Sparse and Low-Rank Matrix Recovery

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Brief introduction of the speakers

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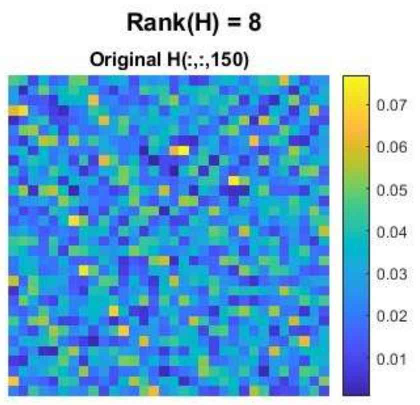
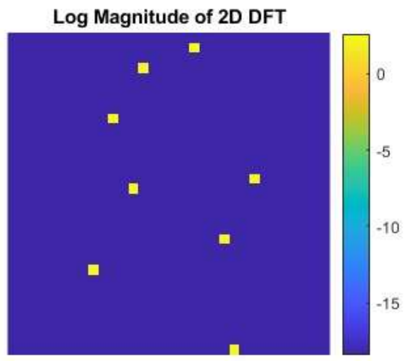
Introduction and Motivation

what are we trying to solve?

why is it an important problem?





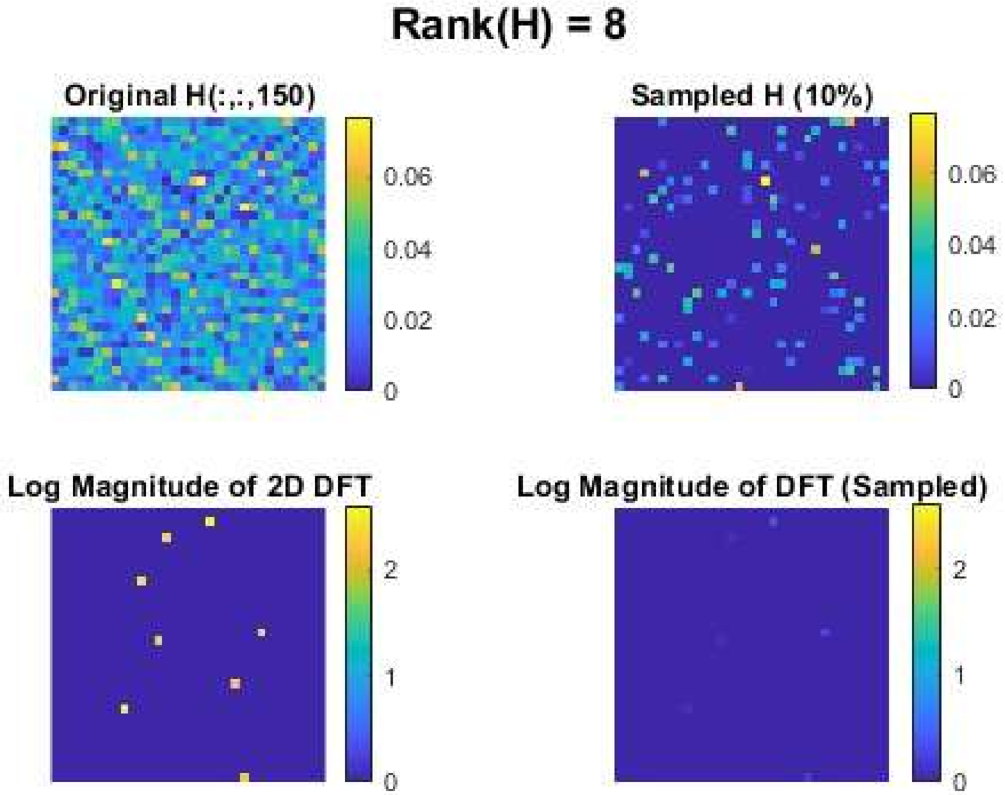
Dataset Description

Sparse and Low-Rank Matrix Recovery

Sparse in the frequency domain

Low-rank in the spatial domain

Subsampling Operator



Sparse in the frequency domain

Low-rank in the spatial domain

Why is it an important problem?

A fundamental challenge in many real-world scenarios.

- In medical imaging, such as MRI, we often want to minimize scan time.
- In wireless communications, bandwidth is limited.

