Control Objectives

1. Study the system (plant) to be controlled and obtain initial information about the control objectives
2. Model the system and simplify the model, if necessary
   (a) Identification of the input and output variables of the process
   (b) Identify the dependencies of each variable, starting with the output, until the system only depends on input variables
   (c) Identify the transmission behaviour between the signals
3. Scale the variables and analyze the resulting model; determine its properties
4. Decide which variables are to be controlled (controlled outputs)
5. Decide on the measurements and manipulated variables: what sensors and actuators will be used and where will they be placed?
6. Select the control configuration
7. Decide on the type of controller to be used
8. Decide on performance specifications, based on the overall control objectives
9. Design a controller
10. Analyze the resulting controlled system to see if the specifications are satisfied; and if they are not satisfied modify the specificiations or the type of controller
11. Simulate the resulting controlled system, either on a computer or a pilot plant
12. Repeat from step 2, if necessary
13. Choose hardware and software and implement the controller
14. Test and validate the control system, and tune the controller on-line, if necessary

Control Objectives

1. Tilt control
2. Height control

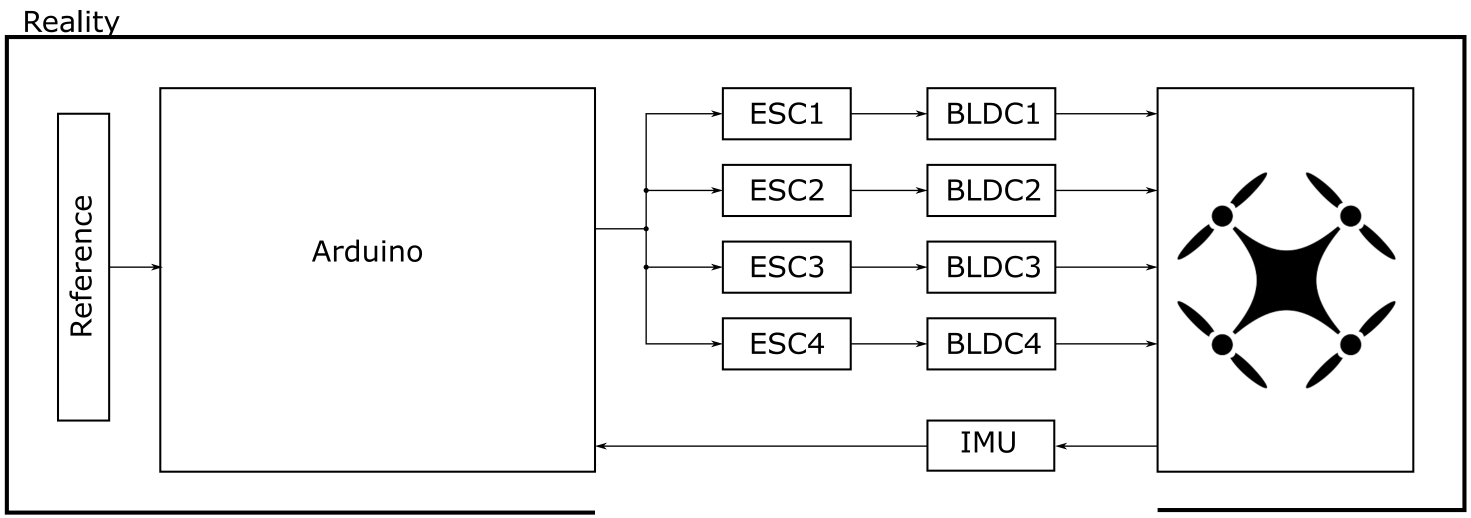

Figure 1: Basic system structure

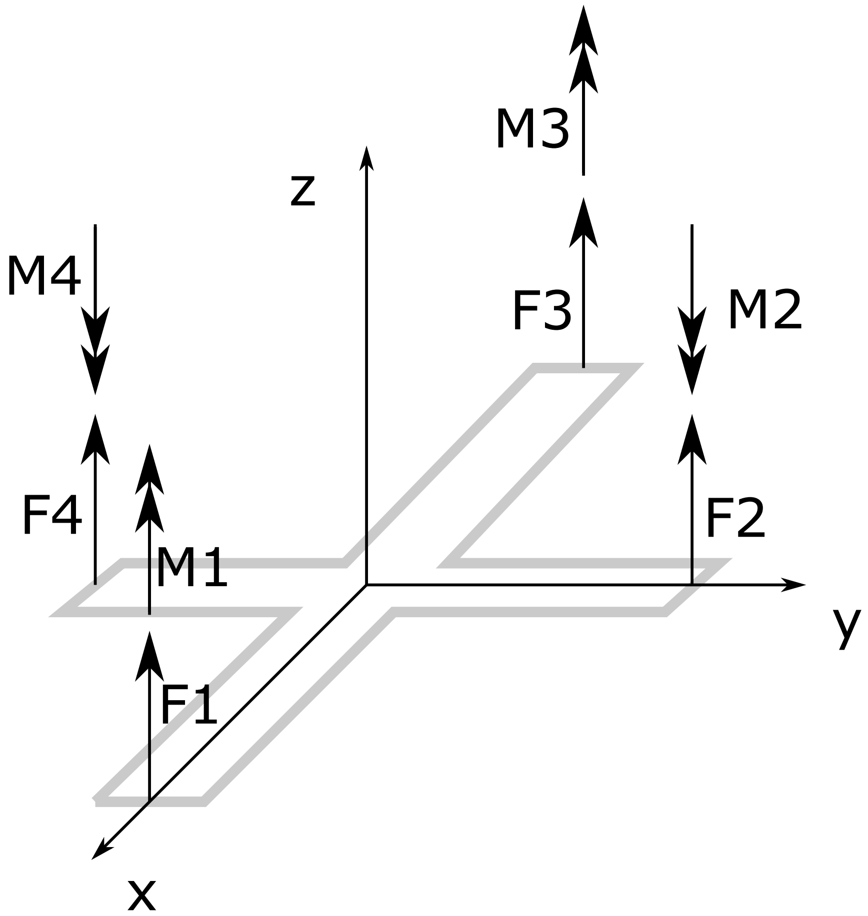

Figure 2: Choice of coordinate system

System Model

Geodetical x,y,z-position

$$\boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Absolute velocity in x,y,z-direction of the body frame

$$\boldsymbol{v} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

Euler Angles

$$\boldsymbol{\Phi} = \begin{pmatrix} \Phi \\ \Theta \\ \Psi \end{pmatrix}$$

Rotational speed around x,y,z-axis of the body frame

$$\boldsymbol{\omega} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

PWM signal for electronic speed controller (ESC)

$$\boldsymbol{u} = \begin{pmatrix} PWM_1 \\ PWM_2 \\ PWM_3 \\ PWM_4 \end{pmatrix}$$

Absolute speed in body frame transformed to inertial frame yields derivative of geodatic coordinates

$$\dot{\boldsymbol{x}} = \underline{T}_b^i \cdot \boldsymbol{v}$$

System Model

Principle of linear momentum of point mass - body fixed frame

$$\begin{aligned} \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} &= \sum \boldsymbol{F} \\ m \cdot \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} &= \sum \boldsymbol{F} \\ m \cdot \left(\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right)_{\omega=0} + \boldsymbol{\omega} \times m \cdot \boldsymbol{v} &= \sum \boldsymbol{F} \\ \Rightarrow \dot{\boldsymbol{v}} &= \frac{1}{m} \cdot \sum \boldsymbol{F} - \boldsymbol{\omega} \times \boldsymbol{v} \end{aligned}$$

with (see Eq. 7 for $\underline{T}_i^b$)

$$\sum \boldsymbol{F} = \underline{T}_i^b \cdot \begin{pmatrix} 0 \\ 0 \\ -m \cdot g \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ F_1 + F_2 + F_3 + F_4 \end{pmatrix}$$

