Matérn Edge GPs

Derived from SDEs on the edge set

- $\mathbf{f}_1 \sim \mathrm{GP}(\mathbf{0}, \mathbf{K}_1)$

- Matérn graph kernel

EVD: $\mathbf{L}_1 = \mathbf{U}_1 \mathbf{\Lambda}_1 \mathbf{U}_1^\top$

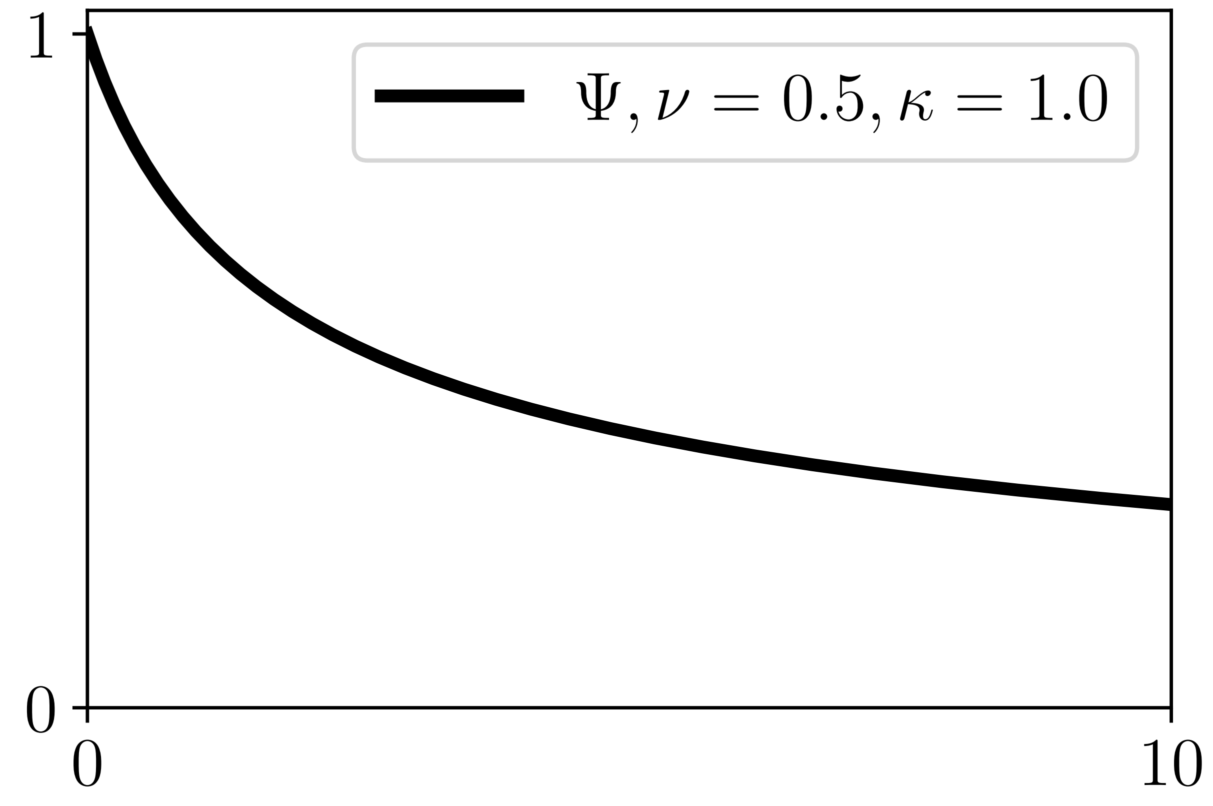
$\Phi(\mathbf{L}_1)\mathbf{f}_1 = \mathbf{w}_1$, with

$$\Phi(\mathbf{L}_1) = \left(\frac{2\nu}{\kappa^2}\mathbf{I} + \mathbf{L}_1\right)^{\frac{\nu}{2}} \text{ and } \mathbf{w}_1 \sim N(\mathbf{0}, \sigma^2\mathbf{I})$$

- The solution gives edge GPs

$$\text{Matérn: } \mathbf{f}_1 \sim \mathrm{GP}\left(0, \left(\frac{2\nu}{\kappa^2}\mathbf{I} + \mathbf{L}_1\right)^{-\nu}\right)$$

$$\text{Diffusion: } \mathbf{f}_1 \sim \mathrm{GP}\left(0, e^{-\frac{\kappa^2}{2}\mathbf{L}_1}\right)$$



- Low-pass in the eigen-spectrum

Smoothness

Node function — 0-form (scalar field)

Edge function — 1-form (vector field)

Divergence

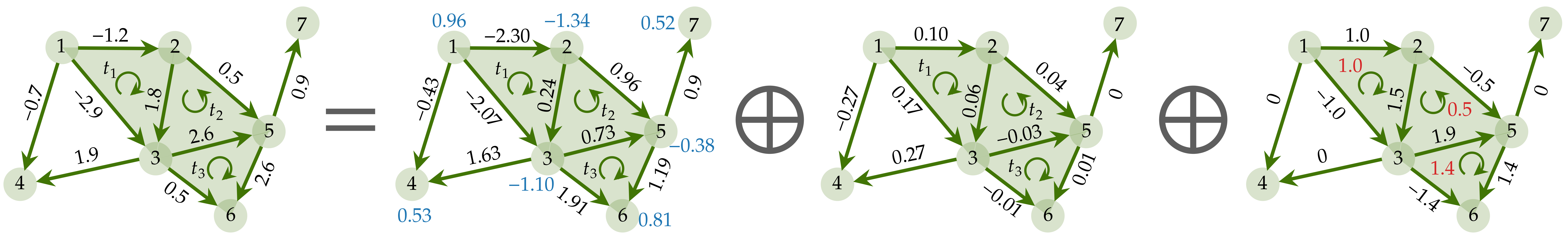
Curl

Hodge decomposition

Lovász et al. 2004; Lim et al. 2020

$$\mathbb{R}^{N_1} = \mathrm{im}(\mathbf{B}_1^\top) \oplus \ker(\mathbf{L}_1) \oplus \mathrm{im}(\mathbf{B}_2)$$

$$\mathbf{f}_1 = \mathbf{f}_G + \mathbf{f}_H + \mathbf{f}_C$$



Edge flow = Gradient flow + Harmonic flow + Curl flow

Gradient flow: Curl-free, irrotational

Harmonic flow: Div- and curl-free

Curl flow: Div-free, solenoidal

Hodge-compositional Edge GP

$$\mathbf{f}_G \sim \mathrm{GP}(\mathbf{0}, \mathbf{K}_G)$$
$$\mathbf{f}_H \sim \mathrm{GP}(\mathbf{0}, \mathbf{K}_H)$$
$$\mathbf{f}_C \sim \mathrm{GP}(\mathbf{0}, \mathbf{K}_C)$$