) - E_1^{(m-1)}(q) = \sum_{s=1}^\infty \left(\frac{q^{s(m+1)}}{1-q^s} - \frac{q^{sm}}{1-q^s} \right) = \sum_{s=1}^\infty -q^{sm} = -\frac{q^m}{1-q^m}, \quad (28)$$

we convert $p_{\lambda,1,m}(q)$ into the following form

$$\begin{aligned} p_{\lambda,1,m}(q) =& 4m+\lambda+1-2E_1^{(m-1)}(q)-2E_1^{(2m-2)}(q)+\frac{2q^m}{1-q^m}+\frac{2q^{2m-1}}{1-q^{2m-1}}+\frac{2q^{2m}}{1-q^{2m}} \\ =& 4(m-1)+\lambda+1-2E_1^{(m-1)}(q)-2E_1^{(2m-2)}(q)+f_{1,m}(q)+4 \\ =& p_{\lambda,1,m-1}(q)+f_{1,m}(q)+4, \end{aligned} \tag{29}$$

where

$$f_{1,m}(q) := \frac{2q^m}{1-q^m}+\frac{2q^{2m-1}}{1-q^{2m-1}}+\frac{2q^{2m}}{1-q^{2m}}. \tag{30}$$

Substituting the (29) into (27), combining the common factors $p_{\lambda,1,m-1}(q)+f_{1,m}(q)$ and applying the identity (26), we see that

$$\begin{aligned} g_{\lambda,m}(q) =& \frac{(1-q^m)^2(1-q^{2m-1})(1-q^{2m})}{q^{4m-1}}\left(p_{\lambda,1,m-1}(q)+f_{1,m}(q)\right) \\ & -\left(2q^{1-3m}+2q^{-2m}-6(q+1)q^{-m}+4q+10q^m-6q^{2m}\right). \end{aligned}$$

Since

$$\frac{(1-q^m)^2(1-q^{2m-1})(1-q^{2m})}{q^{4m-1}}f_{1,m}(q) = 2q^{1-3m}+2q^{-2m}-6(q+1)q^{-m}+4q+10q^m-6q^{2m},$$

we conclude that

$$g_{\lambda,m}(q) = \frac{(1-q^m)^2(1-q^{2m-1})(1-q^{2m})}{q^{4m-1}}p_{\lambda,1,m-1}(q).$$

Therefore

$$\begin{aligned} (-1)^\lambda\sum_{m=0}^{\infty}\frac{q^{m(2m+1)+\lambda m}}{(q;q)_m^2(q;q)_{2m}}g_{\lambda,m}(q) =& (-1)^\lambda\sum_{m=1}^{\infty}\frac{q^{(m-1)(2m-1)+\lambda m}}{(q;q)_{m-1}^2(q;q)_{2m-2}}p_{\lambda,1,m-1}(q) \\ =& (-1)^\lambda q^\lambda\sum_{m=0}^{\infty}\frac{q^{m(2m+1)+\lambda m}}{(q;q)_m^2(q;q)_{2m}}p_{\lambda,1,m}(q) = q^\lambda H^+_{\lambda,1}(q), \end{aligned}$$

as desired. Similarly one proves the q-difference equation for $j=1,2$ and $|q|\neq 1$, using the recursive relation (28) and

$$E_2^{(m)}(q)-E_2^{(m-1)}(q) = \sum_{s=1}^{\infty}\left(\frac{sq^{s(m+1)}}{1-q^s}-\frac{sq^{sm}}{1-q^s}\right) = \sum_{s=1}^{\infty}-sq^{sm} = -\frac{q^m}{(1-q^m)^2}. \tag{31}$$

This completes the proof of Theorem 2. ☐

3.4. **Self-duality.** In this section we prove Theorem 3. Throughout this section we assume $|q|<1$ and give the proof for this case only; the proof for $|q|>1$ is similar and is omitted. Our method can be used to give a systematic proof of the self-duality properties of the q-holonomic modules that appear in the refined quantum modularity conjecture of knot complements or of closed 3-manifolds.

We first compute the determinant of the truncated Wronskian $W_\lambda(q)$. It is well-known (see eg [GK13, Lemma 4.7]) that it satisfies the first order linear q-difference equation

$$\det(W_{\lambda+1}(q))-q\det(W_\lambda(q)) = 0.$$

It follows that $\det(W_\lambda(q)) = q^\lambda c(q)$ for some q-series $c(q)$ independent of λ. We claim that

$$\det(W_\lambda(q)) = 32q^{\lambda+11/4} + O(q^{3\lambda/2}), \tag{32}$$

for all sufficiently large natural numbers λ, which implies that $c(q) = 32q^{11/4}$. To show (32), recall that $W_\lambda(q) = \big(H^+_{\lambda+i,j}(q)\big)_{0\le i,j\le 5}$ when $|q|<1$. The definition of $H^+_{\lambda,j}(q)$ implies that

$$H^\pm_{\lambda,j}(q) = R^\pm_{\lambda,j}(q) + O(q^{3\lambda/2}), \tag{33}$$

where $R^+_{\lambda,j}(q)$ and $R^-_{\lambda,j}(q)$ are given by

$$\begin{aligned}
R^+_{\lambda,j}(q) &= (-1)^\lambda \left(p_{\lambda,j,0}(q) + p_{\lambda,j,1}\frac{q^{\lambda+3}}{(1-q)^4(1+q)}\right), \quad j=0,1,2,\\
R^+_{\lambda,3}(q) &= (-1)^\lambda \frac{q^{1/8}}{(1-q^{1/2})^2}\frac{q^{1+\lambda/2}}{1-q},\\
R^+_{\lambda,4}(q) &= 1 + \frac{q^{\lambda+3}}{(1+q)^3(1-q)^2},\\
R^+_{\lambda,5}(q) &= \frac{q^{1/8}}{(1+q^{1/2})^2}\frac{q^{1+\lambda/2}}{1-q},
\end{aligned} \tag{34}$$

