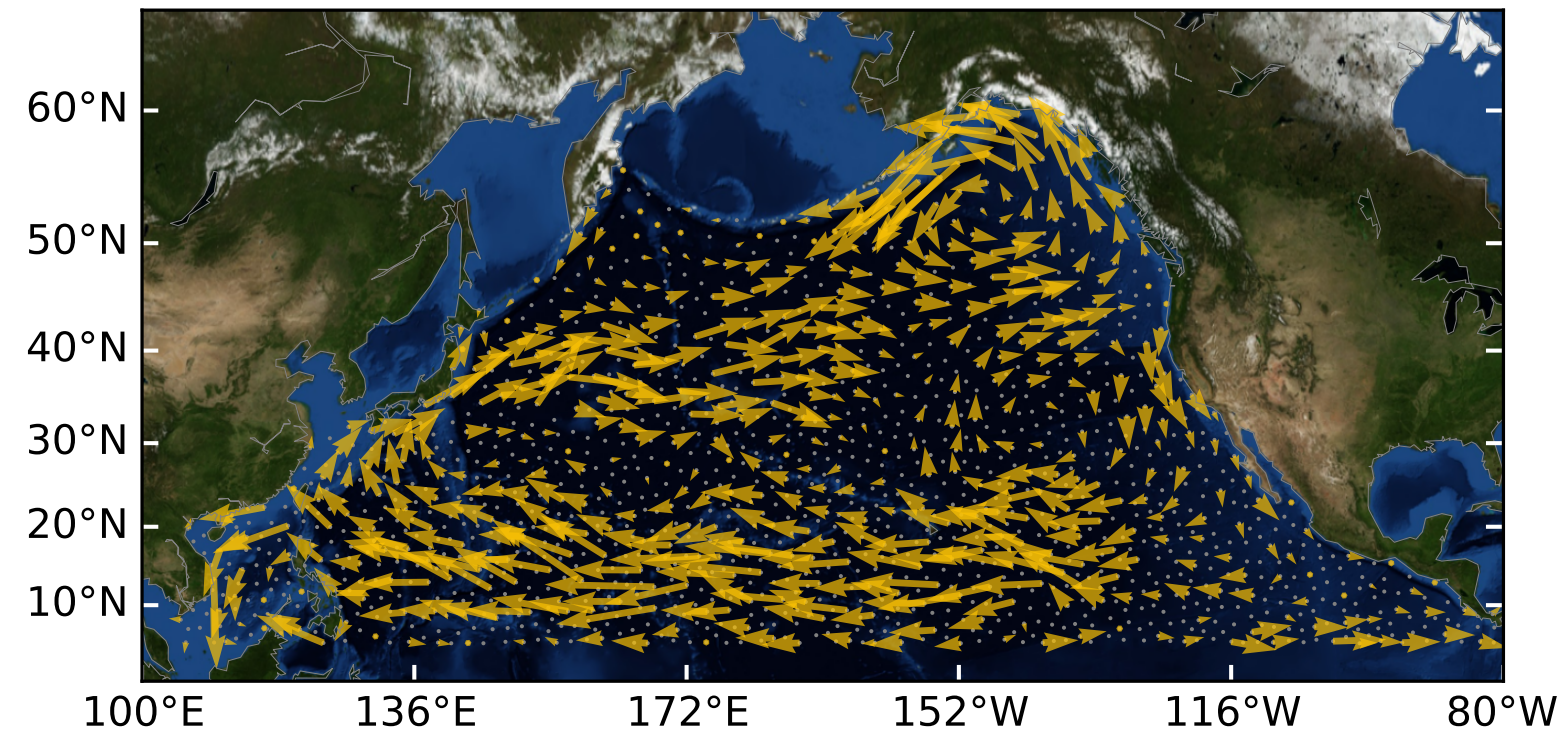
