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Measuring Risk in Fixed Income Portfolios using Yield Curve Models

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1.Introduction

The Value at Risk (VaR) model is a crucial statistical tool that measures potential financial losses within a firm, portfolio, or position over a specific period. It is a fundamental method in risk management and is widely adopted in financial literature and by most financial institutions.

While there is considerable research on using VaR for equity portfolios, its application in fixed income portfolios is less explored. This lack of research is significant given the critical role fixed income securities play in diversified portfolios managed by institutional investors. Our team aims to address this gap by focusing our project on risk management in fixed income portfolios, exploring new dimensions within this field. After reviewing numerous articles and papers, we have chosen to employ term structure factor models known for their effectiveness in predicting yields. Our methodology is efficient and well-suited for managing large arrays of fixed income securities due to its factor-based approach. We are implementing dynamic factor models to predict bond returns and using various GARCH-type models to estimate the covariance matrix of returns. This approach will enable us to generate out-of-sample VaR estimates for a bond portfolio. Our method is versatile, accommodating various yield curve models and their conditional volatility frameworks.

2. Value at Risk (VaR) using Yield Curve Model

In this section, we explore the application of dynamic factor models to the yield curve for calculating Value at Risk (VaR) estimates. Factor models for interest rate term structures provide us with explicit formulas for anticipated yields and their conditional covariance matrix. Using these calculated moments, we demonstrate how to derive the distribution of bond prices and returns. These distributions will subsequently serve as essential inputs for determining the VaR for a bond portfolio.

2.1 Dynamic Factor Models for the Yield Curve

We consider a set of time series of bond yields with N different maturities, $\tau_1, \ldots, \tau_N$. The yield at time t of a security with maturity τ_i is denoted by $y_t(\tau_i)$ for t = 1,...,T. The N × 1 vector of all yields at time t is given by

$$y_t = \left[y_{1,t}, y_{2,t}, \ldots, y_{N,t}\right]', t = 1, \ldots, T.$$

The general specification of the dynamic factor model is given by

$$y_t = \Lambda(\lambda, \tau) f_t + \varepsilon_t, \qquad \epsilon_t \sim NID(0, \Sigma_t), t = 1, \ldots, T,$$

Λ(λ,τ) represents an N×K matrix of factor loadings, f_t denotes a K-dimensional stochastic process, and ϵ_t is an N×1 vector of disturbances, which has a conditional covariance matrix denoted by Σ_t. We impose a restriction on the covariance matrix Σ_t to ensure it is diagonal. Additionally, the dynamic factors, f_t, are described through a specified stochastic process:

$$f_t = \mu + \gamma f_{t-1} + \eta_t, \qquad \eta_t \sim NID(0, \Omega_t), t = 1, \ldots, T,$$

μ represents a K×1 vector of constants, and γ is the K×K transition matrix. Additionally, Ω_t is defined as the conditional covariance matrix for the disturbance vector η_t, which is independent from the vector of residuals ε_t. To calculate the loading matrix Λ(λ,τ), we utilize both the Nelson-Siegel and Svensson models. The formulas for these models are outlined below:

$$y_t(\tau) = \beta_{1t} + \beta_{2t}\left(\frac{1 - e_{-\lambda\tau}}{\lambda\tau}\right) + \beta_{3t}\left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{\lambda\tau}\right).$$

$$y_t(\tau) = \beta_{1t} + \beta_{2t}\left(\frac{1 - e^{-\lambda_1\tau}}{\lambda_1\tau}\right) + \beta_{3t}\left(\frac{1 - e^{-\lambda_1\tau}}{\lambda_1\tau} - e^{-\lambda_1\tau}\right) + \beta_{4t}\left(\frac{1 - e^{-\lambda_2\tau}}{\lambda_2\tau} - e^{-\lambda_2\tau}\right).$$

Here are the yield curves modeled by the Nelson-Siegel and Svensson models.