and

$$\begin{aligned}
R^-_{\lambda,j}(q) &= (-1)^\lambda \left(P_{\lambda,j,0}(q) + P_{\lambda,j,1}(q)\frac{q^{\lambda+2}}{(1-q)^4(1+q)}\right), \quad j=0,1,2,\\
R^-_{\lambda,3}(q) &= (-1)^\lambda \frac{q^{-1/8}}{(1-q^{-1/2})^2}\frac{q^{\lambda/2}}{1-q},\\
R^-_{\lambda,4}(q) &= 1 + \frac{q^{\lambda+2}}{(1+q)^3(1-q)^2},\\
R^-_{\lambda,5}(q) &= \frac{q^{-1/8}}{(1+q^{-1/2})^2}\frac{q^{\lambda/2}}{1-q}.
\end{aligned} \tag{35}$$

Thus,

$$W_\lambda(q) = R_\lambda(q) + O(q^{3\lambda/2})\,, \tag{36}$$

where $R_\lambda(q) = (R_{\lambda+i,j}(q))_{0\le i,j\le 5}$. Since

$$\det(W_\lambda(q)) + O(q^{3\lambda/2}) = \det(R_\lambda(q)) + O(q^{3\lambda/2}) = 32q^{\lambda+11/4} + O(q^{3\lambda/2}) \tag{37}$$

Equation (32) follows. It is noteworthy that the Eisenstein series $\mathcal{E}_2(q)$ which appear in the entries of $R_\lambda(q)$ cancel upon taking the determinant. The same happens in the entries of the matrix (45) below.

This concludes the proof of (9). We next prove the orthogonality property (10) following the method of [GW, Sec 2.5]. By the q-difference equation (11), we have

$$W_{\lambda+1}(q) = A(\lambda,q)W_\lambda(q), \quad W_{-\lambda-1}(q^{-1}) = \tilde{A}(\lambda,q)W_{-\lambda}(q^{-1}), \tag{38}$$

where

$$A(\lambda,q)=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ -q & -2q & 1 & 2(1+q) & q+q^{\lambda+4} & -2 \end{pmatrix} \tag{39}$$

and

$$\tilde{A}(\lambda,q)=A(-\lambda-1,q^{-1})^{-1}=\begin{pmatrix} -2 & q & 2(1+q) & 1+q^{\lambda-2} & -2q & -q\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Consider

$$Q(\lambda,q):=\begin{pmatrix} -12 & 8 & -4 & 2 & 0 & 0\\ 8 & -4 & 2 & 0 & 0 & 0\\ -4 & 2 & 0 & 0 & 0 & 2\\ 2 & 0 & 0 & 0 & 2 & -4\\ 0 & 0 & 0 & 2 & -4 & 8+2q^{\lambda+2}\\ 0 & 0 & 2 & -4 & 8+2q^{\lambda+3} & -12-4q^{\lambda+2}-4q^{\lambda+3} \end{pmatrix}.$$

It is easy to see that the matrices A, Q and $\tilde{A}$ (all with entries in the polynomial ring $\mathbb{Q}[q^{\pm 1},q^{\pm\lambda}]$) satisfy

$$A(\lambda,q)Q(\lambda,q)\tilde{A}(\lambda+5,q)=Q(\lambda+1,q)\,. \tag{40}$$

Note that all matrices above are invertible, with determinants

$$\det(A(\lambda,q)=q, \qquad \det(\tilde{A}(\lambda,q)=q, \qquad \det(Q(\lambda,q)=-64q^{5+2\lambda}\,. \tag{41}$$

Using (38) and (40), we see that

$$W_{\lambda+1}(q)^{-1}Q(\lambda+1,q)\left(W_{-\lambda-6}(q^{-1})^{-1}\right)^T=W_\lambda(q)^{-1}Q(\lambda,q)\left(W_{-\lambda-5}(q^{-1})^{-1}\right)^T,$$

hence $W_\lambda(q)^{-1}Q(\lambda,q)\left(W_{-\lambda-5}(q^{-1})^{-1}\right)^T$ is independent of λ. The claim is that we have

$$W_\lambda(q)^{-1}Q(\lambda,q)\left(W_{-\lambda-5}(q^{-1})^{-1}\right)^T=D:=\begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0\\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{42}$$

Since we have seen that the left-hand side of (42) is independent of λ, it suffices to show that

$$W_\lambda(q)^{-1}Q(\lambda,q)\left(W_{-\lambda-5}(q^{-1})^{-1}\right)^T=D+O(q^{\lambda/2}), \tag{43}$$

for any sufficiently large $\lambda\in\mathbb{N}$. Equation (36), together with (9) gives that

$$W_\lambda(q)^{-1}Q(\lambda,q)\left(W_{-\lambda-5}(q^{-1})^{-1}\right)^T+O(q^{\lambda/2})=R_\lambda(q)^{-1}Q(\lambda,q)\left(R_{-\lambda-5}(q^{-1})^{-1}\right)^T+O(q^{\lambda/2}) \tag{44}$$

and an explicit calculation shows that

$$R_\lambda(q)^{-1}Q(\lambda,q)\left(R_{-\lambda-5}(q^{-1})^{-1}\right)^T + O(q^{\lambda/2}) = D + O(q^{\lambda/2}) \tag{45}$$

where $R_{-\lambda-5}(q^{-1})^{-1}+O(q^{\lambda/2})$ can be computed by multiplying the adjugate of $R_{-\lambda-5}(q^{-1})+O(q^{\lambda/2})$ with the inverse of its determinant (36). Equation (43) follows.

