Set 1: Released 2023/3/6

Name: 李昀升 Li Yunsheng ID: 12232850 Zelmanov's Algebraic Lectures

---

1. Find a basis of the algebra

$$\langle 1, a, x, y \mid ax - xa = x, ay - ya = -y, x^2 = y^2 = 0, xy + yx = 1 \rangle$$

Solution. Let the order be $a < x < y$, then the relations give reductions

$$\begin{cases} xa \to & ax - x \\ ya \to & ay + y \\ x^2 \to & 0 \\ y^2 \to & 0 \\ yx \to & 1 - xy \end{cases}$$

The composable leading monomials along with their possible compositions are

$$\begin{cases} x^2, xa \rightsquigarrow & x^2 a \\ y^2, ya \rightsquigarrow & y^2 a \\ yx, x^2 \rightsquigarrow & yx^2 \\ y^2, yx \rightsquigarrow & y^2 x \end{cases}$$

We have

$$\begin{aligned} (x^2, xa - ax + x)_{x^2a} =& x^2a - x(xa - ax + x) \\ =& xax - x^2 \\ \to& (ax - x)x - 0 = ax^2 - x^2 \to 0. \\ (y^2, ya - ay - y)_{y^2a} =& y^2a - y(ya - ay - y) \\ =& yay + y^2 \\ \to& (ay + y)y + 0 = ay^2 + y^2 \to 0. \\ (yx + xy - 1, x^2)_{yx^2} =& (yx + xy - 1)x - yx^2 \\ =& xyx - x \\ \to& x(1 - xy) - x = -x^2y \to 0. \\ (y^2, yx + xy - 1)_{y^2x} =& y^2x - y(yx + xy - 1) \\ =& -yxy + y \\ \to& -(1 - xy)y + y = xy^2 \to 0. \end{aligned}$$

Therefore the set of irreducible words,

$$Ir = \{a^k x^\epsilon y^\delta \mid k \in \mathbb{N},\ \epsilon, \delta \in \{0, 1\}\},$$

is a basis of the algebra. □

2. Let V be a vector space over a field F. A mapping $N \colon V \to F$ is called a *quadratic form* if

(1) $N(\alpha v) = \alpha^2 N(v)$ for all $\alpha \in F$ and $v \in V$;

(2) $N(v, w) \coloneqq N(v + w) - N(v) - N(w)$ is a bilinear form.

Given a quadratic form N, the algebra $C\ell(V,N) = \langle 1, V \mid v^2 = N(v)\cdot 1,\ v \in V \rangle$ is called the Clifford algebra of the form N.[1] Find a basis of $C\ell(V,N)$.

Solution. Note that we have

$$N(v,v) = N(2v) - 2N(v) = 2N(v).$$

Let $\{v_i\}_{i\in I}$ be a basis of V and I be well-ordered, then

$$C\ell(V,N) = \langle 1, v_i \mid \left(\sum_i a_i v_i\right)^2 = N\left(\sum_i a_i v_i\right) \rangle$$

Since

$$N\left(\sum_i a_i v_i\right) = \frac{1}{2} N\left(\sum_i a_i v_i, \sum_i a_i v_i\right) = \frac{1}{2}\sum_i \sum_j a_i a_j N(v_i, v_j),$$

and

$$\left(\sum_i a_i v_i\right)^2 = \sum_i \sum_j a_i a_j v_i v_j,$$

we see that all relations of the form $\left(\sum_i a_i v_i\right)^2 = N\left(\sum_i a_i v_i\right)$ are generated by

$$v_i^2 = N(v_i), \quad v_i v_j + v_j v_i = N(v_i, v_j), \quad i, j \in I. \tag{1}$$

Conversely, by taking $a_i = 1$ and other coefficients zero we get $v_i^2 = N(v_i)$ for any $i \in I$, and thus $v_i^2 + v_i v_j + v_j v_i + v_j^2 = (v_i + v_j)^2 = N(v_i) + N(v_i, v_j) + N(v_j)$ gives that $v_i v_j + v_j v_i = N(v_i, v_j)$ for any $i, j \in I$. Therefore the relations in eq. (1) defines the same algebra, i.e.