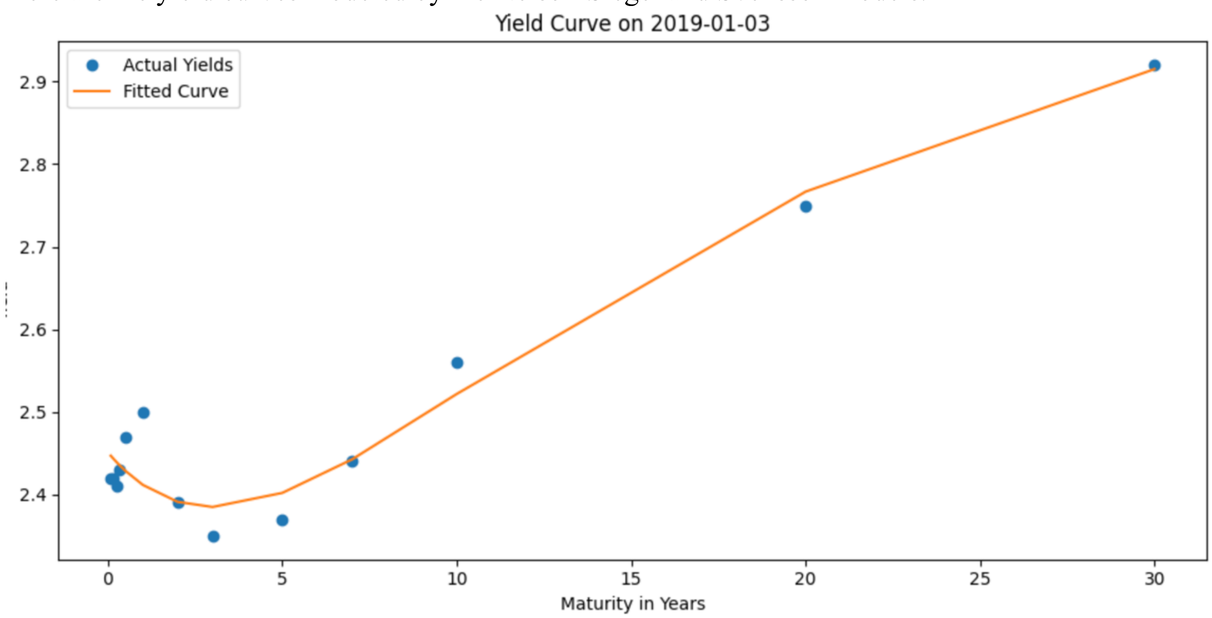

Figure 2.1 Nelson-Siegel Model

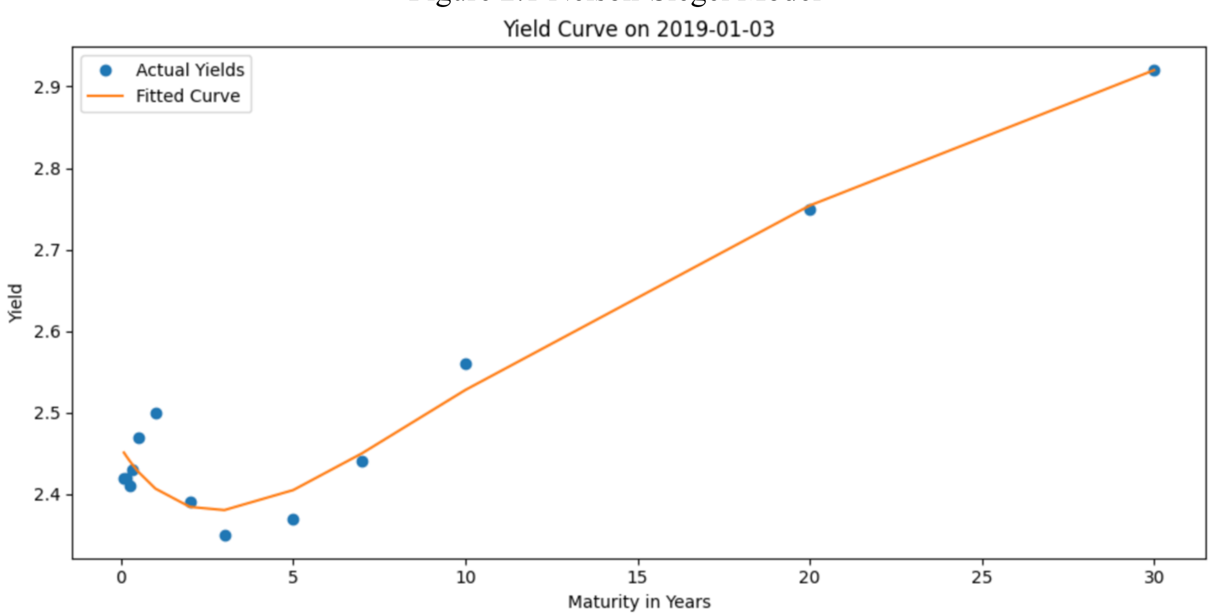

Figure 2.2 Svensson Model

The model seems to capture the general trend of the yield curves quite well for some dates, showing a good fit across different maturities. There are some discrepancies, particularly for short maturities, where the model might not perfectly align with the actual yields. This could be due to model specification or the dynamic nature of financial markets.

2.2 Conditional Covariance of the Factor Models for the Yield Curve

Forecasting the volatility of interest rates is a significant challenge in financial econometrics, especially relevant in fixed income applications. This is crucial because operations such as interest rate hedging and arbitrage are affected by fluctuations in volatility. In these activities, compensating for the market price of interest rate risk often becomes necessary.

To construct Ω_t, the conditional covariance matrix of the factors, we can explore various model specifications. For this project, we have chosen the dynamic conditional correlation model (DCC), and the corresponding formula is provided below:

$$\Omega_t = D_t \psi_t D_t,$$

Additionally, Dt is a K × K diagonal matrix whose diagonal entries are h_(f_kt), representing the conditional variance of the k-th factor. ψ_t is a symmetric correlation matrix with elements ρij,t, where ρii,t = 1, and i, j range from 1 to K. In the DCC model, the conditional correlation ρij,t is defined as follows:

$$\rho_{ij,t} = \frac{q_{ij,t}}{\sqrt{q_{ii,t} q_{jj,t}}},$$

where $q_{ij,t}$, are the elements of Q_t, which follows a GARCH-type dynamics:

$$Q_t = (1 - \alpha - \beta)\bar{Q} + \alpha z_{t-1} z_{t-1}' + \beta Q_{t-1},$$

