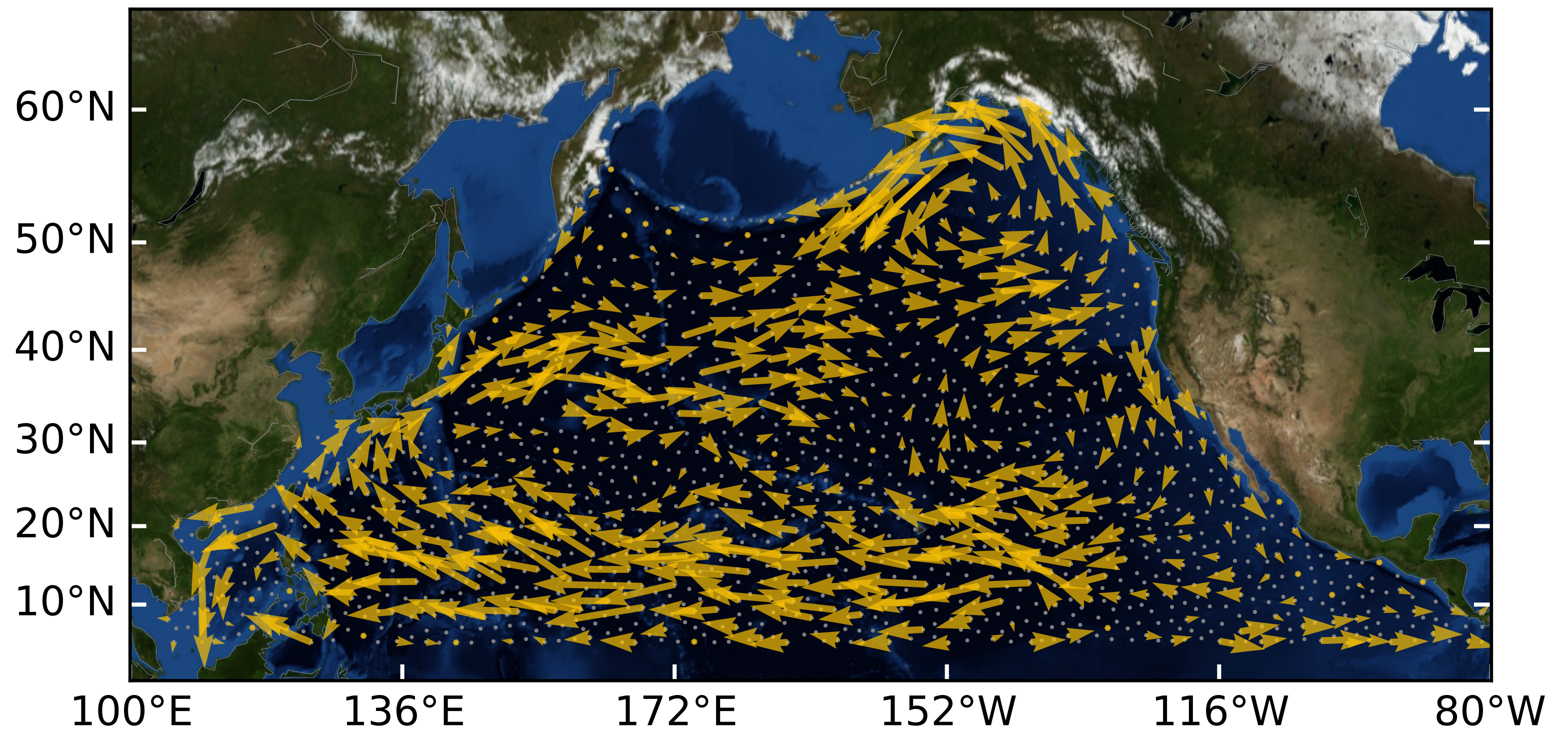
