Topological Schrödinger Bridge Matching

- A rigo formulation of topological SBP
- Investigating optimal TSBP solutions (Gaussian and general cases)
  - Stochastic optimal control on topological domains
  - (Dynamic) optimal transport
- TSB-based learning models
  - Unifies score-matching (diffusion-based), flow-matching (ODE-based) …
  - For generative and matching purposes
  - …. some discussions on possible directions based on energy interpretations

Optimal topological SB

- Schrödinger system characterizes the optimality
- Disintegration of measures: gives us static TSBP (OT formulation)

$$\min D_{KL}(\mathbb{P}_{01} \| \mathbb{Q}_{\mathcal{T}01}) \;\; s.t. \; \mathbb{P}_0 = \rho_0, \mathbb{P}_1 = \rho_1$$

  - An E-OT with transport cost: $\left\| y_1 - \Psi_1 y_0 - \xi_1 \right\|^2_{K_{11}^{-1}}$
  - Lagrange multipliers: gives us a topological Schrödinger system —— iterative proportional fitting (cont. Sinkhorn all.) for half-bridge prob

- Stochastic control: tells us how optimal TSB follows a forward-backward SDE system

$$\mathrm{d}X_t = \left[f_t + g_t Z_t\right] \mathrm{d}t + g_t \mathrm{d}W_t, \quad X_0 \sim \rho_0, \quad Z_t \equiv g_t \nabla \log \varphi_t(X_t)$$
$$\mathrm{d}X_t = \left[f_t - g_t \hat{Z}_t\right] \mathrm{d}t + g_t \mathrm{d}W_t, \quad X_1 \sim \rho_1, \quad \hat{Z}_t \equiv g_t \nabla \log \hat{\varphi}_t(X_t)$$

Enables learning!!!

- Nonlinear Feynman-Kac formula: gives us the dynamics of $\log \varphi_t$ and $\log \hat{\varphi}_t$