Sparse Property of the Data

- In many cases, real-world signals exhibit sparsity in a specific domain
  - Communication signals, natural images

Low-Rank Property of the Data

- Low-rank or approximately low-rank matrices are common in data science.
  - Movie recommendation

02

Analysis Procedures and Methodologies

How are we trying to solve the problem?

Models and analysis procedures

Successive Proximal Gradient Descent

Sparse structure and sparsifying objective

$$\min_{\hat{H}} \|\mathcal{F}(\hat{H})\|_0 \quad \text{subject to} \quad \mathcal{P}_\Omega(\hat{H}) = G$$

$$\Downarrow$$

$$\min_{\hat{H}} \|\mathcal{F}(\hat{H})\|_1 \quad \text{subject to} \quad \mathcal{P}_\Omega(\hat{H}) = G$$

$$\Downarrow$$

$$\min_{\hat{H}} \|\mathcal{F}(\hat{H})\|_1 \quad \text{subject to} \quad \|\mathcal{P}_\Omega(\hat{H}) - G\|_2 \leq \epsilon$$

$$\Downarrow$$

$$\min_{\hat{H}} \|\mathcal{P}_\Omega(\hat{H}) - G\|_2^2 + \lambda\|\mathcal{F}(\hat{H})\|_1$$

Successive Proximal Gradient Descent

Low-rank structure and nuclear norm minimization

$$\min_{\hat{\mathbf{H}}} \left\|\mathcal{P}_\Omega(\hat{\mathbf{H}}) - \mathbf{y}\right\|_2^2 \quad \text{subject to} \quad \operatorname{rank}(\hat{\mathbf{H}}) = R, \mathbf{y} = \mathcal{P}_\Omega(\mathbf{H})$$

$$\Downarrow$$

$$\min_{\hat{\mathbf{H}}} \left\|\mathcal{P}_\Omega(\hat{\mathbf{H}}) - \mathbf{y}\right\|_2^2 + \lambda \left\|\hat{\mathbf{H}}\right\|_* \quad \text{subject to } \mathbf{y} = \mathcal{P}_\Omega(\mathbf{H})$$

Successive Proximal Gradient Descent

Summary

$$\min_{\hat{\mathbf{H}}} \quad \left\|\mathcal{P}_\Omega(\hat{\mathbf{H}}) - \mathbf{y}\right\|_2^2 + \lambda_1 \|\mathbf{X}\|_1 + \lambda_2 \left\|\hat{\mathbf{H}}\right\|_*$$

$$\text{subject to} \quad \mathbf{y} = \mathcal{P}_\Omega(\mathbf{H}), \quad \mathbf{X} = \mathcal{F}(\hat{\mathbf{H}})$$

$$\Downarrow$$

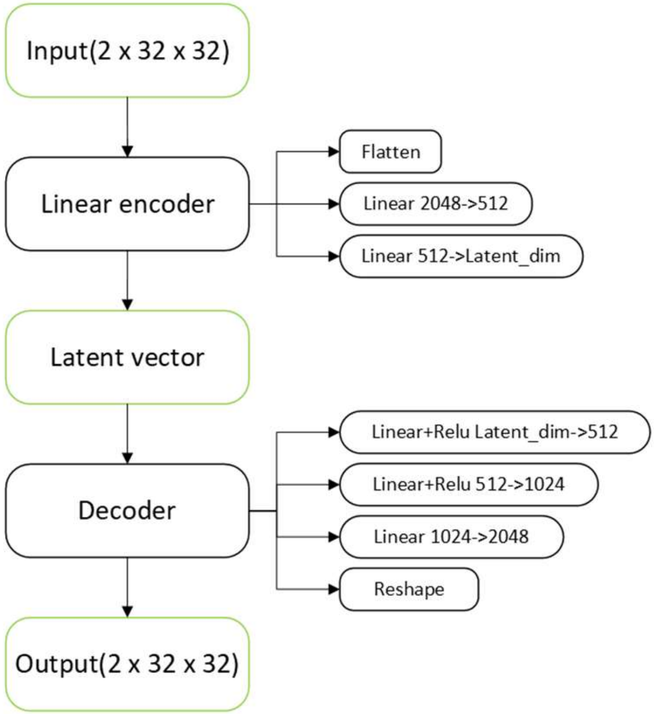
$$\mathbf{P}_{t+1} = \hat{\mathbf{H}}_t - \eta \nabla g(\hat{\mathbf{H}}_t) = \hat{\mathbf{H}}_t - 2\eta \cdot \mathcal{P}_\Omega(\hat{\mathbf{H}}_t - \mathbf{y})$$