Angular velocities in body frame can be transformed to the derivatives of the euler angles via the Kalman transformation matrix (see Eq. 13 for $\underline{V}_b^i$)

$$\dot{\boldsymbol{\Phi}} = \underline{V}_b^i \cdot \boldsymbol{\omega}$$

Principle of momentum - body fixed frame

$$\begin{aligned} \frac{\mathrm{d}L}{\mathrm{d}t} &= \sum \boldsymbol{M} \\ \left(\frac{\mathrm{d}L}{\mathrm{d}t}\right)_{\omega=0} + \boldsymbol{\omega} \times L &= \sum \boldsymbol{M} \\ \Theta_{body} \cdot \left(\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\right)_{\omega=0} + \boldsymbol{\omega} \times (\Theta_{body} \cdot \boldsymbol{\omega}) &= \sum \boldsymbol{M} \\ \Rightarrow \dot{\boldsymbol{\omega}} &= \Theta_{body}^{-1} \cdot \left(\sum \boldsymbol{M} - \boldsymbol{\omega} \times (\Theta_{body} \cdot \boldsymbol{\omega})\right) \end{aligned}$$

with

$$\sum \boldsymbol{M} = \begin{pmatrix} l \cdot (F_2 - F_4) \\ l \cdot (F_3 - F_1) \\ M_1 - M_2 + M_3 - M_4 \end{pmatrix}$$

The general state space $\dot{x} = f(\boldsymbol{x}, \boldsymbol{u})$ therefore looks as follows

$$\begin{pmatrix} \dot{\boldsymbol{x}} \\ \dot{\boldsymbol{v}} \\ \dot{\boldsymbol{\Phi}} \\ \dot{\boldsymbol{\omega}} \end{pmatrix} = \begin{pmatrix} \underline{T}_b^i \cdot v \\ \underline{T}_i^b \cdot \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{1}{m} \sum_{i=1}^4 F_i(u_i) \end{pmatrix} - \boldsymbol{\omega} \times v \\ \underline{V}_b^i \cdot \omega \\ \Theta_{body}^{-1} \cdot \left(\begin{pmatrix} l \cdot (F_2(u_2) - F_4(u_4)) \\ l \cdot (F_3(u_3) - F_1(u_1)) \\ \sum_{i=1}^4 (-1)^{i+1} \cdot M(u_i) \end{pmatrix} - \boldsymbol{\omega} \times (\Theta_{body} \cdot \boldsymbol{\omega}) \right) \end{pmatrix}$$

For a quadrotor frame symmetric to the x- and y-axis, the inertia elements $\Theta_{b12} = \Theta_{b21}$ and $\Theta_{b23} = \Theta_{b32}$ equal 0. With

$$\boldsymbol{\omega} \times (\Theta_{body} \cdot \boldsymbol{\omega}) = \begin{pmatrix} (\Theta_{b31} \cdot \omega_1 + \Theta_{b32} \cdot \omega_2 + \Theta_{b33} \cdot \omega_3) \cdot \omega_2 - (\Theta_{b21} \cdot \omega_1 + \Theta_{b22} \cdot \omega_2 + \Theta_{b23} \cdot \omega_3) \cdot \omega_3 \\ (\Theta_{b11} \cdot \omega_1 + \Theta_{b12} \cdot \omega_2 + \Theta_{b13} \cdot \omega_3) \cdot \omega_3 - (\Theta_{b31} \cdot \omega_1 + \Theta_{b32} \cdot \omega_2 + \Theta_{b33} \cdot \omega_3) \cdot \omega_1 \\ (\Theta_{b21} \cdot \omega_1 + \Theta_{b22} \cdot \omega_2 + \Theta_{b23} \cdot \omega_3) \cdot \omega_1 - (\Theta_{b11} \cdot \omega_1 + \Theta_{b12} \cdot \omega_2 + \Theta_{b13} \cdot \omega_3) \cdot \omega_2 \end{pmatrix}$$
$$= \begin{pmatrix} (\Theta_{b31} \cdot \omega_1 + \Theta_{b33} \cdot \omega_3) \cdot \omega_2 - \Theta_{b22} \cdot \omega_2 \cdot \omega_3 \\ (\Theta_{b11} \cdot \omega_1 + \Theta_{b13} \cdot \omega_3) \cdot \omega_3 - (\Theta_{b31} \cdot \omega_1 + \Theta_{b33} \cdot \omega_3) \cdot \omega_1 \\ \Theta_{b22} \cdot \omega_2 \cdot \omega_1 - (\Theta_{b11} \cdot \omega_1 + \Theta_{b13} \cdot \omega_3) \cdot \omega_2 \end{pmatrix}$$

and the resulting simplifications of the inverse of the inertia matrix (see section C.2) the state space can be stated as follows

$$\Rightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{u} \\ \dot{v} \\ \dot{w} \\ \dot{\Phi} \\ \dot{\Theta} \\ \dot{\Psi} \\ \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} c\Theta \cdot c\Psi \cdot u + (s\Phi \cdot s\Theta \cdot c\Psi - c\Phi \cdot s\Psi) \cdot v + (c\Phi \cdot s\Theta \cdot c\Psi + s\Phi \cdot s\Psi) \cdot w \\ c\Theta \cdot s\Psi \cdot u + (s\Phi \cdot s\Theta \cdot s\Psi + c\Phi \cdot c\Psi) \cdot v + (c\Phi \cdot s\Theta \cdot s\Psi - s\Phi \cdot c\Psi) \cdot w \\ -s\Theta \cdot u + (s\Phi \cdot c\Theta) \cdot v + (c\Phi \cdot c\Theta) \cdot w \\ -g \cdot -s\Theta - \omega_2 \cdot w + \omega_3 \cdot v \\ -g \cdot s\Phi \cdot c\Theta - \omega_3 \cdot u + \omega_1 \cdot w \\ -g \cdot c\Phi \cdot c\Theta - \omega_1 \cdot v + \omega_2 \cdot u + \frac{1}{m} \cdot (F_1(u_1) + F_2(u_2) + F_3(u_3) + F_4(u_4)) \\ p + q \cdot s\Phi \cdot t\theta + r \cdot c\Phi \cdot t\theta \\ q \cdot c\Phi + r \cdot -s\Phi \\ q \cdot \frac{s\Phi}{c\Theta} + r \cdot \frac{c\Phi}{c\Theta} \\ \begin{pmatrix} \frac{\Theta_{b33}}{\Theta_{b11} \cdot \Theta_{b33} - \Theta_{b13}^2} & 0 & -\frac{\Theta_{b13}}{\Theta_{b11} \cdot \Theta_{b33} - \Theta_{b13}^2} \\ 0 & \frac{1}{\Theta_{b22}} & 0 \\ -\frac{\Theta_{b13}}{\Theta_{b11} \cdot \Theta_{b33} - \Theta_{b13}^2} & 0 & \frac{\Theta_{b11}}{\Theta_{b11} \cdot \Theta_{b33} - \Theta_{b13}^2} \end{pmatrix} \cdot \begin{pmatrix} l \cdot (F_2 - F_4) - (\Theta_{b31} \cdot \omega_1 + \Theta_{b33} \cdot \omega_3) \cdot \omega_2 + \Theta_{b22} \cdot \omega_2 \cdot \omega_3 \\ l \cdot (F_3 - F_1) - (\Theta_{b11} \cdot \omega_1 + \Theta_{b13} \cdot \omega_3) \cdot \omega_3 + (\Theta_{b31} \cdot \omega_1 + \Theta_{b33} \cdot \omega_3) \cdot \omega_1 \\ M_1 - M_2 + M_3 - M_4 - \Theta_{b22} \cdot \omega_2 \cdot \omega_1 + (\Theta_{b11} \cdot \omega_1 + \Theta_{b13} \cdot \omega_3) \cdot \omega_2 \end{pmatrix} \end{pmatrix} \quad (1)$$

The esc's and bldc is identified using following input sequence and resulting force.

