习题课 #2 (Sep. 20)

1) 按一行或多行求行列式

2) Cramer 法则解方程组

3) 选讲 Cauchy 行列式.

Def. 余子式 ((i,j) minor) $M_{ij} = |A_{kl}|_{k\neq i, l\neq j}$.

代数余子式 ((i,j) cofactor) $C_{ij} = (-1)^{i+j} M_{ij}$.

Thm. $\sum_j a_{ij} C_{kj} = \delta_{ik} \det(A)$ i.e., $AC^T = \det(A) I$

$A = \det(A) C^{-T}$. $C = \det(A)^{-1} A^{-T}$. $A^{-1} = \det(A)^{-1} C^T$.

Thm. 系数矩阵行列式与线性方程组解的联系.

(n元方程组, n个方程) $\det(A) \neq 0 \Rightarrow$ 有唯一解. 否则无解或有无穷个解.

特别地, 对于齐次方程组. $\det(A) \neq 0 \Rightarrow$ 有唯一零解. 否则有无穷个解.

Thm. (Cramer 法则) 对于 n个方程 n元方程组, $\det(A) \neq 0$ 的情形, 有

唯一解: $\det(A)^{-1} (\det(B_1), \cdots, \det(B_n))^T$.

$$\det(B_j) = \begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & b_1 & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & b_n & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix} = \sum_i C_{ij} b_i$$

矩阵理解: $x = A^{-1} b = \det(A)^{-1} C^T b = \det(A)^{-1} \left(\sum_i C^T(j,i) b_i\right) = \det(A)^{-1} \sum_i C_{ij} b_i$

1) 计算三对角 (Toeplitz) 行列式:
$$\begin{vmatrix} a & b & & & 0 \\ c & a & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & a & b \\ 0 & & & c & a \end{vmatrix}$$

记 n阶行列式值为 D_n.

$D_n = aD_{n-1} - bcD_{n-2}$ for $n > 2$.

$D_2 = a^2 - bc$. $D_1 = a$.

令 η_1,η_2 为 $x^2-ax+bc=0$ 的两个解.

$$\begin{cases} D_n-\eta_1 D_{n-1}=\eta_2(D_{n-1}-\eta_1 D_{n-2})=\eta_2^{n-2}(D_2-\eta_1 D_1)\\ D_n-\eta_2 D_{n-1}=\eta_1(D_{n-1}-\eta_2 D_{n-2})=\eta_1^{n-2}(D_2-\eta_2 D_1)\end{cases}$$

① $\eta_1=\eta_2=\eta$: $\eta^{-n}D_n-\eta^{-(n-1)}D_{n-1}=\eta^{-2}(D_2-\eta D_1)$.

$$\Rightarrow D_n=\eta^n\left(\eta^{-1}D_1+(n-1)\eta^{-2}(D_2-\eta D_1)\right)$$

② $\eta_1\neq\eta_2$. $D_{n-1}=(\eta_2-\eta_1)^{-1}\left(\eta_2^{n-2}(D_2-\eta_1D_1)-\eta_1^{n-2}(D_2-\eta_2D_1)\right)$

2) 计算行列式: $\begin{vmatrix} a & b & b & \cdots & b\\ c & a & & & \vdots \\ \vdots & & \ddots & a & b \\ c & \cdots & & c & a\end{vmatrix}$.

记 n阶行列式值为 D_n. $D_1=a$, $D_2=a^2-bc$

$$D_n=(a-b)D_{n-1}+b\begin{vmatrix}1&1&\cdots&\cdots&1\\ c&a&b&\cdots&b\\ \vdots&&\ddots&&\vdots\\ &&&a&b\\ c&\cdots&&c&a\end{vmatrix}=(a-b)D_{n-1}+b\begin{vmatrix}a-c&b-c&\cdots&b-c\\ c-c & a-c & &\vdots\\ &\ddots&\ddots&b-c\\ &&c-c&a-c\end{vmatrix}$$

$$=(a-b)D_{n-1}+b(a-c)^{n-1}.$$

$$\begin{cases} D_n=(a-b)D_{n-1}+b(a-c)^{n-1}\\ D_n=(a-c)D_{n-1}+c(a-b)^{n-1}\end{cases}$$

① $b\neq c$: $D_{n-1}=(b-c)^{-1}\left(b(a-c)^{n-1}-c(a-b)^{n-1}\right)$

② $b=c$. $(a-b)^{-n}D_n=(a-b)^{-(n-1)}D_{n-1}+b(a-b)^{-1}$

$$\Rightarrow D_n=(a-b)^n\left((a-b)^{-1}D_1+b(a-b)^{-1}(n-1)\right)$$

$$=(a-b)^{n-1}\left(D_1+(n-1)b\right)$$

思考: 对于上述两题 $b=c$ 的情形是否可考虑 $c\to b$ 的极限? 合理性?

