Topological SBP

- A topological domain $\mathcal{T}$, e.g., a graph or a simplicial complex
- Topological stochastic process: $X := (X_t)_{t\in[0,1]} : [0,1] \times \mathcal{X} \to \mathbb{R}^n$
  - $\mathcal{X}$: the node/edge space of $\mathcal{T}$, with n the dimension
  - X follows some unknown dynamics with distr. law: $X \sim \mathbb{P} \to X_t \sim \mathbb{P}_t$ (time-marginal)
- Given the initial and final (empirical) signal distr. $X_0 \sim \rho_0$ and $X_1 \sim \rho_1$

Topological Schrödinger Bridge Problem

$$\min D_{KL}(\mathbb{P} \| \mathbb{Q}_{\mathcal{T}}) \; s.t. \; \mathbb{P}_0 = \rho_0, \mathbb{P}_1 = \rho_1$$

Optimal topological SB

- Schrödinger system characterizes the optimality
- Disintegration of measures: gives us static TSBP (OT formulation)

$$\min D_{KL}(\mathbb{P}_{01} \| \mathbb{Q}_{\mathcal{T}01}) \; s.t. \, \mathbb{P}_0 = \rho_0, \mathbb{P}_1 = \rho_1$$

  - An E-OT with transport cost: $\left\| y_1 - \Psi_1 y_0 - \xi_1 \right\|^2_{K_{11}^{-1}}$
  - Lagrange multipliers: gives us a topological Schrödinger system —— iterative proportional fitting (cont. Sinkhorn alg.) for half-bridge prob
- Stochastic control: tells us how optimal TSB follows a forward-backward SDE system
  - Forward: $dX_t = dY_t + Z_t dt,\ X_0 \sim \rho_0$
  - Backward: $dX_t = dY_t - \tilde{Z}_t dt,\ X_1 \sim \rho_1$
- Nonlinear Feynman-Kac formula: gives us a likelihood

TSB-learning model

Learnable	Trainable
$Z_t \approx Z_t(\theta)$	$l(x_0; \phi)$
$\tilde{Z}_t \approx \tilde{Z}_t(\phi)$	$l(x_1; \theta)$