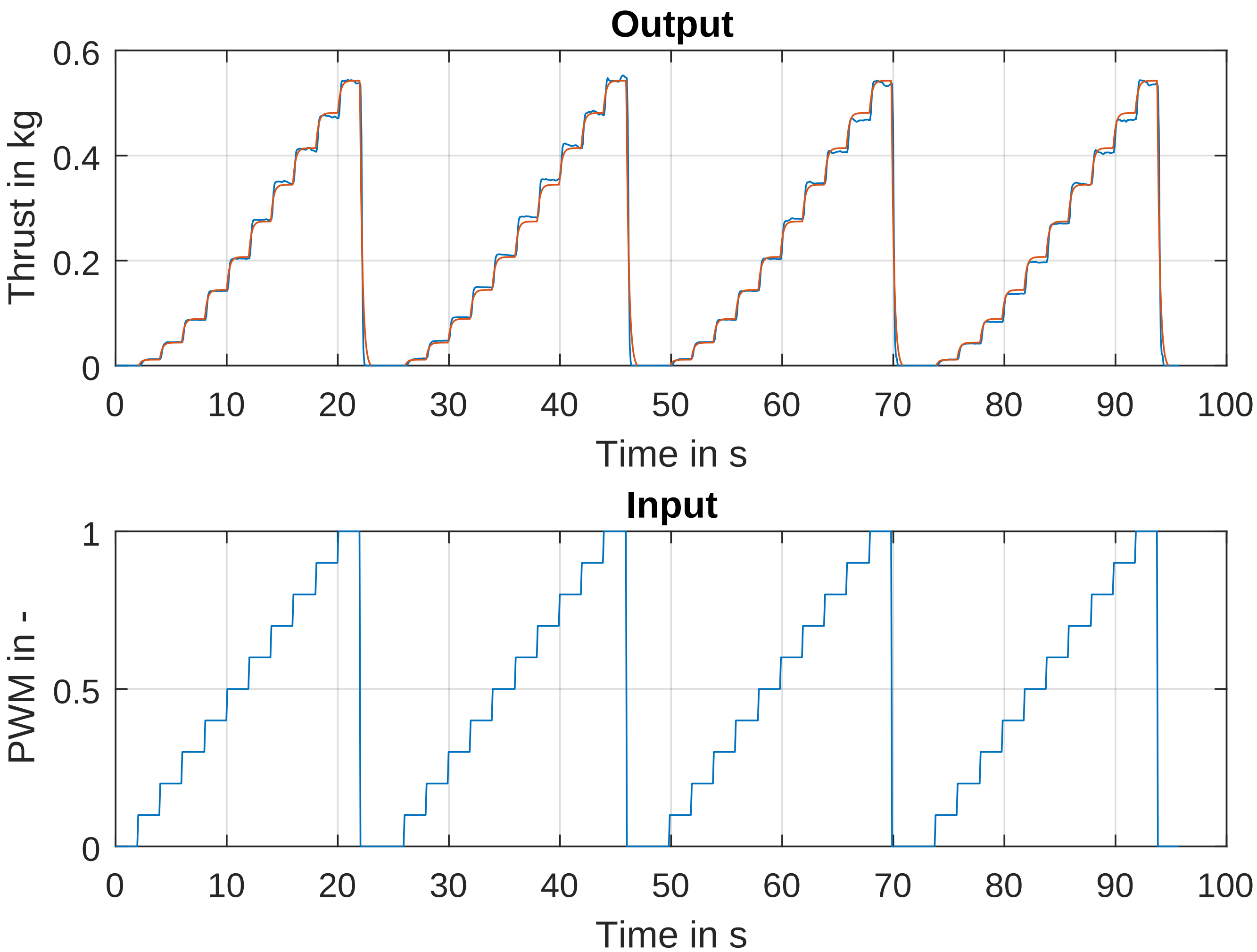

Figure 3: ESC + BLDC Unit - Measured input output behaviour

System Model

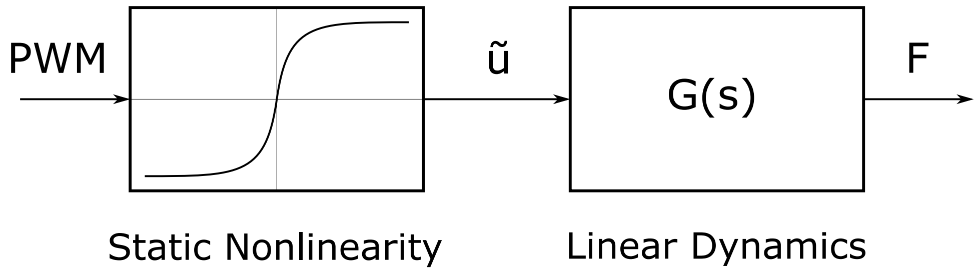

Figure 4: Hammerstein model structure

Assuming a Hammerstein model structure (see Fig. 4) the input force to the quadrotor model is based on the following model structure:

$$F = \frac{b_0}{a_0 - a_1 \cdot z^{-1}} \cdot \tilde{u}$$

$$\tilde{u} = p_3 \cdot u_{\mathrm{PWM}}^3 + p_2 \cdot u_{\mathrm{PWM}}^2 + p_1 \cdot u_{\mathrm{PWM}} + p_0$$

The momentum M is initially assumed to correlate to the thrust via a factor γ

$$M = \gamma \cdot F$$

Model Analysis

Analyze linearized model around setpoint $\boldsymbol{x}_0 = \begin{pmatrix} x_0 & y_0 & z_0 & \mathbf{0} \end{pmatrix}^T$ and $\boldsymbol{F}_0 = \begin{pmatrix} \frac{m \cdot g}{4} & \frac{m \cdot g}{4} & \frac{m \cdot g}{4} & \frac{m \cdot g}{4} \end{pmatrix}^T$

$$\dot{x}_0 = \mathbf{0}$$

$$A = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & g & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -g & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$\boldsymbol{B} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\frac{1}{m} & \frac{1}{m} & \frac{1}{m} & \frac{1}{m} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\multicolumn{4}{c}{\Theta_{body}^{-1} \cdot \begin{pmatrix} 0 & l & 0 & -l \\ -l & 0 & l & 0 \\ \frac{\mathrm{d}M_1}{\mathrm{d}F_1} & -\frac{\mathrm{d}M_2}{\mathrm{d}F_2} & \frac{\mathrm{d}M_3}{\mathrm{d}F_3} & -\frac{\mathrm{d}M_4}{\mathrm{d}F_4} \end{pmatrix}}
\end{pmatrix}$$

$$
= \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\frac{1}{m} & \frac{1}{m} & \frac{1}{m} & \frac{1}{m} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-\frac{\Theta_{b13}}{\Theta_{b11}\cdot\Theta_{b33}-\Theta_{b13}^2}\cdot\frac{\mathrm{d}M_1}{\mathrm{d}F_1} & \frac{\Theta_{b33}\cdot l+\Theta_{b13}\cdot\frac{\mathrm{d}M_2}{\mathrm{d}F_2}}{\Theta_{b11}\cdot\Theta_{b33}-\Theta_{b13}^2} & -\frac{\Theta_{b13}}{\Theta_{b11}\cdot\Theta_{b33}-\Theta_{b13}^2}\cdot\frac{\mathrm{d}M_3}{\mathrm{d}F_3} & \frac{-\Theta_{b33}\cdot l+\Theta_{b13}\cdot\frac{\mathrm{d}M_4}{\mathrm{d}F_4}}{\Theta_{b11}\cdot\Theta_{b33}-\Theta_{b13}^2} \\
-\frac{l}{\Theta_{22}} & 0 & \frac{l}{\Theta_{22}} & 0 \\
\frac{\Theta_{b11}}{\Theta_{b11}\cdot\Theta_{b33}-\Theta_{b13}^2}\cdot\frac{\mathrm{d}M_1}{\mathrm{d}F_1} & \frac{-\Theta_{b13}\cdot l-\Theta_{b11}\cdot\frac{\mathrm{d}M_2}{\mathrm{d}F_2}}{\Theta_{b11}\cdot\Theta_{b33}-\Theta_{b13}^2} & \frac{\Theta_{b11}}{\Theta_{b11}\cdot\Theta_{b33}-\Theta_{b13}^2}\cdot\frac{\mathrm{d}M_3}{\mathrm{d}F_3} & \frac{\Theta_{b13}\cdot l-\Theta_{b11}\cdot\frac{\mathrm{d}M_4}{\mathrm{d}F_4}}{\Theta_{b11}\cdot\Theta_{b33}-\Theta_{b13}^2}
\end{pmatrix}
$$

Tilt, orientation and height dynamics ($\Phi,\ \Theta,\ \Psi,\ z$) are decoupled from remaining system. Only x and y direction are coupled to pitch and roll angles.

