Southern University of Science and Technology

Master's Thesis Proposal

Title: On the Quantum Modularity Conjecture for Knots

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CHAPTER 1 INTRODUCTION

1.1 Overview

In 1995, using the quantum dilogarithm function

$$(x;q)_\infty := \prod_{n=0}^{\infty}(1-xq^n), \quad (|q|<1)$$

R. Kashaev introduced a knot invariant related to a positive integer N, which is denoted as $\langle K\rangle_N$ for a knot K[1]. For any knot K and positive integer N, the invariant $\langle K\rangle_N$ is a complex number such that $\langle K\rangle_N \in \mathbb{Z}[\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}}]$. Kashaev conjectured that, if K is hyperbolic, which means that the complement $S^3 \setminus K$ can be given a hyperbolic structure, then the absolute value of $\langle K\rangle_N$ grows exponentially as N increases. More precisely, the following full asymptotic expansion was conjectured

$$\langle K\rangle_N \sim N^{\frac{3}{2}}\mathrm{e}^{\frac{\mathrm{i}V(K)}{2\pi}N}\Phi^{(K)}\left(\frac{2\pi\mathrm{i}}{N}\right), \quad N\to\infty, \tag{1-1}$$

where $V(K)$ is the hyperbolic volume of $S^3\setminus K$ and $\Phi^{(K)}(\hbar)$ is a divergent power series in $\hbar$[2]. This conjectural expansion is known as the Volume Conjecture.

The Volume Conjecture turns out to be a special case of a more general conjecture, the Quantum Modularity Conjecture. In 2001, H. Murakami and J. Murakami discovered that the Kashaev's invariant $\langle K\rangle_N$ is equal to the evaluation of the colored Jones polynomial $J_N^K(q)$ at $q=\eta_N$, where $\eta_N := \mathrm{e}^{\frac{2\pi\mathrm{i}}{N}}$[3]. Extending $\langle K\rangle_N$ equivariantly to a 1-periodic function $\mathbf{J}^K$ on the rational numbers such that $\mathbf{J}^K\left(-\frac{1}{N}\right) = \langle K\rangle_N$, this observation then motivates a more general conjectural full asymptotic expansion that[4]

$$\mathbf{J}^K\left(\frac{aX+b}{cX+d}\right) \sim (cX+d)^{\frac{3}{2}}\mathrm{e}^{\frac{\mathrm{i}V(K)}{2\pi}\left(X+\frac{d}{c}\right)}\Phi_{a/c}^{(K)}\left(\frac{2\pi\mathrm{i}}{c(cX+d)}\right)\mathbf{J}^K(X), \quad X\to\infty \text{ in } \mathbb{Q}, \tag{1-2}$$

for any matrix $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with $c>0$, where $\Phi_\alpha^{(K)}(\hbar)$ is a power series with algebraic coefficients depending on $\alpha\in\mathbb{Q}/\mathbb{Z}$. This expansion states the Quantum Modularity Conjecture. The case where $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0\end{smallmatrix}\right)$ and $X=N$ of eq. (1-2) implies eq. (1-1)[4], since there are identities

$$\Phi_0^{(K)}(\hbar) = \Phi^{(K)}(\hbar), \quad \mathbf{J}^K(N) = \mathbf{J}^K(0) = 1.$$

The following table gives a brief timeline of the story:

Year	Topic	Contents
1995	Volume Conjecture	$\lvert\langle K\rangle_N\rvert$ grows exponentially as N increases
2001	Murakami & Murakami's Discovery	$\langle K\rangle_N$ is equal to $J_N^K(\eta_N)$
2010	Quantum Modularity Conjecture	Full asymptotic expansion of the extended $\langle K\rangle_N$

Over the past years the Quantum Modularity Conjecture has become one of the most outstanding problems in quantum topology, and during the research into it multiple phenomenons and consequences have been revealed[5-7]. The phenomenons, mostly observed by S. Garoufalidis and D. Zagier in their research of the 4_1 knot, the 5_2 knot and the $(-2,3,7)$ pretzel knot, indicate a close relationship of the conjecture with the Dimofte-Gaiotto-Gukov index and the Anderson-Kashaev state integral, two knot invariants that were introduced in 2011[6,8-9]. The invariants also turned out to be related to the quantum spin network. Most of these relations are given in terms of the corresponding q-series rising from the conjecture, invariants and spin network[6]. A brief introduction of part of the work by Garoufalidis and Zagier will be included in section 1.2.

A family of numerical evidence for the Quantum Modularity Conjecture has been presented by Garoufalidis and Zagier. Although a proof for the 4_1 knot is easy, currently for very few knots a rigorous proof of the Quantum Modularity Conjecture has been given[6,10]. The goal of this project is to investigate the Quantum Modulartiy Conjecture based on the work of Garoufalidis and Zagier by looking into examples whose computation has not been accomplished, for instance the $(-2,3,7)$ pretzel knot.

1.2 Recent Work

In this section a brief summary of the discoveries for the 4_1 knot and 5_2 knot will be given. For the $(-2,3,7)$ pretzel knot, the summary will be presented in chapter 2, followed by the newly obtained results of computations done by the author and An Ni.

1.2.1 The 4_1 Knot

The state integral of the 4_1 knot is a holomorphic function on $\mathbb{C}' := \mathbb{C} \setminus (-\infty, 0]$, defined by[9]eq. (38)

$$Z_{4_1}(\tau) = \int_{\mathbb{R}+\mathrm{i}\varepsilon} \Phi_{\sqrt{\tau}}(x)^2 \mathrm{e}^{-\pi \mathrm{i} x^2}\,\mathrm{d}x, \quad (\tau \in \mathbb{C}')$$

where $\Phi_b(x)$ is the Faddeev's quantum dilogarithm, which is a meromorphic function on the complex plane. The definition and a series of well-studied properties of the Faddeev's quantum dilogarithm can be found in Anderson and Kashaev's paper[9]. For b in the first quadrant of the complex plane, the poles of $\Phi_b(x)$ are

$$\frac{\mathrm{i}(b+b^{-1})}{2}+\mathrm{i}\mathbb{N}b+\mathrm{i}\mathbb{N}b^{-1},$$

which all live in the upper half plane.

Using the method of residues, the state integral of the 4_1 knot when $\operatorname{Im}\tau>0$ (so that $|q|<1$ in the following) can be expanded into a combination of q-series $G_0(q)$ and $G_1(q)$,

$$2\mathrm{i}\left(\frac{\tilde{q}}{q}\right)^{\frac{1}{24}}Z_{4_1}(\tau)=\tau^{\frac{1}{2}}G_1(q)G_0(\tilde{q})-\tau^{-\frac{1}{2}}G_0(q)G_1(\tilde{q}), \tag{1-3}$$

where $q=\mathrm{e}^{2\pi\mathrm{i}\tau}$, $\tilde{q}=\mathrm{e}^{-2\pi\mathrm{i}\tau^{-1}}$. Explicitly, the q-series are given by

$$G_0(q)=\sum_{n=0}^{\infty}(-1)^n\frac{q^{\frac{n(n+1)}{2}}}{(q)_n^2},\quad G_1(q)=\sum_{n=0}^{\infty}\left(1+2n-4\sum_{s=1}^{\infty}\frac{q^{s(n+1)}}{1-q^s}\right)(-1)^n\frac{q^{\frac{n(n+1)}{2}}}{(q)_n^2},$$

where the convention of Pochhammer symbol

$$(q)_n\coloneqq(q;q)_n,\quad (x;q)_n\coloneqq\prod_{i=0}^{n-1}(1-xq^i),\quad (|q|<1)$$

is adapted. The computation, along with that of the 5_2 knot and 1-dimensional state integrals in general, has been given in detail by Garoufalidis and Kashaev[11]. The symmetry that $\Phi_b(x)=\Phi_{b^{-1}}(x)$[9]Appx. A implies that $Z_{4_1}(\tau)=Z_{4_1}(\tau^{-1})$ whenever $\tau\in\mathbb{C}\setminus\mathbb{R}$, hence we can extend $G_0(q)$ and $G_1(q)$ to $|q|>1$ by

$$G_0(q)=G_0(q^{-1}),\quad G_1(q)=-G_1(q^{-1}),\quad (q\in\mathbb{C},\ |q|\neq 1)$$

such that the factorization eq. (1-3) holds for all $\tau\in\mathbb{C}\setminus\mathbb{R}$[6].

