习题课 #3 (Sep. 27)

1) 线性无关性的判别（解方程组、Cramer法则）

2) 线性相关/无关的概念及等价描述 P94.8 P97.12 P106.11.

3) 行列式补充习题

1) n元排列 $(n\geq 2)$ 中奇偶排列数相等.

$-\#\{\text{奇排列}\}+\#\{\text{偶排列}\}=\sum_{i_1\cdots i_n}(-1)^{\tau(i_1\cdots i_n)}=\det(1)_{ij}=0$ for $n\geq 2$.

Def. TFAE: $(\alpha_1,\cdots,\alpha_n\in\mathbb{F}^n)$

a) $\alpha_1,\cdots,\alpha_n$ 任意元素不能由其它向量线性表出

b) $\sum_i k_i\alpha_i=0$ 仅有零解

c) $\det(\alpha_1,\cdots,\alpha_n)\neq 0$.

d) 方程组 $\sum_j(\alpha_j)_i x_j=\lambda x_i$ 有非零解 $\Rightarrow\lambda\neq 0$.

c') $\det(\alpha_1,\cdots,\alpha_n)^T\neq 0$.

d') 方程组 $\sum_j(\alpha_i)_j x_j=\lambda x_i$ 有非零解 $\Rightarrow\lambda\neq 0$.

若 $\alpha_1,\cdots,\alpha_n\in\mathbb{F}^s$, $s\neq n$ 时仅有 a) $\Leftrightarrow$ b).

Rem: $(\alpha_1,\cdots,\alpha_n)$ 线性无关 $\not\Rightarrow$ $(\alpha_1,\cdots,\alpha_n)'$ 线性无关. 仅当 $s=n$ 时有此蕴含关系.

2) 令 $\alpha_1,\cdots,\alpha_s$ 线性无关, $\beta=\sum_i k_i\alpha_i$. 那么

$k_i\neq 0\Rightarrow\alpha_1,\cdots,\alpha_{i-1},\beta,\alpha_{i+1},\cdots,\alpha_s$ 线性无关.

Proof. 令 $\sum_{j\neq i}r_j\alpha_j+r_i\beta=0$. 有

$$\sum_{j\neq i}(r_j+r_ik_j)\alpha_j+r_ik_i\alpha_i=0.$$

由线性无关性：$r_ik_i=r_j+r_ik_j=0.\ \forall j\neq i.$

又$k_i\neq 0$. 于是有 $r_i=0$. 进而 $r_j=0.\ \forall j\neq i.$

3)* (Steinitz替换定理) 令 $\alpha_1,\cdots,\alpha_s\in F^n$. $\beta_1\cdots\beta_t$线性无关，可由其线性表出.

则存在 $j_1,\cdots,j_t$ 使得 $\{\alpha_i\mid i\neq j_l,\forall l=1,\cdots,t\}\cup\{\beta_1,\cdots,\beta_t\}$与$\alpha_1\cdots\alpha_s$等价

Proof. 注意只需说明$\alpha_1,\cdots,\alpha_s$可由$\{\alpha_i\mid i\neq j_l,\forall l=1,\cdots,t\}\cup\{\beta_1,\cdots,\beta_t\}$

线性表出. 以下对t进行归纳. (注意 t至多为 s)

① $t=1$: β_1线性无关即 $\beta_1\neq 0$. 于是必有 $\beta_1=\sum_j k_j\alpha_j$, 且 k_j不全为0.

不妨 $k_1\neq 0$. 下面说明$\alpha_1,\cdots,\alpha_s$可由 $\beta_1,\alpha_2,\cdots,\alpha_s$线性表出.

$\beta_1=\sum_j k_j\alpha_j$ 蕴含着 $\alpha_1=k_1^{-1}(\beta_1-\sum_{j\geq 2}k_j\alpha_j)$. 故命题得证.

② 假设结论对 $t-1$都成立. 根据条件，$\beta_1,\cdots,\beta_t$线性无关,可由$\alpha_1\cdots\alpha_s$线性表出,

那么特别地，$\beta_1,\cdots,\beta_{t-1}$线性无关.也可由$\alpha_1,\cdots,\alpha_s$线性表出.

于是由归纳假设,存在相应的$j_1,\cdots,j_{t-1}$.

不失一般性，不妨 $\beta_1,\cdots,\beta_{t-1},\alpha_t,\cdots,\alpha_s$ 可线性表出$\alpha_1,\cdots,\alpha_s$.