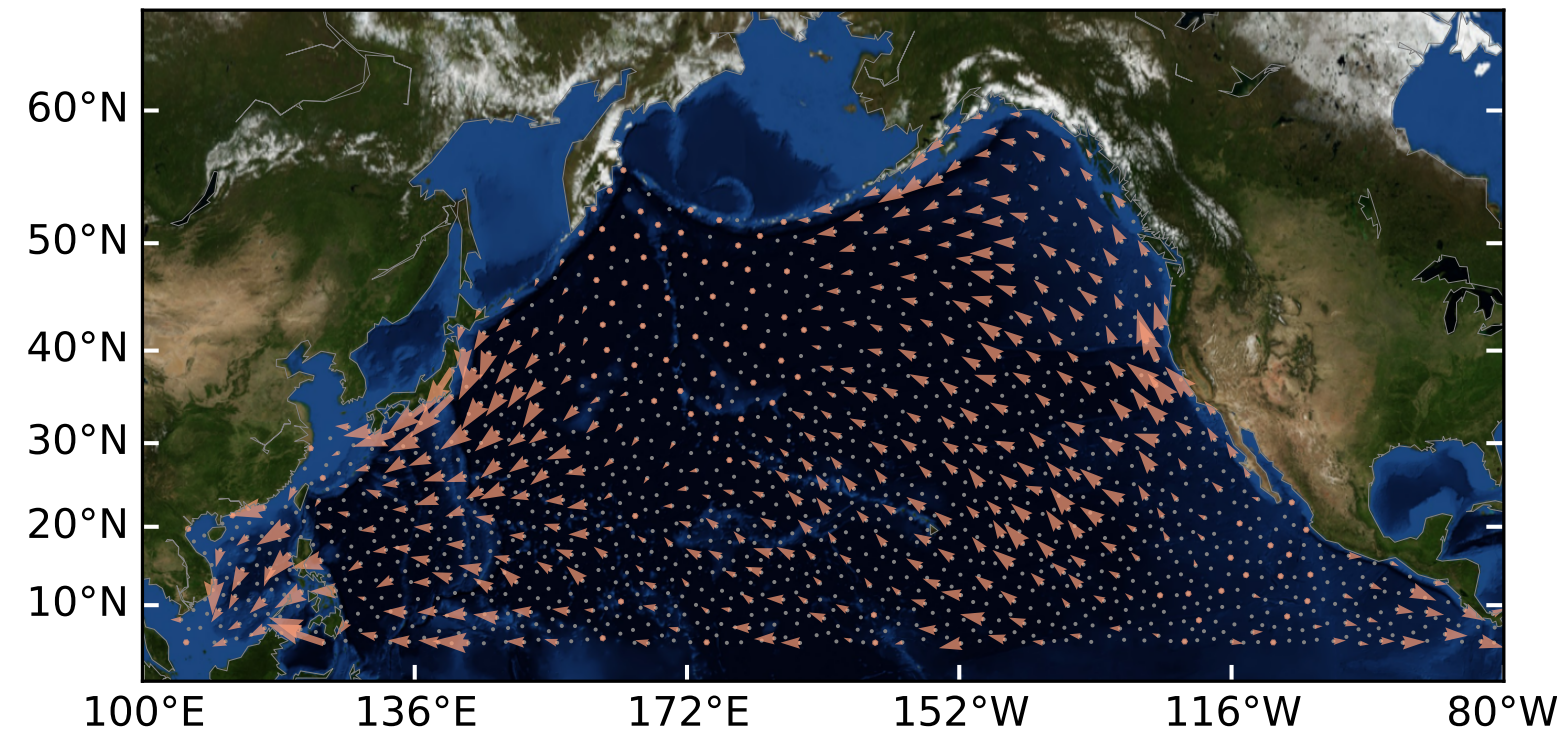