In Garoufalidis and Zagier's recent paper[6], the following observations have been presented:

Let $\widehat{\Phi}_{4_1}(\hbar)$ be defined by

$$\widehat{\Phi}_{4_1}(\hbar)=\mathrm{e}^{\frac{\mathrm{i}V(4_1)}{\hbar}}\Phi^{(4_1)}(\hbar),$$

where $\Phi^{(4_1)}(\hbar)$ is given by eq. (1-1) for $K=4_1$, then

Observation 1: When τ tends to 0 along any ray in the interior of the upper half-plane,

$$G_0(\mathrm{e}^{2\pi\mathrm{i}\tau})\sim\sqrt{\tau}\left(\widehat{\Phi}_{4_1}(2\pi\mathrm{i}\tau)-\mathrm{i}\widehat{\Phi}_{4_1}(-2\pi\mathrm{i}\tau)\right)$$

to all orders in τ.

Observation 2: When τ tends to 0 in a cone in the interior of the upper half-plane

$$G_1(\mathrm{e}^{2\pi\mathrm{i}\tau}) \sim \frac{1}{\sqrt{\tau}}\left(\widehat{\Phi}_{4_1}(2\pi\mathrm{i}\tau) + \mathrm{i}\widehat{\Phi}_{4_1}(-2\pi\mathrm{i}\tau)\right)$$

to all orders in τ.

Observation 3: For $|q| < 1$, we have

$$G_0(q) = (q)_\infty \sum_{n=0}^{\infty}(-1)^n \frac{q^{\frac{n(3n+1)}{2}}}{(q)_n^3} = \frac{1}{(q)_\infty}\sum_{n,m=0}^{\infty}(-1)^{n+m}\frac{q^{\frac{(n+m)(n+m+1)}{2}}}{(q)_n(q)_m},$$

and

$$G_1(q) = \sum_{n=0}^{\infty}(1+6n)(-1)^n \frac{q^{\frac{n(n+1)}{2}}}{(q)_n^2}.$$

The series $\sum_{n=0}^{\infty}(-1)^n \frac{q^{\frac{n(3n+1)}{2}}}{(q)_n^3}$ occurred in Garoufalidis' work on the stability of the coefficients of the evaluation of the regular quantum spin network[12].

Let $\mathrm{Ind}_{4_1}(q)$ denote the Dimofte-Gaiotto-Gukov index of the 4_1 knot, which is also a q-series, then

Observation 4:

$$\mathrm{Ind}_{4_1}(q) = G_0(q)G_1(q).$$

These observations suggest a close relationship between these topics which is still under investigation. They cannot be purely accidental random identities, as similar (extended) relations have also been discovered for the 5_2 knot.

1.2.2 The 5_2 Knot

The state integral of the 5_2 knot is[9]eq. (39)

$$Z_{5_2}(\tau) = \int_{\mathbb{R}+\mathrm{i}\varepsilon} \Phi_{\sqrt{\tau}}(x)^3 \mathrm{e}^{-2\pi\mathrm{i}x^2}\,\mathrm{d}x. \quad (\tau \in \mathbb{C}')$$

Using the method of residues and extending by symmetry, it factorizes into the following form[6]

$$2\mathrm{e}^{\frac{3\mathrm{i}\pi}{4}}\left(\frac{\tilde{q}}{q}\right)^{\frac{1}{8}} Z_{5_2}(\tau) = \tau h_2(\tau)h_0(\tau^{-1}) + 2h_1(\tau)h_1(\tau^{-1}) + \frac{1}{\tau}h_0(\tau)h_2(\tau^{-1}),$$

for $\tau \in \mathbb{C}\setminus\mathbb{R}$, where

$$h_j(\tau) = (\pm 1)^j H_j^{\pm}(\mathrm{e}^{\pm 2\pi\mathrm{i}\tau}), \quad \text{for } \pm\,\mathrm{Im}(\tau) > 0,$$

with q-series $H_j^{\pm}(q)$ given by

$$H_j^{+}(q)=\sum_{m=0}^{\infty} t_m(q)p_m^{(j)}(q),\quad H_j^{-}(q)=\sum_{m=0}^{\infty} T_m(q)P_m^{(j)}(q),\quad (j=0,1,2)$$

where

$$t_m(q)=\frac{q^{m(m+1)}}{(q;q)_m^3},\quad T_m(q)=\frac{(-1)^m q^{m(m+1)/2}}{(q;q)_m^3},$$

and

$$p_m^{(0)}(q)=1,\ p_m^{(1)}(q)=\frac{1+3\mathcal{E}_1(q)}{4}+\sum_{j=1}^{m}\frac{2+q^j}{1-q^j},\ p_m^{(2)}(q)=p_m^{(1)}(q)^2-\frac{3+\mathcal{E}_2(q)}{24}+\sum_{j=1}^{m}\frac{3q^j}{(1-q^j)^2},$$

$$P_m^{(0)}(q)=1,\ P_m^{(1)}(q)=\frac{3\mathcal{E}_1(q)-1}{4}+\sum_{j=1}^{m}\frac{1+2q^j}{1-q^j},\ P_m^{(2)}(q)=P_m^{(1)}(q)^2-\frac{\mathcal{E}_2(q)-3}{24}+\sum_{j=1}^{m}\frac{3q^j}{(1-q^j)^2}.$$

Here $\mathcal{E}_1(q)$ and $\mathcal{E}_2(q)$ are the weight 1 and weight 2 Eisenstein series defined by $\mathcal{E}_1(q)=1-4\sum_{n\geq 1}\frac{q^n}{1-q^n}$ and $\mathcal{E}_2(q)=1-24\sum_{n\geq 1}\frac{q^n}{(1-q^n)^2}$, respectively.

Parallel to the 4_1 knot, the following observations were made[6]:

Let $\widehat{\Phi}_{5_2}$ be the following vector of series

$$\widehat{\Phi}_{5_2}:=\begin{pmatrix}\widehat{\Phi}^{(5_2,\sigma_1)}\\ \widehat{\Phi}^{(5_2,\sigma_3)}\\ \widehat{\Phi}^{(5_2,\sigma_2)}\end{pmatrix},$$

where $\widehat{\Phi}^{(5_2,\sigma_1)}$ is the series for the 5_2 knot in eq. (1-1), $\widehat{\Phi}^{(5_2,\sigma_2)}$ and $\widehat{\Phi}^{(5_2,\sigma_3)}$ are two other series indexed by $\sigma_j\in\mathcal{P}_{5_2}$ where $\mathcal{P}_{5_2}$ coincides with the set of boundary parabolic $\mathrm{SL}_2(\mathbb{C})$-representations of $\pi_1(S^3\setminus 5_2)$. A definition of $\widehat{\Phi}^{(K,\sigma_j)}$ for a knot K was given by T. Dimofte and Garoufalidis[13-14]. Let $h=\begin{pmatrix}\tau^{-1}h_0\\ h_1\\ \tau h_2\end{pmatrix}$, then