To model the conditional variance of the measurement errors ϵ_i, we posit that Σ_t is a diagonal matrix. The diagonal elements of this matrix are represented by $h_{t\varepsilon_i}$, where $h_{t\varepsilon_i}$ denotes the conditional variance of each ϵ_i.

2.3 Expected bond returns and the conditional covariance matrix of bond returns

As we will explore in greater detail later, calculating the Value at Risk (VaR) necessitates deriving estimates for both the expected returns of each bond and the covariance matrix of the bond returns within the portfolio. However, the factor models for the term structure of interest rates that we discussed earlier are specifically aimed at modeling bond yields. Despite this, it is feasible to derive expressions for both the expected bond return and the conditional covariance matrix of bond returns, based on the distribution of the expected yields. The equations previously mentioned show that the distribution of expected yields, y_(t|t-1), follows a normal distribution N(μ_(y,t),Σ_(y,t)), where μ_(y,t) = Λf_(t|t-1) and Σ_(y,t) = ΛΩ_(t|t-1) Λ^' + Σ_(t|t-1). Here, f_(t|t-1) is the one-step-ahead forecast of the factors, and Σ_(t|t-1) and Ω_(t|t-1) represent the one-step-ahead forecasts of the conditional covariance matrices. With this distribution of yields, it is possible to calculate the expected distribution of fixed-maturity bond prices. Considering that the price of a bond at time t, Pt(τ), represents the present value at time t of $1 receivable τ periods in the future, and yt|t-1 represents the one-step-ahead forecast of its continuously compounded zero-coupon nominal yield to maturity, we can derive the vector of expected bond prices Pt|t-1 for all maturities:

