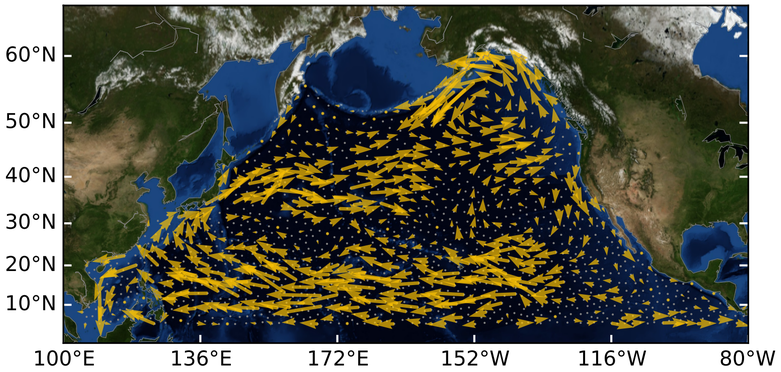
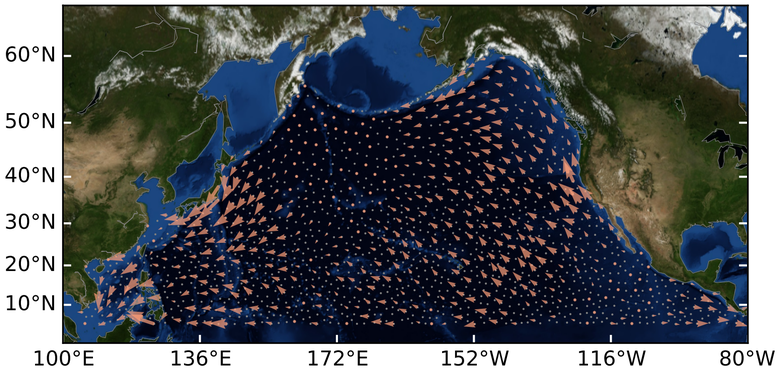
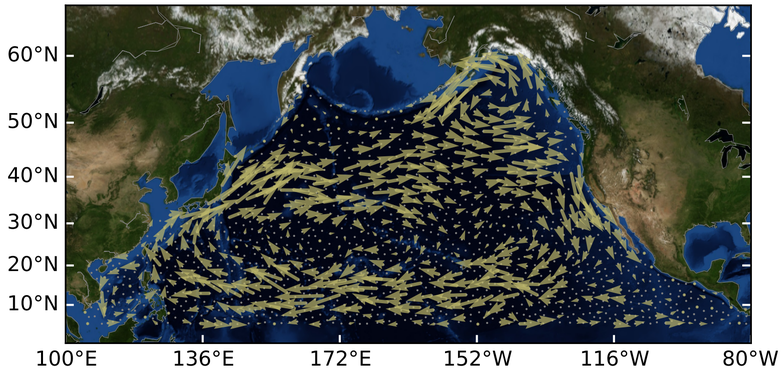
Applications of Hodge decomposition



Ocean currents



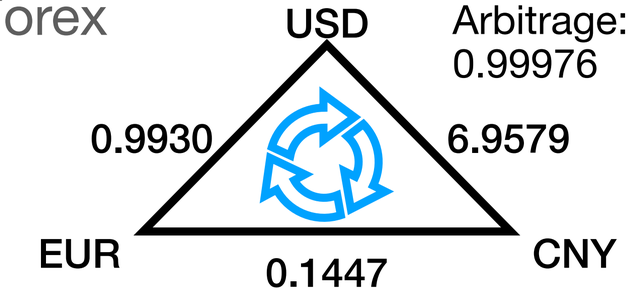
Gradient flow
Curl-free, irrotational



Curl flow
Div-free, solenoidal

Chen, Yu-Chia et al. (2021) "Helmholtzian Eigenmap."

Forex



Arbitrage:
0.99976

$$r^{a/b} r^{b/c} = r^{a/c}$$ Arbitrage-free

$$f_{[a,b]} + f_{[b,c]} - f_{[a,c]} = 0$$ Curl-free

- Water flows (div-free)
- Electrical currents, voltages

- Brain networks (Anand et al. 2022)
- Game theory (Candogan et al. 2011)
- Ranking problems (Jiang et al. 2011)
- Random walks (Strang et al. 2020)
- ...

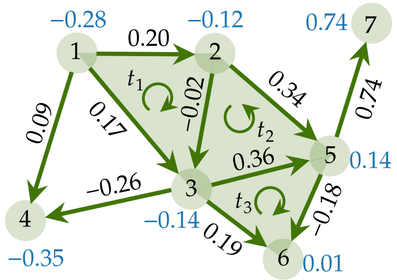
Eigenspace of $\mathbf{L}_1$ spans Hodge subspaces

- **Nonzero** Eigenspace of down Laplacian spans the gradient space

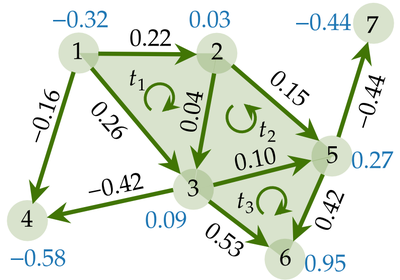
$$\lambda_G = \|\mathbf{B}_1\mathbf{u}_G\|_2^2$$

Gradient eigenvector

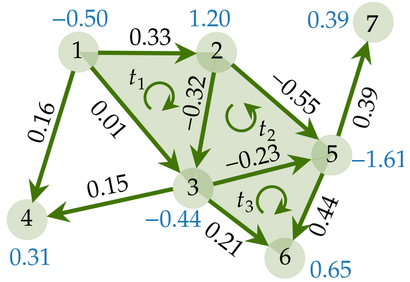
Fourier basis reflecting divergent properties



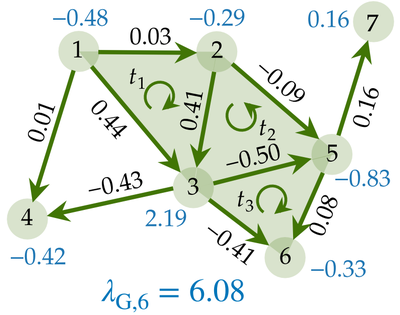
$\lambda_{G,1} = 0.80$



$\lambda_{G,2} = 1.61$



$\lambda_{G,5} = 5.12$



$\lambda_{G,6} = 6.08$

Simplicial Fourier transform

Frequency — eigenvalues

Fourier basis — eigenvectors

$$\lambda_C = \|\mathbf{B}_2^\top\mathbf{u}_C\|_2^2$$

$\lambda_{C,1} = 1.59$

$\lambda_{C,2} = 3.00$

$\lambda_{C,3} = 4.41$

$\lambda_{H,1} = 0$

EVD: $\mathbf{L}_1 = \mathbf{U}_1\mathbf{\Lambda}_1\mathbf{U}_1^\top$

$$\mathbf{U}_1 = [\mathbf{U}_H\ \mathbf{U}_G\ \mathbf{U}_C]$$
$$\text{span}(\mathbf{U}_G) = \text{im}(\mathbf{B}_1^\top)$$