Optimal topological SB

- Schrödinger system characterizes the optimality
- Disintegration of measures: gives us static TSBP (OT formulation)

$$\min D_{KL}(\mathbb{P}_{01} \| \mathbb{Q}_{\mathcal{T}01}) \ s.t. \mathbb{P}_0 = \rho_0, \mathbb{P}_1 = \rho_1$$

  - An E-OT with transport cost: $\| y_1 - \Psi_1 y_0 - \xi_1 \|^2_{K_{11}^{-1}}$
  - Lagrange multipliers: gives us a topological Schrödinger system —— iterative proportional fitting (cont. Sinkhorn all.) for half-bridge prob

- Stochastic control: tells us how optimal TSB follows a forward-backward SDE system

$$\mathrm{d}X_t = \left[f_t + g_t Z_t\right] \mathrm{d}t + g_t \mathrm{d}W_t, \quad X_0 \sim \rho_0, \quad Z_t \equiv g_t \nabla \log \varphi_t(X_t)$$
$$\mathrm{d}X_t = \left[f_t - g_t \hat{Z}_t\right] \mathrm{d}t + g_t \mathrm{d}W_t, \quad X_1 \sim \rho_1, \quad \hat{Z}_t \equiv g_t \nabla \log \hat{\varphi}_t(X_t)$$

Enables learning!!!

- Nonlinear Feynman-Kac formula: gives us the dynamics of $\log \varphi_t$ and $\log \hat{\varphi}_t$

Topological SBP

- A topological domain $\mathcal{T}$, e.g., a graph or a simplicial complex
- Topological stochastic process: $X := (X_t)_{t\in[0,1]} : [0,1] \times \mathcal{X} \to \mathbb{R}^n$
  - $\mathcal{X}$: the node/edge space of $\mathcal{T}$, with n the dimension
  - X follows some unknown dynamics with distr. law: $X \sim \mathbb{P} \to X_t \sim \mathbb{P}_t$ (time-marginal)
- Given the initial and final (empirical) signal distr. $X_0 \sim \rho_0$ and $X_1 \sim \rho_1$

Topological Schrödinger Bridge Problem

$$\min D_{KL}(\mathbb{P} \| \mathbb{Q}_{\mathcal{T}}) \ s.t.\ \mathbb{P}_0 = \rho_0, \mathbb{P}_1 = \rho_1$$