Edge Convolution
Shift-and-Sum

Simplicial shifting

$\mathbf{x}_1$

$\mathbf{L}_{1,u}\mathbf{x}_1$

$\mathbf{L}_{1,u}^2\mathbf{x}_1$

$\mathbf{L}_{1,d}\mathbf{x}_1$

$\mathbf{L}_{1,d}^2\mathbf{x}_1$
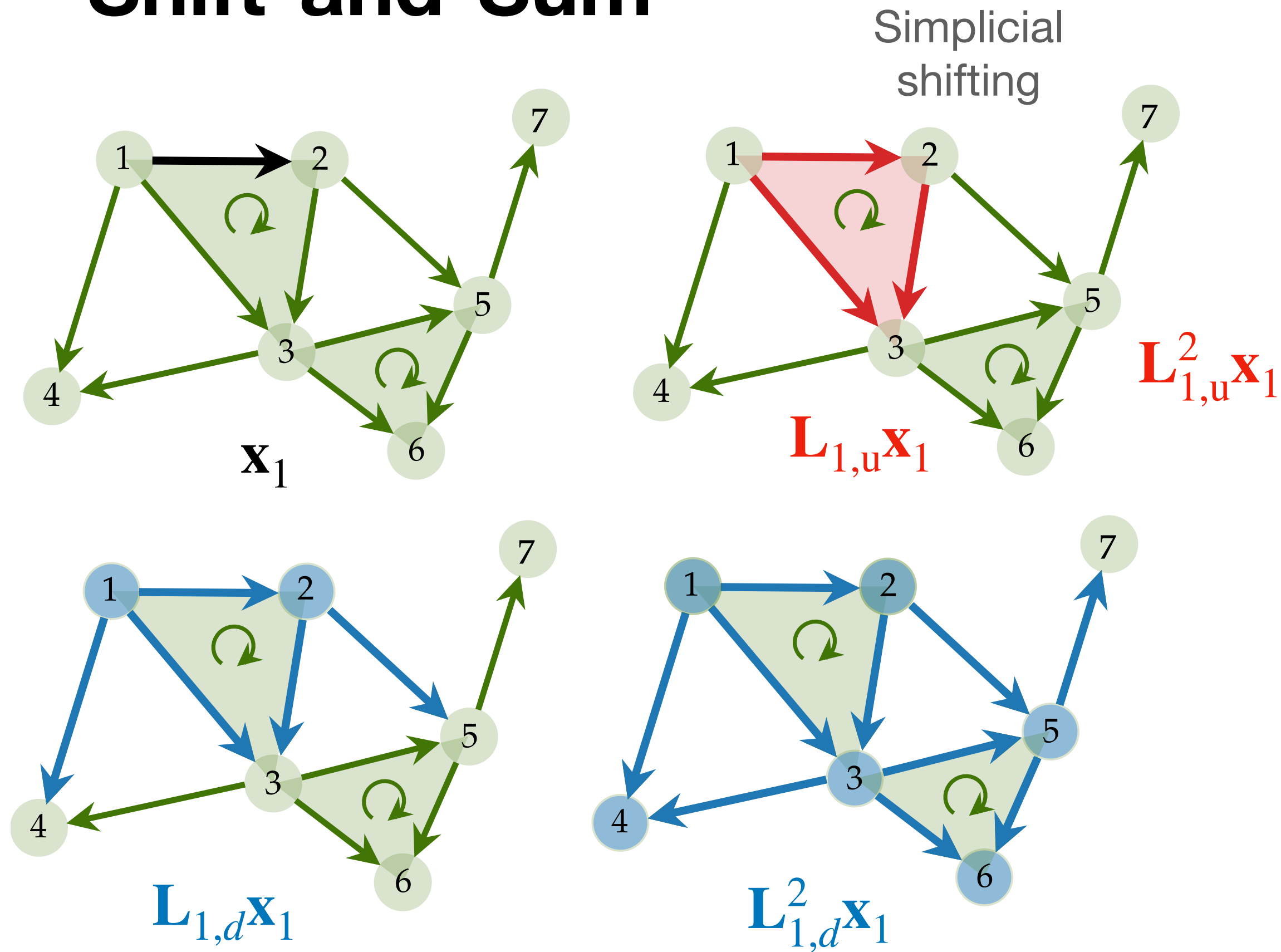
$$[\mathbf{L}_{1,d}\mathbf{f}]_i = \sum_{j\in\{\mathcal{N}_{1,i}\cup i\}}[\mathbf{L}_{1,d}]_{ij}[\mathbf{f}]_j$$

Simplicial locality

Simplicial Convolutional Filter

$$\mathbf{H} := \mathbf{H}(\mathbf{L}_d, \mathbf{L}_u; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_{k=0}^{K_d} \alpha_k \mathbf{L}_d^k + \sum_{k=0}^{K_u} \beta_k \mathbf{L}_u^k$$

