Matérn Edge GPs

Derived from SDEs on the edge set

- $\mathbf{f}_1 \sim \mathrm{GP}(\mathbf{0}, \mathbf{K}_1)$

EVD: $\mathbf{L}_1 = \mathbf{U}_1 \mathbf{\Lambda}_1 \mathbf{U}_1^\top$

- Matérn graph kernel

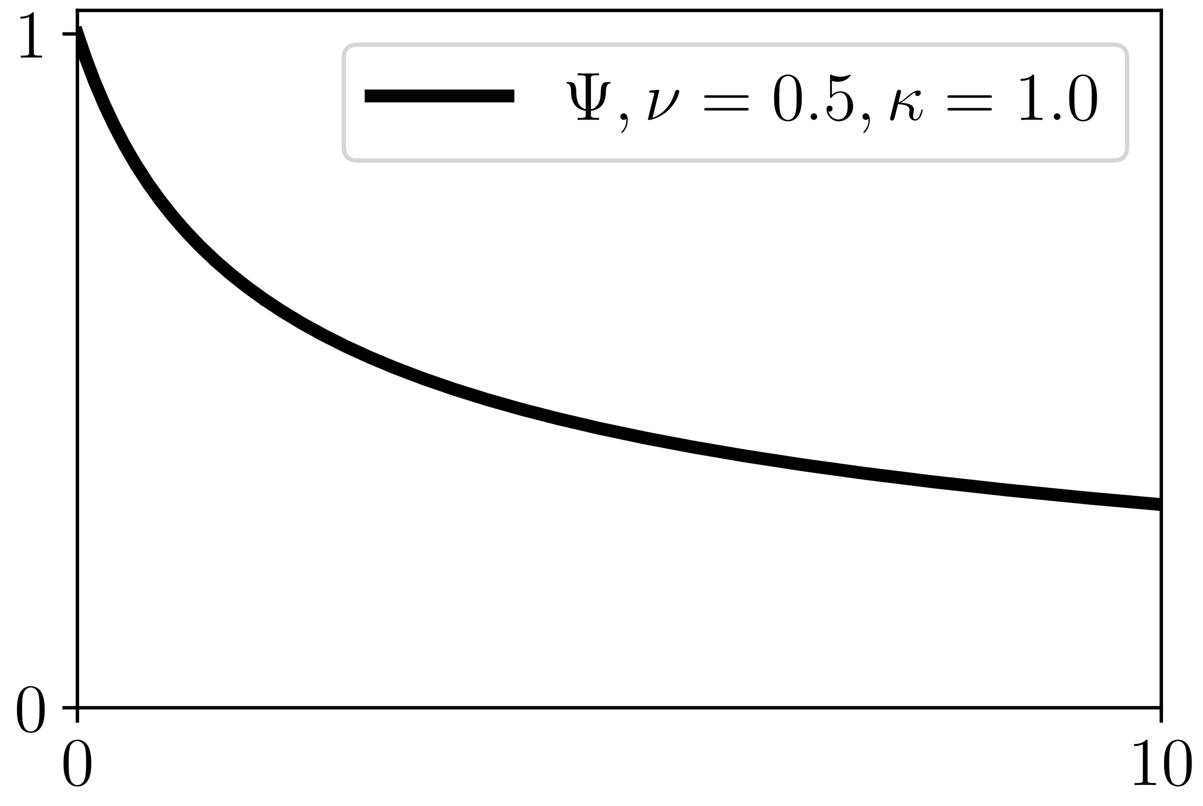
$\Phi(\mathbf{L}_1)\mathbf{f}_1 = \mathbf{w}_1$, with

$$\Phi(\mathbf{L}_1) = \left(\frac{2\nu}{\kappa^2}\mathbf{I} + \mathbf{L}_1\right)^{\frac{\nu}{2}} \text{ and } \mathbf{w}_1 \sim N(\mathbf{0}, \sigma^2 \mathbf{I})$$

- The solution gives edge GPs

$$\text{Matérn: } \mathbf{f}_1 \sim \mathrm{GP}\left(0, \left(\frac{2\nu}{\kappa^2}\mathbf{I} + \mathbf{L}_1\right)^{-\nu}\right)$$

$$\text{Diffusion: } \mathbf{f}_1 \sim \mathrm{GP}\left(0, e^{-\frac{\kappa^2}{2}\mathbf{L}_1}\right)$$