Observation 5:

$$h(\tau)\sim\begin{cases}N_+\widehat{\Phi}(2\pi\mathrm{i}\tau) & \text{when } \arg(\tau)\in(0,0.19)\\ N_-\widehat{\Phi}(2\pi\mathrm{i}\tau) & \text{when } \arg(\tau)\in\left(-\frac{\pi}{2},0\right)\end{cases}$$

where

$$N_+=\begin{pmatrix}1/2 & 1/2 & 1\\ 0 & 1/2 & 1/2\\ -1/12 & 5/12 & -2/3\end{pmatrix},\quad N_-=\begin{pmatrix}-1/2 & -1/2 & 1/2\\ 3/4 & -1/4 & -1/4\\ -13/12 & -1/12 & 1/12\end{pmatrix}.$$

For the index, there is

Observation 6:

$$\mathrm{Ind}_{5_2}(q) = 2H_1^+(q)H_1^-(q).$$

Furthermore, the following quadratic relation for the q-series $H_j^{\pm}$'s was also observed

Observation 7:

$$H_0^+(q)H_2^-(q) - 2H_1^+(q)H_1^-(q) + H_2^+(q)H_0^-(q) = 0.$$

For the 4_1 knot, this could not be seen since it is trivially

$$G_0(q)G_1(q) - G_1(q)G_0(q) = 0,$$

as a consequence of that the 4_1 knot is amphichiral.

1.2.3 The Descendant State Integral

By adding a factor $\mathrm{e}^{2\pi(\lambda\tau^{1/2}-\mu\tau^{-1/2})x}$ to the integrand we obtain the descendant state integral[7]. For example, the descendant state integral of the 4_1 knot is

$$Z_{4_1}^{(\lambda,\mu)}(\tau) = \int_{\mathbb{R}+\mathrm{i}\varepsilon} \Phi_{\sqrt{\tau}}(x)^2 \mathrm{e}^{-\pi\mathrm{i}x^2+2\pi(\lambda\tau^{1/2}-\mu\tau^{-1/2})x}\,\mathrm{d}x. \quad (\lambda,\mu\in\mathbb{Z})$$

By the method of residues and the symmetry, it factorizes as the following,

$$Z_{4_1}^{(\lambda,\mu)}(\tau) = (-1)^{\lambda-\mu+1}\frac{\mathrm{i}}{2}q^{\frac{m}{2}+\frac{1}{24}}\tilde{q}^{\frac{\mu}{2}-\frac{1}{24}}\left(\sqrt{\tau}G_0^{(\mu)}(\tilde{q})G_1^{(\lambda)}(q) - \frac{1}{\sqrt{\tau}}G_1^{(\mu)}(\tilde{q})G_0^{(\lambda)}(q)\right), \tag{1-4}$$

where $G_0^{(k)}$ and $G_1^{(k)}$ are defined by

$$G_0^{(k)}(q) = \sum_{n=0}^{\infty}(-1)^n\frac{q^{\frac{n(n+1)}{2}+kn}}{(q)_n^2}, \quad G_1^{(k)}(q) = \left(1+2k+2n-4\sum_{s=1}^{\infty}\frac{q^{s(n+1)}}{1-q^s}\right)\sum_{n=0}^{\infty}(-1)^n\frac{q^{\frac{n(n+1)}{2}+kn}}{(q)_n^2},$$

for $|q|<1$ and extended to $|q|>1$ by $G_j^{(k)}(q^{-1}) = (-1)^j G_j^{(k)}(q)$. The matrix of these series,

$$w_k(q) = \begin{pmatrix} G_0^{(k)}(q) & G_1^{(k)}(q) \\ G_0^{(k+1)}(q) & G_1^{(k+1)}(q) \end{pmatrix}, \quad (|q|\neq 1)$$

satisfies the following linear q-difference equation[15]:

Theorem 1.1: The matrix $w_k(q)$ is a fundamental solution of the linear q-difference equation

$$y_{k+1}(q) - (2-q^k)y_k(q) + y_{k-1}(q) = 0 \quad (k\in\mathbb{Z}).$$

It has constant determinant

$$\det(w_k(q)) = 2, \tag{1-5}$$

and satisfies the symmetry and orthogonality properties

$$w_k(q^{-1}) = w_{-k}(q)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\frac{1}{2}w_k(q)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}w_k(q^{-1})^T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

for all integers k and for $|q| \neq 1$.

The factorization eq. (1-4) implies, since the left-hand-side is a holomorphic function on $\tau \in \mathbb{C}'$, that the matrix-valued function

$$W_{\lambda,\mu}(\tau) = (w_\mu(\tilde{q})^T)^{-1}\begin{pmatrix} 1/\tau & 0 \\ 0 & 1 \end{pmatrix}w_\lambda(q)^T, \quad (q = \mathrm{e}^{2\pi\mathrm{i}\tau},\ \tilde{q} = \mathrm{e}^{-2\pi\mathrm{i}\tau^{-1}})$$

which is originally defined only for $\tau \in \mathbb{C} \setminus \mathbb{R}$, extends holomorphically to $\tau \in \mathbb{C}'$ for all integers λ and μ.

A similar story of descendants for the 5_2 knot can be found in Garoufalidis and Zagier's recent paper[6]Sec. 4.3.

In the study of the refined quantum modularity conjecture for the 4_1 knot, the following 2-by-2 matrix of asymptotic series was found by Garoufalidis and Zagier[10]

$$\widehat{\mathbf{\Phi}}_{4_1}(\hbar) = \begin{pmatrix} \widehat{\Phi}_{4_1}(\hbar) & \widehat{\Psi}_{4_1}(\hbar) \\ \mathrm{i}\widehat{\Phi}_{4_1}(\hbar) & -\mathrm{i}\widehat{\Psi}_{4_1}(\hbar) \end{pmatrix},$$

where $\widehat{\Psi}_{4_1}(\hbar) = \mathrm{e}^{C/\hbar}\Psi^{(4_1)}(\hbar)$ and $\Psi^{(4_1)}(\hbar)$ is a power series in $\hbar$. Let $Q(\tau)$ be the following matrix of linear combinations of $G_j^{(k)}$'s,

$$Q(\tau) = w_0(q)^T\begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix},$$

then

Observation 8: As $\tau \to 0$ in the upper half-plane, we have:

$$\begin{pmatrix} 1/\sqrt{\tau} & 0 \\ 0 & \sqrt{\tau} \end{pmatrix}Q(\tau) \sim \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\widehat{\mathbf{\Phi}}_{4_1}(2\pi\mathrm{i}\tau).$$

As a consequence, by eq. (1-5) that $\det(Q(\tau)) = 2$ for all τ, it follows that

$$\det(\widehat{\mathbf{\Phi}}_{4_1}(\hbar)) = 1,$$

and[6]

$$\widehat{\mathbf{\Phi}}_{4_1}(-\hbar)\widehat{\mathbf{\Phi}}_{4_1}(\hbar)^T = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}.$$

CHAPTER 2 CURRENT PROGRESS - THE $(-2,3,7)$ PRETZEL KNOT

For the $(-2,3,7)$ pretzel knot, the factorization of the state integral involves 6 pairs of q-series, and some of them are power series in integer powers of $q^{1/2}$, which is different from the case of the 4_1 knot and the 5_2 knot. This new phenomenon is formulated by Garoufalidis and Zagier as the level of knots, and $(-2,3,7)$ is said to have level $N=2$. Writing the 6 pairs of q-series as $H_j^{\pm}(q)$ for $j=0,1,\cdots,5$, Garoufalidis and Zagier found the following[6]:

