

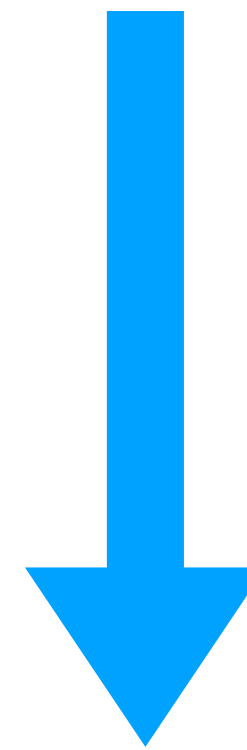
Towards solving TSBP [Lénoard 2014]

Topological Schrödinger Bridge Problem

$$\min D_{KL}(\mathbb{P} \parallel \mathbb{Q}_{\mathcal{T}}) \text{ s.t. } \mathbb{P}_0 = \rho_0, \mathbb{P}_1 = \rho_1$$

Disintegration of Measures

$$\mathbb{P}(\cdot) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathbb{P}^{xy}(\cdot) \mathbb{P}_{01}(dx dy)$$



Static TSBP / E-OT

$$\min D_{KL}(\mathbb{P}_{01} \parallel \mathbb{Q}_{\mathcal{T}01}) \text{ s.t. } \mathbb{P}_0 = \rho_0, \mathbb{P}_1 = \rho_1$$

+

\mathbb{P} shares bridges with $\mathbb{Q}_{\mathcal{T}}$

$$\mathbb{P}^{xy} = \mathbb{Q}_{\mathcal{T}}^{xy}$$

An E-OT with transport cost: $\left\| y_1 - \Psi_1 y_0 - \xi_1 \right\|_{K_{11}^{-1}}^2$

Iterative proportional / Markovian fitting

Stochastic control view

- Equivalently, a variational problem

$$\min_{b_t} \mathbb{E} \left[\frac{1}{2} \int_0^1 \|b(t, X_t)\|^2 dt \right], \quad \text{s.t.} \quad \begin{cases} dX_t = [f_t + g_t b(t, X_t)] dt + g_t dW_t, \\ X_0 \sim \rho_0, X_1 \sim \rho_1. \end{cases}$$

- Initially formulated for classical case [Dai Pro 1991; Pavon 1991; Caluya & Halder 2021]
- Optimal TSB follows a forward-backward SDE system

Forward: $dX_t = dY_t + Z_t dt, X_0 \sim \rho_0$

Backward: $dX_t = dY_t - \tilde{Z}_t dt, X_1 \sim \rho_1$

Enables
learning!!!

- $Z_t \equiv g_t \nabla \log \varphi_t(X_t), \tilde{Z}_t \equiv g_t \nabla \log \tilde{\varphi}_t(X_t)$, depend on a system of PDE pairs (FPE, HJB)
- Nonlinear Feynman-Kac formula: gives us a likelihood [Chen et al. 2022]