3) (Cauchy determinant) 计算 $\left|\frac{1}{x_i+y_j}\right|_{ij}$.

注意到 $\frac{1}{x_i+y_j}-\frac{1}{x_i+y_1}=\frac{y_1-y_j}{(x_i+y_1)(x_i+y_j)}=\frac{1}{x_i+y_1}\cdot\frac{y_1-y_j}{x_i+y_j}$

$$\begin{vmatrix} \frac{1}{x_1+y_1} & \frac{1}{x_1+y_1}\cdot\frac{y_1-y_2}{x_1+y_2} & \cdots & \frac{1}{x_1+y_1}\cdot\frac{y_1-y_n}{x_1+y_n} \\ \vdots & \vdots & & \vdots \\ \frac{1}{x_n+y_1} & \frac{1}{x_n+y_1}\cdot\frac{y_1-y_2}{x_n+y_2} & \cdots & \frac{1}{x_n+y_1}\cdot\frac{y_1-y_n}{x_n+y_n} \end{vmatrix}$$

$$=\frac{\prod_{j>1}(y_1-y_j)}{\prod_i(x_i+y_1)}\begin{vmatrix} 1 & \frac{1}{x_1+y_2} & \cdots & \frac{1}{x_1+y_n} \\ \vdots & \vdots & & \vdots \\ 1 & \frac{1}{x_n+y_2} & \cdots & \frac{1}{x_n+y_n} \end{vmatrix}$$

$$=\frac{\prod_{j>1}(y_1-y_j)}{\prod_i(x_i+y_1)}\begin{vmatrix} 1 & \frac{1}{x_1+y_2} & \cdots & \frac{1}{x_1+y_n} \\ 0 & \frac{x_1-x_2}{(x_2+y_2)(x_1+y_2)} & & \frac{x_1-x_2}{(x_2+y_n)(x_1+y_n)} \\ \vdots & \vdots & & \vdots \\ 0 & \frac{x_1-x_n}{(x_n+y_2)(x_1+y_2)} & & \frac{x_1-x_n}{(x_n+y_n)(x_1+y_n)} \end{vmatrix}$$

$$=\frac{\prod_{j>1}(y_1-y_j)}{\prod_i(x_i+y_1)}\prod_{j>1}(x_j-x_1)\cdot\frac{1}{\prod_{j>1}(x_1+y_j)}\begin{vmatrix} \frac{1}{x_2+y_2} & \cdots & \frac{1}{x_2+y_n} \\ \frac{1}{x_n+y_2} & \cdots & \frac{1}{x_n+y_n} \end{vmatrix}$$

$$=\frac{\prod_{j>1}(x_j-x_1)\prod_{j>1}(y_1-y_j)}{\prod_{i=1\text{或}j=1}(x_i+y_j)}\begin{vmatrix} \frac{1}{x_2+y_2} & \cdots & \frac{1}{x_2+y_n} \\ \frac{1}{x_n+y_2} & \cdots & \frac{1}{x_n+y_n} \end{vmatrix}$$

$$=\prod_{i<j}(x_j-x_i)(y_i-y_j)\prod_{i,j}(x_i+x_j)^{-1}.$$

~~4~~ 求行列式 $\begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1} & \ddots & & \vdots \\ \vdots & & \ddots & a_1 \\ a_1 & \cdots & & a_0 \end{vmatrix}$ （循环方阵） [需要矩阵乘法]

$$A(1,\xi,\cdots,\xi^{n-1})^T = f(\xi)(1,\xi,\cdots,\xi^{n-1})^T \Rightarrow \det(A)=\prod_{\xi^n=1} f(\xi)$$

