Notes on Bott&Tu

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Chapter 1

De Rham Theory

§1 From Manifolds to de Rham Theory on Euclidean Spaces

A review of manifolds

A *topological manifold of dimension* n is a topological space which is second countable, Hausdorff and locally homeomorphic to $\mathbb{R}^n$. Intuitively, one can think of a manifold as a space obtained by gluing countable many copies of $\mathbb{R}^n$ in a nice way. In this point of view, one may also think that a manifold is "established" by embedding $\mathbb{R}^n$'s into it. Note that the Hausdorff condition in the definition is essential , as there exist non-Hausdorff spaces that satisfy the other two conditions, for example the real line with two origins,

$$\mathbb{R} \times \{0, 1\}/((r, 0) \sim (r, 1),\ r \neq 0).$$

A *smooth manifold* is a topological manifold with a *smooth structure* on it. By basic calculus we all know the definition of the smoothness of a map $f : \mathbb{R}^n \to \mathbb{R}^m$. Also, we know that a homeomorphism between Euclidean spaces needs not be smooth (for example $t \mapsto t^{\frac{1}{3}}$ in $\mathbb{R}$). The smooth structure is just about how you embed $\mathbb{R}^n$'s into (open sets of) a manifold, so that this extra information (of choice of homeomorphisms) enables you to judge the smoothness when needed. Also, we need this information to be self-consistent, say if two embeddings $i_\alpha : \mathbb{R}^n \hookrightarrow U_\alpha \subset M$ and $i_\beta : \mathbb{R}^n \hookrightarrow U_\beta \subset M$ have an overlap $U_\alpha \cap U_\beta \neq \varnothing$, then we should expect that a real-valued function f on M is always smooth, no matter we judge it from the point of i_α or i_β, i.e. both $f \circ i_\alpha$ and $f \circ i_\beta$ should be smooth. As f varies among all real-valued functions on $U_\alpha \cap U_\beta$ such that $f \circ i_\alpha$ is smooth, one would see that demanding $f \circ i_\beta$ to be always smooth is equivalent to demanding that

$$i_\alpha^{-1} \circ i_\beta : i_\beta^{-1}(U_\alpha \cap U_\beta) \to i_\alpha^{-1}(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$$

is smooth. By interchanging α and β we see that we must demand that

$$i_\alpha^{-1} \circ i_\beta : i_\beta^{-1}(U_\alpha \cap U_\beta) \to i_\alpha^{-1}(U_\alpha \cap U_\beta)$$

is a diffeomorphism. More oftenly we use the inverse of the embedding, the homeomorphism $\varphi_\alpha \coloneqq i_\alpha^{-1} : U_\alpha \to \mathbb{R}^n$, and call it a *parametrization* or *coordinate chart* (of *coordinate neighborhood* U_α in this case). Two coordinate charts φ_α and φ_β are said to be *smoothly compatible* if the transition

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$

is a diffeomorphism.

By specifying a smoothly compatible way how an open covering of M is parametrized, a *smooth atlas* on M is defined. This is sufficient for us to determine a smooth structure on M, but we could make this definition better. There are numerous smooth atlases on a single manifold M, but many of them determine same smooth structures. To avoid this situation, we let a smooth structure on a manifold be defined by a *maximal smooth atlas* (with respect to inclusion), i.e. a smooth atlas consisted of every possible coordinate chart that is smoothly compatible with all coordinate charts in it. By using maximal smooth atlases instead of smooth ones, two smooth structures on a manifold are the same if and only if their defining atlases are equal. Note that every smooth atlas can be extended to be a maximal smooth atlas, by adding to it all coordinate charts that are smoothly compatible with the original atlas; one should verify that the resulted collection is still a smooth atlas.

Thereby we conclude our definition of a smooth manifold as the following:

Definition 1.1. Smooth Manifold

A smooth manifold M of dimension n is a topologicl manifold of dimension n with a smooth structure which is defined by a maximal smooth atlas, i.e. a maximal collection of smoothly compatible coordinate charts $\{\varphi_\alpha : U_\alpha \to \mathbb{R}^n\}_\alpha$ of open sets U_α's of M, where $\bigcup_\alpha U_\alpha = M$.

Not every topological manifold can be given a smooth structure, as the first example of a non-smoothable manifold is found by Kervaire in 1960, see *A Manifold which does not not admit any Differentiable Structure*. Without further specification, in our case we shall always refer to a smooth manifold (without boundary) by the single word "manifold".

Given a smooth manifold, without further specification, a parametrization, coordinate chart or *smooth chart* of that manifold always refers to a coordinate chart that lies in its (maximal) smooth atlas, so that it is a local characterization of the smooth structure on that manifold (and that's the meaning of the words "parametrization" and "coordinate").

Now we introduce formally how do we judge the smoothness when needed in the most general case:

Definition 1.2. Smooth Map

Given two (smooth) manifolds M and N, a map $F : M \to N$ is smooth if for any $p \in M$ there exists a parametrization φ of a neighborhood $U \ni p$ and ψ of a neighborhood $V \supset F(U) \ni F(p)$ such that the composition $\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V) \subset \mathbb{R}^n$ is smooth (as maps between Euclidean spaces).

Without further specification, the single word "map" between two manifolds always refers to a smooth one, as this is the only kind of maps we are concerned about.

It is immediate from the definition that a smooth map is continuous. As open balls in $\mathbb{R}^n$ are diffeomorphic to $\mathbb{R}^n$ (for example diffeomorphism of the unit open ball centered at the origin is given by $\boldsymbol{x} \mapsto \frac{1}{1-\|\boldsymbol{x}\|}\boldsymbol{x}$), it follows that a continuous map $F : M \to N$ is smooth if and only if for any parametrization φ of U in M and ψ of V in N, the composition

$$\psi \circ F \circ \varphi^{-1} : \varphi(F^{-1}(V) \cap U) \to \psi(V \cap F^{-1}(U))$$

is always smooth. Also, note that replacing $\mathbb{R}^n$ in definition 1.1 by "an open subset of $\mathbb{R}^n$" gives equivalent characterization of smooth structure.

Given a smooth manifold M and an open set U of M, by restricting all coordinate charts of M to U we see that U inherits a smooth structure from M and becomes a smooth manifold. The inclusion $i_U : U \hookrightarrow M$ is clearly smooth (in fact it is far more better than being smooth and injective); this is a special case of submanifold. With this in mind we can give a characterization of smoothness, which is really useful:

Proposition 1.1. Smoothness Is Local

Let M and N be smooth manifolds and $F : M \to N$ a map.

(a) If every point $p \in M$ has a neighborhood U such that the restriction $F|_U = F \circ i_U$ is smooth, then F is smooth.

(b) Conversely, if F is smooth, then its restriction to every open subset is smooth.

As an immediate consequence of this localness, we have the gluing lemma of smooth maps:

Lemma 1.2. Gluing Lemma of Smooth Maps

Let M and N be smooth manifold and $\{U_\alpha\}_\alpha$ be an open cover of M. Suppose that for each α there is a smooth map $F_\alpha : U_\alpha \to N$ agreeing on overlaps $F_\alpha|_{U_\alpha \cap U_\beta} = F_\beta|_{U_\alpha \cap U_\beta}$ for all indexes α and β, then there exists a unique smooth map $F : M \to N$ such that $F|_{U_\alpha} = F_\alpha$ for each α.

Since the composition of smooth functions between Euclidean spaces is again smooth, it follows that the composition of smooth maps between manifolds is still smooth. Also, it is trivial that the identity map on a manifold is smooth. Thus smooth manifolds with smooth maps between them as morphisms form a category, the category of smooth manifolds, denoted as Man or Diff (Diff for differentiable). The isomorphisms in this category are smooth maps with smooth inverse, i.e. the diffeomorphisms.

With the category built, we can now distinguish different smooth manifolds up to isomorphism. A first example is that, even on the most simple manifold, there exist different smooth structures:

Example 1.1

On $\mathbb{R}$ the global parametrization $\varphi : \mathbb{R} \to \mathbb{R} : t \mapsto t^3$ gives a smooth structure that is not the same as the standard smooth structure which is given by the identity on $\mathbb{R}$, since the identity on $\mathbb{R}$ is not a diffeomorphism with the smooth structure given by φ at one side and the standard one at the other side, because $t \mapsto t^{\frac{1}{3}}$ is not smooth.

Without further specification, the smooth structure on $\mathbb{R}^n$ always refers to the standard one, the one given by the identity on $\mathbb{R}^n$.

If one knows about classical differential geometry, then one must know about the *tangent space* at a point of a regular surface in 3-dimensional Euclidean space, which is the plane spanned by the tangent vectors, the *velocity*, of parametrized curves on that surface. For a manifold, which is a generalization of a regular surface, we expect a similar construction to deal with. However, a manifold is not something that is embedded in the Euclidean space[1], so we need to take another approach.