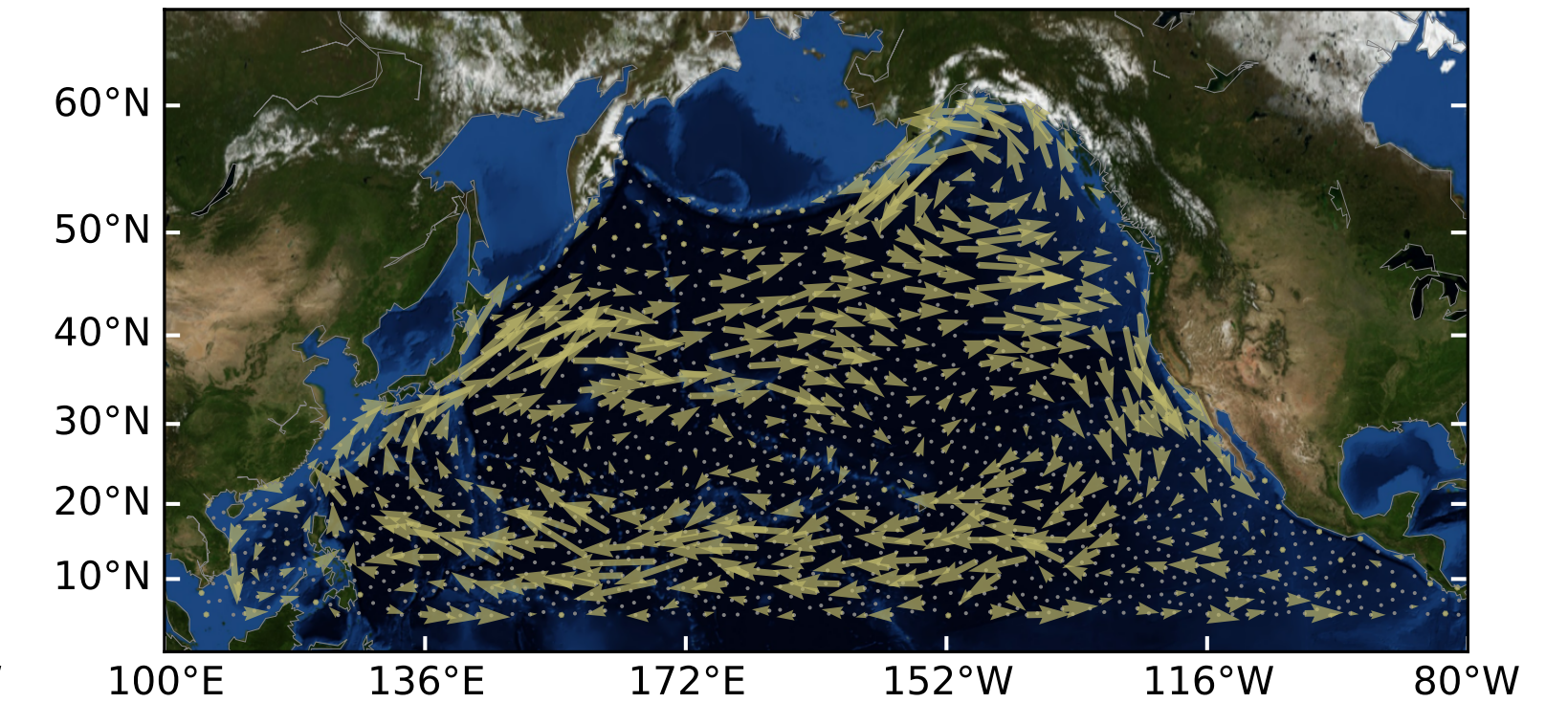
Applications of Hodge decomposition



Ocean currents

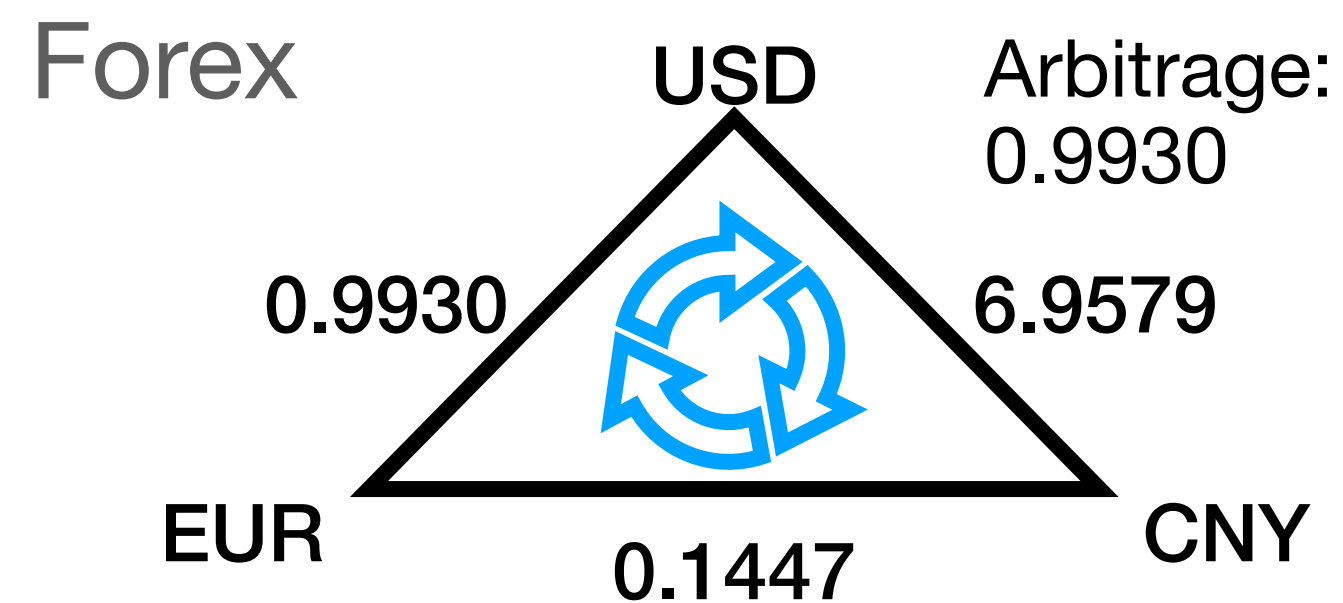


Gradient flow
Curl-free, irrotational



Curl flow
Div-free, solenoidal

Chen, Yu-Chia et al. (2021) "Helmholtzian Eigenmap."



$$r^{a/b} r^{b/c} = r^{a/c} \quad \text{Arbitrage-free}$$

$$f_{[a,b]} + f_{[b,c]} - f_{[a,c]} = 0 \quad \text{Curl-free}$$

- Water flows (div-free)
- Electrical currents, voltages

- Brain networks (Anand et al. 2022)
- Game theory (Candogan et al. 2011)
- Ranking problems (Jiang et al. 2011)
- Random walks (Strang et al. 2020)
- ...

Eigenspace of L_1 spans Hodge subspaces

- **Nonzero** Eigenspace of **down Laplacian** spans the **gradient** space
- **Nonzero** Eigenspace of **up Laplacian** spans the **curl** space
- **Kernel** of Laplacian spans the **harmonic** space