又 β_t可由$\alpha_1,\cdots,\alpha_s$线性表出，进而可设

$$\beta_t=\sum_{j<t}p_j\beta_j+\sum_{j\geq t}\varepsilon_j\alpha_j.$$

由$\beta_1,\cdots,\beta_t$线性无关知 ε_j不全为0. 不妨 $\varepsilon_t\neq 0$.

由上式易知α_t可由 $\beta_1,\cdots,\beta_t,\alpha_{t+1},\cdots,\alpha_s$线性表出.

又已证得 $\beta_1,\cdots,\beta_{t-1},\alpha_t,\cdots,\alpha_s$可线性表出$\alpha_1,\cdots,\alpha_s$,

故最终有 $\beta_1,\cdots,\beta_t,\alpha_{t+1},\cdots,\alpha_s$可线性表出$\alpha_1,\cdots,\alpha_s$.

Rem. 事实上上述证明依赖线性表出关系的传递性

对于 $V=R^n$ (或一般的线性空间), 可定义

$\{\alpha_1,\cdots,\alpha_s\}\leq\{\beta_1,\cdots,\beta_t\}$ 若 $\alpha_1,\cdots,\alpha_s$能由 $\beta_1,\cdots,\beta_t$线性表出.

$(\mathcal{P}_{fin}(V), \le)$ 构成了偏序关系，其中 $\mathcal{P}_{fin}$ 指代有限子集全体.
容易看出 ϕ 为最小元，且这样的序关系并非全序.
思考：$(\mathcal{P}_{fin}(V), \le)$ 中有无最大元？是否唯一？
若记最大元构成集合 $\{A_j\}_j$，$\min\{|A_j|\}_j$ 取值为何？

4) 令 $\alpha_1,\cdots,\alpha_s \in F^n$. 证明 $\alpha_1,\cdots,\alpha_s$ 中任意 r ($r\le\min(n,s)$) 个线性无关 iff.
$\sum_j x_j\alpha_j=0$ 任一非零解非零分量数目大于 r

Proof. $\Rightarrow$) 若不然，不妨设存在 $x_1,\cdots,x_r$ 不全为 0，$x_{r+1},\cdots,x_s=0$ 满足 $\sum_j x_j\alpha_j=0$.
于是 $\sum_{j\le r} x_j\alpha_j=\sum_j x_j\alpha_j=0$. 这与 $\alpha_1,\cdots,\alpha_r$ 线性无关矛盾.
$\Leftarrow$) 若不然，不妨 $\alpha_1,\cdots,\alpha_r$ 线性相关，即存在 $x_1,\cdots,x_r$ 不全为 0，
$\sum_{j\le r} x_j\alpha_j=0$. 令 $x_{r+1},\cdots,x_s=0$ 有 $\sum_j x_j\alpha_j=0$.
且此时 $x_1,\cdots,x_s$ 中非零分量必定不超过 r. 矛盾.

5) 对于 $A=(a_{ij})_{ij}=(\alpha_1,\cdots,\alpha_n)$，证明若对角占优，即
$|a_{ii}|>\sum_{j\ne i}|a_{ij}|$，$\forall i$，则 $\alpha_1,\cdots,\alpha_n$ 线性无关.

Proof. 令 $\sum_j k_j\alpha_j=0$. 若 $k_1,\cdots,k_n$ 不全为 0. 不妨 $|k_1|=\max_j|k_j|$.
根据定义，有 $k_1a_{11}+\sum_{j>1}k_ja_{1j}=0$.
而 $|k_1a_{11}|=\left|\sum_{j>1}k_ja_{1j}\right|\le\sum_{j>1}|k_j|\cdot|a_{1j}|\le|k_1|\sum_{j>1}|a_{1j}|<|k_1|\cdot|a_{11}|$. 矛盾.

6) 计算 $\det(\sum_k a_{ik}-a_{ij})$
令 $A_i=\sum_k a_{ik}$.

$$\begin{vmatrix} A_1-a_{11} & \cdots & A_1-a_{1n} \\ \vdots & & \vdots \\ A_n-a_{n1} & & A_n-a_{nn}\end{vmatrix} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & A_1-a_{11} & \cdots & A_1-a_{1n} \\ \vdots & \vdots & & \\ 0 & A_n-a_{n1} & \cdots & A_n-a_{nn}\end{vmatrix} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ -A_1 & -a_{11} & \cdots & -a_{1n} \\ & & & \\ -A_n & -a_{n1} & \cdots & -a_{nn}\end{vmatrix}$$

$$=\begin{vmatrix} 1-n & 1 & \cdots & 1 \\ 0 & -a_{11} & \cdots & -a_{1n} \\ \vdots & \vdots & & \vdots \\ 0 & -a_{n1} & \cdots & -a_{nn} \end{vmatrix} = (-1)^n(1-n)\det A.$$

