

# Matérn Edge GPs

## Derived from SDEs on the edge set

- $\mathbf{f}_1 \sim \text{GP}(\mathbf{0}, \mathbf{K}_1)$
- Matérn graph kernel

$$\text{EVD: } \mathbf{L}_1 = \mathbf{U}_1 \mathbf{\Lambda}_1 \mathbf{U}_1^\top$$

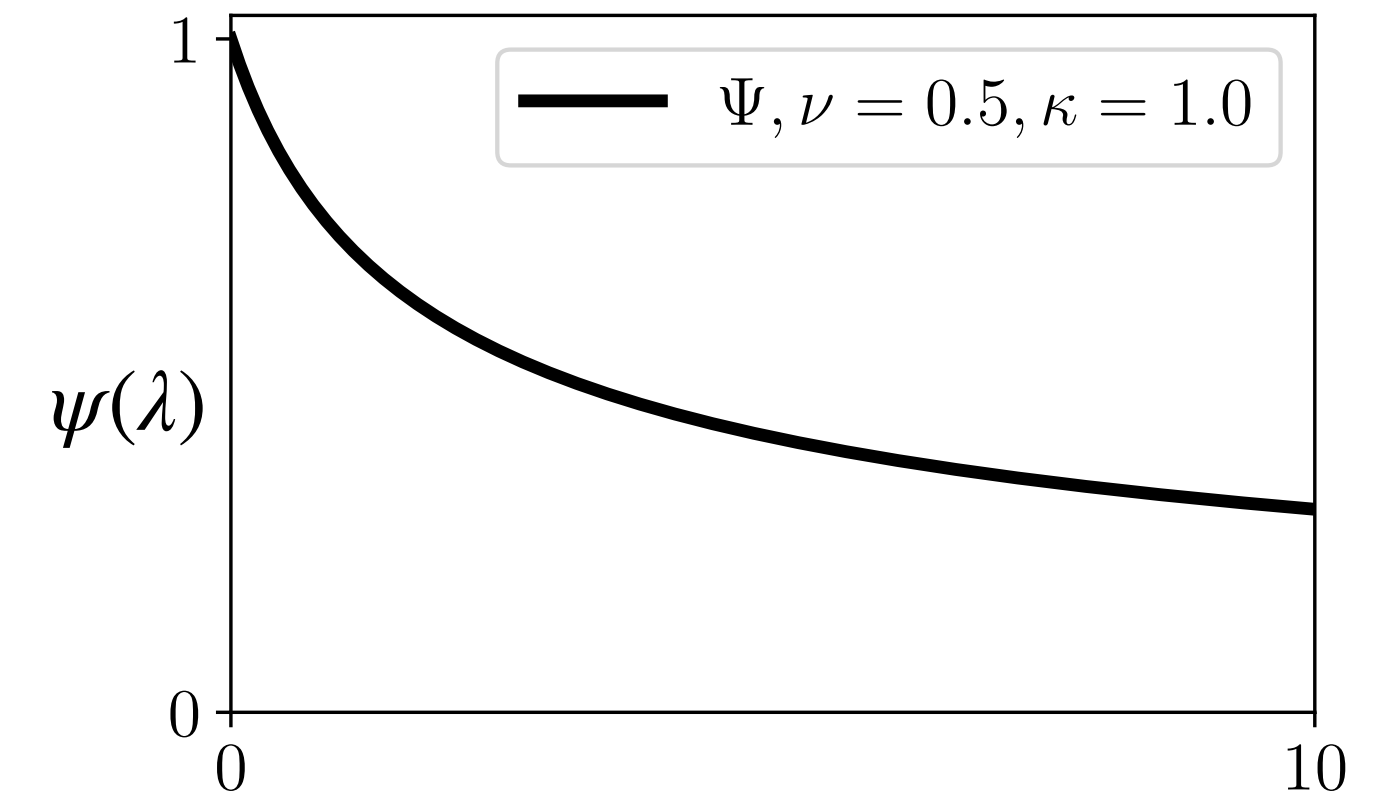
$$\Phi(\mathbf{L}_1) \mathbf{f}_1 = \mathbf{w}_1, \text{ with}$$

$$\Phi(\mathbf{L}_1) = \left( \frac{2\nu}{\kappa^2} \mathbf{I} + \mathbf{L}_1 \right)^{\frac{\nu}{2}} \text{ and } \mathbf{w}_1 \sim N(\mathbf{0}, \sigma^2 \mathbf{I})$$

- The solution gives edge GPs

$$\text{Matérn: } \mathbf{f}_1 \sim \text{GP}\left(0, \left(\frac{2\nu}{\kappa^2} \mathbf{I} + \mathbf{L}_1\right)^{-\nu}\right)$$

$$\text{Diffusion: } \mathbf{f}_1 \sim \text{GP}\left(0, e^{-\frac{\kappa^2}{2} \mathbf{L}_1}\right)$$



- Low-pass in the eigen-spectrum

### Smoothness

Node function — 0-form (scalar field)  
Edge function — 1-form (vector field)

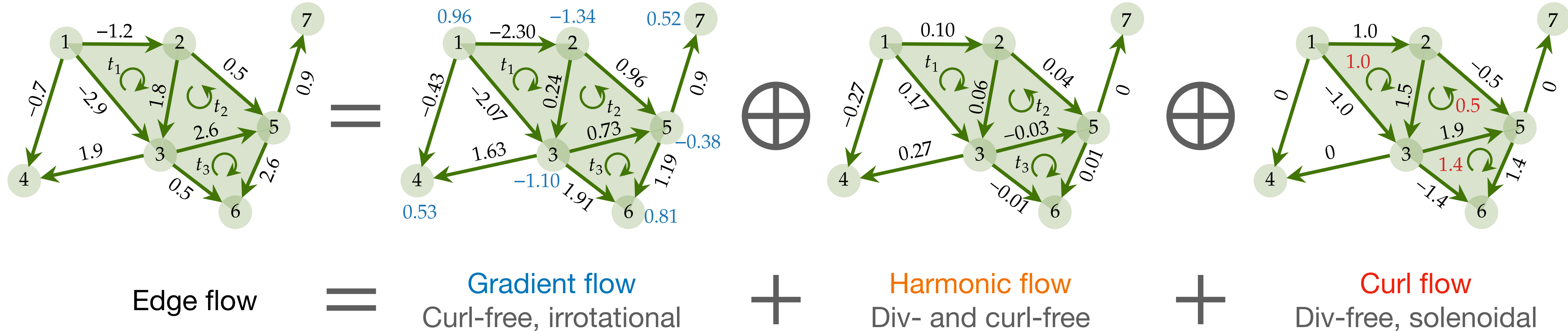
Divergence  
Curl

# Hodge decomposition

Lovász et al. 2004; Lim et al. 2020

$$\mathbb{R}^{N_1} = \text{im}(\mathbf{B}_1^\top) \oplus \text{ker}(\mathbf{L}_1) \oplus \text{im}(\mathbf{B}_2)$$

$$\mathbf{f}_1 = \mathbf{f}_G + \mathbf{f}_H + \mathbf{f}_C$$



Hodge-compositional Edge GP

$$\mathbf{f}_G \sim \text{GP}(\mathbf{0}, \mathbf{K}_G)$$

$$\mathbf{f}_H \sim \text{GP}(\mathbf{0}, \mathbf{K}_H)$$

$$\mathbf{f}_C \sim \text{GP}(\mathbf{0}, \mathbf{K}_C)$$