Observation 9: The relation with the index is given by

$$\mathrm{Ind}_{(-2,3,7)}(q)=H_1^{+}(q)H_1^{-}(q),$$

and the following quadratic relation holds:

$$\frac{1}{2}H_0^{+}(q)H_2^{-}(q)-H_1^{+}(q)H_1^{-}(q)+\frac{1}{2}H_2^{+}(q)H_0^{-}(q)-H_3^{+}(q)H_3^{-}(q)+H_4^{+}(q)H_4^{-}(q)-H_5^{+}(q)H_5^{-}(q)=0.$$

Since the $(-2,3,7)$ pretzel knot has 6 boundary parabolic $\mathrm{SL}_2(\mathbb{C})$ representations, there are 6 series $\{\widehat{\Phi}_{\alpha}^{(\sigma_i)}(\hbar)\}_{j=1}^{6}$. Similar to the case of the 4_1 knot and the 5_2 knot, consider the vector of asymptotic series corresponding to the $(-2,3,7)$ pretzel knot $\widehat{\Phi}_{\alpha}(\hbar):=\left(\widehat{\Phi}_{\alpha}^{(\sigma_i)}(\hbar)\right)_{j=1}^{6}$ and the vector of holomorphic functions $h(\tau):=(h_j(\tau))_{j=1}^{6}$ with weight $(-1,0,1,-1,-1,-1)$, where $h_j(\tau)=(\pm1)^j H_j^{\pm}(\mathrm{e}^{\pm2\pi\mathrm{i}\tau})$ for $\pm\,\mathrm{Im}(\tau)>0$ respectively, then

Observation 10: For any $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$, as $X\in\mathbb{C}\setminus\mathbb{R}$ in a sector near the positive real axis and $X\to\infty$, we have:

$$h|_{\gamma}(X)\sim\rho(\gamma)\begin{pmatrix}0&1&-1&0&-1&-1/2\\0&1&1&0&0&0\\0&2/3&-2/3&0&4/3&1/6\\0&-1&1&0&1&-1/2\\0&0&0&-1/2&-1&0\\2&0&0&-1/2&-1&0\end{pmatrix}\widehat{\Phi}_{\alpha}\left(\frac{2\pi\mathrm{i}}{cX+d}\right)$$

to all orders in $1/X$, where $(h|_{\gamma})(\tau)=\left((c\tau+d)^{(\text{weight of }h_j)}h_j(\gamma\tau)\right)_{j=1}^{6}$, $\gamma\tau=\frac{a\tau+b}{c\tau+d}$, $\alpha=a/c$ and ρ is a complex representation of $\mathrm{SL}_2(\mathbb{Z})$.

Note that since some of $H_j^{\pm}(q)$ are power series in $q^{1/2}$, here $h_j(\tau)$ are 2-periodic, instead of 1-periodic as in the case of the 4_1 knot and the 5_2 knot.

The following two sections present the results of computations on the descendant state integral of the $(-2,3,7)$ pretzel knot, involving its factorization and asymptotic expansion, done by the author and An Ni.

2.1 Factorization of the Descendant State Integral

Recall from Garoufalidis and Zagier's recent paper[6]eq. (51) that the descendant state integral of $(-2,3,7)$ pretzel knot is

$$Z^{(\lambda,\mu)}_{(-2,3,7)}(\tau) = \left(\frac{q}{\tilde{q}}\right)^{-\frac{1}{24}} \int_{\mathbb{R}+\mathrm{i}\frac{c_b}{2}+\mathrm{i}\varepsilon} \Phi_{\sqrt{\tau}}(x)^2 \Phi_{\sqrt{\tau}}(2x-c_b) \mathrm{e}^{-\pi\mathrm{i}(2x-c_b)^2+2\pi(\lambda b-\mu b^{-1})x}\,\mathrm{d}x,$$

where $\lambda,\mu \in \mathbb{Z}$, $\tau = b^2$, $\sqrt{\tau} = b$ and $c_b = \mathrm{i}(b+b^{-1})/2$.

Theorem 2.1: We have:

$$\begin{aligned} &2\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}\left(q^{\frac{\lambda}{2}}\tilde{q}^{\frac{\mu}{2}}\right)^{-1} Z^{(\lambda,\mu)}_{(-2,3,7)}(\tau) \\ &\qquad = -\frac{1}{2\tau}h_0(\lambda,\tau)h_2(\mu,\tau^{-1}) + h_1(\lambda,\tau)h_1(\mu,\tau^{-1}) - \frac{\tau}{2}h_2(\lambda,\tau)h_0(\mu,\tau^{-1}) \\ &\qquad\quad - \mathrm{i}\left(\frac{1}{2}h_3(\lambda,\tau)h_4(\mu,\tau^{-1}) - \frac{1}{2}h_4(\lambda,\tau)h_3(\mu,\tau^{-1}) + h_5(\lambda,\tau)h_5(\mu,\tau^{-1})\right). \end{aligned}$$

In the above theorem,

$$h_j(k,\tau) \coloneqq (\pm 1)^j H^{\pm}_{k,j}(\mathrm{e}^{\pm 2\pi\mathrm{i}\tau}) \text{ for } \pm\operatorname{Im}(\tau) > 0$$

are defined as the following: Recall that

$$\mathcal{E}_2(q) = 1 - 24\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}, \quad E^{(m)}_l(q) = \sum_{s=1}^{\infty}\frac{s^{l-1}q^{s(m+1)}}{1-q^s}.$$

For $j = 0,1,2$:

$$H^+_{\lambda,j}(q) = \sum_{m=0}^{\infty} t_{\lambda,m}(q)p^{(j)}_{\lambda,m}(q), \quad H^-_{\mu,j}(q) = \sum_{n=0}^{\infty} T_{\mu,n}P^{(j)}_{\mu,n}(q),$$

with

$$t_{\lambda,m}(q) = (-1)^{\lambda}\frac{q^{m(2m+1)+\lambda m}}{(q)^2_m(q)_{2m}}, \quad T_{\mu,n}(q) = (-1)^{\mu}\frac{q^{n(n+1)+\mu n}}{(q)^2_n(q)_{2n}},$$

and

$$p_{\lambda,m}^{(0)}(q)=1,\quad p_{\lambda,m}^{(1)}(q)=4m+\lambda+1-2E_1^{(m)}(q)-2E_1^{(2m)}(q),$$
$$p_{\lambda,m}^{(2)}(q)=p_{\lambda,m}^{(1)}(q)^2-2E_2^{(m)}(q)-4E_2^{(2m)}(q)-\frac{1}{3}\mathcal{E}_2(q),$$
$$P_{\mu,n}^{(0)}(q)=1,\quad P_{\mu,n}^{(1)}(q)=2n+\mu+1-2E_1^{(n)}(q)-2E_1^{(2n)}(q),$$
$$P_{\mu,n}^{(2)}(q)=P_{\mu,n}^{(1)}(q)^2+12E_2^{(0)}(q)-\frac{1}{2}-2E_2^{(n)}(q)-4E_2^{(2n)}(q)+\frac{1}{3}\mathcal{E}_2(q),$$