This concludes the proof of Theorem 3. □

4. Stationary phase of the descendant state integral

4.1. **Stationary phase.** In this section we compute the stationary phase of the state integral around its critical points. This is a well-known method of asymptotic analysis that can be found in many classic books (eg., [Olv74]). For convenience, we define a renormalized version of the descendant state integral (3) given by

$$\hat{Z}^{(\lambda,\lambda')}_{(-2,3,7)}(\tau) \;=\; (\tilde{q}/q)^{\frac{1}{24}} Z^{(\lambda,\lambda')}_{(-2,3,7)}(\tau)\,. \tag{46}$$

Throughout this section, we will use the notation

$$\hbar \coloneqq 2\pi i\tau, \qquad \tau = b^2\,, \tag{47}$$

and determine the asymptotic expansion of $\hat{Z}^{(\lambda,\lambda')}_{(-2,3,7)}(\tau)$ as $\hbar \to 0$.

It turns out that there are 6 critical points α

$$(\alpha^3-\alpha-1)(\alpha^3+2\alpha^2-\alpha-1) = 0 \tag{48}$$

in two Galois orbits of the cubic number fields with discriminants -23 and 49, respectively. After a change of parametrization of these number fields (to match with the conventions of [GZ24], these critical points are given by

$$\alpha = -\xi+\xi^2, \qquad \xi^3-\xi^2+1=0 \tag{49a}$$

$$\alpha = -1-\eta, \qquad \eta^3+\eta^2-2\eta-1=0\,. \tag{49b}$$

The next theorem computes the stationary phase expansion of $\hat{Z}^{(\lambda,\lambda')}_{(-2,3,7)}(\tau)$ at each critical point.

Theorem 5. *The stationary phase of $\hat{Z}^{(\lambda,\lambda')}_{(-2,3,7)}(\tau)$ is given by $e^{\frac{2\pi i\lambda'\log\alpha}{\hbar}}\hat{\Phi}^{(\alpha)}(\lambda,\hbar)$, where*

$$\hat{\Phi}^{(\alpha)}(\lambda,\hbar) \;=\; e^{\frac{V_{0,0}(\alpha)}{\hbar}}\Phi^{(\alpha)}(\lambda,\hbar), \qquad \Phi^{(\alpha)}(\lambda,\hbar) \;=\; \frac{\alpha^\lambda}{\sqrt{i\Delta(\alpha)}}\sum_{k=0}^{\infty} c_k(\alpha,\lambda)\hbar^k \tag{50}$$

and

$$\begin{aligned} V_{0,0}(\alpha) &= 2\mathrm{Li}_2(-\alpha)-\mathrm{Li}_2(\alpha^{-2}),\\ \Delta(\alpha) &= -2\alpha^5+12\alpha^3-2\alpha^2-16\alpha-10\,, \end{aligned} \tag{51}$$

and $c_k(\alpha,\lambda)\in\mathbb{Q}(\alpha)[\lambda]$ are polynomials in λ of degree $2k$ with coefficients in $\mathbb{Q}(\alpha)$ with $c_0(\alpha,\lambda)=1$ given explicitly by a formal Gaussian integration.

We have computed 400 coefficients of the above series for $\lambda = 0$ and 40 coefficients for general λ. Since there are two number fields involved, we present the asymptotic series $\hat{\Phi}^{(\sigma)}(\lambda, \hbar)$ separately for each field. For α as in (49a), we have

$$\begin{aligned}\hat{\Phi}^{(\sigma)}(\lambda,\hbar) = \frac{\alpha^{\lambda} e^{\frac{V_{0,0}}{\hbar}}}{\sqrt{i(-6\xi^2+10\xi-4)}}\Bigg(1+\Bigg(\left(-\frac{1}{46}\xi^2-\frac{7}{92}\xi+\frac{3}{92}\right)\lambda^2+\left(\frac{3}{46}\xi^2-\frac{11}{92}\xi+\frac{17}{46}\right)\lambda \\ +\frac{293}{8464}\xi^2+\frac{127}{2116}\xi-\frac{681}{8464}\Bigg)\hbar+O(\hbar^2)\Bigg),\end{aligned} \tag{52}$$

and for α as in (49b), we have

$$\begin{aligned}\hat{\Phi}^{(\sigma)}(\lambda,\hbar) = \frac{\alpha^{\lambda} e^{\frac{V_{0,0}}{\hbar}}}{\sqrt{i(-4\eta^2+2\eta-2)}}\Bigg(1+\Bigg(\left(\frac{1}{28}\eta^2+\frac{1}{14}\eta-\frac{1}{28}\right)\lambda^2+\left(\frac{1}{28}\eta^2-\frac{1}{14}\eta+\frac{3}{14}\right)\lambda \\ +\frac{1}{16}\eta^2+\frac{1}{16}\eta-\frac{17}{168}\Bigg)\hbar+O(\hbar^2)\Bigg).\end{aligned} \tag{53}$$