7) 计算 $\begin{vmatrix} a_{11}-a_{12} & a_{12}-a_{13} & \cdots & a_{1,n-1}-a_{1n} & 1 \\ \vdots & & & \vdots & \vdots \\ a_{n1}-a_{n2} & a_{n2}-a_{n3} & \cdots & a_{n,n-1}-a_{nn} & 1 \end{vmatrix}$

① 原式 $=\begin{vmatrix} a_{11} & a_{12}-a_{13} & \cdots & a_{1,n-1}-a_{1n} & 1 \\ \vdots & & & \vdots & \vdots \\ a_{n1} & a_{n2}-a_{n3} & \cdots & a_{n,n-1}-a_{nn} & 1 \end{vmatrix} + \begin{vmatrix} -a_{12} & -a_{13} & \cdots & -a_{1n} & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ -a_{n2} & -a_{n3} & \cdots & -a_{nn} & 1 \end{vmatrix}.$

$$=\begin{vmatrix} a_{11} & a_{12} & a_{13}-a_{14} & \cdots & a_{1,n-1}-a_{1,n} & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3}-a_{n4} & \cdots & a_{n,n-1}-a_{nn} & 1 \end{vmatrix} + \begin{vmatrix} a_{11} & -a_{13} & -a_{14} & \cdots & -a_{1n} & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & -a_{n3} & -a_{n4} & \cdots & -a_{nn} & 1 \end{vmatrix}$$

$$+\begin{vmatrix} 1 & a_{12} & a_{13} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$$

$$=\begin{vmatrix} a_{11} & \cdots & a_{1,n-1} & 1 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{n,n-1} & 1 \end{vmatrix} + \cdots + \begin{vmatrix} a_{11} & 1 & a_{13} & \cdots \\ \vdots & \vdots & \vdots & \\ a_{n1} & 1 & a_{n3} & \cdots \end{vmatrix} + \begin{vmatrix} 1 & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ 1 & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

$=\sum_j A_{ij}.$

② 原式 $=\sum_{i_1=0}^{1}\sum_{i_2=0}^{1}\cdots\sum_{i_{n-1}=0}^{1}\begin{vmatrix} (-1)^{i_1}a_{1,1+i_1} & \cdots & (-1)^{i_{n-1}}a_{1,n-1+i_{n-1}} & 1 \\ \vdots & & \vdots & \vdots \\ (-1)^{i_1}a_{n,1+i_1} & \cdots & (-1)^{i_{n-1}}a_{n,n-1+i_{n-1}} & 1 \end{vmatrix}$

对求和式中若 $i_k=1$ 而 $i_{k+1}=0$，则求和项为0.（$k=1,2,\cdots,n-2$）

故仅有 $(i_1,\cdots,i_{n-1})=(0,\cdots,0),(0,\cdots,0,1),\cdots,(0,1,\cdots,1)$ 或 $(1,\cdots,1)$

时求和项可能非0. 故而（记 l 为取1的最小序号. 若无则约定为 n）

$$
原式=\sum_{l=1}^{n}\begin{vmatrix} a_{11} & \cdots & a_{1,l-1} & -a_{1,l+1} & \cdots & -a_{1,n} & 1 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{n,l-1} & -a_{n,l+1} & \cdots & -a_{n,n} & 1 \end{vmatrix}
$$

$$
=\sum_{l=1}^{n}\begin{vmatrix} a_{11} & \cdots & a_{1,l-1} & 1 & a_{1,l+1} & \cdots & a_{1,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,l-1} & 1 & a_{n,l+1} & \cdots & a_{n,n} \end{vmatrix}=\sum_{l}\sum_{k}A_{kl}
$$

8) 计算 $\det(\sin(\alpha_k+\alpha_l))_{k,l}$.