$$C\ell(V,N) = \langle 1, v_i \mid v_i^2 = N(v_i),\ v_i v_j + v_j v_i = N(v_i, v_j),\ i, j \in I \rangle.$$

The claim is that $R \coloneqq \{v_i^2 - N(v_i) \mid i \in I\} \cup \{v_i v_j + v_j v_i - N(v_i, v_j) \mid i, j \in I\}$ is closed under compositions. All the possible forms of nontrivial compositions are

$$\begin{cases} v_i^2, v_i v_j \rightsquigarrow & v_i^2 v_j \\ v_i v_j, v_j^2 \rightsquigarrow & v_i v_j^2 \\ v_i v_j, v_j v_k \rightsquigarrow & v_i v_j v_k \end{cases}$$

where $v_i > v_j > v_k$. We have

$$\begin{aligned}(v_i^2 - N(v_i))v_j - v_i(v_i v_j + v_j v_i - N(v_i, v_j)) &= -N(v_i)v_j - v_i v_j v_i + N(v_i, v_j)v_i \\ &\to -N(v_i)v_j + (v_j v_i - N(v_i, v_j))v_i + N(v_i, v_j)v_i \\ &= -N(v_i)v_j + v_j v_i^2 \\ &\to -N(v_i)v_j + N(v_i)v_j = 0.\end{aligned}$$

$$\begin{aligned}(v_i v_j + v_j v_i - N(v_i, v_j))v_j - v_i(v_j^2 - N(v_j)) &= v_j v_i v_j - N(v_i, v_j)v_j + N(v_j)v_i \\ &\to v_j(-v_j v_i + N(v_i, v_j)) - N(v_i, v_j)v_j + N(v_j)v_i \\ &= -v_j^2 v_i + N(v_j)v_i \\ &\to -N(v_j)v_i + N(v_j)v_i = 0.\end{aligned}$$

$$\begin{aligned}&(v_i v_j + v_j v_i - N(v_i, v_j))v_k - v_i(v_j v_k + v_k v_j - N(v_j, v_k)) \\ =& v_j v_i v_k - N(v_i, v_j)v_k - v_i v_k v_j + N(v_j, v_k)v_i \\ \to& v_j(-v_k v_i + N(v_i, v_k)) - N(v_i, v_j)v_k + (v_k v_i - N(v_i, v_k))v_j + N(v_j, v_k)v_i \\ =& -v_j v_k v_i - N(v_i, v_j)v_k + v_k v_i v_j + N(v_j, v_k)v_i \\ \to& (v_k v_j - N(v_j, v_k))v_i - N(v_i, v_j)v_k + v_k(-v_j v_i + N(v_i, v_j)) + N(v_j, v_k)v_i = 0.\end{aligned}$$

These prove the claim. Therefore the set of irreducible words $\{v_{i_1} \cdots v_{i_n} \mid i_1 \le \cdots \le i_n\}$ is a basis of $C\ell(V,N)$. $\square$

[1] Although it is not written in the presentation, the linear relations in V are also moduloed so that operations on $C\ell(V,N)$ are compatible with those on V.

3. Find a normal form in the semigroup

$$\langle\, x, y \mid yx = 1 \,\rangle\,.$$

Solution. Let $x < y$. Since there is only one relation, the reduction system is confluent by definition. The normal forms are words of the form $x^i y^j$, $i, j \geq 0$. For example, the normal form of $y^2 x$ is y and the normal form of $y^4 x^3 y x^3 y^2$ is xy^2.