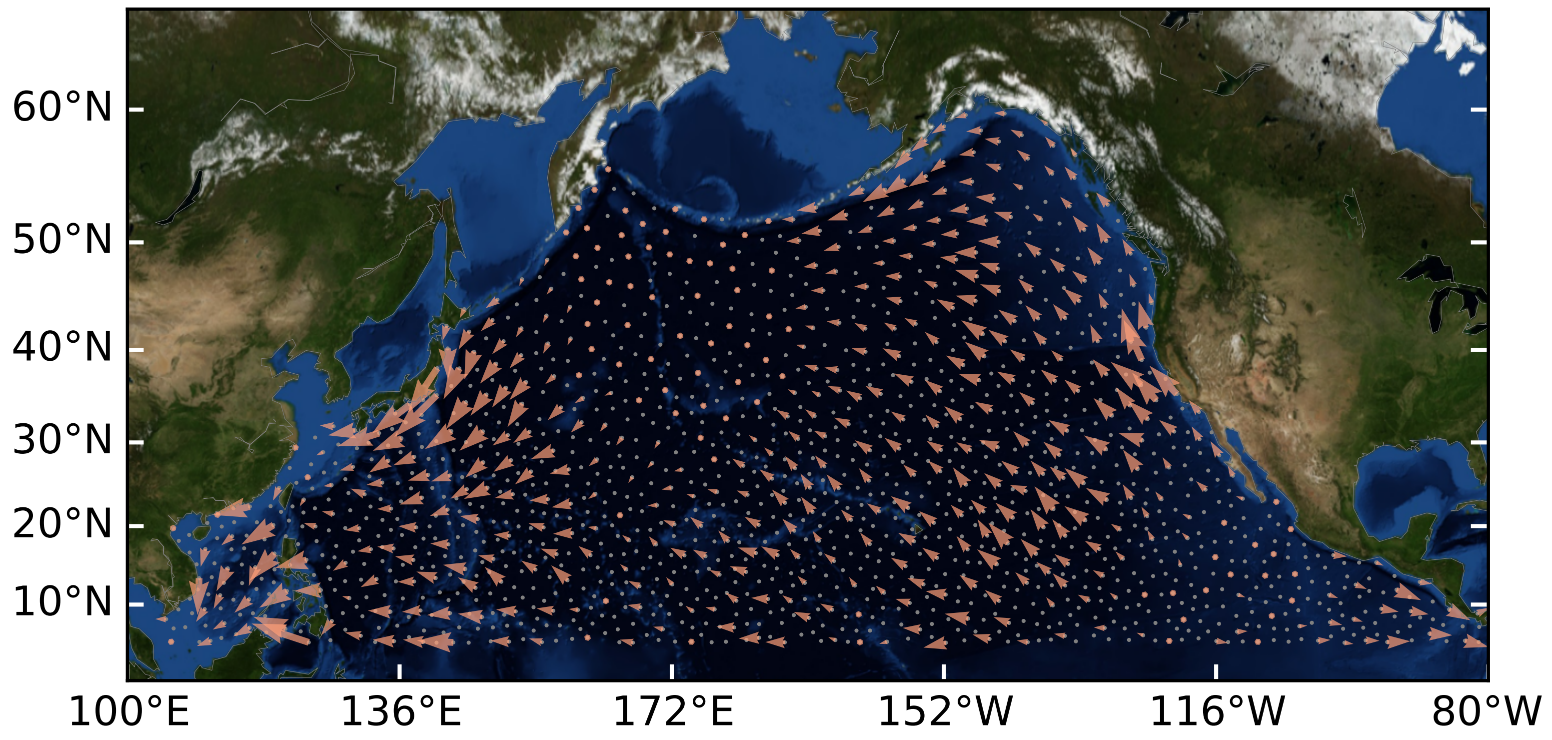
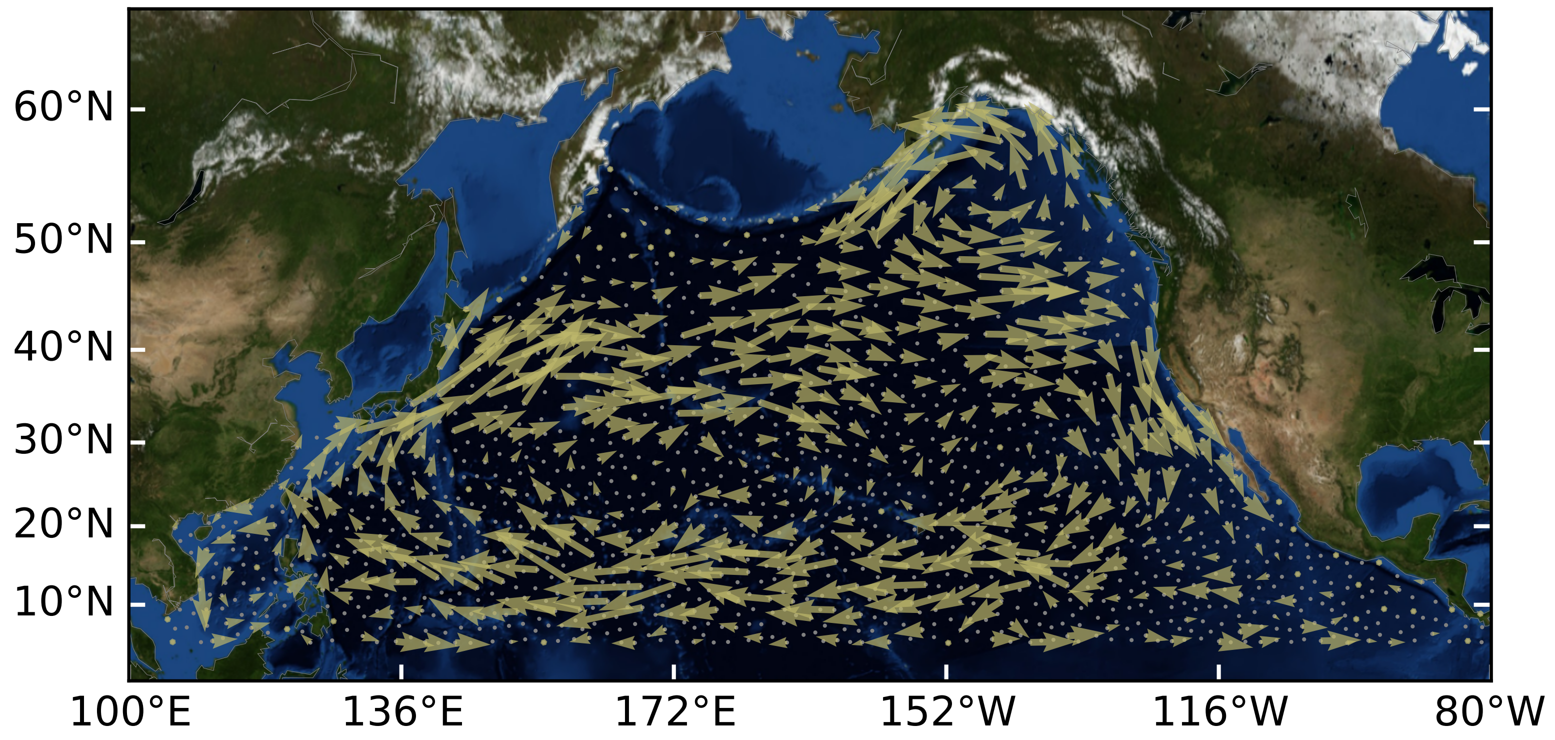
Applications of Hodge decomposition