For $j=3,4,5$:

$$H_{\lambda,3}^{+}(q)=(-1)^{\lambda}\frac{q^{1/8}}{(1-q^{1/2})^2}\sum_{m=0}^{\infty}\frac{q^{(2m+1)(m+1)+\lambda(m+1/2)}}{(q^{3/2};q)_m^2(q)_{2m+1}}\qquad H_{\mu,4}^{-}(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)+\mu n}}{(-q;q)_n^2(q)_{2n}}$$

$$H_{\lambda,4}^{+}(q)=\sum_{m=0}^{\infty}\frac{q^{(2m+1)m+\lambda m}}{(-q;q)_m^2(q)_{2m}}\qquad H_{\mu,3}^{-}(q)=(-1)^{\mu}\frac{q^{-1/8}}{(1-q^{-1/2})^2}\sum_{n=0}^{\infty}\frac{q^{n(n+2)+\mu(n+1/2)}}{(q^{3/2};q)_n^2(q)_{2n+1}}$$

$$H_{\lambda,5}^{+}(q)=\frac{q^{1/8}}{(1+q^{1/2})^2}\sum_{m=0}^{\infty}\frac{q^{(2m+1)(m+1)+\lambda(m+1/2)}}{(-q^{3/2};q)_m^2(q)_{2m+1}}\qquad H_{\mu,5}^{-}(q)=\frac{q^{-1/8}}{(1+q^{-1/2})^2}\sum_{n=0}^{\infty}\frac{q^{n(n+2)+\mu(n+1/2)}}{(-q^{3/2};q)_n^2(q)_{2n+1}}$$

For the above q-hypergeometric series, the following symmetries and quadratic relation are satisfied

$$H_{k,0}^{+}(q^{-1})=H_{-k,0}^{-}(q)\quad H_{k,1}^{+}(q^{-1})=-H_{-k,1}^{-}(q)\quad H_{k,2}^{+}(q^{-1})=H_{-k,2}^{-}(q)$$
$$H_{k,3}^{+}(q^{-1})=-H_{-k,3}^{-}(q)\quad H_{k,4}^{+}(q^{-1})=H_{-k,4}^{-}(q)\quad H_{k,5}^{+}(q^{-1})=-H_{-k,5}^{-}(q).$$

$$\frac{1}{2}H_{k,0}^{+}(q)H_{k,2}^{-}(q)-H_{k,1}^{+}(q)H_{k,1}^{-}(q)+\frac{1}{2}H_{k,2}^{+}(q)H_{k,0}^{-}(q)$$
$$-H_{k,3}^{+}(q)H_{k,3}^{-}(q)+\frac{1}{4}H_{k,4}^{+}(q)H_{k,4}^{-}(q)-H_{k,5}^{+}(q)H_{k,5}^{-}(q)=0. \tag{2-1}$$

When $(\lambda,\mu)=(0,0)$, this factorization can be connected to that in Garoufalidis and Zagier's recent paper[6]eq. (50) using the following identities:

$$\begin{aligned}\frac{(q^{3/2};q)_\infty^2}{(q;q)_\infty^2}\frac{(\tilde{q};\tilde{q})_\infty^2}{(-1;\tilde{q})_\infty^2}&=\frac{\mathrm{e}^{-\frac{\pi \mathrm{i}}{2}}q^{1/8}}{2(1-q^{1/2})^2}\tau,\\ \frac{(-q;q)_\infty^2}{(q;q)_\infty^2}\frac{(\tilde{q};\tilde{q})_\infty^2}{(-\tilde{q}^{-1/2};\tilde{q})_\infty^2}&=\frac{\mathrm{e}^{-\frac{\pi \mathrm{i}}{2}}\tilde{q}^{-1/8}}{2(1-\tilde{q}^{-1/2})^2}\tau,\\ \frac{(-q^{3/2};q)_\infty^2}{(q;q)_\infty^2}\frac{(\tilde{q};\tilde{q})_\infty^2}{(-\tilde{q}^{-1/2};\tilde{q})_\infty^2}&=\frac{\mathrm{e}^{-\frac{\pi \mathrm{i}}{2}}q^{1/8}\tilde{q}^{-1/8}}{(1+q^{1/2})^2(1+\tilde{q}^{-1/2})^2}\tau.\end{aligned} \tag{2-2}$$

Theorem 2.2: The $H_{\lambda,j}^{+}$'s satisfy the following q-difference equation:

$$H_{\lambda+6,j}^{+}(q)+2H_{\lambda+5,j}^{+}(q)-(q+q^{\lambda+4})H_{\lambda+4,j}^{+}(q)-2(q+1)H_{\lambda+3,j}^{+}(q)$$
$$-H_{\lambda+2,j}^{+}(q)+2qH_{\lambda+1,j}^{+}(q)+qH_{\lambda,j}^{+}(q)=0.$$

Therefore, consider the q-difference equation

$$y_{k+6}(q)+2y_{k+5}(q)-(q+q^{k+4})y_{k+4}(q)-2(q+1)y_{k+3}(q)-y_{k+2}(q)+2qy_{k+1}(q)+qy_k = 0,$$

it has a fundamental solution set given by the columns of the following matrix

$$W_k(q) = \begin{cases} W_k^+(q), \quad |q|<1, \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} W_{-k-5}^-(q^{-1}) \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad |q|>1 \end{cases}$$

where the matrices W_k^ϵ with $\epsilon = \pm$ are respectively

$$W_k^\epsilon = \left(H_{k+i,j}^\epsilon(q)\right)_{0\le i,j\le 5}.$$

The determinants of matrices W_k^ϵ satisfy the following recursive equation given by[16]Lemma 4.7

$$\det(W_{k+1}^\epsilon) - q\det(W_k^\epsilon) = 0.$$

Computing the determinant explicitly, we obtain

$$\det(W_k^\epsilon) = 32q^{k+\frac{5}{2}} \cdot \left(q^{\frac{1}{4}}\right)^\epsilon.$$

2.2 Asymptotic Series Expansion

In this section, we compute the asymptotic expansion of the state integral around its critical points using the method of stationary phase approximation. We will see that there are 6 critical points in 2 Galois orbits of 2 number fields $\mathbb{Q}[x]/(x^3-x^2+1)$ and $\mathbb{Q}[x]/(x^3+x^2-2x-1)$. For example, writing $\hbar := 2\pi\mathrm{i}\tau$, we will show that

Proposition 2.1:

$$Z_{(-2,3,7)}(\hbar) \sim \hat{\Phi}^{(\sigma)}(\hbar) := \frac{\mathrm{e}^{\frac{V_{0,0}}{\hbar}}}{\sqrt{\mathrm{i}(-6\xi^2+10\xi-4)}}\Bigg(1+\left(\frac{293}{8464}\xi^2+\frac{127}{2116}\xi-\frac{681}{8464}\right)\hbar + \left(\frac{65537}{6229504}\xi^2-\frac{50607}{6229504}\xi+\frac{2535}{778688}\right)\hbar^2+O(\hbar^3)\Bigg),$$

where σ is any field embedding of $\mathbb{Q}[x]/(x^3-x^2+1)$ into $\mathbb{C}$, $\xi := \sigma(x)$ and

$$V_{0,0} = 2\,\mathrm{Li}_2(-\alpha) - \mathrm{Li}_2(\alpha^{-2}), \quad \alpha = -\xi+\xi^2.$$

Proposition 2.2:

$$Z_{(-2,3,7)}(\hbar) \sim \hat{\Phi}^{(\sigma)}(\hbar) := \frac{\mathrm{e}^{\frac{V_{0,0}}{\hbar}}}{\sqrt{\mathrm{i}(-4\eta^2+2\eta-2)}}\left(1+\left(\frac{293}{8464}\xi^2+\frac{127}{2116}\xi-\frac{681}{8464}\right)\hbar\right.$$
$$\left.+\left(\frac{65537}{6229504}\xi^2-\frac{50607}{6229504}\xi+\frac{2535}{778688}\right)\hbar^2+O(\hbar^3)\right),$$

where σ is any field embedding of $\mathbb{Q}[x]/(x^3+x^2-2x-1)$ into $\mathbb{C}$, $\eta := \sigma(x)$ and

$$V_{0,0} = 2\,\mathrm{Li}_2(-\alpha)-\mathrm{Li}_2(\alpha^{-2}), \quad \alpha = -1-\eta.$$