$$P_{t|t-1} = exp - \tau \otimes y_{t|t-1}$$

⊗ denotes the Hadamard (elementwise) multiplication. Given that $yt\ |t-1$ is normally distributed, $Pt|t{-}1$ is log-normally distributed. Additionally, the log-return for a specific bond with maturity τi can be expressed as follows:

$$r_{i,t} = \log\left(\frac{P_{i,t}}{P_{i,t-1}}\right) = logP_{i,t} - logP_{i,t-1} = -\tau_i(y_{i,t} - y_{i,t-1})$$

This formulation enables us to derive a closed form for both the vector of expected bond returns and their conditional covariance matrix. The expected log-returns for bonds, $\mu_{r_t|t-1}$and their conditional covariance matrix$\Sigma_{r_t|t-1}$, are specified as:

$$\mu_{r_t|t-1} = E_{t-1}[r_t] = -\tau \otimes (E_{t-1}[y_t] - y_{t-1}) = -\tau \otimes \mu_{y,t} \pm \tau \otimes y_{t-1}$$

$$\Sigma_{r_t|t-1} = Var_{t-1}[r_t] = \tau\tau' \otimes \mathrm{Var}_{t-1}[y_t] = \tau\tau' \otimes \left[\Lambda\Omega_{t|t-1}\Lambda' + \Sigma_{t|t-1}\right]$$

These formulas illustrate that it is feasible to achieve closed form expressions for the expected log-returns of bonds and their covariance matrix using yield curve models. These calculations are essential for computing the VaR for a bond portfolio.

2.4 VaR Computation

We now turn to the calculation of Value at Risk (VaR) for bond portfolios using the previously discussed yield curve models. As we will demonstrate, the established closed form expressions for the vector of bond portfolio returns and its covariance matrix, as detailed in Section 2.3, can be directly applied to the VaR computation for bond portfolios. Throughout this paper, our focus is on the portfolio VaR concerning a long position where traders have purchased fixed income securities and are interested in assessing the risk related to potential declines in market prices. This involves evaluating the risk tied to rising bond yields, which typically correspond to falling prices and consequently negative returns. We denote $R_{t+h} = \left(r_{1,t+h}, \dots \dots, r_{N,t+h}\right)'$ as the vector of h-period returns (from time t to t+h) for the N bonds within the portfolio. The return of the bond portfolio is represented as $r_{p,t+h} = w_t' R_{t+h}$, with w_t being the portfolio weight vector determined at time t. The portfolio's VaR at time t, for a specific holding period h and a given confidence level v, is calculated based on the v-quantile of the conditional distribution of the bond portfolio return.

The distribution of bond log-returns is described by its first two conditional moments, the portfolio return can be expressed as follows:

$$r_{p,t+1} = \mu_{p,t+1} + \sigma_{p,t+1} z_{p,t+1}$$

The standardized unexpected returns $z_{p,t+1}$ are independently and identically distributed, with a mean of zero and a variance of one. The conditional mean and standard deviation of the bond portfolio return, $\mu_{p,t+1}$ and $\sigma_{p,t+1}$, are specified as follows:

$$\mu_{p,t+1} = w_t' \mu_{r_{t+1}} \qquad \sigma^2_{p,t+1} = w_t' \Sigma_{r_{t+1}} w_t$$

$\mu_{p,t+1}$ represents the N×1 vector of conditional mean returns for the individual assets and $\Sigma_{r_{t+1}}$ is their N×N conditional covariance matrix, as previously defined. The portfolio VaR is then determined by:

$$VaR_{t+1} = \mu_{p,t+1} + \sigma_{p,t+1} q$$

q is the v-quantile of the distribution of $z_{p,t+1.}$. The straightforward expressions for the vector of bond portfolio returns and its covariance matrix can be directly used in calculating the bond portfolio VaR.