Simplicial Fourier transform

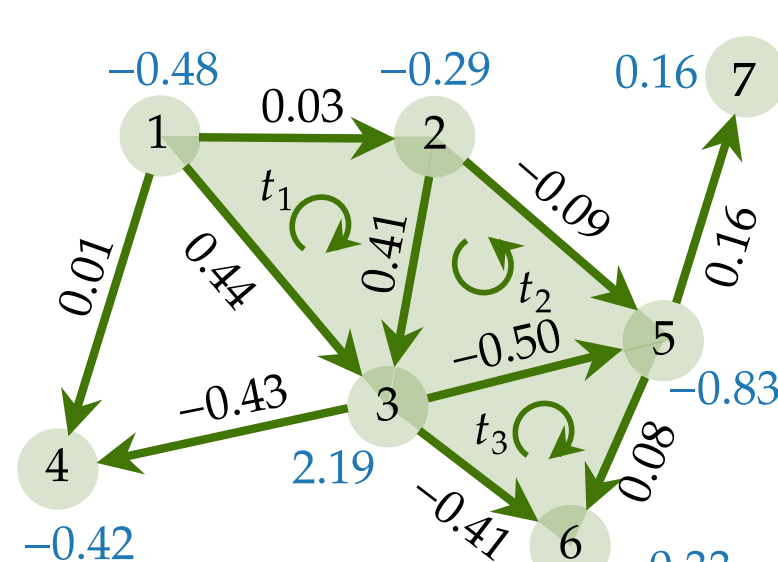
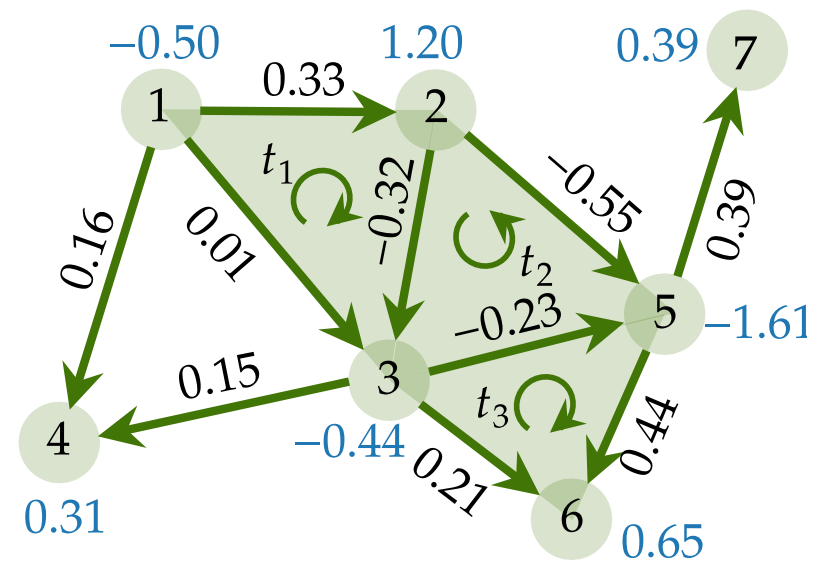
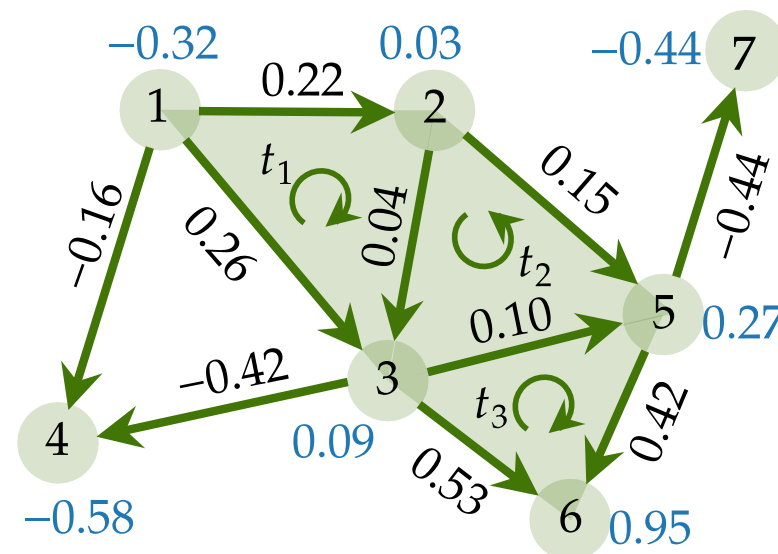
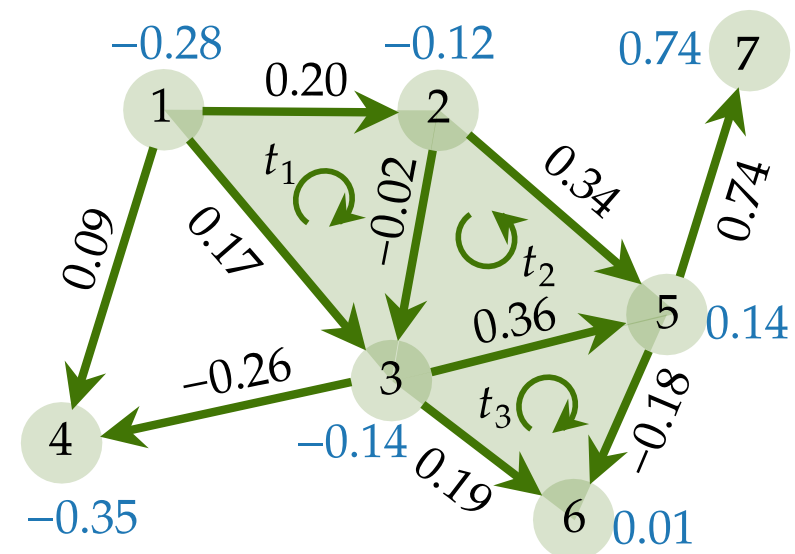
Frequency — eigenvalues

