

# TSHeat examples

- TSHeat-BM:  $dY_t = -cLY_t dt + g dW_t$
- TSHeat-VE (variance-exploding):  $dY_t = -cLY_t dt + \sqrt{d\sigma^2(t)/dt} dW_t$ ,  $\sigma(t) = \sigma_{\min} \left( \frac{\sigma_{\max}}{\sigma_{\min}} \right)^t$
- TSHeat-VP (variance-preserving):  $dY_t = -\left(\frac{1}{2}\beta(t)I + cL\right)Y_t dt + \sqrt{\beta(t)} dW_t$ ,  $\beta(t) = \beta_{\min} + t(\beta_{\max} - \beta_{\min})$
- **Gaussian transition kernels**  $p_{t|0}(y_t | y_0) \sim N(m_t, K_t)$ 
  - Closed-forms for TSHeat-BM, VE
  - Transition matrix of the ODE:  $dY_t = H_t(L)Y_t dt$ 

$$A = \ln \left( \frac{\sigma_{\max}}{\sigma_{\min}} \right) I + cL, \quad \Psi_t = \exp(-cLt)$$

$$\Psi_t = \exp \left( \int_0^t H_\tau d\tau \right)$$

$$K_{t_1, t_2} = \begin{cases} \frac{g^2}{2c} \left( \exp(-cL|t_1 - t_2|) - \exp(-cL(t_1 + t_2)) \right) L^{-1}, & \text{TSHeat-BM} \\ \sigma_{\min}^2 \ln \left( \frac{\sigma_{\max}}{\sigma_{\min}} \right) \exp(-cL(t_1 + t_2)) \left( \exp(2A \min\{t_1, t_2\}) - I \right) A^{-1}, & \text{TSHeat-VE} \end{cases}$$

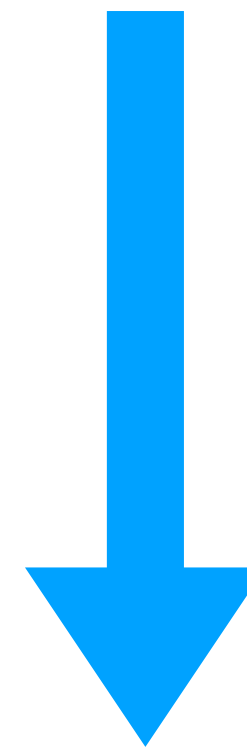
# Towards solving TSBP [Lénoard 2014]

## Topological Schrödinger Bridge Problem

$$\min D_{KL}(\mathbb{P} \parallel \mathbb{Q}_{\mathcal{T}}) \text{ s.t. } \mathbb{P}_0 = \rho_0, \mathbb{P}_1 = \rho_1$$

## Disintegration of Measures

$$\mathbb{P}(\cdot) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathbb{P}^{xy}(\cdot) \mathbb{P}_{01}(dx dy)$$



## Static TSBP / E-OT

$$\min D_{KL}(\mathbb{P}_{01} \parallel \mathbb{Q}_{\mathcal{T}01}) \text{ s.t. } \mathbb{P}_0 = \rho_0, \mathbb{P}_1 = \rho_1$$



$\mathbb{P}$  shares bridges with  $\mathbb{Q}_{\mathcal{T}}$

$$\mathbb{P}^{xy} = \mathbb{Q}_{\mathcal{T}}^{xy}$$

An E-OT with transport cost:  $\left\| y_1 - \Psi_1 y_0 - \xi_1 \right\|_{K_{11}^{-1}}^2$

Iterative proportional / Markovian fitting