We can give more terms when $\lambda = 0$. For α as in (49a), we have

$$\begin{aligned}\hat{\Phi}^{(\sigma)}(0,\hbar) &= \frac{e^{\frac{V_{0,0}}{\hbar}}}{\sqrt{i(-6\xi^2+10\xi-4)}}\Bigg(1+\left(\frac{293}{8464}\xi^2+\frac{127}{2116}\xi-\frac{681}{8464}\right)\hbar \\ &\quad +\left(\frac{65537}{6229504}\xi^2-\frac{50607}{6229504}\xi+\frac{2535}{778688}\right)\hbar^2+O(\hbar^3)\Bigg),\end{aligned} \tag{54}$$

and for α as in (49b), we have

$$\begin{aligned}\hat{\Phi}^{(\sigma)}(0,\hbar) &= \frac{e^{\frac{V_{0,0}}{\hbar}}}{\sqrt{i(-4\eta^2+2\eta-2)}}\Bigg(1+\left(\frac{293}{8464}\xi^2+\frac{127}{2116}\xi-\frac{681}{8464}\right)\hbar \\ &\quad +\left(\frac{65537}{6229504}\xi^2-\frac{50607}{6229504}\xi+\frac{2535}{778688}\right)\hbar^2+O(\hbar^3)\Bigg),\end{aligned} \tag{55}$$

4.2. **Formal Gaussian integration.** Using the identity (19) we convert the descendant state integral into the following form,

$$Z^{(\lambda,\lambda')}_{(-2,3,7)}(\hbar) = \left(\frac{q}{\tilde{q}}\right)^{-\frac{1}{24}} \int_{\mathbb{R}+i\frac{c_b}{2}+i\varepsilon} \Phi_b(x)^2\Phi_b(2x-c_b)e^{-\pi i(2x-c_b)^2+2\pi(\lambda b-\lambda' b^{-1})x}\,\mathrm{d}x \tag{56}$$

$$= \int_{\mathbb{R}+i\frac{c_b}{2}+i\varepsilon} \frac{\Phi_b(x)^2}{\Phi_b(-2x+c_b)} e^{2\pi(\lambda b-\lambda' b^{-1})x}\,\mathrm{d}x, \tag{57}$$

and then apply the approximation [AK14, eq. (65)]

$$\Phi_b\left(\frac{z}{2\pi b}\right) = \exp\left(\sum_{n=0}^{\infty} \hbar^{2n-1}\frac{B_{2n}(1/2)}{(2n)!}\mathrm{Li}_{2-2n}(-e^z)\right).$$

We begin with a change of variables $x \mapsto \frac{z}{2\pi b}$, so that

$$\Phi_b(x)^2 = \Phi_b\left(\frac{z}{2\pi b}\right)^2 \sim \exp\left(\sum_{n=0}^{\infty} \hbar^{2n-1}\frac{B_{2n}(1/2)}{(2n)!}2\mathrm{Li}_{2-2n}(-e^z)\right)$$

and

$$\begin{aligned}\Phi_b(-2x+c_b) = \Phi_b\left(\frac{-2z+2\pi b c_b}{2\pi b}\right) &\sim \exp\left(\sum_{n=0}^{\infty} \hbar^{2n-1}\frac{B_{2n}(1/2)}{(2n)!}\mathrm{Li}_{2-2n}(-e^{-2z+2\pi b c_b})\right)\\ &= \exp\left(\sum_{n=0}^{\infty} \hbar^{2n-1}\frac{B_{2n}(1/2)}{(2n)!}\mathrm{Li}_{2-2n}(e^{-2z+\frac{\hbar}{2}})\right).\end{aligned}$$

Using the identity

$$\mathrm{Li}_{2-2n}(e^{-2z+s}) = \sum_{k=0}^{\infty}\frac{\mathrm{Li}_{2-2n-k}(e^{-2z})}{k!}s^k,$$

we have

$$\mathrm{Li}_{2-2n}(-e^{-2z+\frac{\hbar}{2}}) = \sum_{k=0}^{\infty}\frac{\mathrm{Li}_{2-2n-k}(e^{-2z})}{k!}\left(\frac{\hbar}{2}\right)^k.$$

Collecting the above equalities up, we obtain

$$Z^{(\lambda,\lambda')}_{(-2,3,7)}(\hbar) \sim \frac{i}{\sqrt{2\pi i\hbar}}\int \exp\left(\lambda z + \frac{2\pi i\lambda'}{\hbar}z + V(z,\hbar)\right)\mathrm{d}z,$$