Here are the VaR under different Confidence Level (%):

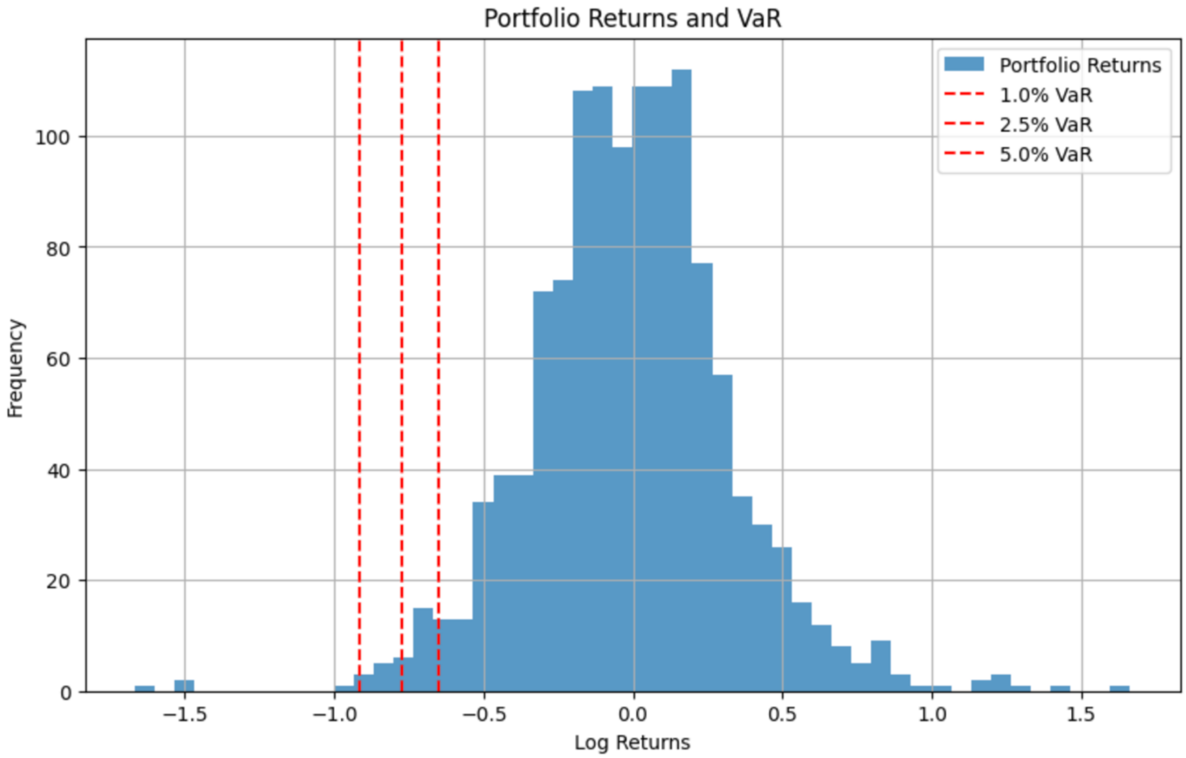

Figure 2.3

3. Estimation Procedure

In this project, we assume the parameter λ_t is constant, which significantly simplifies the estimation process. By setting λ_t as fixed, we are able to estimate β_1t, β_2t, and β_3t using the ordinary least squares (OLS) regression method as described in equation 2.1. Subsequently, the measurement equation is handled as a separate cross-sectional analysis for each time period, using OLS to estimate the factors individually for all time periods. Once the time-series data for the factors are estimated, the next step involves modeling the dynamics of these factors either by applying a joint VAR(1) model or by estimating separate AR(1) models for each factor.