Moreover, for the descendant state integral $Z^{(\lambda,\lambda')}_{(-2,3,7)}(\hbar)$ with $\lambda' = 0$, we will show that

Proposition 2.3:

$$Z^{(\lambda,0)}_{(-2,3,7)}(\hbar) \sim \hat{\Phi}^{(\sigma)}(\hbar,\lambda) := \frac{\mathrm{e}^{\frac{V_{0,0}}{\hbar}+\lambda\alpha}}{\sqrt{\mathrm{i}(-6\xi^2+10\xi-4)}}\left(1+\left(\left(-\frac{1}{46}\xi^2-\frac{7}{92}\xi+\frac{3}{92}\right)\lambda^2+\left(\frac{3}{46}\xi^2-\frac{11}{92}\xi+\frac{17}{46}\right)\lambda\right.\right.$$
$$\left.\left.+\frac{293}{8464}\xi^2+\frac{127}{2116}\xi-\frac{681}{8464}\right)\hbar+O(\hbar^2)\right),$$

where σ is any field embedding of $\mathbb{Q}[x]/(x^3-x^2+1)$ into $\mathbb{C}$, $\xi := \sigma(x)$ and

$$V_{0,0} = 2\,\mathrm{Li}_2(-\alpha)-\mathrm{Li}_2(\alpha^{-2}), \quad \alpha = -\xi+\xi^2.$$

Proposition 2.4:

$$Z^{(\lambda,0)}_{(-2,3,7)}(\hbar) \sim \hat{\Phi}^{(\sigma)}(\hbar,\lambda) = \frac{\mathrm{e}^{\frac{V_{0,0}}{\hbar}+\lambda\alpha}}{\sqrt{\mathrm{i}(-4\eta^2+2\eta-2)}}\left(1+\left(\left(\frac{1}{28}\eta^2+\frac{1}{14}\eta-\frac{1}{28}\right)\lambda^2+\left(\frac{1}{28}\eta^2-\frac{1}{14}\eta+\frac{3}{14}\right)\lambda\right.\right.$$
$$\left.\left.+\frac{1}{16}\eta^2+\frac{1}{16}\eta-\frac{17}{168}\right)\hbar+O(\hbar^2)\right),$$

where σ is any field embedding of $\mathbb{Q}[x]/(x^3+x^2-2x-1)$ into $\mathbb{C}$, $\eta := \sigma(x)$ and

$$V_{0,0} = 2\,\mathrm{Li}_2(-\alpha)-\mathrm{Li}_2(\alpha^{-2}), \quad \alpha = -1-\eta.$$

Throughout this section, we will keep the notation that $\hbar = 2\pi\mathrm{i}\tau$.

2.2.1 The Details of the Asymptotic Expansion

Using the identity that

$$\Phi_b(x)\Phi_b(-x) = \Phi_b(0)^2\mathrm{e}^{\pi\mathrm{i}x^2},$$

we convert the state integral into the following form,

$$Z_{(-2,3,7)}(\hbar) = \left(\frac{q}{\tilde{q}}\right)^{-\frac{1}{24}}\int_{\mathbb{R}+\mathrm{i}\frac{c_b}{2}+\mathrm{i}\varepsilon}\Phi_b(x)^2\Phi_b(2x-c_b)\mathrm{e}^{-\pi\mathrm{i}(2x-c_b)^2}\,\mathrm{d}x = \int_{\mathbb{R}+\mathrm{i}\frac{c_b}{2}+\mathrm{i}\varepsilon}\frac{\Phi_b(x)^2}{\Phi_b(-2x+c_b)}\,\mathrm{d}x,$$

and then apply the approximation[9]eq. (65)

$$\Phi_b\left(\frac{z}{2\pi b}\right) = \exp\left(\sum_{n=0}^{\infty} \hbar^{2n-1}\frac{B_{2n}(1/2)}{(2n)!}\,\mathrm{Li}_{2-2n}(-\mathrm{e}^{z})\right).$$

To start with, apply the change of variables $x \mapsto \frac{z}{2\pi b}$, then

$$\Phi_b(x)^2 = \Phi_b\left(\frac{z}{2\pi b}\right)^2 \sim \exp\left(\sum_{n=0}^{\infty} \hbar^{2n-1}\frac{B_{2n}(1/2)}{(2n)!}2\,\mathrm{Li}_{2-2n}(-\mathrm{e}^{z})\right), \quad (\hbar = 2\pi \mathrm{i} b^2)$$

and

$$\begin{aligned}\Phi_b(-2x+c_b) = \Phi_b\left(\frac{-2z+2\pi b c_b}{2\pi b}\right) &\sim \exp\left(\sum_{n=0}^{\infty} \hbar^{2n-1}\frac{B_{2n}(1/2)}{(2n)!}\,\mathrm{Li}_{2-2n}(-\mathrm{e}^{-2z+2\pi b c_b})\right)\\ &= \exp\left(\sum_{n=0}^{\infty} \hbar^{2n-1}\frac{B_{2n}(1/2)}{(2n)!}\,\mathrm{Li}_{2-2n}(\mathrm{e}^{-2z+\frac{\hbar}{2}})\right).\end{aligned}$$

Using the identity

$$\mathrm{Li}_{2-2n}(\mathrm{e}^{-2z+s}) = \sum_{k=0}^{\infty} \frac{\mathrm{Li}_{2-2n-k}(\mathrm{e}^{-2z})}{k!}s^k,$$

we have

$$\mathrm{Li}_{2-2n}(-\mathrm{e}^{-2z+\frac{\hbar}{2}}) = \sum_{k=0}^{\infty} \frac{\mathrm{Li}_{2-2n-k}(\mathrm{e}^{-2z})}{k!}\left(\frac{\hbar}{2}\right)^k.$$

Collecting the above equalities up, we obtain

$$Z_{(-2,3,7)}(\hbar) \sim \hat{\Phi}(\hbar) \coloneqq \frac{\mathrm{i}}{\sqrt{2\pi \mathrm{i}\hbar}} \int \exp V(z,\hbar)\,\mathrm{d}z,$$

where

$$
\begin{aligned}
V(z,\hbar) &= \sum_{n=0}^{\infty} \hbar^{2n-1}\frac{B_{2n}(1/2)}{(2n)!} 2\,\mathrm{Li}_{2-2n}(-\mathrm{e}^{z}) - \sum_{n,k\geq 0} \hbar^{2n+k-1}\frac{B_{2n}(1/2)}{(2n)!}\frac{\mathrm{Li}_{2-2n-k}(\mathrm{e}^{-2z})}{k!}\frac{1}{2^{k}} \\
&= \sum_{n=0}^{\infty} \hbar^{2n-1}\frac{B_{2n}(1/2)}{(2n)!} 2\,\mathrm{Li}_{2-2n}(-\mathrm{e}^{z}) \\
&\quad - \left(\sum_{n,k\geq 0} \hbar^{2n+2k-1}\frac{B_{2n}(1/2)}{(2n)!}\frac{\mathrm{Li}_{2-2n-2k}(\mathrm{e}^{-2z})}{(2k)!}\frac{1}{2^{2k}} \right. \\
&\quad \left. + \sum_{n,k\geq 0} \hbar^{2n+2k}\frac{B_{2n}(1/2)}{(2n)!}\frac{\mathrm{Li}_{1-2n-2k}(\mathrm{e}^{-2z})}{(2k+1)!}\frac{1}{2^{2k+1}} \right) \\
&= \sum_{n=0}^{\infty} \hbar^{2n-1}\left(\frac{B_{2n}(1/2)}{(2n)!} 2\,\mathrm{Li}_{2-2n}(-\mathrm{e}^{z}) - \sum_{k=0}^{\infty} \frac{B_{2n-2k}(1/2)}{(2n-2k)!(2k)!}\frac{\mathrm{Li}_{2-2n}(\mathrm{e}^{-2z})}{2^{2k}} \right) \\
&\quad + \sum_{n=0}^{\infty} \hbar^{2n}\left(-\sum_{k=0}^{\infty} \frac{B_{2n-2k}(1/2)}{(2n-2k)!(2k+1)!}\frac{\mathrm{Li}_{1-2n}(\mathrm{e}^{-2z})}{2^{2k+1}} \right).
\end{aligned}
$$