Controlled Outputs

The control goal is to keep a stable tilt, fixed height and avoid permanent rotation ($\boldsymbol{x} = z,\ \Phi,\ \Theta,\ \Psi$). As these states are decoupled from the rest, the reduced model

$$\dot{\boldsymbol{x}}_{red} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot \boldsymbol{x}_{red} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{m} & \frac{1}{m} & \frac{1}{m} & \frac{1}{m} \\ \multicolumn{4}{c}{\Theta_{body}^{-1} \cdot \begin{pmatrix} 0 & l & 0 & -l \\ -l & 0 & l & 0 \\ \frac{\mathrm{d}M_1}{\mathrm{d}F_1} & -\frac{\mathrm{d}M_2}{\mathrm{d}F_2} & \frac{\mathrm{d}M_3}{\mathrm{d}F_3} & -\frac{\mathrm{d}M_4}{\mathrm{d}F_4} \end{pmatrix}} \end{pmatrix} \cdot \boldsymbol{F} \tag{2}$$

with $\boldsymbol{x}_{red} = \left(z, \Phi, \Theta, \dot{z}, p, q\right)^T$ can be treated.

Measured and Manipulated Variables

Add overview of sensor and actuator placement.

System Inputs	System Outputs
$\mathrm{PWM}_{\mathrm{esc},1}$	$\frac{\mathrm{d}u}{\mathrm{d}t}$
$\mathrm{PWM}_{\mathrm{esc},2}$	$\frac{\mathrm{d}v}{\mathrm{d}t}$
$\mathrm{PWM}_{\mathrm{esc},3}$	$\frac{\mathrm{d}w}{\mathrm{d}t}$
$\mathrm{PWM}_{\mathrm{esc},4}$	p
	q
	r

Table 1: Inputs and outputs

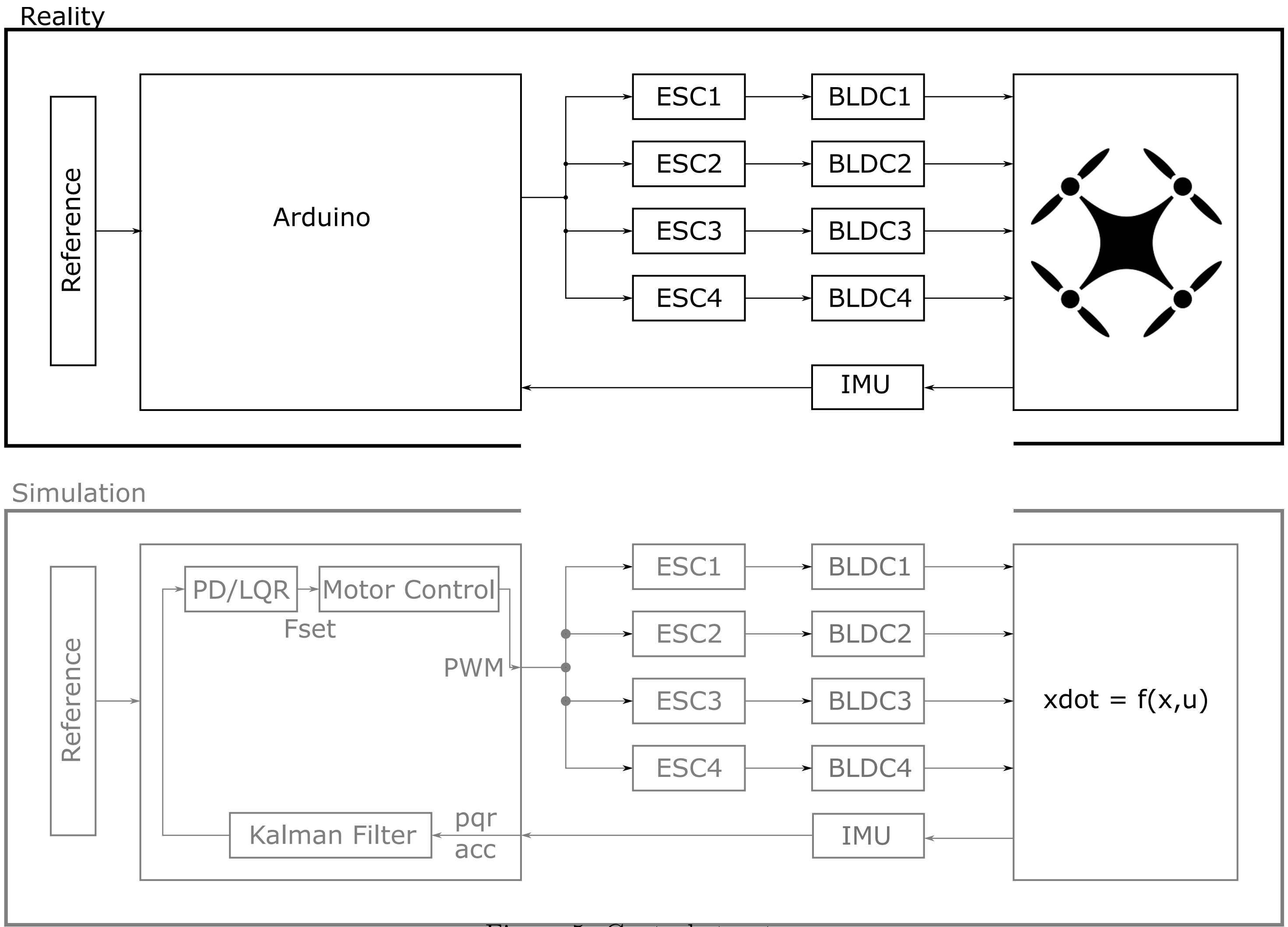

Figure 5: Control structure

Controller Type

LQR + Kalman Filter.

Performance Specification

Tilt of quadrotor remains unchanged.

Controller - PD

By reformulating Eq. 2 as

$$\dot{x}_{red} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot \boldsymbol{x}_{red} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

with

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{m} \quad \frac{1}{m} \quad \frac{1}{m} \quad \frac{1}{m} \\ \Theta_{body}^{-1} \cdot \begin{pmatrix} 0 & l & 0 & -l \\ -l & 0 & l & 0 \\ \frac{\mathrm{d}M_1}{\mathrm{d}F_1} & -\frac{\mathrm{d}M_2}{\mathrm{d}F_2} & \frac{\mathrm{d}M_3}{\mathrm{d}F_3} & -\frac{\mathrm{d}M_4}{\mathrm{d}F_4} \end{pmatrix} \end{pmatrix} \cdot \boldsymbol{F} = E \cdot \boldsymbol{F}$$

the controller can be designed by interpreting three independant SISO systems:

$$\ddot{x}_{i,red} = u_i \qquad i = 1\text{-}4$$

Choosing a PD controller

$$\ddot{x}_{i,red} + k_d \cdot \dot{x}_{i,red} + k_p \cdot x_{i,red} = 0 \quad \Rightarrow \quad s_{1/2} = -\frac{k_d}{2} \pm \sqrt{\left(\frac{k_d}{2}\right)^2 - k_p}$$

and claiming real poles ($k_p = \left(\frac{k_d}{2}\right)^2$), the system's poles can directly prescribed by

$$\Rightarrow s_{1/2} = -\frac{k_d}{2}$$

Controller - PD

In order to obtain $\boldsymbol{F}$ or E can be inverted numerically or the following reformulation can be used

$$\begin{pmatrix} u_1 \cdot m \\ \Theta \cdot \begin{pmatrix} u_2 \\ u_3 \\ u_4 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & l & 0 & -l \\ -l & 0 & l & 0 \\ \frac{\mathrm{d}M_1}{\mathrm{d}F_1} & -\frac{\mathrm{d}M_2}{\mathrm{d}F_2} & \frac{\mathrm{d}M_3}{\mathrm{d}F_3} & -\frac{\mathrm{d}M_4}{\mathrm{d}F_4} \end{pmatrix} \cdot \boldsymbol{F}$$

$$\begin{pmatrix} m & 0 \\ 0 & \Theta \end{pmatrix} \cdot \boldsymbol{u} = \tilde{E} \cdot \boldsymbol{F}$$

$$\begin{pmatrix} \tilde{u}_1 \\ \frac{1}{l} \cdot \tilde{u}_2 \\ \frac{1}{l} \cdot \tilde{u}_3 \\ \frac{\mathrm{d}F}{\mathrm{d}M} \cdot \tilde{u}_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 \end{pmatrix} \cdot \boldsymbol{F}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{l} & 0 & 0 \\ 0 & 0 & \frac{1}{l} & 0 \\ 0 & 0 & 0 & \frac{\mathrm{d}F}{\mathrm{d}M} \end{pmatrix} \cdot \begin{pmatrix} m & 0 \\ 0 & \Theta \end{pmatrix} \cdot \boldsymbol{u} = \hat{E} \cdot \boldsymbol{F}$$

$$\hat{E}^{-1} = \frac{1}{4} \begin{pmatrix} 1 & 0 & -2 & 1 \\ 1 & 2 & 0 & -1 \\ 1 & 0 & 2 & 1 \\ 1 & -2 & 0 & -1 \end{pmatrix}$$