There are many equivalent approaches to this, among which two of them shall be introduced. First we introduce the most intuitive one, the approach of velocity of curves. A *smooth curve* on a manifold M is a smooth map $\gamma : J \to M$ where J is an interval (usually open) in $\mathbb{R}$. As is observed before, the information of the smooth structure on M is encoded in the set of all real-valued smooth functions on M, which is denoted as $C^\infty(M)$. Given a real-valued smooth function f on M, then its composition with a smooth curve $\gamma : J \to M$ is a smooth map between 1-dimensional Euclidean spaces, $f \circ \gamma : J \subset \mathbb{R} \to \mathbb{R}$. On considering the velocity of a curve, we would want to take its derivative with respect to time $t \in J$, which measures how "fast" and in which direction the curve flows (recall the regular surface case in your mind). We cannot take the derivative directly for a smooth curve $\gamma : J \to M$, but the composition $f \circ \gamma$ is good enough for us to take the derivative, which gives velocity of γ "measured" by $f \in C^\infty(M)$. Clearly, "measuring" from a single function gives incomplete information. As we don't want to lose any information about the velocity of γ, we demand that f varies among all smooth functions on M. This gives us a function

$$\begin{aligned}(- \circ \gamma)' \colon C^\infty(M) \times J &\longrightarrow \mathbb{R} \\ (f, t) &\longmapsto (f \circ \gamma)'(t)\end{aligned}$$

More oftenly we specify a time $t_0 \in J$ and talk about the *velocity of* γ *at* t_0, the function

$$\begin{aligned}(- \circ \gamma)'(t_0) \colon C^\infty(M) &\longrightarrow \mathbb{R} \\ f &\longmapsto (f \circ \gamma)'(t_0)\end{aligned}$$

So the definition is concluded as below:

Definition 1.3. Velocity of γ at t_0

Given a manifold M and a smooth curve $\gamma : J \to M$ and $t_0 \in J$, the velocity of γ at t_0, denoted as $\gamma'(t_0)$, is defined to be the function

$$\gamma'(t_0) \coloneqq (- \circ \gamma)'(t_0) : C^\infty(M) \to \mathbb{R} : f \mapsto (f \circ \gamma)'(t_0) = \frac{\mathrm{d}(f \circ \gamma)}{\mathrm{d}t}(t_0).$$

Other common notations for the velocity are

$$\dot{\gamma}(t_0), \quad \frac{\mathrm{d}\gamma}{\mathrm{d}t}(t_0), \quad \left.\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right|_{t=t_0}.$$

Viewing $C^\infty(M)$ as a real linear space (under the point-wise summation $(f+g)(p) = f(p) + g(p)$), then the velocity of γ at t_0 is a linear map satisfying

$$\gamma'(t_0)(fg) = f(\gamma(t_0))\gamma'(t_0)(g) + \gamma'(t_0)(f)g(\gamma(t_0)), \quad \text{for all } f, g \in C^\infty(M), \tag{1.1}$$

since $(fg)' = fg' + f'g$ for smooth functions on $\mathbb{R}$. A linear map $v : C^\infty(M) \to \mathbb{R}$ satisfying that $v(fg) = f(p)v(g) + g(p)v(f)$ for all $f, g \in C^\infty(M)$, where p is a point on M, is called a *derivation at* p; clearly the velocity of γ at t_0 is a derivation at $\gamma(t_0)$. Conversely, it is true that every derivation at p is the velocity of some curve γ at t_0 where $\gamma(t_0) = p$.[2] Noticing that derivations are closed under addition (as linear maps), we see that given a point $p \in M$, the velocities of all curves at the time when it flows through p (if it does) form a linear space, which we define it to be the *tangent space to* M *at* p, denoted as T_pM. This is the approach of the velocity of curves.

[1] Though it can be embedded, due to results of Whitney *Whitney Embedding Theorem*. But for a single manifold there is not a canonical way to do this embedding, so we'd better not use this result to define the tangent space.

[2] This fact is not at all easy to see. Read Chapter 3 of Lee's *Introduction to Smooth Manifolds* if you are curious about the proof.

Since derivation is equivalent to velocity, to simplify the construction we usually use the derivation in substitution of the velocity; this is the second approach introduced. So we conclude the definition of tangent space, using derivation, as below:

Definition 1.4. Tangent Space to M at p

Given a manifold M and a point $p \in M$, the tangent space to M at p, denoted as T_pM, is the linear space of all derivations at p, where a derivation is a linear map $v : C^\infty(M) \to \mathbb{R}$ satisfying that

$$v(fg) = f(p)v(g) + g(p)v(f), \quad \text{for all } f, g \in C^\infty(M).$$

An elements in T_pM is referred to as a *tangent vector of M at p*. Admitting the fact that every derivation is the velocity of some curve, some basic properties of derivation can be derived easily[3]:

Lemma 1.3. Properties of Derivations

Given a manifold M and a point $p \in M$, let $v \in T_pM$ and $f, g \in C^\infty(M)$, then

(a) If f is a constant function, then $vf = 0$.

(b) If $f(p) = g(p) = 0$, then $v(fg) = 0$.

(c) (Local Nature of Derivation) If f and g agree on some neighborhood of p, then $vf = vg$.

Given two manifolds M and N, a smooth map $F : M \to N$ brings smooth curves on M to smooth curves on N, and so does their velocities. By the chain rule, it follows that the smooth map preserves the velocities, thus it induces a map between tangent spaces of M and N. This is proposed as the following:

Definition 1.5. Differential of F at p

Given two manifolds M,N and a smooth map $F : M \to N$, for each $p \in M$ there is a map

$$\mathrm{d}F_p : T_pM \to T_{F(p)}N,$$

named as the *differential of F at p*, which is defined by the formula

$$\mathrm{d}F_p(v)(f) = v(f \circ F),$$

where $v \in T_pM$ and $f \in C^\infty(N)$.

Basic properties of the differential are listed below, which are easy to verify:

Proposition 1.4. Properties of Differentials

Let M, N and P be manifolds and $F : M \to N$, $G : N \to P$ be smooth maps. Let $p \in M$, then

(a) $\mathrm{d}F_p : T_pM \to T_{F(p)}N$ *is linear.*

(b) $\mathrm{d}(G \circ F)_p = \mathrm{d}G_{F(p)} \circ \mathrm{d}F_p : T_pM \to T_{G\circ F(p)}P$.

(c) $\mathrm{d}(1_M)_p = 1_{T_pM} : T_pM \to T_pM$.

(d) If F is a diffeomorphism, then $\mathrm{d}F_p : T_pM \to T_{F(p)}N$ *is an isomorphism and* $(\mathrm{d}F_p)^{-1} = \mathrm{d}(F^{-1})_{F(p)}$.

Before looking deeper into the tangent space, we introduce a technical but powerful tool which is a bridge for us to connect local things and global things on a manifold, the *partition of unity*.

[3] Using this fact to derive properties of derivation is in fact a circular argument, but this shall not stop us from using it to build our intuition. One can prove these formally without this fact after reading about the smooth bump function (proposition 1.6).

Theorem 1.5. Partition of Unity

Let M be a manifold and $\{U_\alpha\}_\alpha$ an arbitrary open cover of M, then there exists a smooth partition of unity subordinate to $\{U_\alpha\}_\alpha$, which is a family of smooth maps $\{\rho_\alpha : M \to \mathbb{R}\}_\alpha$ such that

(i) $0 \le \rho_\alpha(x) \le 1$ *for all* α *and* $x \in M$.

(ii) $\operatorname{supp} \rho_\alpha \subset U_\alpha$ *for each* α.

(iii) $\{\operatorname{supp} \rho_\alpha\}_\alpha$ *is locally finite, which means that every point in M has a neighborhood that intersects only finitely many elements in $\{\operatorname{supp} \rho_\alpha\}_\alpha$. In particular, for every point $x \in M$ there is only finitely many α such that $\rho_\alpha(x) > 0$.*

(iv) $\sum_\alpha \rho_\alpha(x) = 1$ *for all* $x \in M$.

Proof. See Chapter 2, Partition of Unity, of Lee's *Introduction to Smooth Manifolds* if you are curious. □

A base to the proof is that for an open ball B in $\mathbb{R}^n$ and a closed ball B' contained in that open ball, there exists by an explicit formula a *smooth bump function* $f : \mathbb{R}^n \to \mathbb{R}$ such that $f \equiv 1$ on B', $0 < f(x) < 1$ for $x \in B \setminus B'$ and $f \equiv 0$ outside B. Shrink the open ball B for a little bit and we see that we can demand f to satisfy $\operatorname{supp} f \subset B$. Pre-composing a parametrization $\varphi : U \to \mathbb{R}^n$ of a manifold M with f and we have a smooth function $\tilde{f} = f \circ \varphi : U \to \mathbb{R}$ with $\operatorname{supp} \tilde{f}$ contained in U and $\tilde{f} \equiv 1$ on $\varphi^{-1}(B')$ a closed subset of U, and we can extend it by zero to get a global smooth function on M.

For an arbitrary closed subset $A \subset M$ and an arbitrary open subset U containing A, there might not be a parametrization of U, but we can still have a *smooth bump function for A supported in U*, i.e. a smooth function ψ on M with $\operatorname{supp} \psi \subset U$ and $\psi \equiv 1$ on A, using the power of partition of unity: let the open cover of M be $\{U, M \setminus A\}$, and we have a partition of unity subordinate to this open cover. The smooth map supported in U in this partition of unity is exactly the desired bump, since the other map vanishes on A. We list this result below:

Proposition 1.6. Existence of Smooth Bump Function

Let M be a manifold. For any closed subset $A \subset M$ and any open subset U containing A, there exists a smooth bump function for A supported in U, i.e. a smooth function ψ on M with $\operatorname{supp} \psi \subset U$ and $\psi \equiv 1$ on A.