Fourier basis — eigenvectors

$$\lambda_G = \|\mathbf{B}_1 \mathbf{u}_G\|_2^2$$

Gradient eigenvector

Fourier basis reflecting **divergent** properties

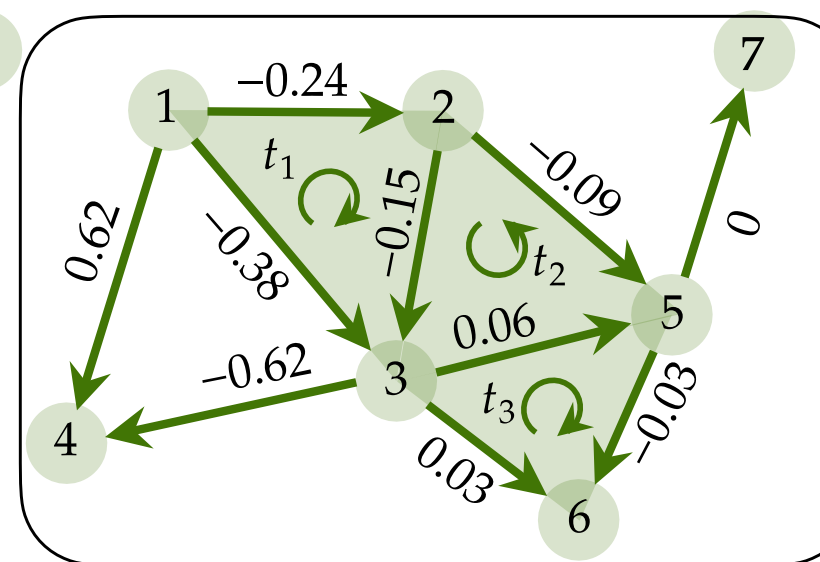
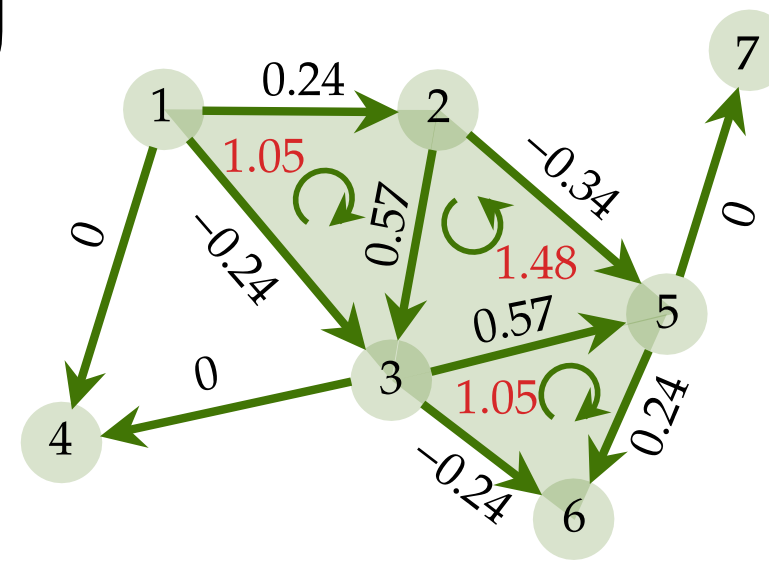
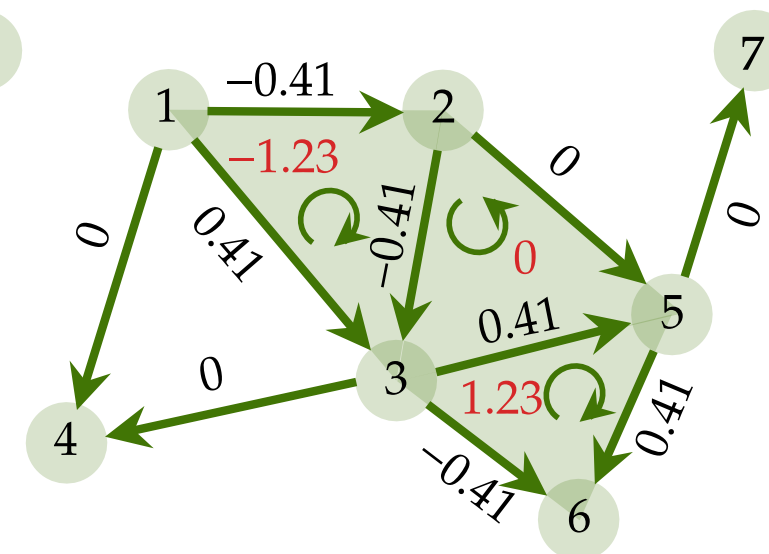
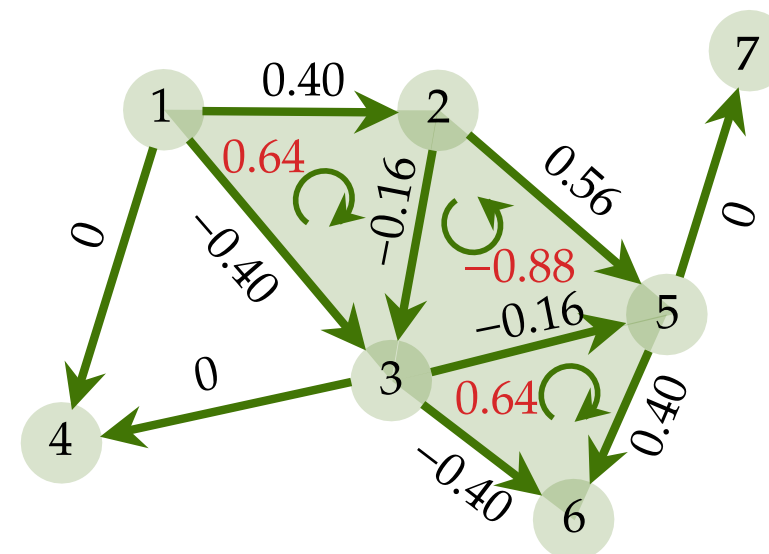


$k = 1$

$$\lambda_C = \|\mathbf{B}_2^T \mathbf{u}_C\|_2^2$$

Curl eigenvector

Fourier basis reflecting **rotational** properties



$$\text{EVD: } \mathbf{L}_1 = \mathbf{U}_1 \mathbf{\Lambda}_1 \mathbf{U}_1^T$$

$$\mathbf{U}_1 = [\mathbf{U}_H \ \mathbf{U}_G \ \mathbf{U}_C]$$

$$\text{span}(\mathbf{U}_H) = \ker(\mathbf{L}_1)$$

$$\text{span}(\mathbf{U}_G) = \text{im}(\mathbf{B}_1^T)$$

$$\text{span}(\mathbf{U}_C) = \text{im}(\mathbf{B}_2)$$

$$\tilde{\mathbf{x}}_k = \mathbf{U}_k^T \mathbf{x}_k, k = 1$$

$$\tilde{\mathbf{x}}_k = [\tilde{\mathbf{x}}_{k,H}^T, \tilde{\mathbf{x}}_{k,G}^T, \tilde{\mathbf{x}}_{k,C}^T]$$