Number Theory: Proof Set 1

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Problem 1

Determine a formula for $\sum_{k=1}^{n} \frac{1}{\sqrt{k}+\sqrt{k+1}}$ as a function of n.

Solution

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{\sqrt{k}+\sqrt{k+1}} &= \sum_{k=1}^{n} \frac{1}{\sqrt{k}+\sqrt{k+1}} \cdot \frac{\sqrt{k}-\sqrt{k+1}}{\sqrt{k}-\sqrt{k+1}} \\
&= \sum_{k=1}^{n} \frac{\sqrt{k}-\sqrt{k+1}}{k-(k+1)} \\
&= \sum_{k=1}^{n} \sqrt{k+1}-\sqrt{k} \\
&= \left(\cancel{\sqrt{2}}-\sqrt{1}\right)+\left(\cancel{\sqrt{3}}-\cancel{\sqrt{2}}\right)+\cdots+\left(\sqrt{n+1}-\cancel{\sqrt{n}}\right) \\
&= \boxed{\sqrt{n+1}-\sqrt{1}}
\end{aligned}$$