For a closed subset $A \subset M$, we say that a map $F : A \to N$ is smooth on A if it has a smooth extension in a neighborhood of A, i.e. if it is a restriction of a smooth map $\tilde{F} : U \to N$ to A where $U \subset M$ is open with $A \subset U$. With $N = \mathbb{R}^n$, multiplying the smooth bump function for A supported in U gives an extension of F to global M, so we have the following lemma:

Lemma 1.7. Extension Lemma for Smooth Functions

Suppose M is a smooth manifold, $A \subset M$ is closed and $f : A \to \mathbb{R}^n$ is a smooth function. For any open subset U containing A, there exists a smooth function $\tilde{f} : M \to \mathbb{R}^k$ such that $\tilde{f}|_A = f$ and $\operatorname{supp} \tilde{f} \subset U$.

With this extension lemma, using the local nature of derivation (lemma 1.3(c)), we can establish the local nature of tangent spaces, as the following states:

Proposition 1.8. The Tangent Space to an Open Submanifold

Let M be a manifold and $U \subset M$ be an open subset with inclusion $\iota : U \hookrightarrow M$. For every $p \in U$, the differential $d\iota_p : T_pU \to T_pM$ is an isomorphism.

Proof. If one admits the fact that derivation is equivalent to velocity of curves, then this is so easy to see. Or one can do this formally using the local nature of derivation, as mentioned above. See Proposition 3.9 of [Lee13] if you just want to read the proof. □

With a direct but solid analysis on $\mathbb{R}^n$, one can see that the tangent space of $\mathbb{R}^n$ at any point $a \in \mathbb{R}^n$ is n dimensional with basis the derivation along coordinates (or, say, the velocity of the coordinate curves), i.e.

$$\left.\frac{\partial}{\partial x^{(1)}}\right|_a, \cdots, \left.\frac{\partial}{\partial x^{(n)}}\right|_a \quad \text{defined by} \quad \left.\frac{\partial}{\partial x^{(i)}}\right|_a (f) = \frac{\partial f}{\partial x^{(i)}}(a), \tag{1.2}$$

where $x^{(i)}$ is the ith coordinate of $\mathbb{R}^n$. For the proof, see Proposition 3.2 of [Lee13]. Let U in proposition 1.8 be a *coordinate neighborhood* with parametrization $\varphi : U \to \mathbb{R}^n$, then we see that T_pM is of dimension n (the dimension of the manifold itself) with basis

$$d(\varphi^{-1})_{\varphi(p)}\left(\frac{\partial}{\partial x^{(1)}}\bigg|_a\right),\cdots,d(\varphi^{-1})_{\varphi(p)}\left(\frac{\partial}{\partial x^{(n)}}\bigg|_a\right). \tag{1.3}$$

For simplicity of notations, we usually claim that $x^{(i)}$'s are the coordinates given by the parametrization $\varphi(p) = (x^{(1)}(p), \cdots, x^{(n)}(p)) \in \mathbb{R}^n$ and write simply $\frac{\partial}{\partial x^{(i)}}\big|_a$ for $d(\varphi^{-1})_{\varphi(p)}\left(\frac{\partial}{\partial x^{(i)}}\big|_a\right)$.

Given two manifolds M and N with dimension m and n respectively, we can equip their product manifold $M \times N$ with a smooth structure given by the products of parametrizations of M and N, i.e. if $\varphi : U \to \mathbb{R}^m$ and $\psi : V \to \mathbb{R}^n$ are parametrizations of M and N, then there is a parametrization $\varphi \times \psi : U \times V \to \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$ on $M \times N$. This makes $M \times N$ a smooth manifold of dimension $m + n$. This construction of product manifolds can be generalized to product finitely many manifolds.

Let $M_1, \cdots, M_k$ be smooth manifolds, then the projections $\pi_j : M_1 \times \cdots \times M_k \to M_j$ are smooth and give a natural isomorphism on the tangent spaces:

Proposition 1.9. The Tangent Space to a Product Manifold

Let $M_1, \cdots, M_k$ be smooth manifolds, and let $\pi_j : M_1 \times \cdots \times M_k \to M_j$ be the projection onto M_j. For any point $p = (p_1, \cdots, p_k) \in M_1 \times \cdots \times M_k$, the map

$$\alpha : T_p(M_1 \times \cdots \times M_k) \to T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k$$

defined by

$$\alpha(v) = (d(\pi_1)_p(v), \cdots, d(\pi_k)_p(v))$$

is an isomorphism.

This can be proved easily by proving the surjectivity of α by constructing a left inverse using the inclusions $\iota_j : M_j \hookrightarrow M_1 \times \cdots \times M_k : m \mapsto (p_1, \cdots, m, \cdots, p_k)$, since

$$\dim T_p(M_1 \times \cdots \times M_k) = \dim M_1 \times \cdots \times M_k = \dim M_1 + \cdots + \dim M_k = \dim(T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k).$$

With proposition 1.9, if one admits the equivalence of derivation and velocity of curves, then one can do a easy "reproof" of the fact that $\dim T_pM = \dim M$ with basis the velocity of coordinate curves:

Reproof. It suffices to do this locally, so we may assume that $M = \mathbb{R}^n = \mathbb{R}_1 \times \cdots \times \mathbb{R}_n$ where $\mathbb{R}_j = \mathbb{R}$. By proposition 1.9, it suffices to show that $\dim T_a\mathbb{R} = 1$ for any $a \in \mathbb{R}$. Since $T_a\mathbb{R}$ is spanned by the velocity of curves in $\mathbb{R}$, this is trivial by the chain rule. □

As we have seen for now, the tangent space at a point of M alone is not very interesting since it is simply $\mathbb{R}^{\dim M}$. It becomes more interesting in two situations: one is when we consider about how a map $F : M \to N$ gives a linear map between tangent spaces in coordinate representations, the other is when we put the tangent spaces at each point of M all together and form a new manifold called the *tangent bundle*. The former is essential for computation (which will appear frequently in latter study), but we would not do it here. One must read Chapter 3, Computations in Coordinates, of [Lee13] for this before reading on (this is essential!). We will only talk about the tangent bundle here.

The tangent bundle of a manifold M, denoted as TM, as mentioned above, is obtained by putting the tangent spaces together:

$$TM = \bigsqcup_{p \in M} T_pM. \tag{1.4}$$

But this is not all: we will give a topology along with a smooth structure on TM, so that it becomes a (smooth) manifold. Let $\pi : TM \to M : (p, v) \mapsto p$ be the projection, this is achieved by claiming what the essential parametrizations are, i.e. we endow it with a smooth atlas, which consists of for each parametrization $\varphi : U \to \mathbb{R}^n$ of $U \subset M$, a parametrization $\tilde{\varphi}$ of $\pi^{-1}(U)$ given by

$$\tilde{\varphi}(p, v) = (\varphi(p), v_1, \cdots, v_n) \in \mathbb{R}^{2n} \tag{1.5}$$

where $p \in U$, $v \in T_pM$ and $v_1, \cdots, v_n \in \mathbb{R}$ are scalars uniquely determined by that $v = \sum_i v_i \frac{\partial}{\partial x_i}$ where x_i's are coordinates corresponding to the parametrization φ.

If one have read about the computations in coordinates in Lee, then one should be able to verify easily that the atlas given above is indeed smooth. Therefore the manifold TM is defined. Note that TM is locally diffeomorphic to $U \times \mathbb{R}^n$, but globally it is not necessarily diffeomorphic to $M \times \mathbb{R}^n$: there may exist twists when gluing the $U \times \mathbb{R}^n$'s up, for example one may consider the tangent bundle of the open Möbius band.

Definition 1.6. Tangent Bundle

The tangent bundle of a manifold M of dimension n, denoted as TM, is the manifold of dimension $2n$

$$TM = \bigsqcup_{p\in M} T_pM,$$

with smooth structure given by, for each parametrizaton $\varphi : U \to \mathbb{R}^n$ of M, a parametrization of $\pi^{-1}(U)$, where $\pi : TM \to M : (p, v) \mapsto p$ is the projection,

$$\tilde{\varphi}(p, v) = (\varphi(p), v_1, \cdots, v_n) \in \mathbb{R}^{2n},$$

where $p \in U$, $v \in T_pM$ and $v_1, \cdots, v_n \in \mathbb{R}$ are scalars uniquely determined by that $v = \sum_i v_i \frac{\partial}{\partial x_i}|_p$ where x_i's are coordinates corresponding to the parametrization φ.

Recall that a map $F : M \to N$ induces a linear map on tangent spaces. Since the tangent bundle is the tangent spaces put together, by putting together the differentials of F (at points) we obtain a map between vector bundles, the *(global) differential* $\mathrm{d}F : TM \to TN$, defined by $\mathrm{d}F(p, v) = (F(p), \mathrm{d}F_p(v))$.

With the local coordinate expression, it is easy to see that $\mathrm{d}F$ is smooth when TM and TN are regarded as manifolds. The properties in proposition 1.4 transfer to global differentials immediately:

Proposition 1.10. Properties of Global Differentials

Let M, N and P be manifolds and $F : M \to N$, $G : N \to P$ be smooth maps, then

(a) $\mathrm{d}F : TM \to TN$ *is smooth.*

(b) $\mathrm{d}(G \circ F) = \mathrm{d}G \circ \mathrm{d}F : TM \to TP$.

(c) $\mathrm{d}1_M = 1_{TM} : TM \to TM$.

(d) *If F is a diffeomorphism, then $\mathrm{d}F : TM \to TN$ is also a diffeomorphism and* $(\mathrm{d}F)^{-1} = \mathrm{d}(F^{-1})$.

With these properties, we see that there is a functor, the *tangent functor* $T : \mathrm{Man} \to \mathrm{Man}$, that sends a manifold to its tangent bundle and a smooth map to its (global) differential. In this point of view, the global differential of F is sometimes denoted as TF.