4. Empirical Application

To demonstrate the usefulness of the proposed estimators for the vector of expected bond returns and their conditional covariance matrix, this section examines the task of forecasting one-step-ahead VaR for an equally-weighted bond portfolio, as detailed in Sections 2.3 and 2.4. As previously mentioned, we focus on the portfolio VaR for a long position where traders hold fixed income securities and aim to assess the risk related to a potential decline in their market values.

4.1 Data

Our dataset comprises five years of daily par yield curve rates for U.S. Treasury bills, spanning from January 2, 2019, to December 29, 2023. This data, which includes a range of maturities from 1 month to 30 years, can be accessed and downloaded from the U.S. Department of Treasury website. The specific maturities available are 1 month, 2 months, 3 months, 4 months, 6 months, 1 year, 2 years, 3 years, 5 years, 7 years, 10 years, 20 years, and 30 years.

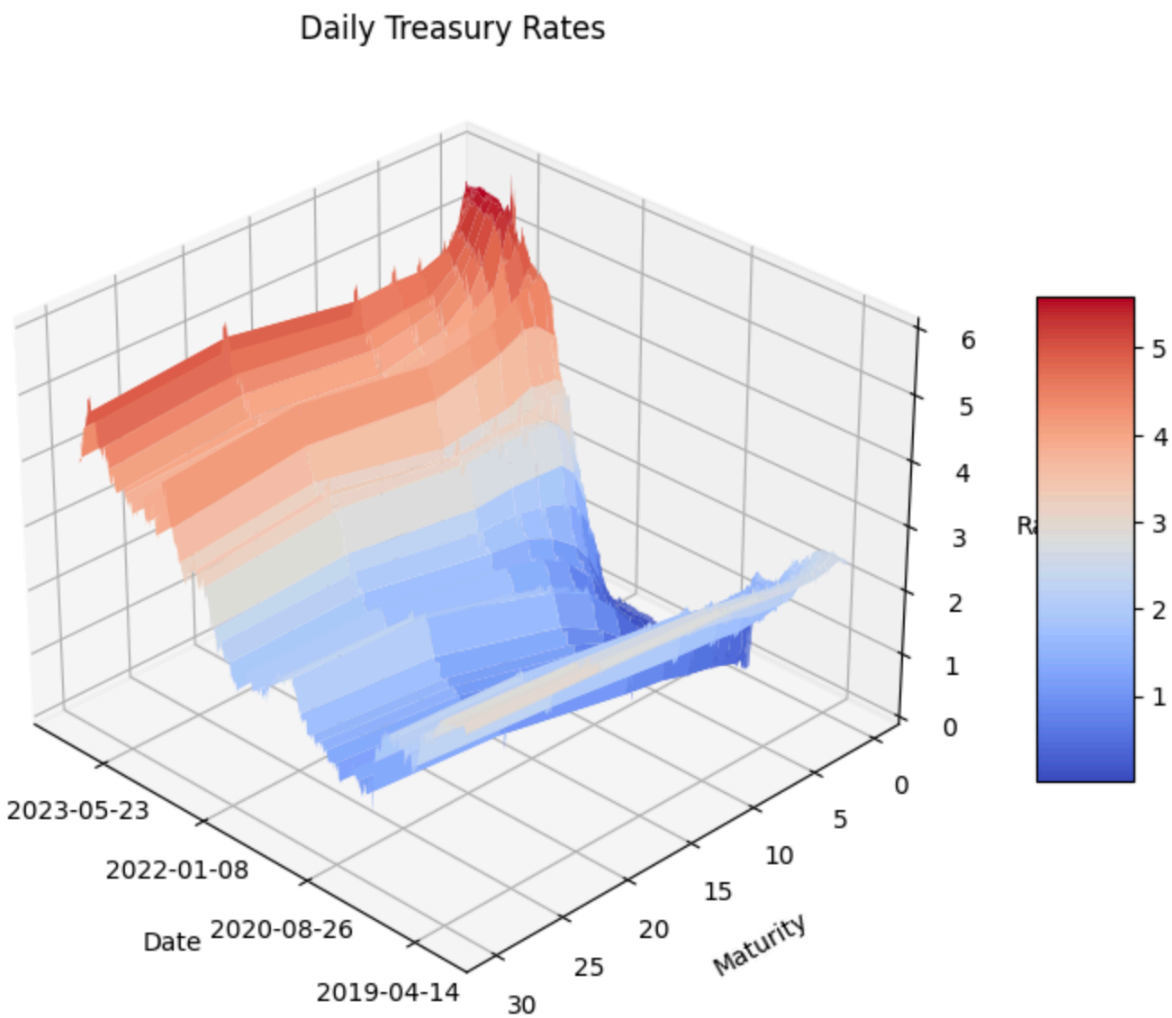