5) 计算 $D=\begin{vmatrix} \sin\theta_1 & \sin2\theta_1 & \cdots & \sin n\theta_1 \\ \vdots & & & \vdots \\ \sin\theta_n & \sin2\theta_n & \cdots & \sin n\theta_n \end{vmatrix}$

$$D=\begin{vmatrix} \frac{1}{2i}(e^{i\theta_1}-e^{-i\theta_1}) & \frac{1}{2i}(e^{2i\theta_1}-e^{-2i\theta_1}) & \cdots & \frac{1}{2i}(e^{ni\theta_1}-e^{-ni\theta_1}) \\ \frac{1}{2i}(e^{i\theta_n}-e^{-i\theta_n}) & \frac{1}{2i}(e^{2i\theta_n}-e^{-2i\theta_n}) & \cdots & \frac{1}{2i}(e^{ni\theta_n}-e^{-ni\theta_n}) \end{vmatrix}$$

$$=\prod_k \frac{1}{2i}(e^{i\theta_k}-e^{-i\theta_k})\begin{vmatrix} 1 & e^{i\theta_1}+e^{-i\theta_1} & \cdots & e^{(n-1)i\theta_1}+e^{(n-3)i\theta_1}e^{-i\theta_1}+\cdots+e^{-(n-1)i\theta_1} \\ \vdots & & & \vdots \\ 1 & e^{i\theta_n}+e^{-i\theta_n} & \cdots & e^{(n-1)i\theta_n}+e^{(n-2)i\theta_n}e^{-i\theta_n}+\cdots+e^{-(n-1)i\theta_n} \end{vmatrix}$$

$$=\prod_k \sin\theta_k \begin{vmatrix} 1 & e^{i\theta_1}+e^{-i\theta_1} & e^{2i\theta_1}+1+e^{-2i\theta_1} & \cdots & e^{(n-1)i\theta_1}+e^{(n-3)i\theta_1}+\cdots+e^{-(n-1)i\theta_1} \\ \vdots \\ 1 & e^{i\theta_n}+e^{-i\theta_n} & e^{2i\theta_n}+1+e^{-2i\theta_n} & \cdots & e^{(n-1)i\theta_n}+e^{(n-3)i\theta_n}+\cdots+e^{-(n-1)i\theta_n} \end{vmatrix}$$

$$=\prod_k \sin\theta_k \begin{vmatrix} 1 & e^{i\theta_1}+e^{-i\theta_1} & e^{2i\theta_1}+e^{-2i\theta_1} & \cdots & e^{(n-1)i\theta_1}+e^{-(n-1)i\theta_1} \\ \vdots & & & & \vdots \\ 1 & e^{i\theta_n}+e^{-i\theta_n} & e^{2i\theta_n}+e^{-2i\theta_n} & \cdots & e^{(n-1)i\theta_n}+e^{-(n-1)i\theta_n} \end{vmatrix}$$

$$=\prod_k \sin\theta_k \begin{vmatrix} 1 & e^{i\theta_1}+e^{-i\theta_1} & (e^{i\theta_1}+e^{-i\theta_1})^2 & \cdots & (e^{i\theta_1}+e^{-i\theta_1})^{n-1} \\ \vdots \\ 1 & e^{i\theta_n}+e^{-i\theta_n} & (e^{i\theta_n}+e^{-i\theta_n})^2 & \cdots & (e^{i\theta_n}+e^{-i\theta_n})^{n-1} \end{vmatrix}$$

$$=\prod_k \sin\theta_k \prod_{i<j} 2(\cos\theta_i-\cos\theta_j)$$

6) 证明：$\frac{d}{dt}\det(a_{ij}(t))_{ij}=\sum_j \det B_j(t)=\sum_{i,j}\frac{d}{dt}a_{ij}(t)\,C_{ij}(t)$　代数余子式　B_j为A仅对第j列求导所得矩阵.

Proof. $\frac{d}{dt}\det(a_{ij}(t))_{ij}=\frac{d}{dt}\sum_{i=1}^{n}(a_{i1}(t)C_{i1}(t))$

$$=\sum_{i=1}^{n}a_{i1}(t)'\,C_{i1}(t)+\sum_{i=1}^{n}a_{i1}(t)\,C_{i1}(t)'$$

$$=\det B_1(t)+\sum_{i=1}^{n}a_{i1}(t)\sum_{j=1}^{n-1}\det D_j^{(i)}(t)\quad\text{(归纳假设)}$$

其中 $D_j^{(i)}(t)$ 为 A 划去i行、1列后仅对第j列求导所得矩阵，这也等同于 $B_{j+1}(t)$ 划去i行、1列对应的矩阵，故而 $\sum_i a_{i1}(t)\det D_j^{(i)}(t)$ 为 $\det B_{j+1}(t)$. 于是原式为 $\sum_j \det B_j(t)$.