Therefore, if we define

$$
\begin{aligned}
V_{2n+1}(z) &= -\sum_{k=0}^{\infty} \frac{B_{2n-2k}(1/2)}{(2n-2k)!(2k+1)!}\frac{\mathrm{Li}_{1-2n}(\mathrm{e}^{-2z})}{2^{2k+1}}, \\
V_{2n}(z) &= \frac{B_{2n}(1/2)}{(2n)!} 2\,\mathrm{Li}_{2-2n}(-\mathrm{e}^{z}) - \sum_{k=0}^{\infty} \frac{B_{2n-2k}(1/2)}{(2n-2k)!(2k)!}\frac{\mathrm{Li}_{2-2n}(\mathrm{e}^{-2z})}{2^{2k}},
\end{aligned}
\tag{2-3}
$$

then $V(z,\hbar) = \sum_{n=0}^{\infty} \hbar^{n-1} V_n(z)$, hence

$$
\hat{\Phi}(\hbar) = \frac{\mathrm{i}}{\sqrt{2\pi\mathrm{i}\hbar}} \int \exp\left(\sum_{n=0}^{\infty} \hbar^{n-1} V_n(z) \right) \mathrm{d}z.
$$

Solving $\frac{\mathrm{d}}{\mathrm{d}z} V_0(z) = 0$, we find that the critical point equation is

$$
(\alpha^3 - \alpha - 1)(\alpha^3 + 2\alpha^2 - \alpha - 1) = 0, \quad (\alpha = \mathrm{e}^{z}).
$$

The expansion $V_n(z) = \sum_{m=0}^{\infty} (z-\alpha)^m V_{n,m}(\alpha)$ at a critical point $\mathrm{e}^{z} = \alpha$ thus gives

$$
\hat{\Phi}(\hbar) = \frac{\mathrm{i}\mathrm{e}^{\frac{V_{0,0}}{\hbar}}}{\sqrt{2\pi\mathrm{i}}} \int \mathrm{d}y \mathrm{e}^{V_{0,2} y^2} \exp\left(\sum_{m\geq 3} \hbar^{\frac{m}{2}-1} y^m V_{0,m} + \sum_{n\geq 1, m\geq 0} \hbar^{n-1+\frac{m}{2}} y^m V_{n,m} \right),
$$

where the change of variables $z \mapsto \alpha + \hbar^{\frac{1}{2}} y$ is applied, and

$$
\begin{aligned}
V_{0,0} &= 2\,\mathrm{Li}_2(-\alpha) - \mathrm{Li}_2(\alpha^{-2}),\\
V_{0,1} &= 0,\\
V_{1,0} &= -\frac{1}{2}\,\mathrm{Li}_1(\alpha^{-2}) = \frac{1}{2}\log(1-\alpha^{-2}),\\
V_{0,2} &= \mathrm{Li}_0(-\alpha) - 2\,\mathrm{Li}_0(\alpha^{-2}) = \frac{-\alpha}{1+\alpha} - \frac{2}{\alpha^2-1} = -\frac{\alpha^2-\alpha+2}{(\alpha-1)(\alpha+1)}\\
&= \alpha^5 - \alpha^4 - 7\alpha^3 + \alpha^2 + 4\alpha + 5,\\
V_{2n,m} &= \frac{1}{m!}\left(\frac{B_{2n}(1/2)}{(2n)!} 2\,\mathrm{Li}_{2-2n-m}(-\alpha) - (-2)^m\,\mathrm{Li}_{2-2n-m}(\alpha^{-2}) \sum_{k=0}^{n} \frac{B_{2n-2k}(1/2)}{(2n-2k)!(2k)!2^{2k}}\right),\\
V_{2n+1,m} &= -\frac{(-2)^m}{m!}\,\mathrm{Li}_{1-2n-m}(\alpha^{-2}) \sum_{k=0}^{n} \frac{B_{2n-2k}(1/2)}{(2n-2k)!(2k+1)!2^{2k+1}}.
\end{aligned}
\tag{2-4}
$$

Note that for $n = 1$ and $m = 0$, we have $\hbar^{n-1+\frac{m}{2}} y^m V_{n,m} = V_{1,0}$. Expand the exponential in the integrand, collect $\hbar$'s and use the formal Gaussian integrals, we obtain

$$
\hat{\Phi}(\hbar) = \frac{\mathrm{e}^{\frac{V_{0,0}}{\hbar}} \mathrm{e}^{V_{1,0}}}{\sqrt{2\mathrm{i}V_{0,2}}} \left(1 + O(\hbar)\right).
$$

If we define $\Delta := \frac{2V_{0,2}}{\mathrm{e}^{2V_{1,0}}}$, then

$$
\hat{\Phi}(\hbar) = \frac{\mathrm{e}^{\frac{V_{0,0}}{\hbar}}}{\sqrt{\mathrm{i}\Delta}} \left(1 + O(\hbar)\right),
$$

with

$$
\Delta = \frac{-2\alpha^2(\alpha^2-\alpha+2)}{(\alpha-1)^2(\alpha+1)^2} = -2\alpha^5 + 12\alpha^3 - 2\alpha^2 - 16\alpha - 10.
$$

2.2.1.1 Explicit Asymptotic Expansion in Number Field of Discriminant -23

Explicitly, when α is a root of $x^3 - x - 1$, we have

$$
\begin{aligned}
V_{1,0} &= \frac{1}{2}\log(1-\alpha^{-2}) = \frac{1}{2}\log(1-\xi^2),\\
V_{0,2} &= -\frac{\alpha^2-\alpha+2}{(\alpha-1)(\alpha+1)} = -3\xi^2 + 2\xi,
\end{aligned}
$$

where ξ is a root of $x^3 - x^2 + 1$, which is related to α by

$$
\xi = 1 - \alpha^2, \quad \alpha = -\xi + \xi^2,
$$

hence

$$\Delta = -6\xi^2 + 10\xi - 4.$$

Computing it out, we obtain

$$Z_{(-2,3,7)}(\hbar) \sim \hat{\Phi}^{(\sigma)}(\hbar) = \frac{e^{\frac{V_{0,0}}{\hbar}}}{\sqrt{i\Delta}}\left(1 + \left(\frac{293}{8464}\xi^2 + \frac{127}{2116}\xi - \frac{681}{8464}\right)\hbar + \left(\frac{65537}{6229504}\xi^2 - \frac{50607}{6229504}\xi + \frac{2535}{778688}\right)\hbar^2 + O(\hbar^3)\right),$$

concluding the proof of proposition 2.1.