If one knows about classical differential geomtry, then one must know about differentiable vector fields on regular surfaces, i.e. the assignments to each point a tangent vector that varies smoothly in sense of parametrizations. A benefit that we make the tangent bundle a manifold is that we can now define a *(smooth) vector field* in a unified way in philosophy of differential manifold: we define vector fields as a special kind of maps between manifolds. Before this, we introduce some conventions:

Definition 1.7. Section

Let $\pi : M \to N$ be a continuous (no need to be smooth) map, a *(global) section of* π is a continuous right inverse for π, i.e. a continuous map $\sigma : N \to M$ such that $\pi \circ \sigma = 1_N$. Intuitively, this σ means that we assign continuously for each point in N an element in M up to π.
A *local section of* π is a continuous map $\sigma : U \to M$ on an open subset U of N such that $\pi \circ \sigma = 1_U$.
If π is smooth, then a *smooth section of* π is σ above demanded to be smooth. Similarly a *smooth local section of* π is defined.

Now we define the vector field as promised:

Definition 1.8. Vector Field

A (global) vector field on a manifold M is a section of the projection map $\pi : TM \to M$. More conceretely, a vector field is a continuous map $X : M \to TM$, usually written $p \mapsto X_p$, such that

$$\pi \circ X = 1_M,$$

or equivalently, $X_p \in T_pM$ for each $p \in M$.
A smooth vector field is a smooth section of the projection π.

For our purpose, we usually ommit the word "smooth" but regard everything to be smooth by default. The vector field we just defined is in fact vector field of tangent vector bundles; we would see what a general vector field is once we meet the definition of a general vector bundle in [BT82].

Given a manifold M and a parametrization $\varphi : U \to \mathbb{R}^n$ of a coordinate neighborhood U of M. Let (x_i) be the coordinates corresponding to φ, then for each $p \in U$, the derivation along coordinates, $\frac{\partial}{\partial x_i}\big|_p$, form a basis of the tangent space T_pM. Let $X : M \to TM$ be a vector field, then its evaluation $X_p \in T_pM$ at each $p \in U$ has unique decomposition up to this basis, say write

$$X_p = \sum_i X_i(p) \frac{\partial}{\partial x_i}\bigg|_p. \tag{1.6}$$

As p varies in U, this decomposition defines n functions $X_i : U \to \mathbb{R}$, called the *component functions of* X in the given chart (or parametrization). Using the convention of *Einstein summation*, the equation above may be rewritten as

$$X_p = X_i(p) \frac{\partial}{\partial x_i}\bigg|_p, \tag{1.7}$$

so we don't need to write the summation symbol again and again.

Using the component functions, we can tell more concretely what a smooth vector field looks like. Recall that the parametrization $\varphi : U \to \mathbb{R}^n$ of M induces a parametrization $\tilde{\varphi} : \pi^{-1}(U) \to \mathbb{R}^{2n}$ of M which maps (p, v) to $(\varphi(p), v_1, \cdots, v_n)$ where the v_i's are given by the decomposition $v = v_i \frac{\partial}{\partial x_i}\big|_p$. Thus by restricting the smooth vector field X to U and post-composing it with $\tilde{\varphi}$, we obtain a smooth function

$$\begin{aligned} \tilde{\varphi} \circ X|_U : U &\longrightarrow \mathbb{R}^{2n} \\ p &\longmapsto (\varphi(p), X_1(p), \cdots, X_n(p)) \end{aligned}$$

It follows that a vector field is smooth if its component functions (with respect to all parametrizations, or a family of parametrizations wich covers M) are smooth. Since $\tilde{p}$ is a parametrization, the converse is also true, thus we obtain a criterion for the smoothness of vector field:

Proposition 1.11. Smoothness Criterion for Vector Fields

Let M be a smooth manifold and $X : M \to TM$ be a rough vector field, i.e. a maybe-non-continous section to $\pi : TM \to M$. Then X is smooth if and only if its component functions with respect to a family of parametrizations which covers M are smooth.

This gives a very good (and useful) characterization of (smooth) vector fields. If one knows about classical differential geometry, then one would find that this is how the generalization of vector fields on manifold be compatible with the classical definition.

Now we are ready to talk about the differential forms, the main object dealt in [BT82]. In short, a *differential k-form*, or just a k-form, is a (continuous) section to $\pi : \Lambda^k T^*M \to M$ where $\Lambda^k T^*M$ is the manifold of *alternating* k-times tensoring of the dual of the tangent bundle. The set of all smooth k-forms is denoted as $\Omega^k(M)$. This is, somehow, too short. Let us explain these one by one.

Let M be a manifold, for each $p \in M$ we consider the *cotangent space* T_p^*M, the dual vector space of the tangent space T_pM; the elements in T_p^*M are called *(tangent) covectors at p*. Let $\varphi : U \to \mathbb{R}^n$ be a parametrization of M with $p \in U$ and (x_i) be the coordinates given by φ. Since $\frac{\partial}{\partial x_i}|_p$ gives a basis of T_pM, its dual basis, denoted as $\mathrm{d}x_i|_p$, which is defined by

$$\mathrm{d}x_i|_p \left(\frac{\partial}{\partial x_j}|_p \right) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

is a baiss of T_p^*M. By unioning T_p^*M up for all $p \in M$ and giving essential parametrizations, we form the dual of the tangent bundle, the *cotangent bundle*:

Definition 1.9. Cotangent Bundle

The cotangent bundle of a manifold M of dimension n, denoted as T^*M, is the manifold of dimension $2n$,

$$T^*M = \bigsqcup_{p \in M} T_p^*M,$$

with smooth structure given by, for each parametrization $\varphi : U \to \mathbb{R}^n$ of M, a parametrization of $\pi^{-1}(U)$ where $\pi : T^*M \to M : (p, \alpha) \mapsto p$ is the projection,

$$\tilde{\varphi}(p, \alpha) = (\varphi(p), a_1, \cdots, a_n) \in \mathbb{R}^{2n},$$

where $p \in U$, $\alpha \in T_p^*M$ and $a_1, \cdots, a_n \in \mathbb{R}$ are scalars uniquely determined by that $\alpha = a_i\, \mathrm{d}x_i|_p$ where (x_i) are coordinates given by φ.

Of course we need to check that given two parametrizations of M, $\varphi : U \to \mathbb{R}^n$ and $\psi : V \to \mathbb{R}^n$, then the composition $\tilde{\psi} \circ \tilde{\varphi}^{-1} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is smooth. Let (x_i) and (y_i) be the coordinates given by φ and ψ respectively, then it suffices to check that the coefficients f_{ij} in the change of basis $\mathrm{d}x_i|_p = f_{ij}(p)\,\mathrm{d}y_j|_p$ varies smoothly with respect to $p \in U \cap V$. By definition of the dual basis, substitute $\frac{\partial}{\partial y_j}|_p$ to both sides and we see that $f_{ij}(p) = \mathrm{d}x_i|_p(\frac{\partial}{\partial y_j}|_p)$, so by the change of coordinates of tangent vectors we have $f_{ij}(p) = \frac{\partial x_i}{\partial y_j}(p)$, the evaluation at p of the partial derivative of the i-th term in codomain of $\varphi \circ \psi^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ to the j-th term in its domain, which is clearly smooth in p.

Let $F : M \to N$ be a smooth map, then the dual map of $\mathrm{d}F : TM \to TN$ gives a map between cotangent bundles, $\mathrm{d}F^* : T^*N \to T^*M$, the *pullback by F*:

Definition 1.10. Pullback of Covectors by F

Let $F : M \to N$ be a map between manifolds, there is a map $\mathrm{d}F^* : T^*N \to T^*M$, called the pullback by F, defined by

$$\mathrm{d}F_p^*(\omega)(v) = \omega(\mathrm{d}F_p(v)),$$

for any $p \in M$, $\omega \in T^*_{F(p)}N$ and $v \in T_pM$.

Similar properties as for the differentials can be derived for the pullbacks:

Proposition 1.12. Properties of Pullbacks of Covectors

Let M, N and P be manifolds and $F : M \to N$, $G : N \to P$ be smooth maps, then

(a) $\mathrm{d}F^* : T^*N \to T^*M$ *is smooth.*

(b) $\mathrm{d}(G \circ F)^* = \mathrm{d}F^* \circ \mathrm{d}G^* : T^*P \to T^*M$.

(c) $\mathrm{d}1_M = 1_{T^*M} : T^*M \to T^*M$.

(d) If F is a diffeomorphism, then $\mathrm{d}F : T^*N \to T^*M$ *is also a diffeomorphism and* $(\mathrm{d}F^*)^{-1} = \mathrm{d}(F^{-1})^*$.

Therefore there is a contravariant functor, the *cotangent functor* $T^* : \mathrm{Man} \to \mathrm{Man}$, that sends a manifold to its cotangent bundle and a smooth map to its pullback mapping. In this point of view, the pullback by F is sometimes denoted as F^*, consistent to the notation for pre-composition.

Now we are going to tensor the cotangent spaces up for our purpose. An advantage that we tensor the dual space of the tangent space instead of tensor directly the tangent space is that we have a better description to the tensor of the dual. Let $T^kT_p^*M$ denote the tensor product of k copies of T_p^*M, it is naturally isomorphic to the set of k-linear maps from the product of k copies of T_pM to $\mathbb{R}$. So instead of dealing with abstract tensors, we are dealing with the concrete k-linear mappings. The word "k-linear" means that the map $\alpha \in T^kT_p^*M$ is linear in k variables, say

$$\alpha(a_1v_1, \cdots, a_kv_k + u) = a_1 \cdots a_{k-1}\alpha(v_1, \cdots, v_{k-1}, u) + a_1 \cdots a_k\alpha(v_1, \cdots, v_k),$$

where $a_1, \cdots, a_k \in \mathbb{R}$ and $v_1, \cdots, v_k, u \in T_pM$. Elements in $T^kT_p^*M$ are named *k-multicovectors (at p of M)*.