where

$$\begin{aligned}V(z,\hbar) &= \sum_{n=0}^{\infty}\hbar^{2n-1}\frac{B_{2n}(1/2)}{(2n)!}2\mathrm{Li}_{2-2n}(-e^z) - \sum_{n,k\geq 0}\hbar^{2n+k-1}\frac{B_{2n}(1/2)}{(2n)!}\frac{\mathrm{Li}_{2-2n-k}(e^{-2z})}{k!}\frac{1}{2^k}\\
&= \sum_{n=0}^{\infty}\hbar^{2n-1}\frac{B_{2n}(1/2)}{(2n)!}2\mathrm{Li}_{2-2n}(-e^z) - \left(\sum_{n,k\geq 0}\hbar^{2n+2k-1}\frac{B_{2n}(1/2)}{(2n)!}\frac{\mathrm{Li}_{2-2n-2k}(e^{-2z})}{(2k)!}\frac{1}{2^{2k}}\right.\\
&\quad \left. + \sum_{n,k\geq 0}\hbar^{2n+2k}\frac{B_{2n}(1/2)}{(2n)!}\frac{\mathrm{Li}_{1-2n-2k}(e^{-2z})}{(2k+1)!}\frac{1}{2^{2k+1}}\right)\\
&= \sum_{n=0}^{\infty}\hbar^{2n-1}\left(\frac{B_{2n}(1/2)}{(2n)!}2\mathrm{Li}_{2-2n}(-e^z) - \sum_{k=0}^{n}\frac{B_{2n-2k}(1/2)}{(2n-2k)!(2k)!}\frac{\mathrm{Li}_{2-2n}(e^{-2z})}{2^{2k}}\right)\\
&\quad + \sum_{n=0}^{\infty}\hbar^{2n}\left(-\sum_{k=0}^{n}\frac{B_{2n-2k}(1/2)}{(2n-2k)!(2k+1)!}\frac{\mathrm{Li}_{1-2n}(e^{-2z})}{2^{2k+1}}\right).\end{aligned}$$

Therefore, if we define

$$\begin{aligned} V_{2n+1}(z) &= -\sum_{k=0}^{\infty} \frac{B_{2n-2k}(1/2)}{(2n-2k)!(2k+1)!} \frac{\mathrm{Li}_{1-2n}(e^{-2z})}{2^{2k+1}}, \\ V_{2n}(z) &= \frac{B_{2n}(1/2)}{(2n)!} 2\mathrm{Li}_{2-2n}(-e^z) - \sum_{k=0}^{n} \frac{B_{2n-2k}(1/2)}{(2n-2k)!(2k)!} \frac{\mathrm{Li}_{2-2n}(e^{-2z})}{2^{2k}}, \end{aligned} \tag{58}$$

then $V(z,\hbar) = \sum_{n=0}^{\infty} \hbar^{n-1} V_n(z)$, hence

$$\hat{Z}^{(\lambda,\lambda')}_{(-2,3,7)}(\hbar) \sim \frac{i}{\sqrt{2\pi i \hbar}} \int \exp\left(\lambda z + \frac{2\pi i \lambda'}{\hbar} z + \sum_{n=0}^{\infty} \hbar^{n-1} V_n(z)\right) \mathrm{d}z.$$

Solving $\frac{\mathrm{d}}{\mathrm{d}z}\left(2\pi i \lambda' z + V_0(z)\right) = 0$, we find that the critical point equation is

$$(\alpha^3 - \alpha - 1)(\alpha^3 + 2\alpha^2 - \alpha - 1) = 0, \quad (\alpha = e^z). \tag{59}$$

The expansion $V_n(z) = \sum_{m=0}^{\infty} (z - \log\alpha)^m V_{n,m}(\log\alpha)$ at a critical point $z = \log\alpha$ thus gives

$$\begin{aligned} \hat{Z}^{(\lambda,\lambda')}_{(-2,3,7)}(\hbar) &\sim \frac{i\alpha^{\lambda} e^{\frac{V_{0,0}+2\pi i\lambda' \log\alpha}{\hbar}}}{\sqrt{2\pi i}} \int \mathrm{d}y e^{V_{0,2}y^2} \exp\left(\lambda \hbar^{\frac{1}{2}} y + \sum_{m\geq 3} \hbar^{\frac{m}{2}-1} y^m V_{0,m} + \sum_{n\geq 1, m\geq 0} \hbar^{n-1+\frac{m}{2}} y^m V_{n,m}\right) \\ &=: e^{\frac{2\pi i \lambda' \log\alpha}{\hbar}} \hat{\Phi}(\lambda,\hbar). \end{aligned} \tag{60}$$

where the change of variables $z \mapsto \log\alpha + \hbar^{\frac{1}{2}} y$ is applied, and

$$\begin{aligned} V_{0,0} &= 2\mathrm{Li}_2(-\alpha) - \mathrm{Li}_2(\alpha^{-2}), \\ V_{0,1} &= -2\pi i \lambda', \\ V_{1,0} &= -\frac{1}{2}\mathrm{Li}_1(\alpha^{-2}) = \frac{1}{2}\log(1-\alpha^{-2}), \\ V_{0,2} &= \mathrm{Li}_0(-\alpha) - 2\mathrm{Li}_0(\alpha^{-2}) = -\frac{\alpha^2-\alpha+2}{(\alpha-1)(\alpha+1)} = \alpha^5 - \alpha^4 - 7\alpha^3 + \alpha^2 + 4\alpha + 5, \\ V_{2n,m} &= \frac{1}{m!}\left(\frac{B_{2n}(1/2)}{(2n)!} 2\mathrm{Li}_{2-2n-m}(-\alpha) - (-2)^m \mathrm{Li}_{2-2n-m}(\alpha^{-2}) \sum_{k=0}^{n} \frac{B_{2n-2k}(1/2)}{(2n-2k)!(2k)!2^{2k}}\right), \\ V_{2n+1,m} &= -\frac{(-2)^m}{m!} \mathrm{Li}_{1-2n-m}(\alpha^{-2}) \sum_{k=0}^{n} \frac{B_{2n-2k}(1/2)}{(2n-2k)!(2k+1)!2^{2k+1}}. \end{aligned} \tag{61}$$

