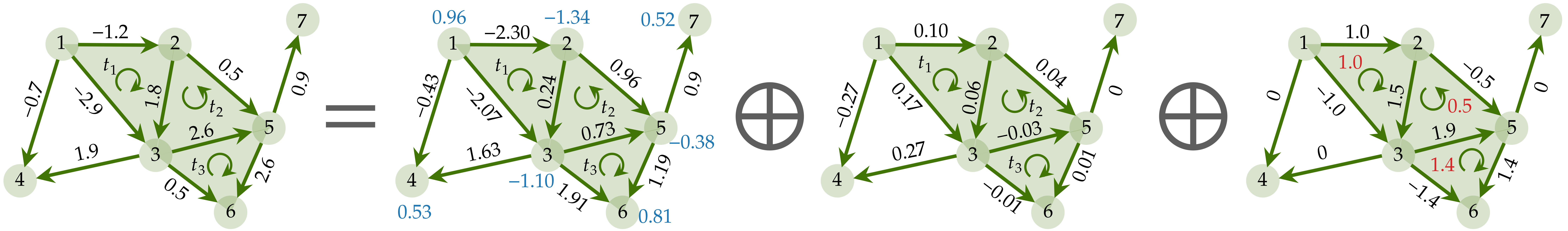
# Hodge decomposition

Lovász et al. 2004; Lim et al. 2020

$$\mathbb{R}^{N_1} = \mathrm{im}(\mathbf{B}_1^\top) \oplus \ker(\mathbf{L}_1) \oplus \mathrm{im}(\mathbf{B}_2)$$

$$\mathbf{f}_1 = \mathbf{f}_G + \mathbf{f}_H + \mathbf{f}_C$$




Edge flow = Gradient flow (Curl-free, irrotational) + Harmonic flow (Div- and curl-free) + Curl flow (Div-free, solenoidal)

**Hodge-compositional Edge GP**

$$\mathbf{f}_G \sim \mathrm{GP}(\mathbf{0}, \mathbf{K}_G)$$
$$\mathbf{f}_H \sim \mathrm{GP}(\mathbf{0}, \mathbf{K}_H)$$
$$\mathbf{f}_C \sim \mathrm{GP}(\mathbf{0}, \mathbf{K}_C)$$

# Hodge-compositional Edge GPs

## Curl-free, div-free GPs

$\mathbf{f}_G \sim \mathrm{GP}(\mathbf{0}, \mathbf{K}_G)$
$\mathbf{f}_H \sim \mathrm{GP}(\mathbf{0}, \mathbf{K}_H)$
$\mathbf{f}_C \sim \mathrm{GP}(\mathbf{0}, \mathbf{K}_C)$

- Gradient kernel $\mathbf{K}_G = \mathbf{U}_G \Psi_G(\mathbf{\Lambda}_G)\mathbf{U}_G^\top$; Curl kernel $\mathbf{K}_C = \mathbf{U}_C \Psi_C(\mathbf{\Lambda}_C)\mathbf{U}_C^\top$
- Matérn family: $\Psi_\square(\mathbf{\Lambda}_\square) = \sigma_\square^2 \left(\frac{2\nu_\square}{\kappa_\square^2}\mathbf{I} + \mathbf{\Lambda}_\square\right)^{-\nu_\square}, \quad \square = \mathrm{H}, \mathrm{G}, \mathrm{C}$
- Also as solutions of SDEs, e.g.,

$\Phi_C(\mathbf{L}_{1,\mathrm{u}})\, \mathbf{f}_1 = \mathbf{w}_C$, with curl noise $\mathbf{w}_C \sim N(0, \sigma_C^2 \mathrm{U}_C \mathbf{U}_C^\top)$ and

$$\Phi(\mathbf{L}_{1,u}) = \left(\frac{2\nu_C}{\kappa_C^2}\mathbf{I} + \mathbf{L}_{1,u}\right)^{\frac{\nu_C}{2}} \text{ or } \Phi(\mathbf{L}_{1,u}) = e^{\frac{\kappa_C^2}{4}\mathbf{L}_{1,u}}$$