Among the maps in $T^kT_p^*M$, there is a special kind of mappings we are concerned about, the *alternating* (or *skew-symmetric*) ones. An element α in $T^kT_p^*M$ is said to be alternating if swiching the order of two vectors gives a negative sign before the result of its evalutaion, i.e.

$$\alpha(v_1, \cdots, v_i, \cdots, v_j, v_k) = -\alpha(v_1, \cdots, v_j, \cdots, v_i, v_k), \tag{1.8}$$

for any vectors $v_1, \cdots, v_k \in T_pM$ whenever $i \neq j$. The set of all alternating elements in $T^kT_p^*M$ is denoted as $\Lambda^kT_p^*M$. Note that if $k \leq 1$, then $\Lambda^kT_p^*M = T^kT_p^*M$ since the alternating statement is vacuously true. For $k = 0$, the empty tensor is dual of the empty Cartesian product of T_pM which is a singleton, so

$$\Lambda^0T_p^*M = T^0T_p^*M = \{f : \{*\} \to \mathbb{R}\} = \mathbb{R}.$$

For $k = 1$, there is simply $\Lambda^1T_p^*M = T_p^*M$.

Again, union $\Lambda^kT_p^*M$ up for all $p \in M$ and we would obtain a manifold after giving essential parametrizations. But before we give the parametrizations, we need to find a natural basis for $\Lambda^kT_p^*M$ (so that the atlas we give by this basis is smooth). This is given by the wedge product:

Definition 1.11. Wedge Product

Let $\alpha_1, \cdots, \alpha_k \in T_p^*M$. Their wedge product $\alpha_1 \wedge \cdots \wedge \alpha_k$ is the alternating k-linear map in $\Lambda^k T_p^*M$ given by

$$\alpha_1 \wedge \cdots \wedge \alpha_k(v_1, \cdots, v_k) = \det \begin{pmatrix} \alpha_1(v_1) & \cdots & \alpha_1(v_k) \\ \vdots & \ddots & \vdots \\ \alpha_k(v_1) & \cdots & \alpha_k(v_k) \end{pmatrix}.$$

Note that if we have $\alpha_1 \wedge \cdots \wedge \alpha_i \in \Lambda^i T_p^*M$ and $\beta_1 \wedge \cdots \wedge \beta_j \in \Lambda^j T_p^*M$, then we can "wedge them up" and get an element

$$\alpha_1 \wedge \cdots \wedge \alpha_i \wedge \beta_1 \wedge \cdots \wedge \beta_j$$

in $\Lambda^{i+j} T_p^*M$, just as the equation above gives. By the properties of determinant we see that the wedge product is linear in each term, i.e.

$$\alpha_1 \wedge \cdots \wedge (\lambda\alpha_i + \beta) \wedge \cdots \wedge \alpha_k = \lambda(\alpha_1 \wedge \cdots \wedge \alpha_i \wedge \cdots \wedge \alpha_k) + \alpha_1 \wedge \cdots \wedge \beta \wedge \cdots \wedge \alpha_k. \tag{1.9}$$

We can extend this linearlity so that the wedge product becomes a bi-linear pairing, $\wedge : \Lambda^i T_p^*M \times \Lambda^j T_p^*M \to \Lambda^{i+j} T_p^*M$, as the following:

$$(\lambda\alpha_1 \wedge \cdots \wedge \alpha_i + \beta_1 \wedge \cdots \wedge \beta_i) \wedge \gamma_1 \wedge \cdots \wedge \gamma_j := \lambda(\alpha_1 \wedge \cdots \wedge \alpha_i \wedge \gamma_1 \wedge \cdots \wedge \gamma_j) + \beta_1 \wedge \cdots \wedge \beta_i \wedge \gamma_1 \wedge \cdots \wedge \gamma_j. \tag{1.10}$$

Do we need to worry about the elements in $\Lambda^i T_p^*M$ that might not be of a linear combination of wedge products of covectors, since the formula above only defines the wedge between those of the form of linear combinations of wedge products? The answer is no, because by basic linear algebra, the wedge products span the whole $\Lambda^i T_p^*M$. In fact, we have our promised natural basis:

Proposition 1.13. A Basis for $\Lambda^k T_p^*M$

If $\alpha_1, \cdots, \alpha_n$ is a basis for T_p^*M, then

$$\{\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_k} \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$$

is a basis for $\Lambda^k T_p^*M$. In particular, if (x_i) are coordinates given by a parametrization of a neighborhood of p, then

$$\{\mathrm{d}x_{i_1}|_p \wedge \cdots \wedge \mathrm{d}x_{i_k}|_p \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$$

is a basis for $\Lambda^k T_p^*M$. Therefore $\Lambda^k T_p^*M$ is of dimension $\binom{n}{k}$.

Proof. See Proposition 14.8 of [Lee13]. □

With this, we can "wedge up" any two alternating multi-covectors by the extended linearlity. Some properties of this wedging follow directly from the definition:

Proposition 1.14. Properties of the Wedge Product

Suppose ω, η, ξ are alternating multicovectors at p of M.

(a) *Bilinearity: For $a, b \in \mathbb{R}$,*

$$(a\omega + b\eta) \wedge \xi = a(\omega \wedge \xi) + b(\eta \wedge \xi),$$
$$\xi \wedge (a\omega + b\eta) = a(\xi \wedge \omega) + b(\xi \wedge \eta).$$

(b) *Associativity:*

$$\omega \wedge (\eta \wedge \xi) = (\omega \wedge \eta) \wedge \xi.$$

(c) *Anticommutativity: If $\omega \in \Lambda^k T_p^*M$ and $\eta \in \Lambda^k T_p^*M$, then*

$$\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega.$$

To simplify the notation of the basis for $\Lambda^k T_p^*M$ in proposition 1.13, we write $\mathrm{d}x^{\mathcal{J}}|_p$ for $\mathrm{d}x_{i_1}|_p \wedge \cdots \wedge \mathrm{d}x_{i_k}|_p$ where $\mathcal{J} = (i_1, \cdots, i_k)$ is an ordered k-tuple. Now we union $\Lambda^k T_p^*M$ up to give the promised manifold:

Definition 1.12. Bundle of Alternating k-multicovectors

Let $\mathcal{J}_1, \cdots, \mathcal{J}_{\binom{n}{k}}$ be a fixed enumeration of elements in $\{(i_1, \cdots, i_k) \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$. The bundle of alternating k-multicovectors of a manifold M of dimension n, denoted as $\Lambda^k T^*M$, is the manifold of dimension $n + \binom{n}{k}$,

$$\Lambda^k T^*M = \bigsqcup_{p\in M} \Lambda^k T_p^*M,$$

with smooth structure given by, for each parametrization $\varphi : U \to \mathbb{R}^n$ of M, a parametrization of $\pi^{-1}(U)$ where $\pi : \Lambda^k T^*M \to M : (p, \omega) \mapsto p$ is the projection,

$$\tilde{\varphi}(p, \omega) = (\varphi(p), a_1, \cdots, a_{\binom{n}{k}}),$$

where $p \in U$, $\omega \in \Lambda^k T_p^*M$ and $a_1, \cdots, a_{\binom{n}{k}} \in \mathbb{R}$ are scalars uniquely determined by that $\omega = a_i \,\mathrm{d}x^{\mathcal{J}_i}|_p$ where (x_i) are coordinates given by φ.

Since the coefficients in the change of basis of covectors varies smoothly as we have seen in discussion of the cotangent bundle definition 1.9 and the wedge product is multi-linear, it follows that the atlas given by the $\tilde{\varphi}$'s is indeed smooth.

Finally we can define the differential k-form:

Definition 1.13. Differential k-form

A differential k-form on M, or just a k-form on M, is a section of $\pi : \Lambda^k T^*M \to M$. For our purpose, we need only the smooth forms, i.e. the smooth sections, and we denote the set of all smooth k-forms on M by $\Omega^k(M)$.

Note that since $\Lambda^0 T_p^*(M) = \mathbb{R}$, a section of $\Lambda^0 T^*(M)$ can be seen as a real-valued function on M, so $\Omega^0(M) = C^\infty(M)$. By point-wise summation and scalar multiplication, we see that $\Omega^k(M)$ is a real linear space. Note that this linear space is usually infinite-dimensional, instead of having dimension $\dim \Lambda^k T_p^*(M) = \binom{n}{k}$.

Let $\omega : M \to \Lambda^k T^*M$ be a k-form, (x_i) be coordinates given by a parametrization φ of $U \subset M$ and $p \in U$. By proposition 1.13 we have at p

$$\omega_p = \omega_{\mathcal{J}_i}(p) \,\mathrm{d}x^{\mathcal{J}_i}|_p. \tag{1.11}$$

Let p vary in U and we get for each $\mathcal{J}_i$ a function $\omega_{\mathcal{J}_i} : U \to \mathbb{R}$, the coefficients of ω with respect to the parametrization φ. A similar smoothness criterion for k-forms to that for vector fields (proposition 1.11) can be established by applying the parametrizations of $\Lambda^k T^*M$:

Proposition 1.15. Smoothness Criterion for k-forms

Let M be a manifold and $\omega : M \to \Lambda^k T^*M$ be a rough k-form. ω is smooth if and only if its coefficients with respect to a family of parametrizations which covers M all vary smoothly in their coordinate neighborhoods.

