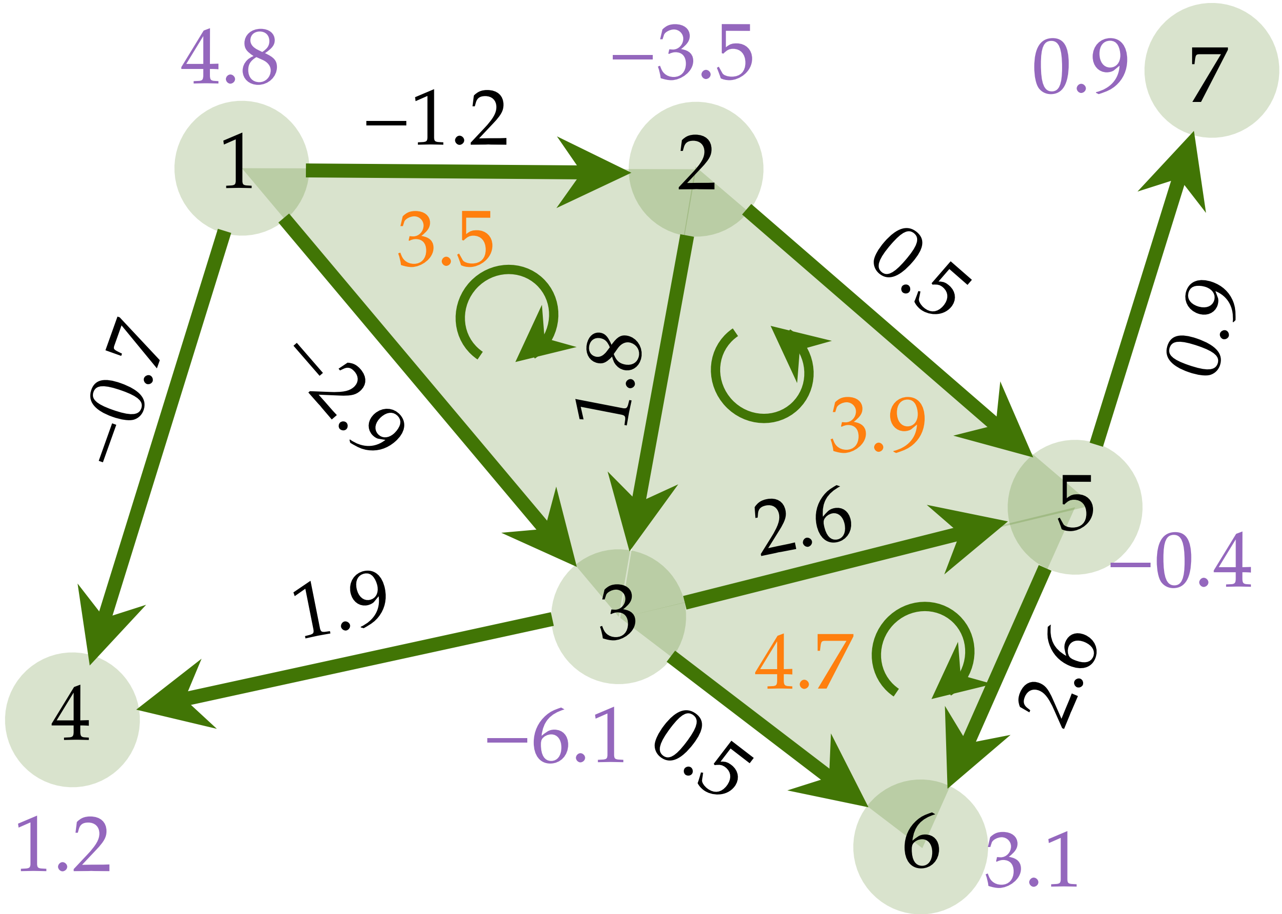
Incidence & Laplacians

1st and 2nd order Discrete Derivatives



Gradient of node signal: $\mathbf{B}_1^\top \mathbf{v}$ $\quad [\mathbf{f}_\text{G}]_{[i,j]} = [\mathbf{v}]_j - [\mathbf{v}]_i$

Divergence of edge flows: $[\mathbf{B}_1 \mathbf{f}]_{[i]} = \sum_{j<i} \mathbf{f}_{[j,i]} - \sum_{i<k} \mathbf{f}_{[i,k]}$

Net-flow = in_flow - out_flow

Curl of edge flows: $[\mathbf{B}_2^\top \mathbf{f}]_t = \mathbf{f}_{[i,j]} + \mathbf{f}_{[j,k]} - \mathbf{f}_{[i,k]}$, for $t = [i, j, k]$

Net-circulation in triangles

$$[\mathbf{B}_1 \mathbf{f}]_5 = 0.5 + 2.6 - (0.9 + 2.6) = -0.4$$

$$[\mathbf{B}_2^\top \mathbf{f}]_{[1,2,3]} = -1.2 + 1.8 - (-2.9) = 3.5$$

Hodge Laplacians = Grad Div + Curl* Curl

Hodge Laplacian: $\mathbf{L}_1 = \mathbf{B}_1^\top \mathbf{B}_1 + \mathbf{B}_2 \mathbf{B}_2^\top$

$$\Delta_1 = \nabla(\nabla\cdot) + \nabla\times(\nabla\times)$$

Simplicial Signal Smoothness

Variations in terms of faces and cofaces

- For edge flows
  - Total divergence $\|\mathbf{B}_1\mathbf{f}\|_2^2 = \mathbf{f}^\top\mathbf{L}_d\mathbf{f}$
  - Total curl $\|\mathbf{B}_2^\top\mathbf{f}\|_2^2 = \mathbf{f}^\top\mathbf{L}_u\mathbf{f}$
- For node signal: $\|\mathbf{B}_1^\top\mathbf{v}\|_2^2 = \mathbf{v}^\top\mathbf{L}_0\mathbf{v}$
- For general simplicial signals
  - Lower variation
  - Upper variation