进一步地：$\sum_j \det B_j(t)=\sum_j\sum_i a_{ij}(t)'\,C_{ij}(t)$ (按第j列展开).

7) (加边法) 计算

$$D_1=\begin{vmatrix}1&\cdots&1\\x_1&\cdots&x_n\\\vdots&&\vdots\\x_1^{k-1}&\cdots&x_n^{k-1}\\x_1^{k+1}&\cdots&x_n^{k+1}\\\vdots&&\vdots\\x_1^{n}&\cdots&x_n^{n}\end{vmatrix}\qquad D_2=\begin{vmatrix}1+x_1&1+x_1^2&\cdots&1+x_1^n\\\vdots&&&\vdots\\1+x_n&1+x_n^2&\cdots&1+x_n^n\end{vmatrix}$$

$$D_3=\det(t+a_{ij})_{ij}$$

D_1为行列式 $\begin{vmatrix}1&1&\cdots&1\\y&x_1&&x_n\\\vdots&\vdots&&\vdots\\y^n&x_1^n&&x_n^n\end{vmatrix}$ 中 y^k 前系数的 $(-1)^k$ 倍.

而上述行列式可写为 $\prod_{j=1}^{n}(x_j-y)\prod_{j<l}(x_l-x_j)$

y^k前系数为 $(-1)^k\sum_{i_1<\cdots<i_{n-k}}\prod_{p=1}^{n-k}x_{i_p}\prod_{j<l}(x_l-x_j)$

于是 $D_1=\sum_{i_1<\cdots<i_{n-k}}\prod_{p=1}^{n-k}x_{i_p}\prod_{j<l}(x_l-x_j)=\sigma_{n-k}\prod_{j<l}(x_l-x_j)$

$$D_2=\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ -1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots \\ -1 & x_n & x_n^2 & \cdots & x_n^n \end{vmatrix} = -\begin{vmatrix} -1 & 1 & 1 & \cdots & 1 \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{vmatrix}$$

$$= -\begin{vmatrix} -2 & 0 & \cdots & 0 \\ +1 & x_1 & \cdots & x_1^n \\ \vdots & \vdots \\ +1 & x_n & \cdots & x_n^n \end{vmatrix} - \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & x_1 & \cdots & x_1^n \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \cdots & x_n^n \end{vmatrix}$$

$$= +2\begin{vmatrix} x_1 & \cdots & x_1^n \\ \vdots & & \vdots \\ x_n & \cdots & x_n^n \end{vmatrix} - \prod_k (x_k-1) \prod_{i<j} (x_j-x_i)$$

$$= \left(2\prod_k x_k - \prod_k (x_k-1)\right)\prod_{i<j}(x_j-x_i).$$

$D_3 = \det A + t\sum_{i,j} C_{ij}$ ← 代数余子式.

8) $\begin{cases} (\lambda-2)x_1-3x_2-2x_3=0, \\ -x_1+(\lambda-8)x_2-2x_3=0, \\ 2x_1+14x_2+(\lambda+3)x_3=0. \end{cases}$ 何时有非零解.

$$\begin{vmatrix} \lambda-2 & -3 & -2 \\ -1 & \lambda-8 & -2 \\ 2 & 14 & \lambda+3 \end{vmatrix} = (\lambda-2)(\lambda^2-5\lambda+4) -3\lambda-9+28+2(6+2\lambda-16)$$

$$= (\lambda-2)(\lambda-1)(\lambda-4)+\lambda-1 = (\lambda-1)(\lambda-3)^2.$$

$\lambda=1$或3时有非零解.

思考: 对于n元齐次方程组(n个方程), 使得

$$\begin{cases} a_{11}x_1+\cdots+a_{1n}x_n=\lambda x_1 \\ \vdots \\ a_{n1}x_1+\cdots+a_{nn}x_n=\lambda x_n \end{cases}$$

有非零解的λ是否可能无穷多个? 若有限, 至多几个?

补充. 分片线性函数拟合、样条函数插值

上一节已证存在唯一至多 n 次多项式恰好经过事先固定的点 $\{(x_k,y_k)\}_{k=0}^{n}$ $(n+1)$ 个. $\{x_k\}_{k=0}^{n}$ 互不相同. [多项式拟合] (利用Vandermonde矩阵非奇异)

在实际应用中. 多项式拟合有潜在问题 ① 计算代价高 ② Runge现象.