Erasing the sign for p in eq. (1.11) and we have the convention of local coordinate representation of forms:

$$\omega|_U = \omega_{\mathcal{J}_i} \,\mathrm{d}x^{\mathcal{J}_i}, \tag{1.12}$$

with $\omega_{\mathcal{J}_i}$'s all smooth.

With the criterion, we see that $\Omega^k(M)$ can be seen as a $C^\infty(M)$-module. Also, we can wedge forms point-wisely, i.e. for $\omega \in \Omega^i(M)$ and $\eta \in \Omega^j(M)$ we put $\omega \wedge \eta \in \Omega^{i+j}(M)$ to be the form given by $(\omega \wedge \eta)_p \coloneqq \omega_p \wedge \eta_p$. This wedging thus makes $\Omega^*(M) \coloneqq \bigoplus_{k\geq 0} \Omega^k(M)$ a graded $C^\infty(M)$-algebra. In this point of view, for a k-form the integer k is called the *degree* of that form. Note that multiplying to a form an element in $C^\infty(M)$ is exactly the same as wedging it with that element seen in $\Omega^0(M) = C^\infty(M)$.

Given a map $F : M \to N$, then it induces a map between the graded algebras $\Omega^*(M)$ and $\Omega^*(N)$ by extending the definition of pullbacks (definition 1.10). For now we give firstly how it induces a map between $\Omega^k(M)$ and $\Omega^k(N)$:

Definition 1.14. Pullback of Forms by F

Let $F : M \to N$ be a map between manifolds, there is a map $F^* : \Omega^k(N) \to \Omega^k(M)$, the pullback by F, defined by

$$F^*(\omega)_p(v_1, \cdots, v_k) = \omega_{F(p)}(\mathrm{d}F_p v_1, \cdots, \mathrm{d}F_p v_k),$$

for any $\omega \in \Omega^k(N)$, $p \in M$ and $v_1, \cdots, v_k \in T_pM$.

Several properties of this pullback can be derived easily from the definition:

Lemma 1.16. Properties of Pullbacks of Forms

Let $F : M \to N$ and $G : N \to P$ be two maps.

(a) $F^* : \Omega^k(N) \to \Omega^k(M)$ *is linear over* $\mathbb{R}$.

(b) $F^*(f(\omega \wedge \eta)) = (f \circ F)(F^*\omega) \wedge (F^*\eta)$ *for any* $\omega \in \Omega^i(N)$, $\eta \in \Omega^j(N)$ *and* $f \in C^\infty(N) = \Omega^0(N)$.

(c) $(G \circ F)^* = F^* \circ G^*$.

(d) $1_M^* = 1_{\Omega^k(M)}$.

(e) *If F is a diffeomorphism, then* $F^* : \Omega^k(N) \to \Omega^k(M)$ *is an isomorphism and* $(F^*)^{-1} = (F^{-1})^*$.

Since the pullback preserves the degree of forms and is commutative with wedging, by taking the direct sum we get a homomorphism of graded $\mathbb{R}$-algebras,

$$F^* \coloneqq \bigoplus_{k \geq 0} F^* : \Omega^*(N) \to \Omega^*(M).$$

It follows that Ω^* can be viewed as a contravariant functor from Man to the category of graded rings.

Now we are ready to talk about the de Rham theory on Euclidean spaces.

The de Rham Theory on Euclidean Spaces

For $\mathbb{R}^n$, it is easy to describe the algebra $\Omega^*(\mathbb{R}^n)$: we have a global coordinate system (x_i) given by the identity parametrization $1_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$, so the coordinate representation in eq. (1.12) can be taken globally on $\mathbb{R}^n$. Hence any element ω in $\Omega^k(\mathbb{R}^n)$ can be written uniquely as

$$\omega = \sum_i f_{\mathcal{J}_i} \, \mathrm{d}x^{\mathcal{J}_i}. \tag{1.13}$$

If one knows about the exterior algebra, then it is easy to see that $\Omega^*(\mathbb{R}^n)$ is isomorphic to the tensor $C^\infty(\mathbb{R}^n) \otimes_{\mathbb{R}} \Lambda_{\mathbb{R}}[\mathrm{d}x_1, \cdots, \mathrm{d}x_n]$.

To establish a cohomology theory, we need to decide a chain complex to work with. For de Rham theory, the complex consists of the smooth forms, i.e. it is of the form

$$0 \to \Omega^0(M) \to \Omega^1(M) \to \cdots \to \Omega^{\dim M}(M) \to \Omega^{\dim M+1}(M) = 0 \to 0 \to \cdots, \tag{1.14}$$

where the sets of forms $\Omega^k(M)$ are considered to be real-linear spaces.

Of course we need to know what the maps $\Omega^k(M) \to \Omega^{k+1}(M)$ are. These maps are called *differential maps* in convention of cohomology theory. For our situation, they are denoted by d and named *exterior differentiation*. Since $\Omega^k(M)$'s are regarded as linear spaces, these d should be linear. To define them, we consider firstly $M = \mathbb{R}^n$ on which the standard coordinates $x_1, \cdots, x_n$ are give; it turns out that our definition would not depend on the choice of coordinates, thus will glue up to give maps $\Omega^k(M) \to \Omega^{k+1}(M)$ whatever the manifold M is. A main philosophy of [BT82] is just like the way we define this d: we first do the discussion locally, in which we need only consider the case $M = \mathbb{R}^n$. After that we see how the results can pass from local to global, giving results on a general manifold (usually with some restrict conditions that help us to do this passing, though).

Now we define $\mathrm{d} : \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$. We want our map be linear, so it suffices to define it on the basis elements, which are of the form $f \, \mathrm{d}x^{\mathcal{J}} = f \, \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$ where $f \in C^\infty(\mathbb{R}^n)$ (recall eq. (1.12)). The defining formula is simply

$$\mathrm{d}(f \, \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}) \coloneqq \sum_{j=1}^n \frac{\partial f}{\partial x_j} \mathrm{d}x_j \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}. \tag{1.15}$$

Recall the formal formula in calculus $\mathrm{d}f = \sum_j \frac{\partial f}{\partial x_j} \mathrm{d}x_j$, eq. (1.15) is nothing but

$$\mathrm{d}(f \, \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}) \coloneqq \mathrm{d}f \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}. \tag{1.16}$$

Since for any $r \in \mathbb{R}$, $\frac{\partial (rf)}{\partial x_j} = r \frac{\partial f}{\partial x_j}$, the claim of d to be linear leads to no illness. Therefore d for $M = \mathbb{R}^n$ is defined.

For this d to make our chain eq. (1.14) a chain complex, we need to verify that $d^2 = 0$. we have by definition

$$\mathrm{d}^2(f \, \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}) = \mathrm{d}\left(\sum_{j=1}^n \frac{\partial f}{\partial x_j} \mathrm{d}x_j \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}\right) = \sum_{j,l} \frac{\partial^2 f}{\partial x_j \partial x_l} \mathrm{d}x_l \wedge \mathrm{d}x_j \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}. \tag{1.17}$$

Since $\frac{\partial^2 f}{\partial x_j \partial x_l} = \frac{\partial^2 f}{\partial x_l \partial x_j}$ while $\mathrm{d}x_l \wedge \mathrm{d}x_j = -\mathrm{d}x_j \wedge \mathrm{d}x_l$, as j, l varies in the summation, all terms cancels out so the result is zero, concluding that $d^2 = 0$.

For a cohomology theory we want a map $F : \mathbb{R}^n \to \mathbb{R}^m$ induces a chain map, i.e. maps between $\Omega^k(\mathbb{R}^n)$ and $\Omega^k(\mathbb{R}^m)$ for each k that commute with the differential d. For de Rham theory, the pullback of F as we defined in definition 1.14, $F^* : \Omega^k(\mathbb{R}^m) \to \Omega^k(\mathbb{R}^n)$, is readily a chain map. To show this we need the following property of d:

Proposition 1.17. Antiderivation

Let $\omega \in \Omega^k(\mathbb{R}^n)$ and $\eta \in \Omega^l(\mathbb{R}^n)$, then

$$\mathrm{d}(\omega \wedge \eta) = (\mathrm{d}\omega) \wedge \eta + (-1)^k \omega \wedge (\mathrm{d}\eta).$$

Proof. The proof of Proposition 14.23(b) in [Lee13] is good enough for reading. □

Using this, noticing that for any $f \in C^\infty(\mathbb{R}^m)$,

$$\begin{aligned} \mathrm{d}(f \circ F) &= \frac{\partial (f \circ F)}{\partial x_i} \mathrm{d}x_i \\ &= \left(\frac{\partial f}{\partial y_j} \circ F \right) \frac{\partial F_j}{\partial x_i} \mathrm{d}x_i = \left(\frac{\partial f}{\partial y_j} \circ F \right) F^* \mathrm{d}y_j = F^* \mathrm{d}f \end{aligned}$$

by the chain rule, we show that F^* is indeed a chain map:

$$F^*\Big(\mathrm{d}\left(f \,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}\right)\Big) = F^* \mathrm{d}f \wedge F^* \mathrm{d}x_{i_1} \wedge \cdots \wedge F^* \mathrm{d}x_{i_k}, \tag{1.18}$$

$$\begin{aligned} \mathrm{d}\Big(F^*\left(f \,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}\right)\Big) &= \mathrm{d}\Big((f \circ F) \wedge F^* \mathrm{d}x_{i_1} \wedge \cdots \wedge F^* \mathrm{d}x_{i_k}\Big) \\ &\overset{\text{proposition 1.17}}{=\!=\!=} \mathrm{d}(f \circ F) \wedge F^* \mathrm{d}x_{i_1} \wedge \cdots \wedge F^* \mathrm{d}x_{i_k} \\ &\overset{\mathrm{d}(f \circ F) = F^* \mathrm{d}f}{=\!=\!=} F^* \mathrm{d}f \wedge F^* \mathrm{d}x_{i_1} \wedge \cdots \wedge F^* \mathrm{d}x_{i_k}. \end{aligned} \tag{1.19}$$

Therefore $F^* \mathrm{d} = \mathrm{d}F^*$ as desired.

