TSHeat examples

- TSHeat-BM: $dY_t = -cLY_t dt + gdW_t$
- TSHeat-VE (variance-exploding): $dY_t = -cLY_t dt + \sqrt{d\sigma^2(t)/dt}\, dW_t$, $\sigma(t) = \sigma_{\min}\left(\frac{\sigma_{\max}}{\sigma_{\min}}\right)^t$
- TSHeat-VP (variance-preserving): $dY_t = -(\frac{1}{2}\beta(t)I + cL)Y_t dt + \sqrt{\beta(t)}dW_t$, $\beta(t) = \beta_{\min} + t(\beta_{\max} - \beta_{\min})$
- Gaussian transition kernels $p_{t|0}(y_t \,|\, y_0) \sim N(m_t, K_t)$
  - Closed-forms for TSHeat-BM, VE
  - Transition matrix of the ODE: $dY_t = H_t(L)Y_t dt$

$$\Psi_t = \exp\left(\int_0^t H_\tau d\tau\right)$$

$$A = \ln\left(\frac{\sigma_{\max}}{\sigma_{\min}}\right) I + cL, \quad \Psi_t = \exp(-cLt)$$

$$K_{t_1,t_2} = \begin{cases} \frac{g^2}{2c}\left(\exp(-cL\,|\,t_1 - t_2\,|) - \exp(-cL(t_1+t_2))\right) L^{-1}, & \text{TSHeat-BM} \\ \sigma_{\min}^2 \ln\left(\frac{\sigma_{\max}}{\sigma_{\min}}\right) \exp(-cL(t_1+t_2))\left(\exp(2A\min\{t_1,t_2\}) - I\right) A^{-1}, & \text{TSHeat-VE} \end{cases}$$

Towards solving TSBP [Lénoard 2014]

Topological Schrödinger Bridge Problem

$$\min D_{KL}(\mathbb{P} \| \mathbb{Q}_{\mathscr{T}}) \; s.t. \; \mathbb{P}_0 = \rho_0, \mathbb{P}_1 = \rho_1$$

Disintegration of Measures

$$\mathbb{P}(\cdot) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathbb{P}^{xy}(\cdot)\, \mathbb{P}_{01}(dx\, dy)$$

Static TSBP / E-OT

$$\min D_{KL}(\mathbb{P}_{01} \| \mathbb{Q}_{\mathscr{T}01}) \; s.t. \; \mathbb{P}_0 = \rho_0, \mathbb{P}_1 = \rho_1$$

+

$\mathbb{P}$ shares bridges with $\mathbb{Q}_{\mathscr{T}}$

$$\mathbb{P}^{xy} = \mathbb{Q}^{xy}_{\mathscr{T}}$$

An E-OT with transport cost: $\left\| y_1 - \Psi_1 y_0 - \xi_1 \right\|^2_{K_{11}^{-1}}$

Iterative proportional / Markovian fitting