$$\begin{aligned}\hat{\mathbf{H}}_{t+1} &= \operatorname{prox}_{\eta h}(\mathbf{P}_{t+1}) \\ &= \arg\min_{\hat{\mathbf{H}}} \left\{ \frac{1}{2\eta} \left\|\hat{\mathbf{H}} - \mathbf{P}_{t+1}\right\|_F^2 + \lambda \left\|\hat{\mathbf{H}}\right\|_* \right\}\end{aligned}$$

$$u_{\text{opt}}[i] = \begin{cases} w[i] - \eta\lambda, & \text{if } w[i] > \eta\lambda \\ w[i] + \eta\lambda, & \text{if } w[i] < -\eta\lambda \\ 0, & \text{otherwise} \end{cases}$$

Autoencoder

The matrix H is low-rank and sparse in the 2D DFT domain. This structure matches well with the strengths of an autoencoder.



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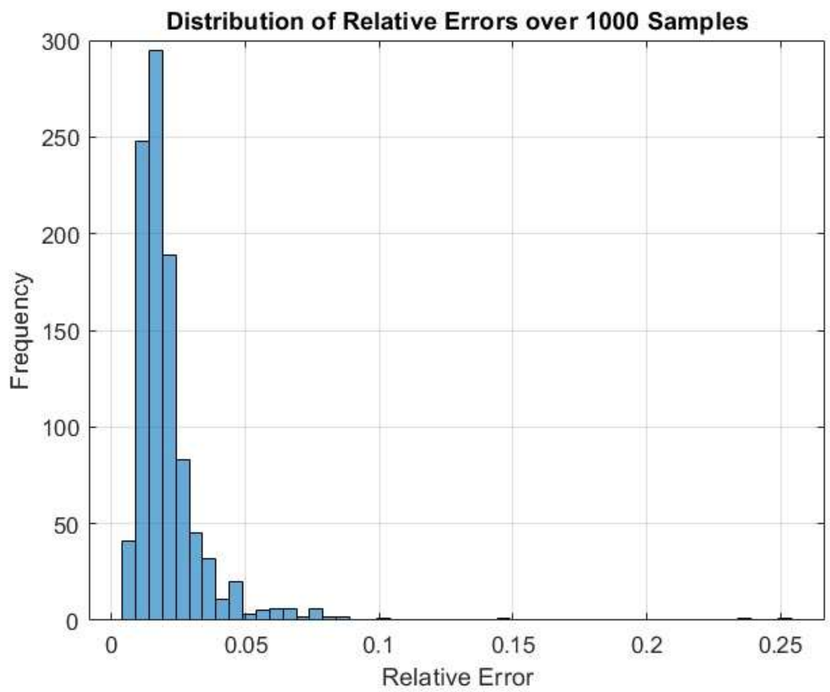
Results