Remark 1.1. Exterior Differentiation on General Manifolds

Before we talk about the de Rham theory on $\mathbb{R}^n$, we pause here for a second to see how this d for $\mathbb{R}^n$ passes to give a definition of d for any manifold M. Given an open subset U of M, the pullback of the inclusion $U \hookrightarrow M$ gives an isomorphism $\Lambda^k T_p^* M \cong \Lambda^k T_p^* U$ for any $p \in U$ (as a consequence of that $T_p U \cong T_p M$). Union these isomorphisms up and we obtain a smooth embedding $\Lambda^k T^* U \hookrightarrow \Lambda^k T^* M$. Now a form ω on U, which is a section $\omega : U \to \Lambda^k T^* U$, so post-composition by the embedding $\Lambda^k T^* U \hookrightarrow \Lambda^k T^* M$ makes $\omega : U \to \Lambda^k T^* U \hookrightarrow \Lambda^k T^* M$ a local section of $\pi : \Lambda^k T^* M \to M$. In this point of view, for any open covering $\{U_\alpha\}$ of M and a family of k-forms $\{\omega_\alpha \in \Omega^k(U_\alpha)\}$ seeing as local sections $\omega_\alpha : U_\alpha \to \Lambda^k T^* M$, if the restrictions of the forms to each overlap agrees, say $\omega_\alpha|_{U_\alpha \cap U_\beta} = \omega_\beta|_{U_\alpha \cap U_\beta}$ for any α, β such that $U_\alpha \cap U_\beta \neq \varnothing$, then by the gluing lemma (lemma 1.2) they glue up to give a global section $\omega : M \to \Lambda^k T^* M$ in $\Omega^k(M)$. Therefore if we want to determine the map $\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$, we can determine it "piece by piece" and then check if they agree on overlaps.

Let M be covered by a family of coordinate neighborhoods $\{U_\alpha\}$ with corresponding parametrizations $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$. Let $i_\alpha : U_\alpha \hookrightarrow M$ be the inclusions, the idea of the piece-wise definition of d is displayed in the diagram below:

$$\begin{tikzcd} \Omega^k(M) \arrow[rrr, dashed] \arrow[d, "i_\alpha^*"'] & & & \Omega^{k+1}(M) \arrow[d, "i_\alpha^*"] \\ \Omega^k(U_\alpha) & \Omega^k(\mathbb{R}^n) \arrow[l, "\varphi_\alpha^*"', "\cong"] \arrow[r, "\mathrm{d}"] & \Omega^{k+1}(\mathbb{R}^n) \arrow[r, "\varphi_\alpha^*", "\cong"'] & \Omega^{k+1}(U_\alpha) \end{tikzcd}$$

To express this by words, given a form $\omega \in \Omega^k(M)$, we restrict it to U_α for each α and apply the bottom line in the diagram to get a form $\varphi_\alpha^* \mathrm{d}(\varphi_\alpha^*)^{-1}(\omega|_{U_\alpha}) \in \Omega^{k+1}(U_\alpha)$. As α varies, we get a family of forms $\{\varphi_\alpha^* \mathrm{d}(\varphi_\alpha^*)^{-1}(\omega|_{U_\alpha})\}$ sub-coordinate to the covering $\{U_\alpha\}$. If they agree on overlaps, then they glue up to give a form $\Omega^{k+1}(M)$ and defines our map $\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$. The linearity of d follows from its linearity locally in each U_α (since the linear structure is defined point-wisely).

Now it remains only to check that $\{\varphi_\alpha^* \mathrm{d}(\varphi_\alpha^*)^{-1}(\omega|_{U_\alpha})\}$ agree on overlaps. Let $U_\alpha \cap U_\beta \neq \varnothing$, we need to show that $\varphi_\alpha^* \mathrm{d}(\varphi_\alpha^*)^{-1} = \varphi_\beta^* \mathrm{d}(\varphi_\beta^*)^{-1}$, which is equivalent to $(\varphi_\alpha \circ \varphi_\beta^{-1})^* \mathrm{d}(\varphi_\alpha^*)^{-1} = \mathrm{d}(\varphi_\beta^*)^{-1}$. Since

$\varphi_\alpha \circ \varphi_\beta^{-1}$ is a map between Euclidean spaces, $(\varphi_\alpha \circ \varphi_\beta^{-1})^*$ commutes with d and thus

$$(\varphi_\alpha \circ \varphi_\beta^{-1})^* \,\mathrm{d}(\varphi_\alpha^*)^{-1} = \mathrm{d}(\varphi_\alpha \circ \varphi_\beta^{-1})^*(\varphi_\alpha^*)^{-1} = \mathrm{d}(\varphi_\beta^*)^{-1}, \tag{1.20}$$

as promised. The properties of d for $\mathbb{R}^n$ also pass to general manifolds since d is determined "locally".

Proposition 1.18. Properties of Exterior Differentiation

Let M be a smooth manifold. The exterior differentiation $\mathrm{d} : \Omega^k(M) \to \Omega^k(M)$ satisfies the following properties:

(a) d *is $\mathbb{R}$-linear.*

(b) (Antiderivation) Let $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$, then

$$\mathrm{d}(\omega \wedge \eta) = (\mathrm{d}\omega) \wedge \eta + (-1)^k \omega \wedge (\mathrm{d}\eta).$$

(c) $d^2 := d \circ d \equiv 0$.

(d) (Naturality) d *commutes with pullbacks, i.e. given $F : M \to N$, then*

$$F^* \,\mathrm{d} = \mathrm{d}F^*,$$

whenever this composition makes sense.

The naturality of exterior differentiation can be translated as being a natural transform from the functor Ω^k and Ω^{k+1}:

$$\begin{tikzcd}
\Omega^k(M) \arrow[r, "\mathrm{d}"] \arrow[d, "F^*"'] & \Omega^{k+1}(M) \arrow[d, "F^*"] \\
\Omega^k(N) \arrow[r, "\mathrm{d}"'] & \Omega^{k+1}(N)
\end{tikzcd}$$

It follows that the pullback of forms gives a chain map between chain complexes $\Omega^*(M)$ and $\Omega^*(N)$. Therefore we can define a functor $\Omega^* : \mathrm{Man} \to K(\mathrm{Vect}_\mathbb{R})$ that sends a manifold M to the chain complex $\Omega^*(M)$ and a map F to its induced chain map between chain complexes.

The cohomology of a chain complex C^*

$$0 \xrightarrow{\partial_{-1}} C^0 \xrightarrow{\partial_0} C^1 \to \cdots \to C^n \xrightarrow{\partial_n} C^{n+1} \xrightarrow{\partial_{n+1}} \cdots \tag{1.21}$$

is defined to be the quotients $H^k(C^*) := \frac{\ker \partial_k}{\operatorname{Im} \partial_{k-1}}$. Thereby the definition of our de Rham cohomology is

Definition 1.15. de Rham Cohomology of $\mathbb{R}^n$

Forms in the kernel of the exterior differentiation d are said to be *closed.* Forms in the image of d are said to be *exact.* Since d is linear over $\mathbb{R}$, the closed k-forms and exact k-forms form two subspaces of $\Omega^k(\mathbb{R}^n)$ for each k. The *k-th de Rham cohomology of $\mathbb{R}^n$* (or the *de Rham cohomology of $\mathbb{R}^n$ at degree or dimension k*) is defined as

$$H^k_{DR}(\mathbb{R}^n) := \frac{\text{the space of closed } k\text{-forms}}{\text{the space of exact } k\text{-forms}}.$$

Usually we omit the subscript DR and write simply H^k for the de Rham cohomology, if the context is clear. Note that since $d^2 = 0$, the subspaces spanned by exact forms are contained by those spanned by closed forms.

Remark 1.2. de Rham Cohomology of M

If one has read through remark 1.1, then one is readily to define the de Rham cohomology of a general manifold M – simply replace $\mathbb{R}^n$ with M in definition 1.15.

Remark 1.3. Diffeomorphism Invariant

Under the chain map condition that $\mathrm{d}F^* = F^*\,\mathrm{d}$, we see that $F^* : \Omega^k(N) \to \Omega^k(M)$ sends closed forms to closed forms and exact forms to exact ones. Therefore the pullback induces a well-defined homomorphism of cohomologies,

$$\begin{aligned} F^*\colon H^k(N) &\longrightarrow H^k(M) \\ [\omega] &\longmapsto [F^*\omega] \end{aligned}$$

If F^* is a diffeomorphism, it is easy to see that the induced map of its inverse F^{-1} between cohomologies is the inverse of $F^* : H^k(N) \to H^k(M)$, since $(F^{-1})^*([F^*\omega]) = [(F^{-1})^*F^*\omega] = [\omega]$ (lemma 1.16(e)). Therefore diffeomorphism induces natural isomorphism between cohomologies, so two diffeomorphic manifolds have the same cohomology.
Aside: The word "same" means in fact not only $H^k(M) \cong H^k(N)$ for each k, but also $H^*(M) \cong H^*(N)$ where $H^*(M) := \oplus_k H^k(M)$ is made to be a ring by the "wedging" $[\omega] \wedge [\eta] := [\omega \wedge \eta]$. It is easy to see that the wedging is well-defined by the antiderivation of d. We will not consider the cohomology as a ring here in this section, though.