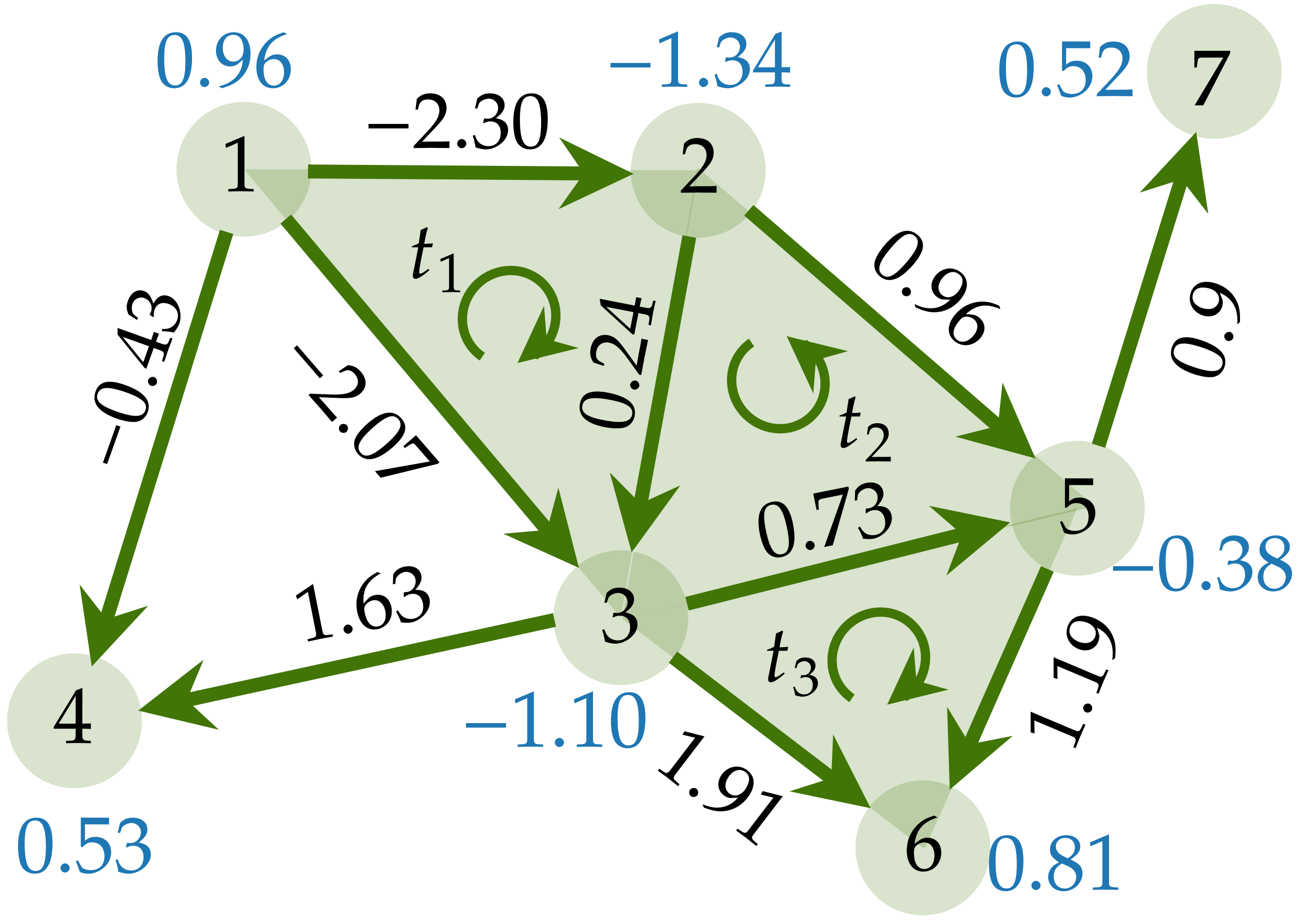
- Low-pass in the eigen-spectrum

Node function — 0-form (scalar field)
Edge function — 1-form (vector field)
We can dive deeper

Incidence & Laplacians

1st and 2nd order Discrete Derivatives

- Node signal $\mathbf{v}$
- Edge flows $\mathbf{f}$



Gradient of node signal: $[\mathbf{f}_{\mathrm{G}}]_{[i,j]} = [\mathbf{B}_1^\top \mathbf{v}]_{[i,j]} = [\mathbf{v}]_j - [\mathbf{v}]_i$

Divergence of edge flows: $[\mathbf{B}_1\mathbf{f}]_{[i]} = \sum_{j<i} \mathbf{f}_{[j,i]} - \sum_{i<k} \mathbf{f}_{[i,k]}$

Curl of edge flows: $[\mathbf{B}_2^\top \mathbf{f}]_t = \mathbf{f}_{[i,j]} + \mathbf{f}_{[j,k]} - \mathbf{f}_{[i,k]}$, for $t = [i,j,k]$

$$[\mathbf{B}_1^\top \mathbf{v}]_{[1,2]} = -1.34 - 0.96 = -2.30$$