Sampling gradient and curl edge GPs

Proof. We focus on the case of gradient GPs. First, we can decompose the gradient kernel in terms of $\boldsymbol{U}_1 = [\boldsymbol{U}_H\ \boldsymbol{U}_G\ \boldsymbol{U}_C]$ as

$$\boldsymbol{K}_G = \boldsymbol{U}_1 \begin{pmatrix} \mathbf{0} & & \\ & \Psi_G(\boldsymbol{\Lambda}_G) & \\ & & \mathbf{0} \end{pmatrix} \boldsymbol{U}_1^\top. \tag{B.9}$$

From a vector $\boldsymbol{v} = (v_1, \ldots, v_{N_1})^\top$ of variables following independent normal distribution, we can draw a random sample of gradient function as

$$\boldsymbol{f}_G = \boldsymbol{U}_1 \mathrm{diag}([\mathbf{0}, \Psi_G^{\frac{1}{2}}(\boldsymbol{\Lambda}_G), \mathbf{0}])\boldsymbol{v} \tag{B.10}$$

where $\mathrm{diag}([\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}])$ is the diagonal matrix with $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})^\top$ on its diagonal.

Therefore, their curls are

$$\mathrm{curl}\, \boldsymbol{f}_G = \boldsymbol{B}_2^\top \boldsymbol{U}_1 \mathrm{diag}([\mathbf{0}, \Psi_G^{\frac{1}{2}}(\boldsymbol{\Lambda}_G), \mathbf{0}]) = \boldsymbol{B}_2^\top \boldsymbol{U}_G \Psi_G^{\frac{1}{2}}(\boldsymbol{\Lambda}_G) = \mathbf{0}. \tag{B.11}$$

Likewise, we can show the samples of a curl GP are div-free.

Posterior distribution of Hodge components

$$
\begin{bmatrix} \boldsymbol{f}_H(\boldsymbol{x}) \\ \boldsymbol{f}_H(\boldsymbol{x}^*) \\ \boldsymbol{f}_G(\boldsymbol{x}) \\ \boldsymbol{f}_G(\boldsymbol{x}^*) \\ \boldsymbol{f}_C(\boldsymbol{x}) \\ \boldsymbol{f}_C(\boldsymbol{x}^*) \\ \boldsymbol{f}_1(\boldsymbol{x}) \\ \boldsymbol{f}_1(\boldsymbol{x}^*) \end{bmatrix} \sim \mathcal{N}\left(\mathbf{0}, \begin{bmatrix} \boldsymbol{K}_H & \boldsymbol{K}_H^* & & & & & \boldsymbol{K}_H & \boldsymbol{K}_H^* \\ \boldsymbol{K}_H^{*\top} & \boldsymbol{K}_H^{**} & & & & & \boldsymbol{K}_H^* & \boldsymbol{K}_H^{**} \\ & & \boldsymbol{K}_G & \boldsymbol{K}_G^* & & & \boldsymbol{K}_G & \boldsymbol{K}_G^* \\ & & \boldsymbol{K}_G^{*\top} & \boldsymbol{K}_G^{**} & & & \boldsymbol{K}_G^* & \boldsymbol{K}_G^{**} \\ & & & & \boldsymbol{K}_C & \boldsymbol{K}_C^* & \boldsymbol{K}_C & \boldsymbol{K}_C^* \\ & & & & \boldsymbol{K}_C^{*\top} & \boldsymbol{K}_C^{**} & \boldsymbol{K}_C^* & \boldsymbol{K}_C^{**} \\ \boldsymbol{K}_H & \boldsymbol{K}_H^{*\top} & \boldsymbol{K}_G & \boldsymbol{K}_G^{*\top} & \boldsymbol{K}_C & \boldsymbol{K}_C^{*\top} & \boldsymbol{K}_1 & \boldsymbol{K}_1^* \\ \boldsymbol{K}_H^{*\top} & \boldsymbol{K}_H^{**} & \boldsymbol{K}_G^{*\top} & \boldsymbol{K}_G^{**} & \boldsymbol{K}_C^{*\top} & \boldsymbol{K}_C^{**} & \boldsymbol{K}_1^{*\top} & \boldsymbol{K}_1^{**} \end{bmatrix} \right) \tag{B.26}
$$

where we represent the kernel matrices by $\boldsymbol{K}_1 = k_1(\boldsymbol{x}, \boldsymbol{x}), \boldsymbol{K}_1^* = k_1(\boldsymbol{x}, \boldsymbol{x}^*)$ and $\boldsymbol{K}_1^{**} = k_1(\boldsymbol{x}^*, \boldsymbol{x}^*)$, and likewise for the other kernel matrices. Given this joint distribution, we can obtain the posterior distributions of the three Hodge components as follows

$$
\boldsymbol{f}_H(\boldsymbol{x}^*) | \boldsymbol{f}_1(\boldsymbol{x}) \sim \mathcal{N}\Big(\boldsymbol{K}_H^{*\top} \boldsymbol{K}_1^{-1} \boldsymbol{f}_1(\boldsymbol{x}), \boldsymbol{K}_H^{**} - \boldsymbol{K}_H^{*\top} \boldsymbol{K}_1^{-1} \boldsymbol{K}_H^* \Big) \tag{B.27a}
$$

$$
\boldsymbol{f}_G(\boldsymbol{x}^*) | \boldsymbol{f}_1(\boldsymbol{x}) \sim \mathcal{N}\Big(\boldsymbol{K}_G^{*\top} \boldsymbol{K}_1^{-1} \boldsymbol{f}_1(\boldsymbol{x}), \boldsymbol{K}_G^{**} - \boldsymbol{K}_G^{*\top} \boldsymbol{K}_1^{-1} \boldsymbol{K}_G^* \Big) \tag{B.27b}
$$

$$
\boldsymbol{f}_C(\boldsymbol{x}^*) | \boldsymbol{f}_1(\boldsymbol{x}) \sim \mathcal{N}\Big(\boldsymbol{K}_C^{*\top} \boldsymbol{K}_1^{-1} \boldsymbol{f}_1(\boldsymbol{x}), \boldsymbol{K}_C^{**} - \boldsymbol{K}_C^{*\top} \boldsymbol{K}_1^{-1} \boldsymbol{K}_C^* \Big) \tag{B.27c}
$$

From these posterior distributions, we can directly obtain the means and the uncertainties of the Hodge components of the predicted edge function.