2.2.1.2 Explicit Asymptotic Expansion in Number Field of Discriminant 49

When α is a root of $x^3 + 2x^2 - x - 1$, we have

$$V_{1,0} = \frac{1}{2}\log(1 - \alpha^{-2}) = \frac{1}{2}\log(\eta^2 + \eta - 2),$$
$$V_{0,2} = -\frac{\alpha^2 - \alpha + 2}{(\alpha - 1)(\alpha + 1)} = -\eta^2 - 3\eta + 3,$$

where η is a root of $x^3 + x^2 - 2x - 1$, which is related to α by

$$\alpha = -1 - \eta,$$

hence

$$\Delta = -4\eta^2 + 2\eta - 2.$$

Computing it out, we obtain

$$Z_{(-2,3,7)}(\hbar) \sim \hat{\Phi}^{(\sigma)}(\hbar) = \frac{e^{\frac{V_{0,0}}{\hbar}}}{\sqrt{i\Delta}}\left(1 + \left(\frac{1}{16}\eta^2 + \frac{1}{16}\eta - \frac{17}{168}\right)\hbar + \left(\frac{23}{5376}\eta^2 + \frac{43}{10752}\eta + \frac{85}{225792}\right)\hbar^2 + O(\hbar^3)\right),$$

concluding the proof of proposition 2.2.

2.2.2 Asymptotic Expansion of the Descendant State Integral

Recalling that the descendant State Integral of $(-2,3,7)$ pretzel knot is

$$Z^{(\lambda,\lambda')}_{(-2,3,7)}(\hbar)=\left(\frac{q}{\tilde{q}}\right)^{-\frac{1}{24}}\int_{\mathbb{R}+\mathrm{i}\frac{c_b}{2}+\mathrm{i}\varepsilon}\Phi_b(x)^2\Phi_b(2x-c_b)\mathrm{e}^{-\pi\mathrm{i}(2x-c_b)^2+2\pi(\lambda b-\lambda' b^{-1})x}\,\mathrm{d}x$$
$$=\int_{\mathbb{R}+\mathrm{i}\frac{c_b}{2}+\mathrm{i}\varepsilon}\frac{\Phi_b(x)^2}{\Phi_b(-2x+c_b)}\mathrm{e}^{2\pi(\lambda b-\lambda' b^{-1})x}\,\mathrm{d}x,$$

we compute its asymptotic expansion when $\lambda'=0$.

Apply the change of variables $x\mapsto\frac{z}{2\pi b}$, then $\mathrm{e}^{2\pi\lambda bx}\mapsto\mathrm{e}^{\lambda z}$. Similar as before, we obtain

$$Z^{(\lambda,0)}_{(-2,3,7)}(\hbar)\sim\hat{\Phi}(\hbar,\lambda):=\frac{1}{\sqrt{2\pi\mathrm{i}\hbar}}\int\exp\left(\lambda z+V(z,\hbar)\right)\mathrm{d}z,$$

with the same $V(z,\hbar)=\sum_{n=0}^{\infty}\hbar^{n-1}V_n(z)=\sum_{n=0}^{\infty}\hbar^{n-1}\sum_{m=0}^{\infty}(z-\alpha)^m V_{n,m}(\alpha)$ as in eqs. (2-3) and (2-4). The critical point equation stays the same, which is

$$(\alpha^3-\alpha-1)(\alpha^3+2\alpha^2-\alpha-1)=0,\quad(\alpha=\mathrm{e}^z).$$

Therefore, applying the change of variables $z\mapsto\alpha+\hbar^{\frac{1}{2}}y$, we obtain

$$\hat{\Phi}(\hbar,\lambda)=\frac{\mathrm{i}\mathrm{e}^{\frac{V_{0,0}}{\hbar}+\lambda\alpha}}{\sqrt{2\pi\mathrm{i}}}\int\mathrm{d}y\mathrm{e}^{V_{0,2}y^2}\exp\left(\lambda\hbar^{\frac{1}{2}}y+\sum_{m\geq3}\hbar^{\frac{m}{2}-1}y^mV_{0,m}+\sum_{n\geq1,m\geq0}\hbar^{n-1+\frac{m}{2}}y^mV_{n,m}\right).$$

Expand and apply the formal Gaussian integrals, we obtain

$$\hat{\Phi}(\hbar,\lambda)=\frac{\mathrm{e}^{\frac{V_{0,0}}{\hbar}+\lambda\alpha}}{\sqrt{\mathrm{i}\Delta}}\left(1+O(\hbar)\right),$$

where

$$\Delta:=\frac{2V_{0,2}}{\mathrm{e}^{2V_{1,0}}}=\frac{-2\alpha^2(\alpha^2-\alpha+2)}{(\alpha-1)^2(\alpha+1)^2}=-2\alpha^5+12\alpha^3-2\alpha^2-16\alpha-10.$$

Explicitly, when α is a root of x^3-x-1, we have

$$Z^{(\lambda,0)}_{(-2,3,7)}(\hbar)\sim\hat{\Phi}(\hbar,\lambda)=\frac{\mathrm{e}^{\frac{V_{0,0}}{\hbar}+\lambda\alpha}}{\sqrt{\mathrm{i}\Delta}}\left(1+\left(\left(-\frac{1}{46}\xi^2-\frac{7}{92}\xi+\frac{3}{92}\right)\lambda^2+\left(\frac{3}{46}\xi^2-\frac{11}{92}\xi+\frac{17}{46}\right)\lambda\right.\right.$$
$$\left.\left.+\frac{293}{8464}\xi^2+\frac{127}{2116}\xi-\frac{681}{8464}\right)\hbar+O(\hbar^2)\right),$$

where ξ is a root of x^3-x^2+1, which is related to α by

$$\xi=1-\alpha^2,\quad\alpha=-\xi+\xi^2,$$

and hence

$$\Delta=-6\xi^2+10\xi-4.$$

This gives proposition 2.3.

When α is a root of x^3+2x^2-x-1, we have

$$Z^{(\lambda,0)}_{(-2,3,7)}(\hbar) \sim \hat{\Phi}(\hbar,\lambda) = \frac{\mathrm{e}^{\frac{V_{0,0}}{\hbar}+\lambda\alpha}}{\sqrt{\mathrm{i}\Delta}}\Bigg(1+\Bigg(\left(\frac{1}{28}\eta^2+\frac{1}{14}\eta-\frac{1}{28}\right)\lambda^2+\left(\frac{1}{28}\eta^2-\frac{1}{14}\eta+\frac{3}{14}\right)\lambda$$
$$+\frac{1}{16}\eta^2+\frac{1}{16}\eta-\frac{17}{168}\Bigg)\hbar+O(\hbar^2)\Bigg),$$

where η is a root of x^3+x^2-2x-1, which is related to α by

$$\alpha = -1-\eta,$$

and hence

$$\Delta = -4\eta^2+2\eta-2.$$

This gives proposition 2.4.

CHAPTER 3 OBJECTIVES

To summarize, the following table presents the main properties observed so far for the 4_1 knot and the 5_2 knot.

Number	Properties of State Integral	4_1 Knot	5_2 Knot
(I)	Relation with QMC	Observations 1 and 2	Observation 5
(II)	Relation with Index	Observation 4	Observation 6
(III)	Quadratic Relation	Trivial	Observation 7
(IV)	q-Difference Equation	Theorem 1.1	See G & Z[6]Sec. 4.3
(V)	Relation of Descendant with QMC	Observation 8	Not Found Yet

For the $(-2,3,7)$ pretzel knot, (I), (II) and (III) have been done by observations 9 and 10; furthermore, the quadratic relation also holds up to a normalization for the descendant state integral as is shown in eq. (2-1). (IV) has been given from the author and An Ni's computation by theorem 2.2.

With these understood, the next step is to numerically investigate the asymptotic expansions in section 2.2, analyze their behaviour as τ approaches to 0 along different rays and try to find a similar relation as (V). After that, we will explore the possibility of finding new relations on the $(-2,3,7)$ pretzel knot, and move on to compute other examples.

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