Note that for $n=1$ and $m=0$, we have $\hbar^{n-1+\frac{m}{2}} y^m V_{n,m} = V_{1,0}$. Expand the exponential in the integrand, collect $\hbar$'s and use the formal Gaussian integrals, we obtain

$$\hat{\Phi}(\lambda,\hbar) = \frac{\alpha^{\lambda} e^{\frac{V_{0,0}}{\hbar}} e^{V_{1,0}}}{\sqrt{2iV_{0,2}}} (1 + O(\hbar)) = \frac{\alpha^{\lambda} e^{\frac{V_{0,0}}{\hbar}}}{\sqrt{i\Delta}} (1 + O(\hbar))$$

where

$$\Delta := \frac{2V_{0,2}}{e^{2V_{1,0}}} = \frac{-2\alpha^2(\alpha^2-\alpha+2)}{(\alpha-1)^2(\alpha+1)^2} = -2\alpha^5 + 12\alpha^3 - 2\alpha^2 - 16\alpha - 10\,.$$

This concludes the proof of Theorem 5. $\square$

When α satisfies (49a), we have

$$V_{1,0} = \frac{1}{2}\log(1-\xi^2), \qquad V_{0,2} = -3\xi^2 + 2\xi, \qquad \Delta = -6\xi^2 + 10\xi - 4 \tag{62}$$

whereas when α satisfies (49b), we have

$$V_{1,0} = \frac{1}{2}\log(\eta^2+\eta-2), \qquad V_{0,2} = -\eta^2 - 3\eta + 3, \qquad \Delta = -4\eta^2 + 2\eta - 2\,. \tag{63}$$

Computing out the formal Gaussian integrals in (60), we obtain (53) and (55).

5. Analytic aspects

In this last section we discuss analytic aspects of matrix-valued holomorphic quantum modular forms. We have already introduced the descendant state integrals which are analytic functions of $\tau \in \mathbb{C}'$, as well as the collection of q-series $H_j^{\pm}(q)$ which are holomorphic functions of τ (where $q = e^{2\pi i\tau}$) in the upper half plane $\mathrm{Im}(\tau) > 0$. In this section we discuss the radial asymptotics of these holomorphic functions as τ tends to zero in a fixed ray (i.e., $\arg(\tau) = \theta \in (-\pi, \pi)$ is fixed). Naturally, one expects these asymptotics to be given in terms of the formal power series $\hat{\Phi}^{(\alpha)}(\lambda, \hbar)$ of Section 4, up to some elementary constants.

All results that we report in this section are numerical, and void of proofs.

5.1. **Asymptotic expansion of the q-series.** Fixing a ray $\arg(\tau) = \theta$, we first computed numerically the values of the series (5) when $\tau = e^{i\theta}/N$ for $N = 800, \dots, 1000$ to high precision in `pari` using the inductive definition of $p_{0,0,m}(q)$ and $P_{0,0,n}(q)$ obtained easily from their definition (22).

We then used Richardson and Zagier's extrapolation methods which are explained in [GZ24] and in great detail in [Whe23], to extrapolate numerically from this data the coefficients of their asymptotic expansion. Properly normalized, these are algebraic numbers in a known number field (one of the cubic fields of (49a)– (49b)) that are known to high accuracy, which can then be recognized exactly. Having done so, the coefficients that we found ought to match one of the $\hat{\Phi}^{(\sigma)}(\hbar)$ series, up to some elementary factors, for some value of σ, which of course depends on the ray.

For example, when $\arg(\tau) = \pi/5$, we found numerically the following relation between the radial asymptotics of the q-series (5) and the asymptotic series $\hat{\Phi}$, where $\hbar = 2\pi i\tau$ and $q = e^{2\pi i\tau}$.

$$
\begin{aligned}
H^+_{0,0}(q) &= \left(\frac{q}{\tilde{q}}\right)^{1/24} \tau e^{\frac{\pi i}{4}} \hat{\Phi}^{(\sigma_1)}(\hbar), & H^-_{0,0}(q) &= \left(\frac{q}{\tilde{q}}\right)^{-1/24} \tau e^{\frac{\pi i}{4}} \hat{\Phi}^{(\sigma_2)}(-\hbar) \\
H^+_{0,1}(q) &= \left(\frac{q}{\tilde{q}}\right)^{1/24} e^{\frac{\pi i}{4}} \hat{\Phi}^{(\sigma_1)}(\hbar) & H^-_{0,1}(q) &= \left(\frac{q}{\tilde{q}}\right)^{-1/24} e^{\frac{\pi i}{4}} \hat{\Phi}^{(\sigma_2)}(-\hbar) \\
H^+_{0,2}(q) &= \left(\frac{q}{\tilde{q}}\right)^{1/24} \frac{2}{3\tau} e^{\frac{\pi i}{4}} \hat{\Phi}^{(\sigma_1)}(\hbar) & H^-_{0,2}(q) &= \left(\frac{q}{\tilde{q}}\right)^{-1/24} \frac{5}{6\tau} e^{\frac{\pi i}{4}} \hat{\Phi}^{(\sigma_2)}(-\hbar) \\
H^+_{0,3}(q) &= \left(\frac{q}{\tilde{q}}\right)^{1/24} \frac{1}{2} e^{-\frac{\pi i}{4}} \hat{\Phi}^{(\sigma_1)}(\hbar) & H^-_{0,3}(q) &= \left(\frac{q}{\tilde{q}}\right)^{-1/24} \frac{1}{2} e^{-\frac{\pi i}{4}} \hat{\Phi}^{(\sigma_2)}(-\hbar) \\
H^+_{0,4}(q) &= \tilde{q}^{-\frac{7}{8}} \left(\frac{q}{\tilde{q}}\right)^{1/24} 2e^{-\frac{\pi i}{4}} \hat{\Phi}^{(\sigma_6)}(\hbar) & H^-_{0,4}(q) &= \tilde{q}^{\frac{7}{8}} \left(\frac{q}{\tilde{q}}\right)^{-1/24} 2e^{-\frac{\pi i}{4}} \hat{\Phi}^{(\sigma_3)}(\hbar) \\
H^+_{0,5}(q) &= \tilde{q}^{-\frac{7}{8}} \left(\frac{q}{\tilde{q}}\right)^{1/24} e^{-\frac{\pi i}{4}} \hat{\Phi}^{(\sigma_6)}(\hbar) & H^-_{0,5}(q) &= \tilde{q}^{\frac{7}{8}} \left(\frac{q}{\tilde{q}}\right)^{-1/24} e^{-\frac{\pi i}{4}} \hat{\Phi}^{(\sigma_3)}(\hbar)\,.
\end{aligned} \tag{64}
$$

