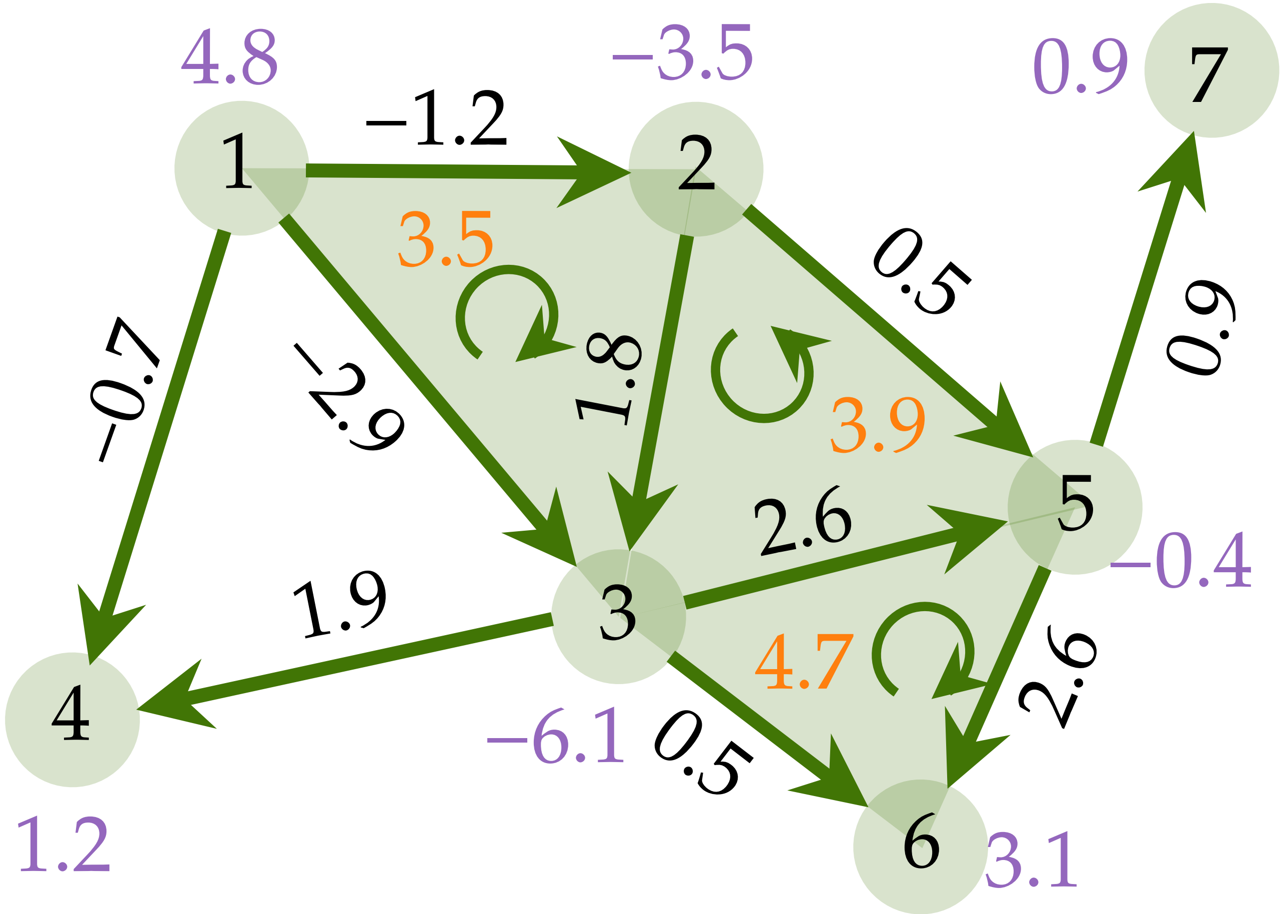
Incidence & Laplacians

1st and 2nd order Discrete Derivatives



Gradient of node signal: $\mathbf{B}_1^\top \mathbf{v}$ $[\mathbf{f}_G]_{[i,j]} = [\mathbf{v}]_j - [\mathbf{v}]_i$

Divergence of edge flows: $[\mathbf{B}_1\mathbf{f}]_{[i]} = \sum_{j<i} \mathbf{f}_{[j,i]} - \sum_{i<k} \mathbf{f}_{[i,k]}$

Net-flow = in_flow - out_flow

Curl of edge flows: $[\mathbf{B}_2^\top \mathbf{f}]_t = \mathbf{f}_{[i,j]} + \mathbf{f}_{[j,k]} - \mathbf{f}_{[i,k]}$, for $t = [i,j,k]$

Net-circulation in triangles

$$[\mathbf{B}_1\mathbf{f}]_5 = 0.5 + 2.6 - (0.9 + 2.6) = -0.4$$

$$[\mathbf{B}_2^\top \mathbf{f}]_{[1,2,3]} = -1.2 + 1.8 - (-2.9) = 3.5$$

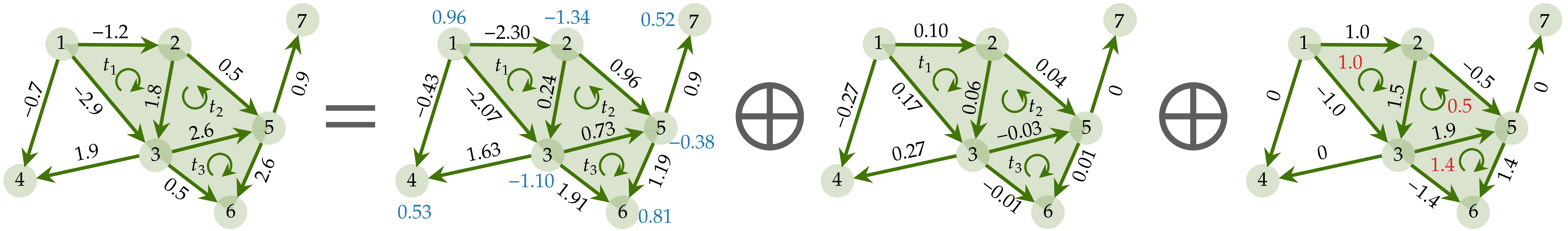
Hodge Laplacians = Grad Div + Curl* Curl

Hodge Laplacian: $\mathbf{L}_1 = \mathbf{B}_1^\top \mathbf{B}_1 + \mathbf{B}_2 \mathbf{B}_2^\top$

Hodge decomposition

Lovász et al. 2004; Lim et al. 2020

$$\mathbb{R}^{N_1} = \mathrm{im}(\mathbf{B}_1^\top) \oplus \ker(\mathbf{L}_1) \oplus \mathrm{im}(\mathbf{B}_2)$$
$$\mathbf{f}_1 = \mathbf{f}_G + \mathbf{f}_H + \mathbf{f}_C$$



Edge flow = Gradient flow (Curl-free, irrotational) + Harmonic flow (Div- and curl-free) + Curl flow (Div-free, solenoidal)

Hodge-compositional Edge GP

$$\mathbf{f}_G \sim \mathrm{GP}(\mathbf{0}, \mathbf{K}_G)$$
$$\mathbf{f}_H \sim \mathrm{GP}(\mathbf{0}, \mathbf{K}_H)$$
$$\mathbf{f}_C \sim \mathrm{GP}(\mathbf{0}, \mathbf{K}_C)$$