System identification (laboratory)

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Exercise title： Parametric offline identification: ARMA & ARX

Group： 1

Reporter： Li Zeyu

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Professional： Automatic Control and Robotics

Full-time First-cycle

studies ID number： 150586

Colleges： WARiE

1 ARMA model

The Auto regressive Moving Average (ARMA) model for time series yt is a deterministic model with an input time

$$yt = \frac{B(q^{-1})}{A(q^{-1})} u(t-d) \text{ where -d is a delay}$$

The Arma model is contructed from the a and b coefficients and the respective data values
The formula

$$yt = \frac{b_1 q^{-1} + b_2 q^{-2} + b_3 q^{-3} + b_4 q^{-4} + b_5 q^{-5} + b_6 q^{-6}}{1 + a_1 q^{-1} + a_2 q^{-2} + a_3 q^{-3} + a_4 q^{-4} + a_5 q^{-5} + a_6 q^{-6}}$$

We write down the model, the cost function of least square estimation and its gradient as following:

Using

$$y_t = \underline{\varphi}_t^T \underline{\theta}$$

for N measured input and output samples, we can write vector - matrix expression:

$$\underline{y}_N = \mathbf{\Phi}_N \underline{\theta}$$

where

$$\underline{y}_N = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} \quad \mathbf{\Phi}_N = \begin{bmatrix} \underline{\varphi}_1^T \\ \underline{\varphi}_2^T \\ \vdots \\ \underline{\varphi}_N^T \end{bmatrix} \quad \underline{\theta} = \begin{bmatrix} a_1 \\ \vdots \\ a_{nA} \\ b_0 \\ b_1 \\ \vdots \\ b_{nB} \end{bmatrix}$$

In the vector-matrix expression of ARMA model

$$\underline{y}_N = \mathbf{\Phi}_N \underline{\theta} \tag{1}$$

there is no disturbance, therefore we can calculate vector $\underline{\theta}$ from the above equation. In the case when N equals to the number of parameters - by solving directly (1). If N is higher, we can:

- multiply both sides of (1) by $\mathbf{\Phi}_N^T$, to obtain square matrix $\mathbf{\Phi}_N^T \mathbf{\Phi}_N$

$$\mathbf{\Phi}_N^T \underline{y}_N = \mathbf{\Phi}_N^T \mathbf{\Phi}_N \underline{\theta}$$

- multiply both sides by $(\mathbf{\Phi}_N^T \mathbf{\Phi}_N)^{-1}$ from the left

$$(\mathbf{\Phi}_N^T \mathbf{\Phi}_N)^{-1} \mathbf{\Phi}_N^T \underline{y}_N = (\mathbf{\Phi}_N^T \mathbf{\Phi}_N)^{-1} \mathbf{\Phi}_N^T \mathbf{\Phi}_N \underline{\theta}$$

Right hand side is just $\underline{\theta}$, therefore we obtain formula for $\underline{\theta}$:

$$\hat{\underline{\theta}} = (\mathbf{\Phi}_N^T \mathbf{\Phi}_N)^{-1} \mathbf{\Phi}_N^T \underline{y}_N \tag{2}$$

Let us consider the case for an ARMA(6,6) model without delay, such that d = 0 and b0 = 0. The time series is given in the data file 15_ARMA.mat. The parameters vector, θ had these values :

Theta =

-0.5000
0.6000
0.3000
0.2000
-0.2000
0.3000
-8.8000
5.2000
-4.9000
1.6000
-0.7000
0.2000

For the parameters, finally obtain a discrete transfer function of model with sampling time of 0.1

G_d =

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     -8.8 z^5 + 5.2 z^4 - 4.9 z^3 + 1.6 z^2 - 0.7 z + 0.2
  ---------------------------------------------------------
  z^6 - 0.5 z^5 + 0.6 z^4 + 0.3 z^3 + 0.2 z^2 - 0.2 z + 0.3
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2 ARX model

The transfer function form of the ARX model specification is

$$y_t = \frac{B(q^{-1})}{A(q^{-1})} u_{t-d} + \frac{1}{A(q^{-1})} \xi_t,$$

In the ARX model, we include the disturbance, the Least square method in this model is based on the minimizing of the squared errors, Vn between the measured output of the system and the measured output of the model. We take into account that we can't measure the disturbance hence we use the one step ahead method. The calculations of the parameters, θ is similar to the ARMA model above.

$$\mathrm{Vn} = \left(\mathrm{yn} - \varphi \mathrm{n}\hat{\theta}\right)^T (\mathrm{yn} - \varphi \mathrm{n}\hat{\theta})$$

The Akaike information criterion suggests that the best result of the least square estimation of the ARX process is obtained when one chooses the optimal order *n*, according to

$$\mathrm{AIC}(n) = \ln V_N(n) + \frac{4n}{N},$$

Where VN(n) is the error of the least square estimation at optimal parameter with model order n.When we calculate the AIC values for each of the assumed order, in our case m=10 we can compare the values for each of these AIC values.

Vn 1 - 10

Vn	99.9114
Vn1	1.5575e+06
Vn10	99.9114
Vn2	5.5893e+05
Vn3	8.2665e+03
Vn4	7.5646e+03
Vn5	1.9468e+03
Vn6	101.1387
Vn7	100.8989
Vn8	100.8407
Vn9	100.5266

AIC m= 1 - 10

aic	4.6547
AIC_1	14.2626
AIC_10	4.6446
AIC_2	13.2418
AIC_3	9.0320
AIC_4	8.9473
AIC_5	7.5940
AIC_6	4.6406
AIC_7	4.6423
AIC_8	4.6458
AIC_9	4.6467

The AIC which is lowest is for the order 6. Meaning our order for the ARX has order 6. It is represented by this transfer function with its parameters with a sampling time of 0.2.

model =

$$\frac{0.8034 z^5 + 0.5016 z^4 + 0.9005 z^3 + 0.5022 z^2 + 0.3009 z + 0.1005}{z^6 + 7.799 z^5 - 6.173 z^4 + 0.8726 z^3 - 0.2926 z^2 - 0.1959 z + 0.0049}$$