Here σ_j for $j = 1, \dots, 6$ are the six roots of the polynomial (48) with the numerical values

$$\sigma_1 = -0.662 - 0.562i, \qquad \sigma_2 = -0.662 + 0.562i, \qquad \sigma_3 = 1.325 \tag{65}$$

corresponding to the field (49a) and

$$\sigma_4 = -2.247, \qquad \sigma_5 = -0.555, \qquad \sigma_6 = 0.802\,, \tag{66}$$

corresponding to the field (49b), respectively.

Note that inserting the asymptotics (64) to the quadratic relation (11), one simply obtains that $0 = 0$.

5.2. **Further aspects.** As we explained briefly in the introduction, matrix-valued holomorphic quantum modular forms are complicated objects with conjectural analytic and arithmetic properties. In the present paper, we focused on the algebraic aspects of these objects. Our paper does not include the following analytic aspects of the matrix-valued holomorphic quantum modular form of the $(-2, 3, 7)$-pretzel knot:

- Asymptotics of the state integral as τ tends to zero in a fixed ray. A detailed analysis of the corresponding thimbles will surely identify those asymptotics with $\mathbb{Z}$-linear combinations of the series $\tilde{q}^{\lambda'}\hat{\Phi}(\lambda, \hbar)$.
- Borel resummation of the factorially divergent series $\hat{\Phi}(\lambda, \hbar)$, and identification of their Stokes phenomenon in terms of the series $H_{\lambda,j}(q)$. Without doubt, the Borel resummation coincides, up to elementary factors, with the descendant state integral itself.
- Asymptotics of the q-series $H_{\lambda,j}(q)$ when τ tends to zero in a fixed ray. This can be deduced combining Theorem 4 with the asymptotics of the state integrals themselves.

The paper also not include the arithmetic aspects related to the matrix of Habiro-like elements. Those can be obtained by the factorization of the descendant state integral (3) at rational points, following [GK15].

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Appendix A. Comparison of the q-series with [GZ23]

Recall the q-series $H_k^+(q)$ and $H_k^-(q)$ for $|q|<1$ from Equations (142) and (143) of [GZ23]. When $k=0,1,2$, the series $H_k^{\pm}(q)$ coincide with $H_{0,k}^{\pm}(q)$, whereas when $k=3,4,5$, they are given by

$$\begin{aligned} H_3^+(q) &= \frac{(q^{3/2};q)_\infty^2}{(q;q)_\infty^2}\sum_{m=0}^\infty \frac{q^{(2m+1)(m+1)}}{(q^{3/2};q)_m^2(q;q)_{2m+1}} & H_4^-(q) &= \frac{(q;q)_\infty^2}{(-1;q)_\infty^2}\sum_{n=0}^\infty \frac{q^{n(n+1)}}{(-q;q)_n^2(q;q)_{2n}} \\ H_4^+(q) &= \frac{(-q;q)_\infty^2}{(q;q)_\infty^2}\sum_{m=0}^\infty \frac{q^{(2m+1)m}}{(-q;q)_m^2(q;q)_{2m}} & H_3^-(q) &= \frac{(q;q)_\infty^2}{(q^{-1/2};q)_\infty^2}\sum_{n=0}^\infty \frac{q^{n(n+2)}}{(q^{3/2};q)_n^2(q;q)_{2n+1}} \\ H_5^+(q) &= \frac{(-q^{3/2};q)_\infty^2}{(q;q)_\infty^2}\sum_{m=0}^\infty \frac{q^{(2m+1)(m+1)}}{(-q^{3/2};q)_m^2(q;q)_{2m+1}} & H_5^-(q) &= \frac{(q;q)_\infty^2}{(-q^{-1/2};q)_\infty^2}\sum_{n=0}^\infty \frac{q^{n(n+2)}}{(-q^{3/2};q)_n^2(q;q)_{2n+1}}. \end{aligned} \tag{67}$$

The comparison between the above series with the ones in our paper are given as follows.

