Towards solving TSBP [Lénoard 2014]

Topological Schrödinger Bridge Problem

$$\min D_{KL}(\mathbb{P}\|\mathbb{Q}_{\mathcal{T}})\ s.t.\ \mathbb{P}_0=\rho_0, \mathbb{P}_1=\rho_1$$

Disintegration of Measures

$$\mathbb{P}(\cdot)=\int_{\mathbb{R}^n\times\mathbb{R}^n}\mathbb{P}^{xy}(\cdot)\,\mathbb{P}_{01}(dx\,dy)$$

Static TSBP / E-OT

$$\min D_{KL}(\mathbb{P}_{01}\|\mathbb{Q}_{\mathcal{T}01})\ s.t.\ \mathbb{P}_0=\rho_0, \mathbb{P}_1=\rho_1$$

+

$\mathbb{P}$ shares bridges with $\mathbb{Q}_{\mathcal{T}}$

$$\mathbb{P}^{xy}=\mathbb{Q}_{\mathcal{T}}^{xy}$$

An E-OT with transport cost: $\| y_1-\Psi_1 y_0-\xi_1 \|^2_{K_{11}^{-1}}$

Iterative proportional / Markovian fitting

Stochastic control view

- Equivalently, a variational problem

$$\min_{b_t} \mathbb{E}\left[\frac{1}{2}\int_0^1 \|b(t, X_t)\|^2\, dt\right], \quad \text{s.t.} \begin{cases} dX_t = [f_t + g_t b(t, X_t)]\, dt + g_t\, dW_t, \\ X_0 \sim \rho_0, X_1 \sim \rho_1\,. \end{cases}$$

- Initially formulated for classical case [Dai Pro 1991; Pavon 1991; Caluya & Halder 2021]
- Optimal TSB follows a forward-backward SDE system

Forward: $dX_t = dY_t + Z_t dt,\ X_0 \sim \rho_0$

Backward: $dX_t = dY_t - \tilde{Z}_t dt,\ X_1 \sim \rho_1$

Enables learning!!!

- $Z_t \equiv g_t \nabla \log \varphi_t(X_t)$, $\tilde{Z}_t \equiv g_t \nabla \log \tilde{\varphi}_t(X_t)$, depend on a system of PDE pairs (FPE, HJB)
- Nonlinear Feynman-Kac formula: gives us a likelihood [Chen et al. 2022]