$\tilde{\boldsymbol{u}}$ are multiplied by the mass m and inertia matrix Θ, respectively, and is applied. If Ψ is not to be explicitely controlled u_4, it can be tried to simply set it to $\sum M_i = 0$. For the purpose of control design, the bldc dynamics are assumed sufficiently fast and only the nonlinearity is considered. The static gain of the transfer function for a step response is $K = \dfrac{b_0}{a_0 - a_1}$.

Controller - LQR

$$
\begin{aligned}
\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} &= \begin{pmatrix} c\Theta \cdot c\Psi & c\Theta \cdot s\Psi & -s\Theta \\ s\Phi \cdot s\Theta \cdot c\Psi - c\Phi \cdot s\Psi & s\Phi \cdot s\Theta \cdot s\Psi + c\Phi \cdot c\Psi & s\Phi \cdot c\Theta \\ c\Phi \cdot s\Theta \cdot c\Psi + s\Phi \cdot s\Psi & c\Phi \cdot s\Theta \cdot s\Psi - s\Phi \cdot c\Psi & c\Phi \cdot c\Theta \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix} \\
&= \begin{pmatrix} s\Theta \\ -s\Phi \cdot c\Theta \\ -c\Phi \cdot c\Theta \end{pmatrix} \cdot g \\
\Rightarrow \tan\Phi &= \frac{\dot{v}}{\dot{w}} \\
\Rightarrow \dot{v}^2 + \dot{w}^2 &= c\Theta^2 \cdot g^2 \\
\Rightarrow \tan\Theta &= \frac{\dot{u}}{\sqrt{\dot{v}^2 + \dot{w}^2}}
\end{aligned}
$$

Show observer simulation results and results based on measurement data (in comparison to madgwick).

Moving Coordinate Systems

Show closed loop control results. State feedback gains, Q, R.

Derivative of vectors in rotating coordinate systems:

$$\frac{\mathrm{d}\boldsymbol{\rho}}{\mathrm{d}t} = \boldsymbol{\omega} \times \boldsymbol{\rho} + \left(\frac{\partial \boldsymbol{\rho}}{\partial t}\right)_{\omega=0} \tag{3}$$

Absolute velocity and acceleration of a moving point in a moving and rotating coordinate system:

$$\boldsymbol{r} = \boldsymbol{r_0} + \boldsymbol{\rho} \tag{4}$$

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{r_0}}{\mathrm{d}t} + \boldsymbol{\omega} \times \boldsymbol{\rho} + v_{rel} \tag{5}$$

$$\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = \underbrace{\frac{\mathrm{d}^2\boldsymbol{r}_0}{\mathrm{d}t^2} + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})}_{Leading\ \ Acceleration} + \underbrace{2\boldsymbol{\omega} \times v_{rel}}_{\substack{Coriolis \\ Acceleration}} + a_{rel} \tag{6}$$

Notes

- Eq. 6 helps formulate simpler correlations for acceleration of moving points in moving and rotating coordinate systems
- For known trajectories (e.g. circular paths), Eq. 6 already states the acceleration and the acting forces can be determined (instead of applying newton's law to determine the acceleration)
- Eq. 6 and Eq. 5 state the absolute acceleration and velocity
- In rotating coordinate systems, Eq. 5 cannot be obtained by integrating Eq. 6, as the implicit acceleration of the rotation must be discounted first
- The second addend of Eq. 3 can be integrated to yield the absolute velocity in the rotating coordinate system. The dynamics of the COS through the mental rotation gets lost. In order to reconstruct the absolute path, this information needs to be restored

Example 1 - Circular path

$$\boldsymbol{a} = \begin{pmatrix} -\omega \cdot R^2 \\ 0 \end{pmatrix}, \qquad \boldsymbol{v} = \begin{pmatrix} 0 \\ \omega \cdot R \end{pmatrix}$$

Integration over one time step T yields the absolute velocity and position:

$$\boldsymbol{v}+ = \begin{pmatrix} 0 \\ \omega \cdot R \end{pmatrix} + T \cdot \begin{pmatrix} -\omega \cdot R^2 \\ 0 \end{pmatrix} = \begin{pmatrix} -T \cdot \omega \cdot R^2 \\ \omega \cdot R \end{pmatrix}$$
$$\boldsymbol{r}+ = \begin{pmatrix} R \\ 0 \end{pmatrix} + T \cdot \begin{pmatrix} 0 \\ \omega \cdot R \end{pmatrix} = \begin{pmatrix} R \\ T \cdot \omega \cdot R \end{pmatrix}$$

Now, or the direction of F (description in absolute coordinates) or the COS needs to be adapted. Using Eq. 3, it follows:

$$a_{rel} = \frac{\mathrm{d}v}{\mathrm{d}t} - \omega \times \boldsymbol{v} = \begin{pmatrix} -\omega \cdot R^2 \\ 0 \end{pmatrix} - \begin{pmatrix} -\omega^2 \cdot R \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Thus, neither the velocity, nor the position will change. This is correct within the relative COS. However, information is lost and in order to reconstruct the absolute path, this information needs to be recovered.

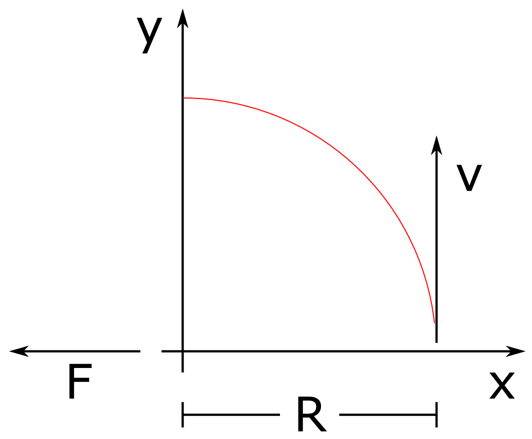

Figure 6: Basics coordinate systems - Intuition 1

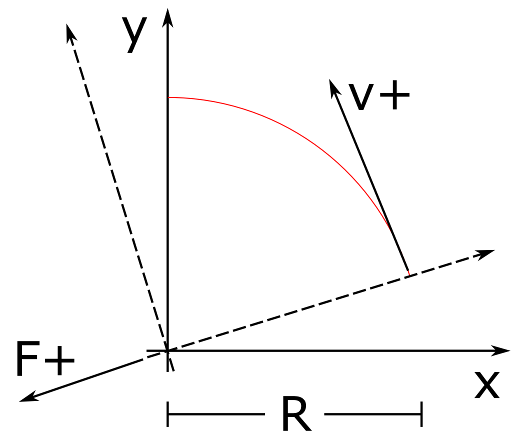

Figure 7: Basics coordinate systems - Intuition 2

Moving Coordinate Systems

Example 2 - Circular path
Applying Eq.6 to the body fixed coordinate system of Fig. 8 (point mass in origin):

$$\begin{aligned}\frac{\mathrm{d}^2\boldsymbol{r}_0}{\mathrm{d}t^2} + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) &= \frac{\mathrm{d}^2\boldsymbol{r}_0}{\mathrm{d}t^2}\\ 2\boldsymbol{\omega} \times v_{rel} &= 0\\ a_{rel} &= 0\\ \Rightarrow \frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} &= \frac{\mathrm{d}^2\boldsymbol{r}_0}{\mathrm{d}t^2}\end{aligned}$$