Lemma 6. We have:

$$\begin{aligned} H_{0,3}^+(q) &= \frac{q^{1/8}}{(1-q^{1/2})^2}\frac{(q;q)_\infty^2}{(q^{3/2};q)_\infty^2}H_3^+(q) & H_{0,3}^-(q) &= \frac{q^{-1/8}}{(1-q^{-1/2})^2}\frac{(q^{-1/2};q)_\infty^2}{(q;q)_\infty^2}H_3^-(q) \\ H_{0,4}^+(q) &= \frac{(q;q)_\infty^2}{(-q;q)_\infty^2}H_4^+(q) & H_{0,4}^-(q) &= \frac{(-1;q)_\infty^2}{(q;q)_\infty^2}H_4^-(q) \\ H_{0,5}^+(q) &= \frac{q^{1/8}}{(1+q^{1/2})^2}\frac{(q;q)_\infty^2}{(-q^{3/2};q)_\infty^2}H_5^+(q) & H_{0,5}^-(q) &= \frac{q^{-1/8}}{(1+q^{-1/2})^2}\frac{(-q^{-1/2};q)_\infty^2}{(q;q)_\infty^2}H_5^-(q) \end{aligned} \tag{68}$$

Proof. We need to show the following identities:

$$\begin{aligned} \frac{(q^{3/2};q)_\infty^2}{(q;q)_\infty^2}\frac{(\tilde{q};\tilde{q})_\infty^2}{(-1;\tilde{q})_\infty^2} &= \frac{e^{-\frac{\pi i}{2}}q^{1/8}}{2(1-q^{1/2})^2}\tau, \\ \frac{(-q;q)_\infty^2}{(q;q)_\infty^2}\frac{(\tilde{q};\tilde{q})_\infty^2}{(-\tilde{q}^{-1/2};\tilde{q})_\infty^2} &= \frac{e^{-\frac{\pi i}{2}}\tilde{q}^{-1/8}}{2(1-\tilde{q}^{-1/2})^2}\tau, \\ \frac{(-q^{3/2};q)_\infty^2}{(q;q)_\infty^2}\frac{(\tilde{q};\tilde{q})_\infty^2}{(-\tilde{q}^{-1/2};\tilde{q})_\infty^2} &= \frac{e^{-\frac{\pi i}{2}}q^{1/8}\tilde{q}^{-1/8}}{(1+q^{1/2})^2(1+\tilde{q}^{-1/2})^2}\tau. \end{aligned} \tag{69}$$

The modularity of the Dedekind η-function implies that

$$\frac{(q;q)_\infty}{(\tilde{q};\tilde{q})_\infty} = e^{\frac{\pi i}{4}}\left(\frac{q}{\tilde{q}}\right)^{\frac{1}{24}}\frac{1}{\sqrt{\tau}}, \tag{70}$$

hence

$$\frac{(q^{1/2};q^{1/2})_\infty}{(\tilde{q}^2;\tilde{q}^2)_\infty} = e^{\frac{\pi i}{4}}\frac{\tilde{q}^{1/12}}{q^{1/48}}\left(\sqrt{\frac{\tau}{2}}\right)^{-1}.$$

Since

$$\begin{aligned}(q^2;q^2)_\infty =&(q;q)_\infty(-q;q)_\infty\\ (q^{1/2};q^{1/2})_\infty =&(q^{1/2};q)_\infty(q;q)_\infty\end{aligned}$$

it follows that

$$\begin{aligned}\frac{(q^{3/2};q)_\infty}{(q;q)_\infty}\frac{(\tilde{q};\tilde{q})_\infty}{(-1;\tilde{q})_\infty} =&\frac{1}{2(1-q^{1/2})}\frac{(q^{1/2};q)_\infty}{(-\tilde{q};\tilde{q})_\infty}\frac{(\tilde{q};\tilde{q})_\infty}{(q;q)_\infty}\\ =&\frac{1}{2(1-q^{1/2})}\frac{(q^{1/2};q^{1/2})_\infty}{(\tilde{q}^2;\tilde{q}^2)_\infty}\frac{(\tilde{q};\tilde{q})^2_\infty}{(q;q)^2_\infty}\\ =&\frac{1}{2(1-q^{1/2})}e^{-\frac{\pi i}{4}}q^{1/16}\sqrt{2\tau}.\end{aligned}$$

Therefore

$$\frac{(q^{3/2};q)^2_\infty}{(q;q)^2_\infty}\frac{(\tilde{q};\tilde{q})^2_\infty}{(-1;\tilde{q})^2_\infty} = \frac{e^{-\frac{\pi i}{2}}q^{1/8}}{2(1-q^{1/2})^2}\tau.$$

The proof for the rest two identities is similar. □

We end this appendix with a remark that the collection of q-hypergeometric series $H^{\pm}_{\lambda,j}(q)$ defined and convergent for $|q|<1$ extend to $|q|>1$ and satisfy the symmetry

$$H^{+}_{\lambda,j}(q^{-1}) = (-1)^{\delta_j}H^{-}_{-\lambda,j}(q), \quad j=0,1,2 \qquad (|q|\neq 1) \tag{71}$$

This extension is possible since

- the terms of (21) satisfy $t_{\lambda,m}(q^{-1}) = T_{-\lambda,m}(q)$,
- the Eisenstein series $\mathcal{E}_1(q)$ and $\mathcal{E}_2(q)$ can be extended to $|q|>1$ satisfying $\mathcal{E}_j(q) = -\mathcal{E}_j(q^{-1})$ for $j=1,2$ [GK15, Remark 19],
- consequently, the terms (22) satisfy $p_{k,j,m}(q) = (-1)^j P_{-k,j,m}(q^{-1})$ for $j=0,1,2$. This follows from the identities

$$E_1^{(0)}(q) = \frac{1-\mathcal{E}_1(q)}{4} \qquad E_2^{(0)}(q) = \frac{1-\mathcal{E}_2(q)}{24}, \tag{72}$$

  and the recursive relations (28) and (31).

The symmetry for $j=3,4,5$ is obvious.

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