Ocean currents



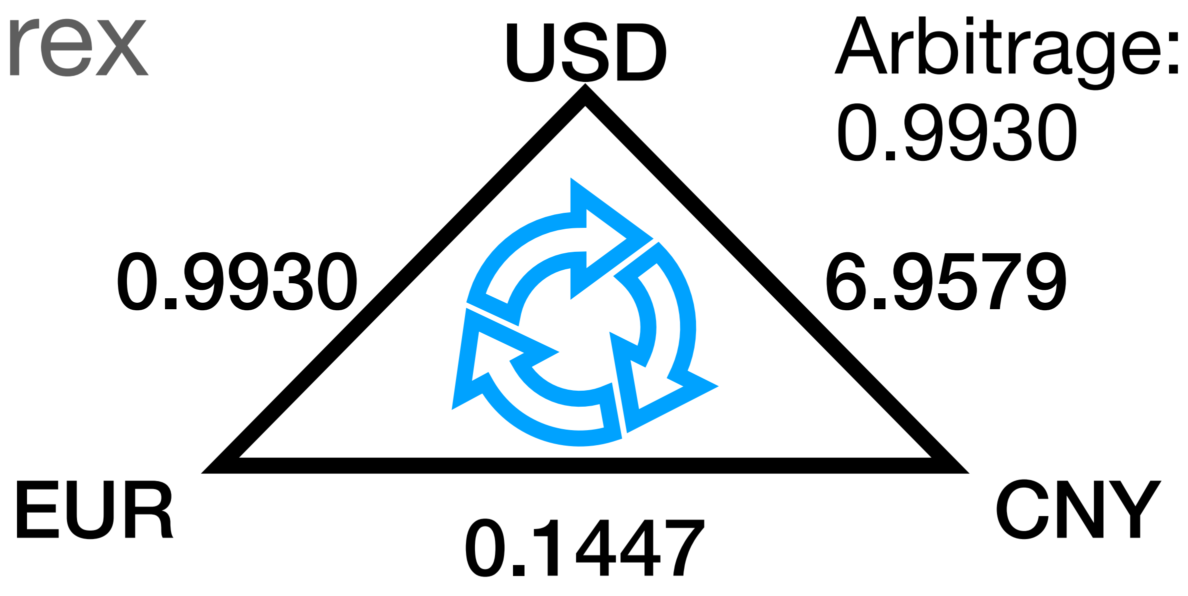
Gradient flow
Curl-free, irrotational



Curl flow
Div-free, solenoidal

Chen, Yu-Chia et al. (2021) "Helmholtzian Eigenmap."

Forex



Arbitrage:
0.9930

$$r^{a/b} r^{b/c} = r^{a/c}$$ Arbitrage-free

$$f_{[a,b]} + f_{[b,c]} - f_{[a,c]} = 0$$ Curl-free

- Water flows (div-free)
- Electrical currents, voltages

- Brain networks (Anand et al. 2022)
- Game theory (Candogan et al. 2011)
- Ranking problems (Jiang et al. 2011)
- Random walks (Strang et al. 2020)
- …

Eigenspace of $\mathbf{L}_1$ spans Hodge subspaces

- **Nonzero** Eigenspace of down Laplacian spans the gradient space
- **Nonzero** Eigenspace of up Laplacian spans the curl space
- Kernel of Laplacian spans the harmonic space