Figure 4.1 Daily Treasury Rates

4.2 Results

In this section, the paper outlines the outcomes of backtesting the VaR estimates derived from the vector of expected bond returns and the specified conditional covariance matrix as discussed previously. Following traditional backtesting methodologies, we compute various VaRs for the relevant factor estimates using the established Nelson-Siegel Model and Svensson Model. We model the factor dynamics with AR(1) and VAR(1) methods, respectively, and use the DCC-GARCH approach for modeling the covariance matrices of the factors.

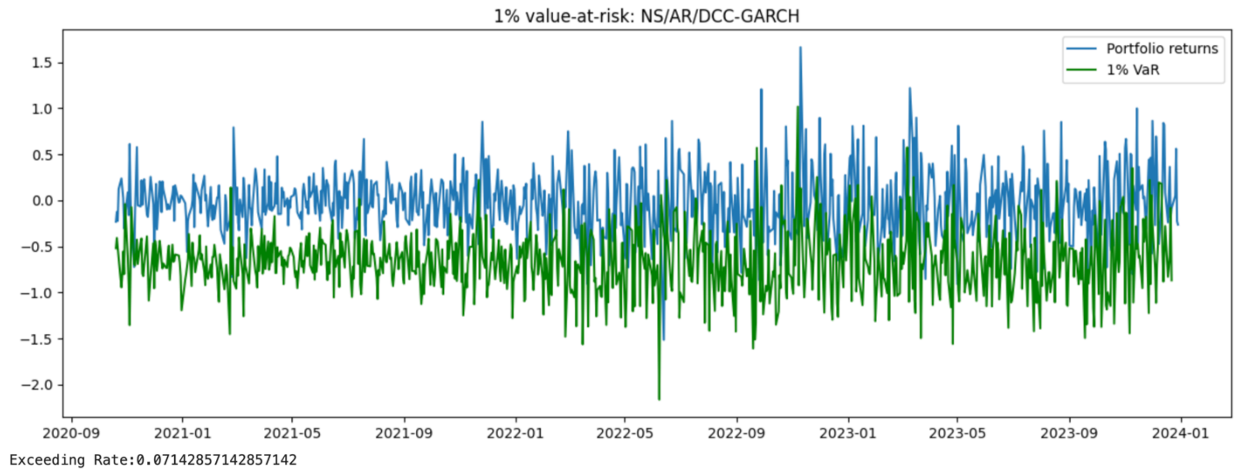

Figure 4.2 1% value−at−risk: Nelson-Siegel Model /AR/DCC−GARCH

This configuration, using the Nelson-Siegel model, Autoregressive (AR), and Dynamic Conditional Correlation GARCH (DCC-GARCH), has an exceeding rate of approximately 7.14%. This indicates that in 7.14% of instances, the actual losses exceeded the forecasted VaR, showing good predictive performance in risk forecasting.