In fact, for any word w, we can always write w as $w = x^i v y^j$ for some $i, j \geq 0$ and v such that either v is empty or v begins with y and ends with x. Count the numbers of y and x in v and denote them respectively as $n(y)$ and $n(x)$. If $n(y) \geq n(x)$, then the normal form of w is $w = x^i y^{j+n(y)-n(x)}$. If $n(y) < n(x)$, then the normal form of w is $w = x^{i+n(x)-n(y)} y^j$. □

4. Prove that every associative algebra A has a presentation $A = \langle\, X \mid R = 0 \,\rangle$, such that the set R is closed with respect to compositions.

Proof. Let $A = \langle\, X \mid R = 0 \,\rangle$ be an arbitrary presentation of A. Write $R_0 \coloneqq R$ and define inductively that

$$R_n \coloneqq R_{n-1} \cup \{(f, g)_w \mid f, g \in R \text{ are composable}\}.$$

Then take the union $R_\infty \coloneqq \bigcup_{i=0}^{\infty} R_i$. Since $I(R) = I(R_\infty)$, where $I(R)$ denotes the ideal generated by R in $F\,\langle\, X \,\rangle$, it is clear that $A = \langle\, X \mid R = 0 \,\rangle = \langle\, X \mid R_\infty = 0 \,\rangle$. For any two elements f, g in R_∞, there must be an integer $N \in \mathbb{N}$ such that $f, g \in R_N$, hence $(f, g)_w \in R_{N+1} \subset R_\infty$, concluding that R_∞ is closed under compositions. □

5. Let L be a Lie algebra. Let $U(L)$ be the universal enveloping algebra of L. Prove that if $a, b \in U(L)$ are nonzero elements then $ab \neq 0$.

Proof. Take a basis $\{e_i\}_{i \in I}$ of L and there exists $\gamma_{ij}^k \in F$ for any $i, j \in I$ such that

$$[e_i, e_j] = \sum_k \gamma_{ij}^k e_k.$$

By construction, let $X = \{x_i\}_{i \in I}$ and we have

$$U(L) = \langle\, X \mid x_i x_j - x_j x_i - \sum_k \gamma_{ij}^k x_k = 0 \,\rangle\,.$$

By the Poincare-Birkhoff-Witt theorem, give an order with the minimality condition on I and then the irreducible words $\{x_{i_1} \cdots x_{i_n} \mid i_1 \leq \cdots \leq i_k\}$ form a basis of $U(L)$. Thus every nonzero elements $a, b \in U(L)$ can be written as finite summations

$$a = \sum_i \alpha_i x_{I_i}, \quad b = \sum_j \beta_j x_{J_j},$$

where I_i and J_j are finite arrays with increasing entries in I and all α_i's and β_j's are nonzero scalars. We have

$$ab = \left(\sum_i \alpha_i x_{I_i}\right)\left(\sum_j \beta_j x_{J_j}\right) = \sum_{i,j} \alpha_i \beta_j x_{I_i} x_{J_j} = \sum_{i,j} \left(\alpha_i \beta_j x_{|I_i \cdot J_j|} + r_{ij}(\boldsymbol{x})\right), \tag{2}$$

where $|I_i \cdot J_j|$ denotes the array obtained by putting the entries of I_i and J_j together and then rearrange them into an increasing order. Clearly $\overline{r_{ij}(\boldsymbol{x})} < x_{|I_i \cdot J_j|}$ since its length is smaller.

Note that if $x_{I_i} < x_{I_{i'}}$ and $x_{J_j} < x_{J_{j'}}$, then $x_{|I_i \cdot J_j|} < x_{|I_{i'} \cdot J_{j'}|}$. Let

$$x_{I_{i_0}} \coloneqq \max_i \{x_{I_i}\}, \quad x_{J_{j_0}} \coloneqq \max_j \{x_{J_j}\},$$

then $x_{|I_{i_0} \cdot J_{j_0}|}$ is the leading monomial of ab and it only appears once in the summation in the very right of eq. (2). Therefore if we expand ab as a linear combination of irreducible words, the coefficient of $x_{|I_{i_0} \cdot J_{j_0}|}$ is exactly $\alpha_{i_0} \beta_{j_0} \neq 0$, concluding that $ab \neq 0$. □