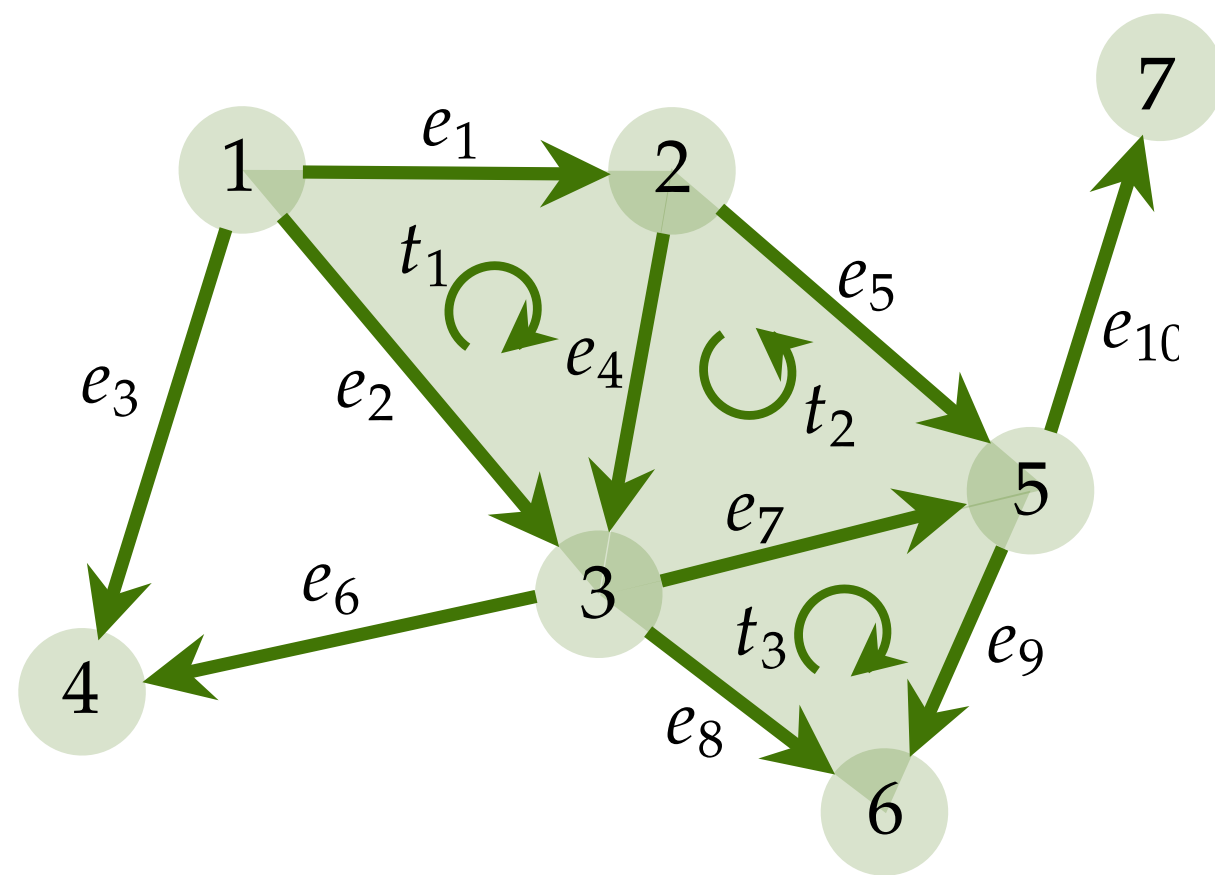


# Algebraic reps. of simplicial 2-complex

## Incidences & Laplacians



Node-to-Edge

$$\mathbf{B}_1 = \begin{matrix} & \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{matrix} & \begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix},$$

Edge-to-Faces

$$\mathbf{B}_2 = \begin{matrix} & \begin{matrix} t_1 & t_2 & t_3 \end{matrix} \\ \begin{matrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \\ e_8 \\ e_9 \\ e_{10} \end{matrix} & \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{matrix}$$

Graph Laplacian:  $\mathbf{L}_0 = \mathbf{B}_1 \mathbf{B}_1^\top$

1-Hodge Laplacian:  $\mathbf{L}_1 = \underbrace{\mathbf{B}_1^\top \mathbf{B}_1}_{\text{Down}} + \underbrace{\mathbf{B}_2 \mathbf{B}_2^\top}_{\text{Up}} := \mathbf{L}_{1,d} + \mathbf{L}_{1,u}$

# GPs on graphs

## Modeling node functions

- $\mathbf{f}_0 \sim \text{GP}(\mathbf{0}, \mathbf{K}_0)$  (Borovitskiy et al. 2021)
- Matérn graph kernel

$$\Phi(\mathbf{L}_0)\mathbf{f}_0 = \mathbf{w}_0, \text{ with}$$

$$\Phi(\mathbf{L}_0) = \left( \frac{2\nu}{\kappa^2} \mathbf{I} + \mathbf{L}_0 \right)^{\frac{\nu}{2}} \text{ and } \mathbf{w}_0 \sim N(\mathbf{0}, \sigma^2 \mathbf{I})$$

- The solution has kernel

$$\mathbf{K}_0 = \sigma^2 \sum_{n=0}^{N_0-1} \psi(\lambda_n) \mathbf{u}_n \mathbf{u}_n^\top, \quad K_0 = \sigma^2 \left( \frac{2\nu}{\kappa^2} \mathbf{I} + \mathbf{L}_0 \right)^{-\nu}$$

$$\psi(\lambda) = \begin{cases} \left( \frac{2\nu}{\kappa^2} + \lambda \right)^{-\nu} & \nu < \infty, \text{ Matern} \\ e^{-\frac{\kappa^2}{2}\lambda} & \nu = \infty, \text{ Diffusion} \end{cases}$$

## GPs from Euclidean to non-Euclidean

### GP in Euclidean settings

Function on a set  $f: X \rightarrow \mathbb{R}$

$$f \sim \text{GP}(\mu, k)$$

- Predictive distribution  $f|_{\mathbf{y}}$

- Matérn GP family, e.g., diffusion

$$k(x, x') = \sigma^2 \exp\left( -\frac{d(x, x')^2}{2\kappa^2} \right)$$

- Distance-based: geometry-aware, but not well-defined for manifolds, graphs ...

- Instead, as solutions of SDEs (Whittle (1963); Lindgren et al. (2011))

$$\left( \frac{2\nu}{\kappa^2} - \Delta \right)^{\frac{\nu}{2} + \frac{d}{4}} f = w$$

-  $\Delta$ : Laplacian,  $w$ : white noise

- implicit, generalizable, domain-aware

- explicit for some domains