故时常考虑局部化逼近方案: 令 $x_0<x_1<\cdots<x_n$.

在每一区间 $[x_{k-1},x_k]$ 使用至多 s 次多项式 $\varphi_k(x)$ 逼近, 使得

$\varphi_k(x_{k-1})=y_{k-1}$, $\varphi_k(x_k)=y_k$. $k=1,2,\cdots,n$. 且在分段处有 r 阶连续性:

$\varphi_k^{(l)}(x_k)=\varphi_{k+1}^{(l)}(x_k)$, $l=0,1,\cdots,r$.

问题: ① 若 $s=1$, $r=0$ (分片线性函数) 证明存在唯一 $\{\varphi_k\}_{k=1}^{n}$ 满足上述条件.

② 若 $s=3$, $r=2$ (cubic spline) 存在唯一性是否仍能保证?

* 将条件转化为线性方程组:

n 个至多 s 次多项式 $\rightarrow$ $n(s+1)$ 个未知元.

插值条件 $\rightarrow$ $2n$ 个方程. 光滑性条件 $\rightarrow$ $(n-1)r$ 个方程.

仅当 $2n+(n-1)r\geq n(s+1)$, 即 $(1+r-s)n\geq r$.

当 $s=r+1$ 时, 需要额外至少 r 个方程才能保证唯一性.

记 $\varphi_k'(x_k)=m_k$. 则由 $\varphi_k(x_k)=y_k$, $\varphi_k'(x_k)=m_k$, $\varphi_k(x_{k-1})=y_{k-1}$, $\varphi_k'(x_{k-1})=m_{k-1}$

可唯一确定 $\varphi_k(x)=y_{k-1}h_k(x)+y_k\tilde{h}_k(x)+m_{k-1}g_k(x)+m_k\tilde{g}_k(x)$.

$$\varphi_k''(x_k)=\varphi_{k+1}''(x_k)\Rightarrow y_{k-1}h_k''(x_k)+y_k\tilde{h}_k''(x_k)+m_{k-1}g_k''(x_k)+m_k\tilde{g}_k''(x_k)$$
$$=y_k h_{k+1}''(x_k)+y_{k+1}\tilde{h}_{k+1}''(x_k)+m_k g_{k+1}''(x_k)+m_{k+1}\tilde{g}_{k+1}''(x_k)$$

组成关于 $m_1,\cdots,m_n$ 的三对角矩阵.

对于自然样条 ($\varphi_1''(x_0)=\varphi_n''(x_n)=0$). 系数矩阵为

$$\begin{pmatrix} 2 & 1 & & & \\ 1-\lambda_1 & 2 & \lambda_1 & & \\ & & \ddots & & \\ & & 1-\lambda_{n-1} & 2 & \lambda_{n-1} \\ & & & 1 & 2 \end{pmatrix}$$ 行列式不为0.

9) $a_{ij}\begin{cases}>0, & i=j,\\ <0, & i\neq j.\end{cases}$ $\sum_j a_{ij}>0$. 证明 $\det(a_{ij})_{ij}>0$.

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22}-a_{21}a_{11}^{-1}a_{12} & a_{23}-a_{21}a_{11}^{-1}a_{13} & \cdots & a_{2n}-a_{21}a_{11}^{-1}a_{1n} \\ \vdots & & & \\ a_{n2}-a_{n1}a_{11}^{-1}a_{12} & a_{n3}-a_{n1}a_{11}^{-1}a_{13} & \cdots & a_{nn}-a_{n1}a_{11}^{-1}a_{1n} \end{vmatrix}$$

第 $(i-1)$ 行对角元为 $a_{ii}-a_{i1}a_{11}^{-1}a_{1i}>-a_{i1}-a_{i1}a_{11}^{-1}a_{1i}$

$$=-a_{11}^{-1}a_{i1}(a_{11}+a_{1i})>0.$$

第 $(i-1)$ 行所有元素之和为 $\sum_{j=2}^{n}a_{ij}-a_{i1}a_{11}^{-1}a_{1j}>-a_{i1}-\sum_{j=2}^{n}a_{i1}a_{11}^{-1}a_{1j}$

$$=-a_{11}^{-1}a_{i1}\left(a_{11}+\sum_{j=2}^{n}a_{1j}\right)>0.$$

第 $(i-1)$ 行非对角元为 $a_{ij}-a_{i1}a_{11}^{-1}a_{1j}<0-0=0.$

依照归纳法立即得证.