Simplicial Fourier transform

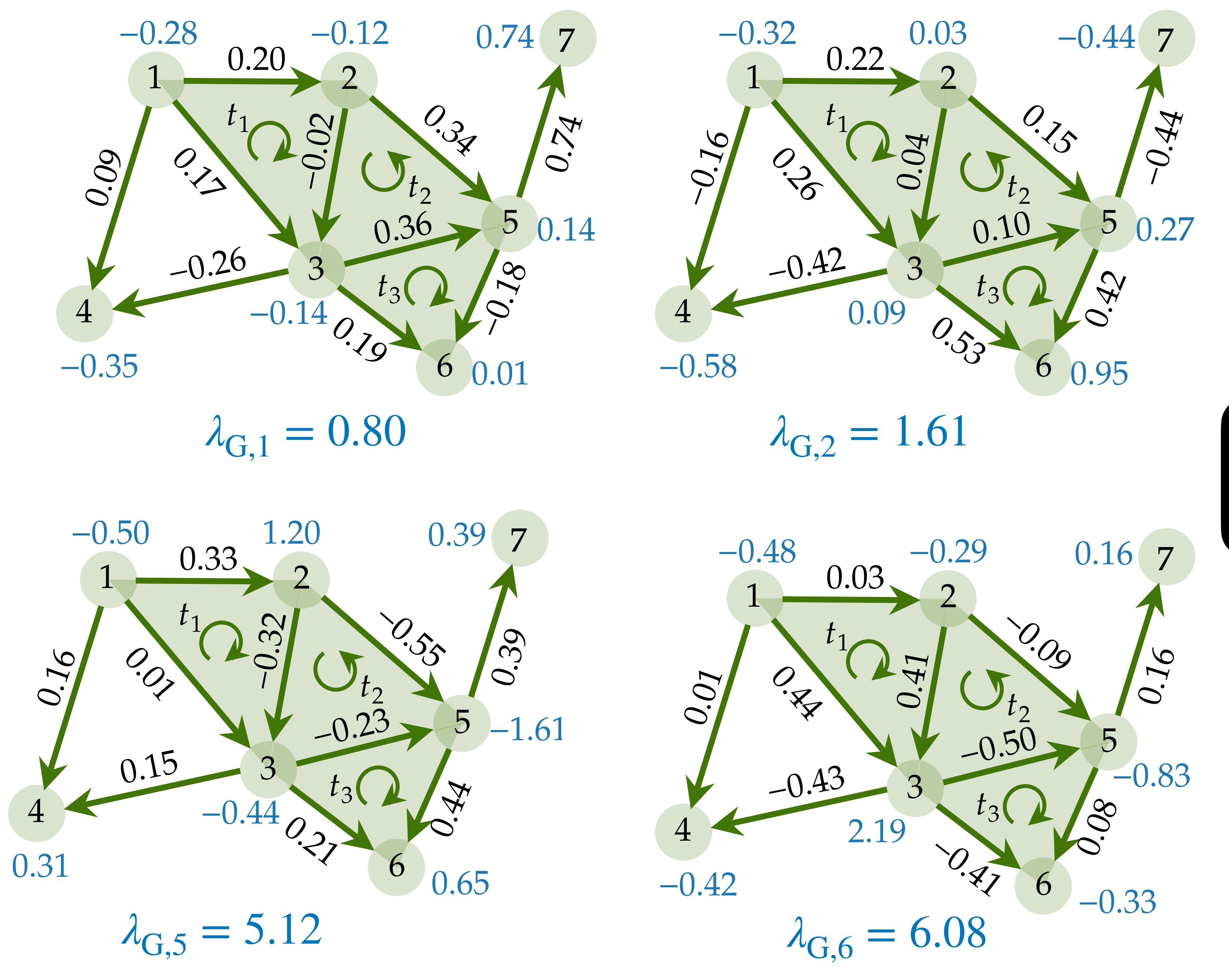
Frequency — eigenvalues

Fourier basis — eigenvectors

$\lambda_G = \|\mathbf{B}_1 \mathbf{u}_G\|_2^2$

Gradient eigenvector

Fourier basis reflecting divergent properties

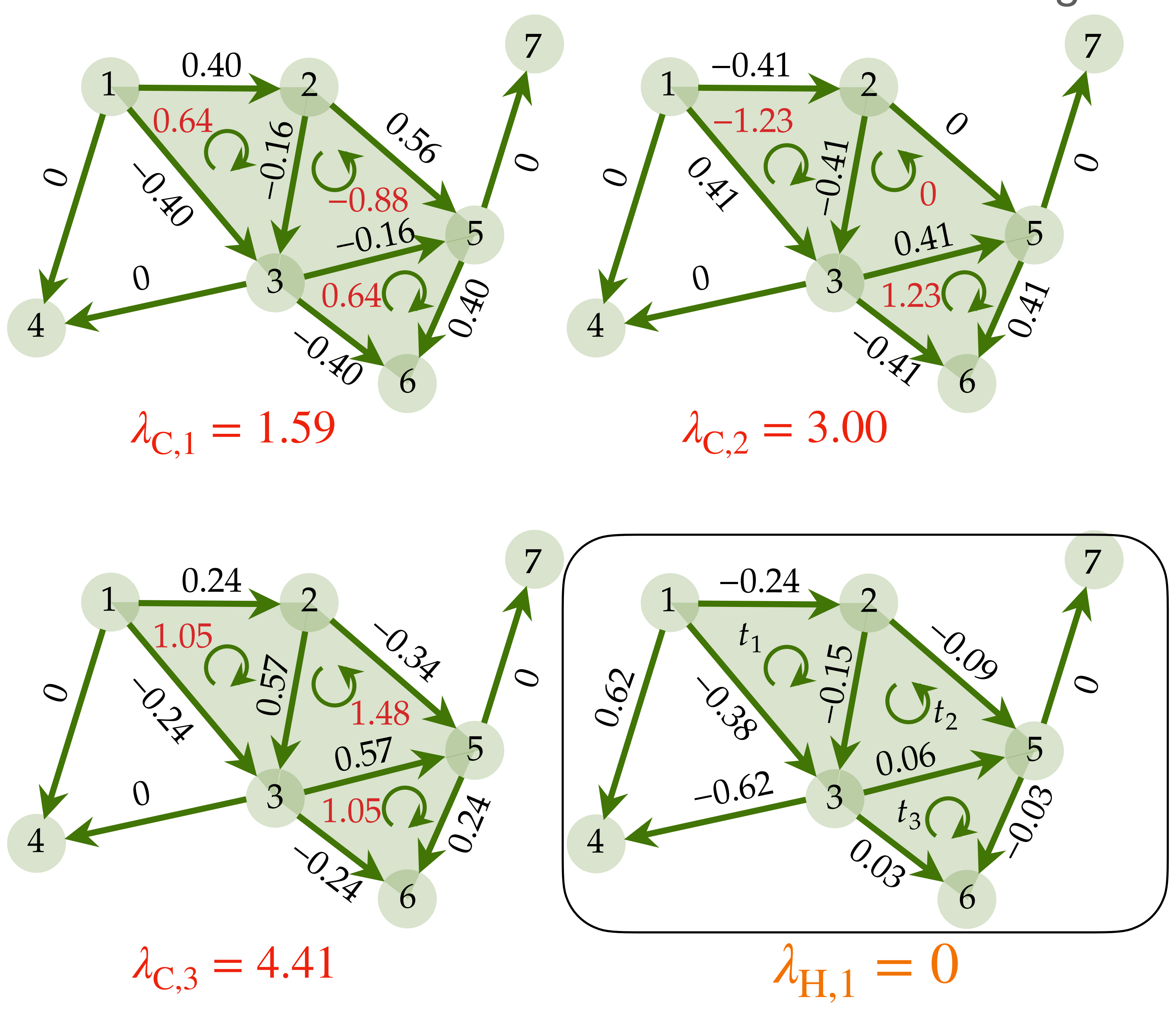


$k = 1$

$\lambda_C = \|\mathbf{B}_2^\top \mathbf{u}_C\|_2^2$

Curl eigenvector

Fourier basis reflecting rotational properties



EVD: $\mathbf{L}_1 = \mathbf{U}_1 \boldsymbol{\Lambda}_1 \mathbf{U}_1^\top$

$\mathbf{U}_1 = [\mathbf{U}_H \ \mathbf{U}_G \ \mathbf{U}_C]$
$\mathrm{span}(\mathbf{U}_H) = \ker(\mathbf{L}_1)$
$\mathrm{span}(\mathbf{U}_G) = \mathrm{im}(\mathbf{B}_1^\top)$
$\mathrm{span}(\mathbf{U}_C) = \mathrm{im}(\mathbf{B}_2)$

- $\tilde{\mathbf{x}}_k = \mathbf{U}_k^\top \mathbf{x}_k, \ k = 1$
- $\tilde{\mathbf{x}}_k = [\tilde{\mathbf{x}}_{k,\mathrm{H}}^\top, \tilde{\mathbf{x}}_{k,\mathrm{G}}^\top, \tilde{\mathbf{x}}_{k,\mathrm{C}}^\top]$