Hodge decomposition
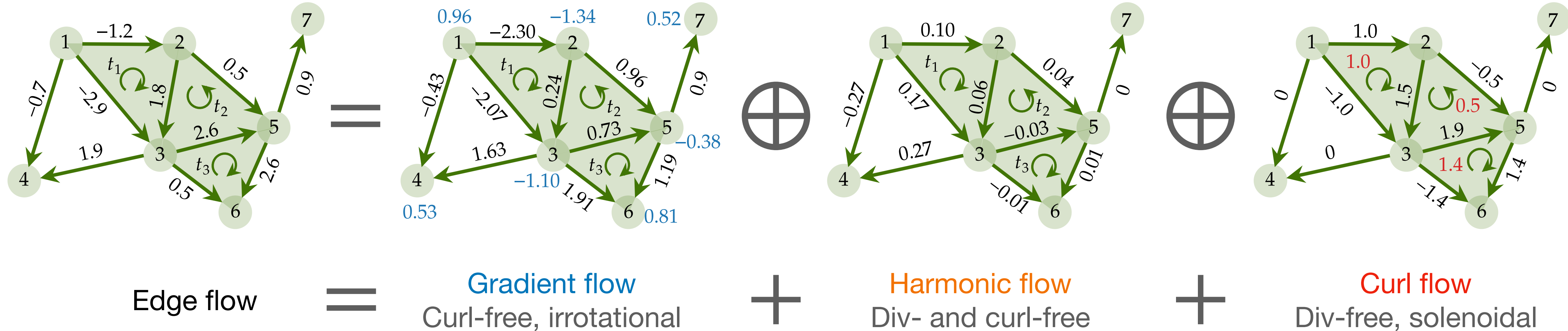
Lovász et al. 2004; Lim et al. 2020

$$\mathbb{R}^{N_1} = \mathrm{im}(\mathbf{B}_1^\top) \oplus \ker(\mathbf{L}_1) \oplus \mathrm{im}(\mathbf{B}_2)$$

$$\mathbf{f}_1 = \mathbf{f}_G + \mathbf{f}_H + \mathbf{f}_C$$

Edge flow $=$ **Gradient flow** $+$ **Harmonic flow** $+$ **Curl flow**
Curl-free, irrotational   Div- and curl-free   Div-free, solenoidal

Hodge-compositional Edge GP

$$\mathbf{f}_G \sim \mathrm{GP}(\mathbf{0}, \mathbf{K}_G)$$
$$\mathbf{f}_H \sim \mathrm{GP}(\mathbf{0}, \mathbf{K}_H)$$
$$\mathbf{f}_C \sim \mathrm{GP}(\mathbf{0}, \mathbf{K}_C)$$

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Hodge-compositional Edge GPs
Curl-free, div-free GPs

$$\mathbf{f}_G \sim \mathrm{GP}(\mathbf{0}, \mathbf{K}_G)$$
$$\mathbf{f}_H \sim \mathrm{GP}(\mathbf{0}, \mathbf{K}_H)$$
$$\mathbf{f}_C \sim \mathrm{GP}(\mathbf{0}, \mathbf{K}_C)$$

- Gradient kernel $\mathbf{K}_G = \mathbf{U}_G \Psi_G(\Lambda_G) \mathbf{U}_G^\top$;   Curl kernel $\mathbf{K}_C = \mathbf{U}_C \Psi_C(\Lambda_C) \mathbf{U}_C^\top$

- Matérn family: $\Psi_\square(\Lambda_\square) = \sigma_\square^2 \left(\dfrac{2\nu_\square}{\kappa_\square^2} \mathbf{I} + \Lambda_\square \right)^{-\nu_\square}, \quad \square = \mathrm{H}, \mathrm{G}, \mathrm{C}$

- Also as solutions of SDEs, e.g.,

$$\Phi_C(\mathbf{L}_{1,\mathrm{u}}) \mathbf{f}_1 = \mathbf{w}_C, \text{ with curl noise } \mathbf{w}_C \sim N(0, \sigma_C^2 \mathbf{U}_C \mathbf{U}_C^\top) \text{ and}$$

$$\Phi(\mathbf{L}_{1,u}) = \left(\frac{2\nu_C}{\kappa_C^2} \mathbf{I} + \mathbf{L}_{1,u} \right)^{\frac{\nu_C}{2}} \text{or } \Phi(\mathbf{L}_{1,u}) = e^{\frac{\kappa_C^2}{4} \mathbf{L}_{1,u}}$$