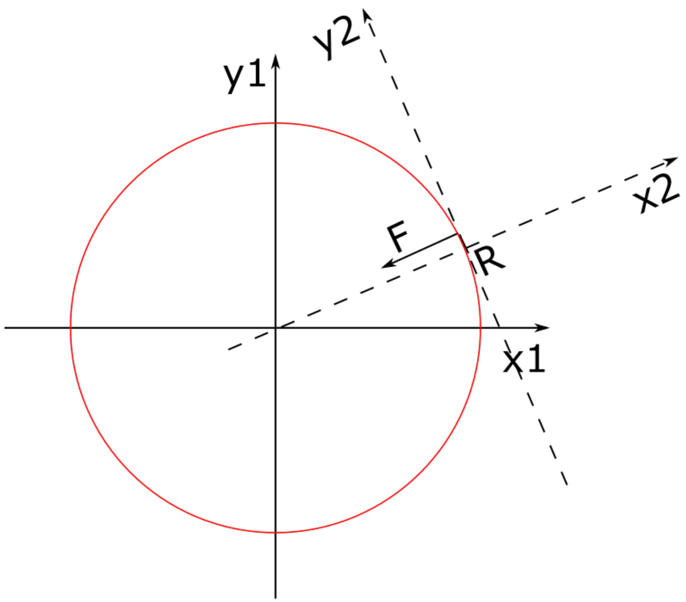

Figure 8: Basics coordinate systems - Intuition 1

Newton's law:

$$\begin{aligned} m \cdot \frac{\mathrm{d}^2(\boldsymbol{r})_1}{\mathrm{d}t^2} &= \sum_i F_i = F \qquad \text{(inertial frame)}\\ \neq m \cdot \frac{\mathrm{d}^2(\boldsymbol{r})_2}{\mathrm{d}t^2} &= 0 \qquad \text{(body frame)}\end{aligned}$$

In the inertial frame, direction changes. In the body frame, the velocity does not change:

$$\neq m \cdot \frac{\mathrm{d}^2(\boldsymbol{r})_2}{\mathrm{d}t^2} = -\begin{pmatrix}0\\0\\\omega\end{pmatrix} \times v + F$$

xyz-Convention

Rotation around x-axis with Φ

$$T(\Phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\Phi & \sin\Phi \\ 0 & -\sin\Phi & \cos\Phi \end{pmatrix}$$

Rotation around y-axis with Θ

$$T(\Theta) = \begin{pmatrix} \cos\Theta & 0 & -\sin\Theta \\ 0 & 1 & \\ \sin\Theta & 0 & \cos\Theta \end{pmatrix}$$

Rotation around z-axis with Ψ

$$T(\Psi) = \begin{pmatrix} \cos\Psi & \sin\Psi & 0 \\ -\sin\Psi & \cos\Psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Orthogonal transformation matrix from inertial to body frame

$$\underline{T}_i^b = \begin{pmatrix} c\Theta\cdot c\Psi & c\Theta\cdot s\Psi & -s\Theta \\ s\Phi\cdot s\Theta\cdot c\Psi - c\Phi\cdot s\Psi & s\Phi\cdot s\Theta\cdot s\Psi + c\Phi\cdot c\Psi & s\Phi\cdot c\Theta \\ c\Phi\cdot s\Theta\cdot c\Psi + s\Phi\cdot s\Psi & c\Phi\cdot s\Theta\cdot s\Psi - s\Phi\cdot c\Psi & c\Phi\cdot c\Theta \end{pmatrix} \tag{7}$$

Orthogonal transformation matrix from body to inertial frame

$$\underline{T}_b^i = \begin{pmatrix} c\Theta\cdot c\Psi & s\Phi\cdot s\Theta\cdot c\Psi - c\Phi\cdot s\Psi & c\Phi\cdot s\Theta\cdot c\Psi + s\Phi\cdot s\Psi \\ c\Theta\cdot s\Psi & s\Phi\cdot s\Theta\cdot s\Psi + c\Phi\cdot c\Psi & c\Phi\cdot s\Theta\cdot s\Psi - s\Phi\cdot c\Psi \\ -s\Theta & s\Phi\cdot c\Theta & c\Phi\cdot c\Theta \end{pmatrix} \tag{8}$$

Karman rotation matrix.

$$\omega = \underline{T}_2^b \cdot \begin{pmatrix} \dot{\Phi} \\ 0 \\ 0 \end{pmatrix} + \underline{T}_1^b \cdot \begin{pmatrix} 0 \\ \dot{\Theta} \\ 0 \end{pmatrix} + \underline{T}_i^b \cdot \begin{pmatrix} 0 \\ 0 \\ \dot{\Psi} \end{pmatrix} \tag{9}$$

$$\omega = \underline{V}_i^b \cdot \dot{\mathbf{\Phi}} \tag{10}$$

$$\underline{V}_b^i = \left(\underline{V}_i^b\right)^{-1} \tag{11}$$

$$\underline{V}_i^b = \begin{pmatrix} 1 & 0 & -s\Theta \\ 0 & c\Phi & s\Phi \cdot c\Theta \\ 0 & -s\Phi & c\Phi \cdot c\Theta \end{pmatrix} \tag{12}$$

$$\underline{V}_b^i = \begin{pmatrix} 1 & s\Phi \cdot t\Theta & c\Phi \cdot t\Theta \\ 0 & c\Phi & -s\Phi \\ 0 & \frac{s\Phi}{c\Theta} & \frac{c\Phi}{c\Theta} \end{pmatrix} \tag{13}$$

Model linearization

Linearization around $\boldsymbol{x}_0$: $\dot{\boldsymbol{x}} = f(\boldsymbol{x}_0) + \frac{\mathrm{d}f}{\mathrm{d}\boldsymbol{x}} \cdot \boldsymbol{x} + \frac{\mathrm{d}f}{\mathrm{d}\boldsymbol{F}} \cdot \boldsymbol{F}$

$$\frac{\mathrm{d}f}{\mathrm{d}\left(\boldsymbol{x} \quad \boldsymbol{v}\right)^T} = \begin{pmatrix} 0 & 0 & 0 & c\Theta \cdot c\Psi & s\Phi \cdot s\Theta \cdot c\Psi - c\Phi \cdot s\Psi & c\Phi \cdot s\Theta \cdot c\Psi + s\Phi \cdot s\Psi \\ 0 & 0 & 0 & c\Theta \cdot s\Psi & s\Phi \cdot s\Theta \cdot s\Psi + c\Phi \cdot c\Psi & c\Phi \cdot s\Theta \cdot s\Psi - s\Phi \cdot c\Psi \\ 0 & 0 & 0 & -s\Theta & s\Phi \cdot c\Theta & c\Phi \cdot c\Theta \\ 0 & 0 & 0 & 0 & r & -q \\ 0 & 0 & 0 & -r & 0 & p \\ 0 & 0 & 0 & q & -p & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\frac{\mathrm{d}f}{\mathrm{d}\boldsymbol{\Phi}} = \begin{pmatrix} \left(s\Phi s\Psi + c\Phi s\Theta c\Psi\right) v + \left(c\Phi s\Psi - s\Phi s\Theta c\Psi\right) w & -s\Theta c\Psi u + s\Phi c\Theta c\Psi v + c\Phi c\Theta c\Psi w & -c\Theta s\Psi u + \left(-c\Phi c\Psi - s\Phi s\Theta s\Psi\right) v + \left(s\Phi c\Psi - c\Phi s\Theta s\Psi\right) w \\ \left(-s\Phi c\Psi + c\Phi s\Theta s\Psi\right) v + \left(-c\Phi c\Psi - s\Phi s\Theta s\Psi\right) w & -s\Theta s\Psi u + s\Phi c\Theta s\Psi v + c\Phi c\Theta s\Psi w & c\Theta c\Psi u + \left(-c\Phi s\Psi + s\Phi s\Theta c\Psi\right) v + \left(c\Phi s\Theta c\Psi + s\Phi s\Psi\right) w \\ c\Phi c\Theta v - s\Phi c\Theta w & -c\Theta u - s\Phi s\Theta v - c\Phi s\Theta w & 0 \\ 0 & g \cdot c\Theta & 0 \\ -g \cdot c\Phi c\Theta & g \cdot s\Phi \cdot c\Theta & 0 \\ g \cdot s\Phi c\Theta & g \cdot c\Phi \cdot s\Theta & 0 \\ -c\Phi t\Theta q - s\Phi t\Theta r & \frac{s\Phi}{c\Theta^2} q + \frac{c\Phi}{c\Theta^2} r & 0 \\ -s\Phi q - c\Phi r & 0 & 0 \\ \frac{c\Phi}{c\Theta} q - \frac{s\Phi}{c\Theta} r & \frac{s\Phi}{c\Theta^2} s\Theta q + \frac{c\Phi}{c\Theta^2} s\Phi r & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Model linearization

