Optimal TSB - Gaussian case

- Consider Gaussian end distributions $\rho_0 \sim N(\mu_0, \Sigma_0), \rho_1 \sim N(\mu_1, \Sigma_1)$

- Results 1: time-marginal $X_t \sim \mathbb{P}_t = N(\mu_t, \Sigma_t)$

$$X_t = \bar{R}_t X_0 + R_t X_1 + \xi_t - R_t \xi_1 + \Gamma_t Z$$

- $R_t = K_{t1} K_{11}^{-1}, \quad \bar{R}_t = \Psi_t - R_t \Psi_1, \quad \Gamma_t := \mathrm{Cov}[Y_t \mid (Y_0, Y_1)] = K_{tt} - K_{t1} K_{11}^{-1} K_{1t}$

- Stochastic interpolant expression [Albergo et al. 2023]

Proof ideas: [Bunne et al. 2023]

- Disintegration of measures: static Gaussian TSBP + recent result on Gaussian E-OT [Janati et al. 2020]

- Reciprocal property [Föllmer 1988]: $\mathbb{P}$ shares the bridge with the reference $\mathbb{Q}_{\mathcal{T}}$

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- Results 2: characterize $\mathbb{P}$ in terms of Itô differential

$$\mathrm{d}X_t = f_\tau(t, X_t; L)\,\mathrm{d}t + g_t\,\mathrm{d}W_t, \quad \text{where} \quad f_\tau(t, x; L) = S_t^\top \Sigma_t^{-1}(x - \mu_t) + \dot{\mu}_t$$

- S_t depends on the transition kernel
- Generalize the recent result [Bunne et al. 2023]
- Optimal $\mathbb{P}$ is in the law of a specific SDE class
- Infinitesimal generator + some tricks (central identity of quantum field theory)
- Can be used as a better (stronger-biased) reference process