Now we see some basic examples:

Example 1.2

(i) $\mathbb{R}^0$ is a single point, thus $\Omega^0(\mathbb{R}^0) = \mathbb{R}$ and $\Omega^k(\mathbb{R}^0) = 0$ for $k > 0$. Immediately,

$$H^k(\mathbb{R}^0) = \begin{cases} \mathbb{R} & k = 0, \\ 0 & k > 0. \end{cases}$$

(ii) We shall see that for any n,

$$H^k(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k = 0, \\ 0 & k > 0, \end{cases}$$

once we prove the Poincaré lemma. For now, a direct computation can tell this for the case $n = 1$, see example 1.5(b) of [BT82].

Remark 1.4

For any connected manifold M, closed forms f in $\Omega^0(M) = C^\infty(M)$ are exactly the constant functions on M, being constant in each coordinate neighborhood U of M by basic calculus ($\frac{\partial f}{\partial x_i} = 0$ for all i). It follows that $H^0(M) = \mathbb{R}$ for any connected manifold M, in particular for $M = \mathbb{R}^n$ as in the preceding example.

Remark 1.5

Let M and N be two manifolds of dimension n, then we can form a new manifold $M \sqcup N$ by union M and N disjointly (and claim that M and N are both open as subsets of $M \sqcup N$). In view of the first paragraph of remark 1.1, we have $\Omega^k(M \sqcup N) = \Omega^k(M) \oplus \Omega^k(N)$ via a natural identification. It follows that $H^k(M \sqcup N) = H^k(M) \oplus H^k(N)$, since finite direct sums of vector spaces commute with quotients.

Compact supports

For de Rham cohomology, we can consider the chain complex formed by not all but a special kind of forms, for instance the forms with compact supports. Notice that a form $\omega \in \Omega^k(\mathbb{R}^n)$ is a function on $\mathbb{R}^n$ whose value at each point $p \in \mathbb{R}^n$ is ω_p in a linear space $\Lambda^k T_p^*\mathbb{R}^n$, we can tell if there is $\omega_p = 0$ or not. The *support* of the form ω is thus defined as $\operatorname{supp}\omega := \overline{\{p \in \mathbb{R}^n \mid \omega_p \neq 0\}}$. The set of compactly supported k-forms on $\mathbb{R}^n$ is denoted as $\Omega_c^k(\mathbb{R}^n)$. Since $\operatorname{supp} d\omega \subset \operatorname{supp}\omega$, the exterior differentiation restricts to be a map $d : \Omega_c^k(\mathbb{R}^n) \to \Omega_c^{k+1}(\mathbb{R}^n)$, thereby we obtain a chain complex

$$0 \to \Omega_c^0(\mathbb{R}^n) \to \Omega_c^1(\mathbb{R}^n) \to \cdots \to \Omega_c^n(\mathbb{R}^n) \to 0 \to \cdots . \tag{1.22}$$

Definition 1.16. de Rham Cohomology with Compact Supports of $\mathbb{R}^n$

The *de Rham cohomology with compact supports of* $\mathbb{R}^n$ is defined as the cohomology of the above chain complex (eq. (1.22)), and is denoted as $H_c^k(\mathbb{R}^n)$.

Remark 1.6. de Rham Cohomology with Compact Supports of M

If one has read through remark 1.1, then one would see that replacing $\mathbb{R}^n$ with M in definition 1.16 gives the de Rham cohomology with compact supports of a general manifold M.

Remark 1.7. Diffeomorphism Invariant

One should note that the pullback of forms does not essentially preserve compact forms, for instance let $\pi : \mathbb{R} \to \mathbb{R}^0 = \{*\}$ be the constant projection, every (nonzero) real-valued function f on $\mathbb{R}^0$ which are 0-forms is compactly supported, but the pullback $\pi^* f = f \circ \pi$ is a nonzero constant function on $\mathbb{R}$ whose support is the whole non-compact $\mathbb{R}$. Therefore there needs not be a chain map induced by a general map $F : M \to N$. However, for special kinds of maps we can still obtain a chain map, for instance the *proper* maps, the maps whose preimages of compact sets are still compact. Further discussion of this will be taken in discussion of the Mayer-Vietoris sequence.
Does the preceding paragraph imply that the compact cohomology may not be a diffeomorphism invariant? The answer is no. Since a diffeomorphism is automatically proper, it does induce a chain map between the complex of compact supported forms, and the arguments in remark 1.3 apply, telling that it induces a natural isomorphism between compact cohomologies.

Again we see some basic examples:

Example 1.3

(i) Since $\mathbb{R}^0$ is a single point, $\Omega_c^0(\mathbb{R}^0) = \mathbb{R}$ and $\Omega_c^k(\mathbb{R}^0) = 0$ for $k > 0$. Immediately,

$$H_c^k(\mathbb{R}^0) = \begin{cases} \mathbb{R} & k = 0, \\ 0 & k > 0. \end{cases}$$

(ii) We shall see that for any n,

$$H_c^k(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k = n, \\ 0 & k \neq n, \end{cases}$$

once we prove the Poincaré lemma for compact cohomology. Again, a direct computation can tell this for the case $n = 1$ by now, see example 16(b) of [BT82].

Remark 1.8

For a compact connected manifold M, $\Omega_c^k(M) = \Omega^k(M)$ for all k, so we have $H_c^0(M) = H^0(M) = \mathbb{R}$ by remark 1.4. For a non-compact connected manifold M, the only compactly supported constant function on M is the zero mapping, thus $H_c^0(M) = 0$.

We now give a solution to exercise 1.7 of [BT82], as an end to this section:

Exercise 1.1

Compute $H^*_{DR}(\mathbb{R}^2 - P - Q)$ where P and Q are two points in $\mathbb{R}^2$. Find the closed forms that represent the cohomology classes.

Solution. Clearly $H^0_{DR}(\mathbb{R}^2 - P - Q) = \mathbb{R}$. Note that moving P and Q around is an diffeomorphism, we may assume that $P = (0, 0)$ and $Q = (2, 0)$.
For the computation of $H^1_{DR}(\mathbb{R}^2 - P - Q)$, let γ_1 be the unit circle and γ_2 be the unit circle centered at $Q = (2, 0)$ and we claim that the linear map $H^1_{DR}(\mathbb{R}^2 - P - Q) \to \mathbb{R}^2 : [\omega] \mapsto (\int_{\gamma_1} \omega, \int_{\gamma_2} \omega)$ is an isomorphism.
This map is well-defined, vanishing on exact forms as $\int_\gamma \mathrm{d}f = \int_a^b \mathrm{d}(f \circ \gamma) = f(\gamma(b)) - f(\gamma(a)) \overset{\gamma(a)=\gamma(b)}{=\!=\!=\!=} 0$.
Also there clearly exist forms such that $\int_{\gamma_1} \omega \neq \int_{\gamma_2} \omega$, so by a reflection along $x = 1$ we see the surjectivity of the linear map. If we uses the argument principle in complex analysis then we can find easily that the closed form that represents $(1, 0) \in \mathbb{R}^2$ is $\frac{1}{2\pi} \frac{xdy - ydx}{x^2 + y^2}$ and its translation from $(0, 0)$ to $(2, 0)$ represents $(0, 1) \in \mathbb{R}^2$.
For injectivity, given a closed 1-form ω such that $\int_{\gamma_1} \omega = \int_{\gamma_2} \omega = 0$, we claim that the integration of ω on any loop ℓ in $\mathbb{R}^2 - P - Q$ must be zero by Stokes' theorem in basic calculus.[a] By Jordan curve theorem a loop must encircle a bounded region Σ in $\mathbb{R}^2$. If the loop does not encircle either P or Q, then we apply Stokes' theorem to the region Σ, getting $0 = \int_\Sigma d\omega = \int_\ell \omega$. A same argument shows that for any circle C centered at P we have $\int_C \omega = 0$ on considering N to be the region between C and γ_1 along with C and γ_1; the same result holds for Q. So for any loop that encircles P or Q (or both of them)

we take sufficiently small circles C_P or C_Q around P or Q that are disjoint from ℓ and apply Stokes' theorem with Σ being the region between the circles and the loop along with the circles and the loop, obtaining that $0 = \int_\Sigma d\omega - \left(\int_{C_P} \omega\right) - \left(\int_{C_Q} \omega\right) = \int_\ell \omega$, where the parameters mean that the term in the parameter may appear or may not. Now we can define a smooth function f on $\mathbb{R}^2 - P - Q$ by integrate ω along a path $\gamma_x \subset \mathbb{R}^2 - P - Q$ from a base-point x_0 to another point x. This f is well-defined, being independent of the choice of the path γ_x by the above result on loops. Apply the mean-value property of integration and we see that $df = \omega$ as desired. □

[a] We will see soon that there is a similar Stokes' theorem for general manifolds

§2 The Mayer-Vietoris Argument

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