$$
\det(\sin(\alpha_k+\alpha_l))_{k,l}=\left(\frac{1}{2i}\right)^n\begin{vmatrix} e^{i(\alpha_1+\alpha_1)}-e^{-i(\alpha_1+\alpha_1)} & \cdots & e^{i(\alpha_1+\alpha_n)}-e^{-i(\alpha_1+\alpha_n)} \\ \vdots & & \vdots \\ e^{i(\alpha_n+\alpha_1)}-e^{-i(\alpha_n+\alpha_1)} & \cdots & e^{i(\alpha_n+\alpha_n)}-e^{-i(\alpha_n+\alpha_n)} \end{vmatrix}
$$

$$
=\sum_{p_1=\pm1}\cdots\sum_{p_n=\pm1}\left(\frac{1}{2i}\right)^n\begin{vmatrix} p_1e^{+ip_1(\alpha_1+\alpha_1)} & \cdots & p_ne^{ip_n(\alpha_1+\alpha_n)} \\ \vdots & & \vdots \\ p_1e^{+ip_1(\alpha_n+\alpha_1)} & \cdots & p_ne^{ip_n(\alpha_n+\alpha_n)} \end{vmatrix}
$$

其中每一求和项为 $p_1\cdots p_n\begin{vmatrix} e^{ip_1(\alpha_1+\alpha_1)} & \cdots & e^{ip_n(\alpha_1+\alpha_n)} \\ \vdots & & \vdots \\ e^{ip_1(\alpha_n+\alpha_1)} & \cdots & e^{ip_n(\alpha_n+\alpha_n)} \end{vmatrix}$

若 $p_1=\cdots=p_n=p$，则

$$
上式=p^n\begin{vmatrix} 1 & 0 & \cdots & 0 \\ e^{ip\alpha_1} & e^{ip(\alpha_1+\alpha_1)} & \cdots & e^{ip(\alpha_1+\alpha_n)} \\ \vdots & \vdots & & \vdots \\ e^{ip\alpha_n} & e^{ip(\alpha_n+\alpha_1)} & \cdots & e^{ip(\alpha_n+\alpha_n)} \end{vmatrix}=p^n\begin{vmatrix} 1 & -e^{ip\alpha_1} & \cdots & -e^{ip\alpha_n} \\ e^{ip\alpha_1} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ e^{ip\alpha_n} & 0 & \cdots & 0 \end{vmatrix}
$$

若$n\geq 2$，则上述行列式必为0.

对于一般的$p_1,\cdots,p_n=\pm 1$，若$n\geq 3$，则至少有2个p_k同号，不妨$p_1=p_2=p$.

$$\begin{vmatrix} e^{ip_1(\alpha_1+\alpha_1)} & \cdots & e^{ip_n(\alpha_1+\alpha_n)} \\ \vdots & & \\ e^{ip_1(\alpha_n+\alpha_1)} & \cdots & e^{ip_n(\alpha_n+\alpha_n)} \end{vmatrix} = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ e^{ip\alpha_1} & e^{ip(\alpha_1+\alpha_1)} & \cdots & e^{ip_n(\alpha_1+\alpha_n)} \\ \vdots & \vdots & & \vdots \\ e^{ip\alpha_n} & e^{ip(\alpha_n+\alpha_1)} & \cdots & e^{ip_n(\alpha_n+\alpha_n)} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -e^{ip\alpha_1} & -e^{ip\alpha_2} & 0 & \cdots & 0 \\ e^{ip\alpha_1} & 0 & 0 & e^{ip_3(\alpha_1+\alpha_3)} & \cdots & e^{ip_n(\alpha_1+\alpha_n)} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ e^{ip\alpha_n} & 0 & 0 & e^{ip_3(\alpha_n+\alpha_3)} & \cdots & e^{ip_n(\alpha_n+\alpha_n)} \end{vmatrix} = 0.$$

故而原行列式在$n\geq 3$时值为0.

另有 $n=1$ 时 行列式值为 $\sin 2\alpha_1$

$n=2$ 时行列式值为 $\sin 2\alpha_1 \sin 2\alpha_2 - \sin^2(\alpha_1+\alpha_2) = -\sin^2(\alpha_1-\alpha_2)$

9) 计算 $\begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ \mu a_{n-1} & a_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_1 \\ \mu a_1 & \cdots & \mu a_{n-1} & a_0 \end{vmatrix}$. μ为一常数.

不妨仅考虑$\mu\neq 0$. 取 $\alpha=(1,\xi,\cdots,\xi^{n-1})$.

$A\alpha=(f(\xi),\xi f(\xi),\cdots,\xi^{n-1}f(\xi))=f(\xi)\alpha$ for $\xi^n=\mu$.

故而 $\det A=\prod_{\xi^n=\mu} f(\xi)$.