$$\frac{\mathrm{d}f}{\mathrm{d}\boldsymbol{\omega}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -w & v \\ w & 0 & -u \\ -v & u & 0 \\ 1 & s\Phi t\Theta & c\Phi t\Theta \\ 0 & c\Phi & -s\Phi \\ 0 & \frac{s\Phi}{c\Theta} & \frac{c\Phi}{c\Theta} \\ \multicolumn{3}{c}{\Theta_{body}^{-1} \cdot \begin{pmatrix} -\Theta_{b31} \cdot q + \Theta_{b21} \cdot r & -\Theta_{b32} \cdot q - \Theta_{b31} \cdot p + (\Theta_{b22} - \Theta_{b33}) \cdot r & \Theta_{b21} \cdot p + (\Theta_{b22} - \Theta_{b33}) \cdot q + 2 \cdot \Theta_{b23} \cdot r \\ 2 \cdot \Theta_{b31} \cdot p + \Theta_{b32} \cdot q + (\Theta_{b33} - \Theta_{b11}) \cdot r & -\Theta_{b12} \cdot r + \Theta_{b32} \cdot p & (\Theta_{b33} - \Theta_{b11}) \cdot p - \Theta_{b12} \cdot q - 2 \cdot \Theta_{b13} \cdot r \\ -2 \cdot \Theta_{b21} \cdot p + (\Theta_{b11} - \Theta_{b22}) \cdot q - \Theta_{b23} \cdot r & (\Theta_{b11} - \Theta_{b22}) \cdot p + 2 \cdot \Theta_{b12} \cdot q + \Theta_{b13} \cdot r & -\Theta_{b23} \cdot p + \Theta_{b13} \cdot q \end{pmatrix}} \end{pmatrix}$$

$$\frac{\mathrm{d}f}{\mathrm{d}\boldsymbol{F}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{m} & \frac{1}{m} & \frac{1}{m} & \frac{1}{m} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \multicolumn{4}{c}{\Theta_{body}^{-1} \cdot \begin{pmatrix} 0 & l & 0 & -l \\ -l & 0 & l & 0 \\ \frac{\mathrm{d}M_1}{\mathrm{d}F_1} & -\frac{\mathrm{d}M_2}{\mathrm{d}F_2} & \frac{\mathrm{d}M_3}{\mathrm{d}F_3} & -\frac{\mathrm{d}M_4}{\mathrm{d}F_4} \end{pmatrix}} \end{pmatrix}$$

Model linearization

$$
\frac{\mathrm{d}(-\boldsymbol{\omega} \times (\Theta_{body} \cdot \boldsymbol{\omega}))}{\mathrm{d}\boldsymbol{\omega}} = \begin{pmatrix} -\Theta_{b31} \cdot q + \Theta_{b21} \cdot r & -\Theta_{b32} \cdot q - \Theta_{b31} \cdot p - \Theta_{b33} \cdot r + \Theta_{b22} \cdot r & -\Theta_{b33} \cdot q + \Theta_{b21} \cdot p + \Theta_{b22} \cdot q + 2 \cdot \Theta_{b23} \cdot r \\ -\Theta_{b11} \cdot r + 2 \cdot \Theta_{b31} \cdot p + \Theta_{b32} \cdot q + \Theta_{b33} \cdot r & -\Theta_{b12} \cdot r + \Theta_{b32} \cdot p & -\Theta_{b11} \cdot p - \Theta_{b12} \cdot q - 2 \cdot \Theta_{b13} \cdot r + \Theta_{b33} \cdot p \\ -2 \cdot \Theta_{b21} \cdot p - \Theta_{b22} \cdot q - \Theta_{b23} \cdot r + \Theta_{b11} \cdot q & -\Theta_{b22} \cdot p + 2 \cdot \Theta_{b12} \cdot q + \Theta_{b11} \cdot p + \Theta_{b13} \cdot r & -\Theta_{b23} \cdot p + \Theta_{b13} \cdot q \end{pmatrix}
$$

$$
= \begin{pmatrix} -\Theta_{b31} \cdot q + \Theta_{b21} \cdot r & -\Theta_{b32} \cdot q - \Theta_{b31} \cdot p + (\Theta_{b22} - \Theta_{b33}) \cdot r & \Theta_{b21} \cdot p + (\Theta_{b22} - \Theta_{b33}) \cdot q + 2 \cdot \Theta_{b23} \cdot r \\ 2 \cdot \Theta_{b31} \cdot p + \Theta_{b32} \cdot q + (\Theta_{b33} - \Theta_{b11}) \cdot r & -\Theta_{b12} \cdot r + \Theta_{b32} \cdot p & (\Theta_{b33} - \Theta_{b11}) \cdot p - \Theta_{b12} \cdot q - 2 \cdot \Theta_{b13} \cdot r \\ -2 \cdot \Theta_{b21} \cdot p + (\Theta_{b11} - \Theta_{b22}) \cdot q - \Theta_{b23} \cdot r & (\Theta_{b11} - \Theta_{b22}) \cdot p + 2 \cdot \Theta_{b12} \cdot q + \Theta_{b13} \cdot r & -\Theta_{b23} \cdot p + \Theta_{b13} \cdot q \end{pmatrix}
$$

$$
\overset{\Theta_{b12}=\Theta_{b23}=0}{=} \begin{pmatrix} -\Theta_{b31} \cdot q & -\Theta_{b31} \cdot p + (\Theta_{b22} - \Theta_{b33}) \cdot r & (\Theta_{b22} - \Theta_{b33}) \cdot q \\ 2 \cdot \Theta_{b31} \cdot p + (\Theta_{b33} - \Theta_{b11}) \cdot r & 0 & (\Theta_{b33} - \Theta_{b11}) \cdot p - 2 \cdot \Theta_{b13} \cdot r \\ (\Theta_{b11} - \Theta_{b22}) \cdot q & (\Theta_{b11} - \Theta_{b22}) \cdot p + \Theta_{b13} \cdot r & \Theta_{b13} \cdot q \end{pmatrix}
$$

$$
\Theta_{body}^{-1} \cdot \frac{\mathrm{d}(-\boldsymbol{\omega} \times (\Theta_{body} \cdot \boldsymbol{\omega}))}{\mathrm{d}\boldsymbol{\omega}} \overset{\Theta_{b12}=\Theta_{b23}=0}{=}
$$

$$
\begin{pmatrix} \frac{-\Theta_{33} \cdot \Theta_{b31} \cdot q + \Theta_{13} \cdot (\Theta_{b22} - \Theta_{b11}) \cdot q}{\Theta_{11} \cdot \Theta_{33} - \Theta_{13}^2} & \frac{\Theta_{33} \cdot (-\Theta_{b31} \cdot p + (\Theta_{b22} - \Theta_{b33}) \cdot r) + \Theta_{13} \cdot ((\Theta_{b22} - \Theta_{b11}) \cdot p - \Theta_{b33} \cdot r)}{\Theta_{11} \cdot \Theta_{33} - \Theta_{13}^2} & \frac{\Theta_{33} \cdot (\Theta_{b22} - \Theta_{b33}) \cdot q - \Theta_{b13}^2 \cdot q}{\Theta_{11} \cdot \Theta_{33} - \Theta_{13}^2} \\ \frac{1}{\Theta_{22}} \cdot (-\Theta_{b11} \cdot r + 2 \cdot \Theta_{b31} \cdot p) & 0 & \frac{1}{\Theta_{22}} \cdot (-2 \cdot \Theta_{b13} \cdot r + \Theta_{b33} \cdot p) \\ \frac{\Theta_{13}^2 \cdot q + \Theta_{11} \cdot (\Theta_{b11} - \Theta_{b22}) \cdot q}{\Theta_{11} \cdot \Theta_{33} - \Theta_{13}^2} & \frac{\Theta_{13} \cdot (\Theta_{b31} \cdot p + (\Theta_{b33} - \Theta_{b22}) \cdot r) + \Theta_{11} \cdot ((\Theta_{b11} - \Theta_{b22}) \cdot p + \Theta_{b33} \cdot r)}{\Theta_{11} \cdot \Theta_{33} - \Theta_{13}^2} & \frac{\Theta_{13} \cdot (\Theta_{b33} - \Theta_{b22}) \cdot q + \Theta_{b11} \cdot \Theta_{b13} \cdot q}{\Theta_{11} \cdot \Theta_{33} - \Theta_{13}^2} \end{pmatrix}
$$

