

Sampling gradient and curl edge GPs

Proof. We focus on the case of gradient GPs. First, we can decompose the gradient kernel in terms of $U_1 = [U_H \ U_G \ U_C]$ as

$$K_G = U_1 \begin{pmatrix} \mathbf{0} & & \\ & \Psi_G(\Lambda_G) & \\ & & \mathbf{0} \end{pmatrix} U_1^\top. \quad (\text{B.9})$$

From a vector $\mathbf{v} = (v_1, \dots, v_{N_1})^\top$ of variables following independent normal distribution, we can draw a random sample of gradient function as

$$\mathbf{f}_G = U_1 \text{diag}([\mathbf{0}, \Psi_G^{\frac{1}{2}}(\Lambda_G), \mathbf{0}]) \mathbf{v} \quad (\text{B.10})$$

where $\text{diag}([\mathbf{a}, \mathbf{b}, \mathbf{c}])$ is the diagonal matrix with $(\mathbf{a}, \mathbf{b}, \mathbf{c})^\top$ on its diagonal.

Therefore, their curls are

$$\text{curl } \mathbf{f}_G = \mathbf{B}_2^\top U_1 \text{diag}([\mathbf{0}, \Psi_G^{\frac{1}{2}}(\Lambda_G), \mathbf{0}]) = \mathbf{B}_2^\top U_G \Psi_G^{\frac{1}{2}}(\Lambda_G) = \mathbf{0}. \quad (\text{B.11})$$

Likewise, we can show the samples of a curl GP are div-free.

Posterior distribution of Hodge components

$$\begin{bmatrix} f_H(\mathbf{x}) \\ f_H(\mathbf{x}^*) \\ f_G(\mathbf{x}) \\ f_G(\mathbf{x}^*) \\ f_C(\mathbf{x}) \\ f_C(\mathbf{x}^*) \\ f_1(\mathbf{x}) \\ f_1(\mathbf{x}^*) \end{bmatrix} \sim \mathcal{N} \left(\mathbf{0}, \begin{bmatrix} K_H & K_H^* & & & & & & \\ K_H^{*\top} & K_H^{**} & & & & & & \\ & & K_G & K_G^* & & & & \\ & & K_G^{*\top} & K_G^{**} & & & & \\ & & & & K_C & K_C^* & & \\ & & & & K_C^{*\top} & K_C^{**} & & \\ & & & & K_C^{*\top} & K_C^{**} & K_1 & K_1^* \\ K_H & K_H^{*\top} & K_G & K_G^{*\top} & K_C & K_C^{*\top} & K_1^{*\top} & K_1^{**} \\ K_H^{*\top} & K_H^{**} & K_G^{*\top} & K_G^{**} & K_C^{*\top} & K_C^{**} & K_1^* & K_1^{**} \end{bmatrix} \right) \quad (\text{B.26})$$

where we represent the kernel matrices by $K_1 = k_1(\mathbf{x}, \mathbf{x})$, $K_1^* = k_1(\mathbf{x}, \mathbf{x}^*)$ and $K_1^{**} = k_1(\mathbf{x}^*, \mathbf{x}^*)$, and likewise for the other kernel matrices. Given this joint distribution, we can obtain the posterior distributions of the three Hodge components as follows

$$f_H(\mathbf{x}^*) | f_1(\mathbf{x}) \sim \mathcal{N} \left(K_H^{*\top} K_1^{-1} f_1(\mathbf{x}), K_H^{**} - K_H^{*\top} K_1^{-1} K_H^* \right) \quad (\text{B.27a})$$

$$f_G(\mathbf{x}^*) | f_1(\mathbf{x}) \sim \mathcal{N} \left(K_G^{*\top} K_1^{-1} f_1(\mathbf{x}), K_G^{**} - K_G^{*\top} K_1^{-1} K_G^* \right) \quad (\text{B.27b})$$

$$f_C(\mathbf{x}^*) | f_1(\mathbf{x}) \sim \mathcal{N} \left(K_C^{*\top} K_1^{-1} f_1(\mathbf{x}), K_C^{**} - K_C^{*\top} K_1^{-1} K_C^* \right) \quad (\text{B.27c})$$

From these posterior distributions, we can directly obtain the means and the uncertainties of the Hodge components of the predicted edge function.