- Efficient, distributed

- Expressive power (Cayley-Hamilton thm)
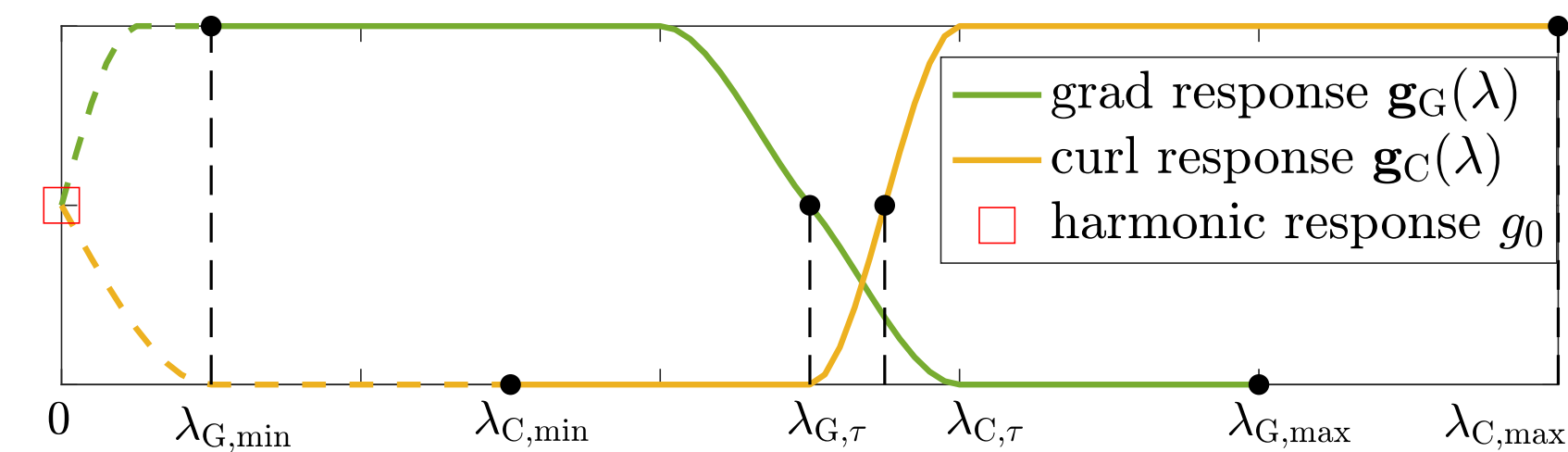
- Hodge-invariant operator

$$\mathbf{H}_1\mathbf{x}_1 = \mathbf{H}_1\big|_{\text{im}(\mathbf{B}_1^\top)}\mathbf{x}_{1,G} + \mathbf{H}_1\big|_{\text{im}(\mathbf{B}_2)}\mathbf{x}_{1,C} + \mathbf{H}_1\big|_{\text{ker}(\mathbf{L}_1)}\mathbf{x}_{1,H}$$

Hodge subspaces are invariant under $\mathbf{H}$

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Edge Convolutions on SCs
Pointwise Multiplication at frequencies

$$\begin{cases} \tilde{H}_{\mathrm{H}}(\lambda) = \alpha_0 + \beta_0, & \text{for } \lambda \in \mathcal{Q}_{\mathrm{H}}, \\ \tilde{H}_{\mathrm{G}}(\lambda) = \sum_{k=0}^{K_d} \alpha_k \lambda^k, & \text{for } \lambda \in \mathcal{Q}_{\mathrm{G}}, \\ \tilde{H}_{\mathrm{C}}(\lambda) = \sum_{k=0}^{K_u} \beta_k \lambda^k, & \text{for } \lambda \in \mathcal{Q}_{\mathrm{C}} \end{cases}$$

simplicial frequencies — *gradient freq.* λ_G
curl freq. λ_C
$\lambda_{G,2} = \lambda_{C,1}$ — *harmonic freq.* λ_H

gradient flow — $\mathbf{H}_1|_{\mathrm{im}(\mathbf{B}_1^\top)} : \mathrm{im}(\mathbf{B}_1^\top) \to \mathrm{im}(\mathbf{B}_1^\top)$

curl flow — $\mathbf{H}_1|_{\mathrm{im}(\mathbf{B}_2)} : \mathrm{im}(\mathbf{B}_2) \to \mathrm{im}(\mathbf{B}_2)$

harmonic flow — $\mathbf{H}_1|_{\mathrm{ker}(\mathbf{L}_1)} : \mathrm{ker}(\mathbf{L}_1) \to \mathrm{ker}(\mathbf{L}_1)$

grad response $\mathbf{g}_G(\lambda)$
curl response $\mathbf{g}_C(\lambda)$
harmonic response g_0

0 $\quad \lambda_{G,\min} \quad \lambda_{C,\min} \quad \lambda_{G,\tau} \quad \lambda_{C,\tau} \quad \lambda_{G,\max} \quad \lambda_{C,\max}$

Why two sets of coefficients instead of one set?

$\mathbf{x}_{1,G}$ $\quad \tilde{\mathbf{x}}_{1,G}$ $\quad \mathbf{H}_1|_{\mathrm{im}(\mathbf{B}_1^\top)}$ $\quad \tilde{h}_G(\lambda)$ $\quad \mathrm{im}(\mathbf{B}_1^\top)$

$\mathrm{im}(\mathbf{B}_1^\top)$ $\quad \times$

$\mathbf{x}_{1,H}$ $\quad \tilde{\mathbf{x}}_{1,H}$ $\quad \mathbf{H}_1|_{\mathrm{ker}(\mathbf{L}_1)}$ $\quad \tilde{h}_H(\lambda)$ $\quad \mathrm{ker}(\mathbf{L}_1)$

$\mathrm{ker}(\mathbf{L}_1)$ $\quad \times$

$\mathbf{x}_{1,C}$ $\quad \tilde{\mathbf{x}}_{1,C}$ $\quad \mathbf{H}_1|_{\mathrm{im}(\mathbf{B}_2)}$ $\quad \tilde{h}_C(\lambda)$ $\quad \mathrm{im}(\mathbf{B}_2)$

$\mathrm{im}(\mathbf{B}_2)$ $\quad \times$

$\lambda_{G,2} = \lambda_{C,1}$