Successive PGD and autoencoder

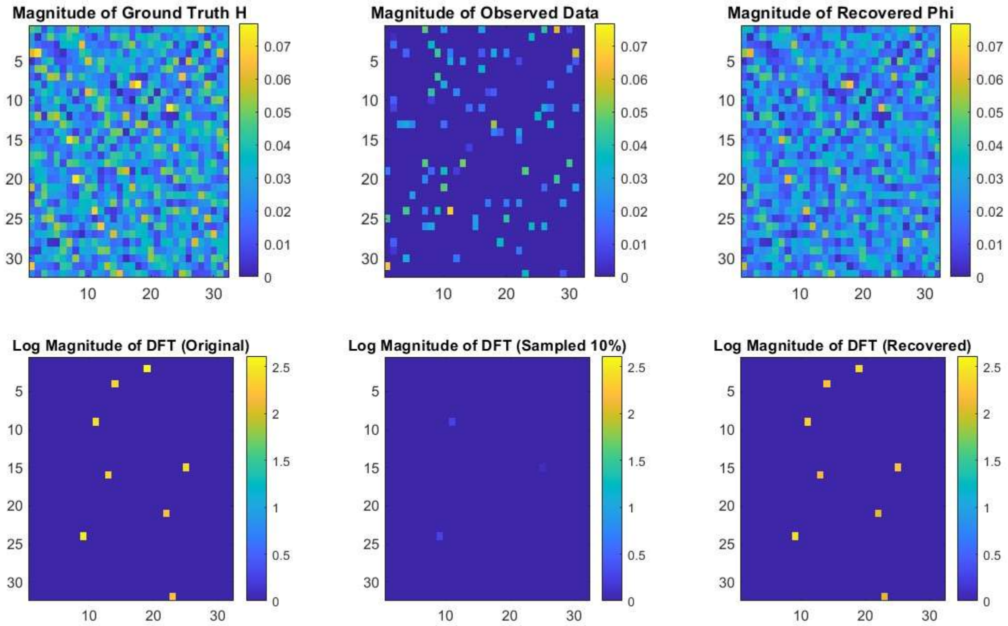
Successive Proximal Gradient Descent

After careful hyperparameter tuning of $\lambda_1, \lambda_2, \eta$



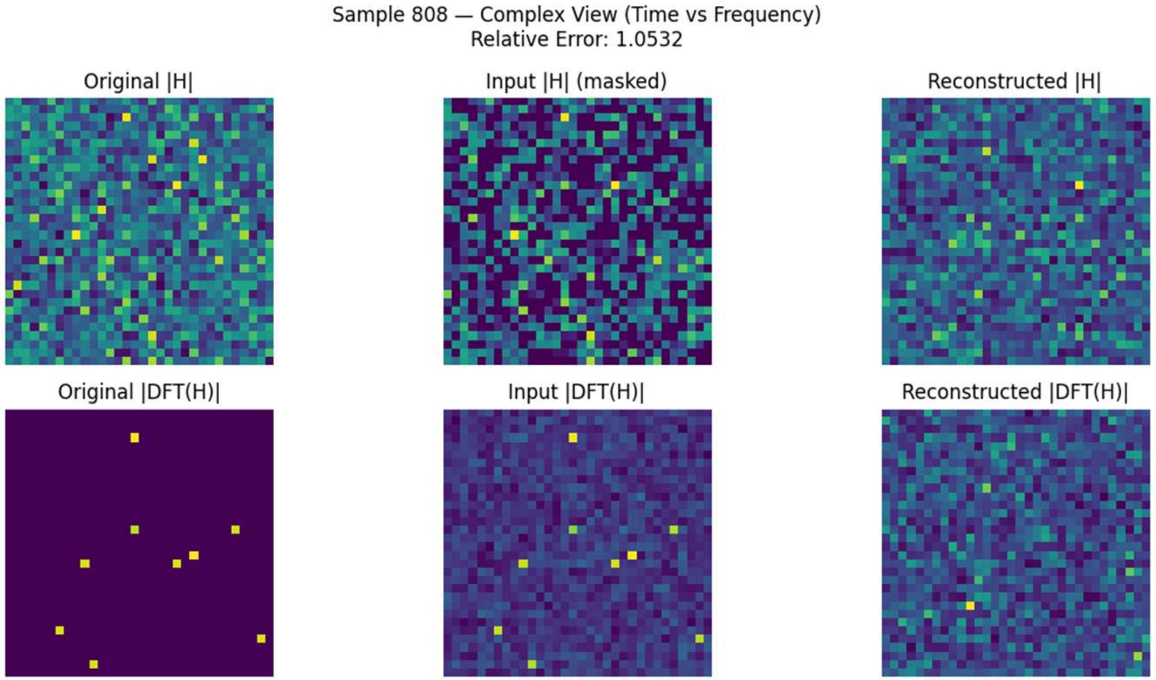
Successive Proximal Gradient Descent

After careful hyperparameter tuning of $\lambda_1, \lambda_2, \eta$



Autoencoder

Poor performance due to limited training data



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Discussions and Conclusions

Successive PGD and autoencoder

Discussions

A fundamental challenge in many real-world scenarios.

- Successive PGD method consistently achieves low relative reconstruction.
- The autoencoder performs significantly worse, which is due to limited data set.

Successive PGD

- It requires much smaller training
- It can directly exploit known structural priors
    - Manual tuning of hyperparameters

Autoencoder

- It can learn complex nonlinear structures from data
    - Availability of large and diverse training datasets

Conclusions

In a setting with limited data set like ours, classical model-based methods such as successive PGD clearly outperform data-driven approaches.

However, with access to a much larger dataset, the autoencoder may be able to match or even surpass successive PGD, provided it learns the relevant structures effectively.

Thank you for your attention.