$$
\Theta_{body}^{-1} \cdot \boldsymbol{M} \overset{\Theta_{b12}=\Theta_{b23}=0}{=} \begin{pmatrix} -\frac{\Theta_{b13}}{\Theta_{b11} \cdot \Theta_{b33} - \Theta_{b13}^2} \cdot \frac{\mathrm{d}M_1}{\mathrm{d}F_1} & \frac{\Theta_{b33} \cdot l + \Theta_{b13} \cdot \frac{\mathrm{d}M_2}{\mathrm{d}F_2}}{\Theta_{b11} \cdot \Theta_{b33} - \Theta_{b13}^2} & -\frac{\Theta_{b13}}{\Theta_{b11} \cdot \Theta_{b33} - \Theta_{b13}^2} \cdot \frac{\mathrm{d}M_3}{\mathrm{d}F_3} & \frac{-\Theta_{b33} \cdot l + \Theta_{b13} \cdot \frac{\mathrm{d}M_4}{\mathrm{d}F_4}}{\Theta_{b11} \cdot \Theta_{b33} - \Theta_{b13}^2} \\ -\frac{l}{\Theta_{22}} & 0 & \frac{l}{\Theta_{22}} & 0 \\ \frac{\Theta_{b11}}{\Theta_{b11} \cdot \Theta_{b33} - \Theta_{b13}^2} \cdot \frac{\mathrm{d}M_1}{\mathrm{d}F_1} & \frac{-\Theta_{b13} \cdot l - \Theta_{b11} \cdot \frac{\mathrm{d}M_2}{\mathrm{d}F_2}}{\Theta_{b11} \cdot \Theta_{b33} - \Theta_{b13}^2} & \frac{\Theta_{b11}}{\Theta_{b11} \cdot \Theta_{b33} - \Theta_{b13}^2} \cdot \frac{\mathrm{d}M_3}{\mathrm{d}F_3} & \frac{\Theta_{b13} \cdot l - \Theta_{b11} \cdot \frac{\mathrm{d}M_4}{\mathrm{d}F_4}}{\Theta_{b11} \cdot \Theta_{b33} - \Theta_{b13}^2} \end{pmatrix}
$$

Inertia matrix - Inverse

Inertia matrix

$$\Theta_{body} = \begin{pmatrix} \Theta_{b11} & \Theta_{b12} & \Theta_{b13} \\ \Theta_{b21} & \Theta_{b22} & \Theta_{b23} \\ \Theta_{b31} & \Theta_{b32} & \Theta_{b33} \end{pmatrix}$$

Derivation inverse

$$\det \Theta_{body} = \Theta_{b11} \cdot \Theta_{b22} \cdot \Theta_{b33} + 2 \cdot \Theta_{b12} \cdot \Theta_{b23} \cdot \Theta_{b13} - \Theta_{b13}^2 \cdot \Theta_{b22} - \Theta_{b23}^2 \cdot \Theta_{b11} - \Theta_{b12}^2 \cdot \Theta_{b33}$$

Adjunct matrix

$$\begin{aligned}
\Theta_{body}^{11} &= \Theta_{b22} \cdot \Theta_{b33} - \Theta_{b23} \cdot \Theta_{b32} = \Theta_{b22} \cdot \Theta_{b33} - \Theta_{b23}^2 \\
\Theta_{body}^{12} &= \Theta_{b21} \cdot \Theta_{b33} - \Theta_{b23} \cdot \Theta_{b31} \\
\Theta_{body}^{13} &= \Theta_{b21} \cdot \Theta_{b32} - \Theta_{b22} \cdot \Theta_{b31} \\
\Theta_{body}^{22} &= \Theta_{b11} \cdot \Theta_{b33} - \Theta_{b13}^2 \\
\Theta_{body}^{21} &= \Theta_{b12} \cdot \Theta_{b33} - \Theta_{b13} \cdot \Theta_{b23} \\
\Theta_{body}^{23} &= \Theta_{b11} \cdot \Theta_{b32} - \Theta_{b12} \cdot \Theta_{b31} \\
\Theta_{body}^{31} &= \Theta_{b12} \cdot \Theta_{b23} - \Theta_{b13} \cdot \Theta_{b22} \\
\Theta_{body}^{32} &= \Theta_{b11} \cdot \Theta_{b23} - \Theta_{b13} \cdot \Theta_{b21} \\
\Theta_{body}^{33} &= \Theta_{b11} \cdot \Theta_{b22} - \Theta_{b12}^2
\end{aligned}$$

$$\Theta_{body,adj}^{T} = \begin{pmatrix} \Theta_{body}^{11} & -\Theta_{body}^{21} & \Theta_{body}^{31} \\ -\Theta_{body}^{12} & \Theta_{body}^{22} & -\Theta_{body}^{32} \\ \Theta_{body}^{13} & -\Theta_{body}^{23} & \Theta_{body}^{33} \end{pmatrix}$$

Inertia matrix - Inverse

Inverse matrix

$$\Theta_{body}^{-1} = \frac{1}{\det \Theta_{body}} \cdot \begin{pmatrix} \Theta_{b22} \cdot \Theta_{b33} - \Theta_{b23}^2 & \Theta_{b13} \cdot \Theta_{b23} - \Theta_{b12} \cdot \Theta_{b13} & \Theta_{b12} \cdot \Theta_{b23} - \Theta_{b13} \cdot \Theta_{b22} \\ \Theta_{b23} \cdot \Theta_{b31} - \Theta_{b21} \cdot \Theta_{b33} & \Theta_{b11} \cdot \Theta_{b33} - \Theta_{b13}^2 & \Theta_{b13} \cdot \Theta_{b21} - \Theta_{b11} \cdot \Theta_{b23} \\ \Theta_{b21} \cdot \Theta_{b32} - \Theta_{b22} \cdot \Theta_{b31} & \Theta_{b12} \cdot \Theta_{b31} - \Theta_{b11} \cdot \Theta_{b32} & \Theta_{b11} \cdot \Theta_{b22} - \Theta_{b12}^2 \end{pmatrix}$$

If quadrotor is symmetric about x and y axis, $\Theta_{b12} = \Theta_{b21} = \Theta_{b23} = \Theta_{b32} = 0$ holds

$$\Rightarrow \Theta_{body}^{-1} = \frac{1}{\Theta_{b11} \cdot \Theta_{b22} \cdot \Theta_{b33} - \Theta_{b22} \cdot \Theta_{b13}^2} \cdot \begin{pmatrix} \Theta_{b22} \cdot \Theta_{b33} & 0 & -\Theta_{b22} \cdot \Theta_{b13} \\ 0 & \Theta_{b11} \cdot \Theta_{b33} - \Theta_{b13}^2 & 0 \\ -\Theta_{b22} \cdot \Theta_{b13} & 0 & \Theta_{b11} \cdot \Theta_{b22} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\Theta_{b33}}{\Theta_{b11} \cdot \Theta_{b33} - \Theta_{b13}^2} & 0 & -\frac{\Theta_{b13}}{\Theta_{b11} \cdot \Theta_{b33} - \Theta_{b13}^2} \\ 0 & \frac{1}{\Theta_{b22}} & 0 \\ -\frac{\Theta_{b13}}{\Theta_{b11} \cdot \Theta_{b33} - \Theta_{b13}^2} & 0 & \frac{\Theta_{b11}}{\Theta_{b11} \cdot \Theta_{b33} - \Theta_{b13}^2} \end{pmatrix}$$