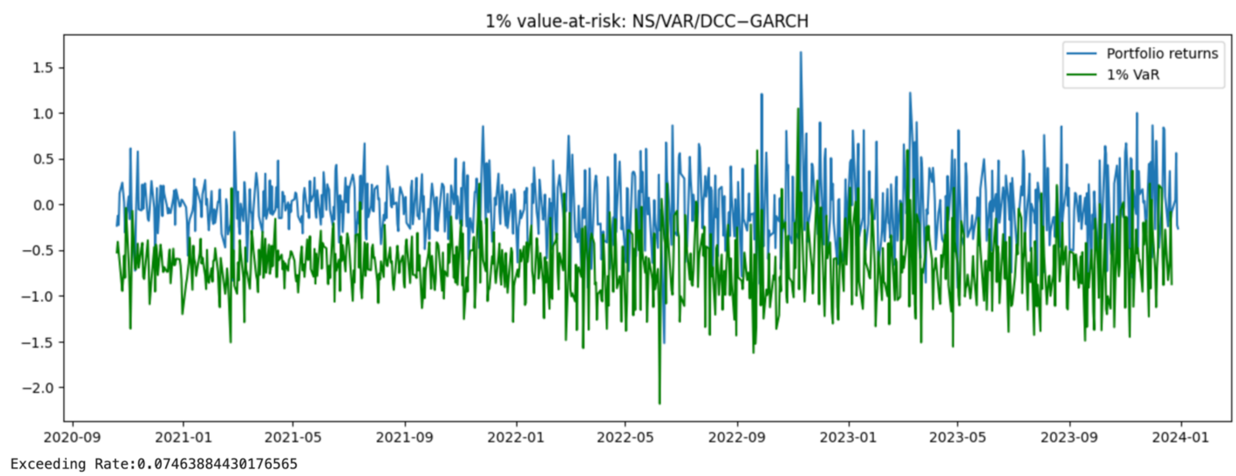

Figure 4.3 1% value−at−risk: Nelson-Siegel Model /VAR/DCC−GARCH

This model configuration employs the Nelson-Siegel model, Vector Autoregression (VAR), and DCC-GARCH. The exceeding rate is around 7.46%, slightly higher than the AR version, suggesting a slight decrease in predictive accuracy in this setup.

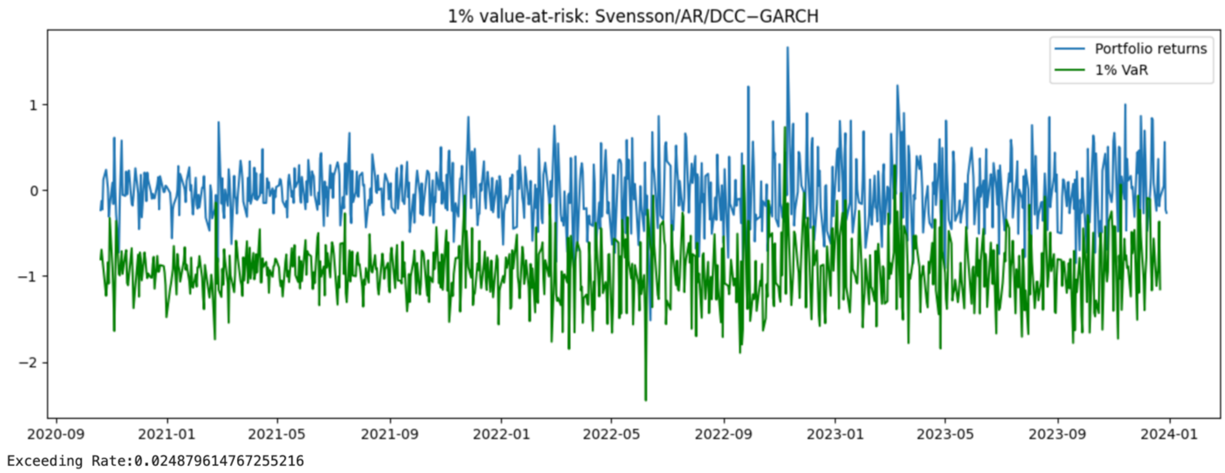

Figure 4.4 1% value−at−risk: Svensson/AR/DCC−GARCH

The combination of the Svensson model, AR, and DCC-GARCH shows a lower exceeding rate of about 2.49%. This indicates that this model configuration performs more accurately in predicting extreme risk events, offering a more conservative risk estimate.

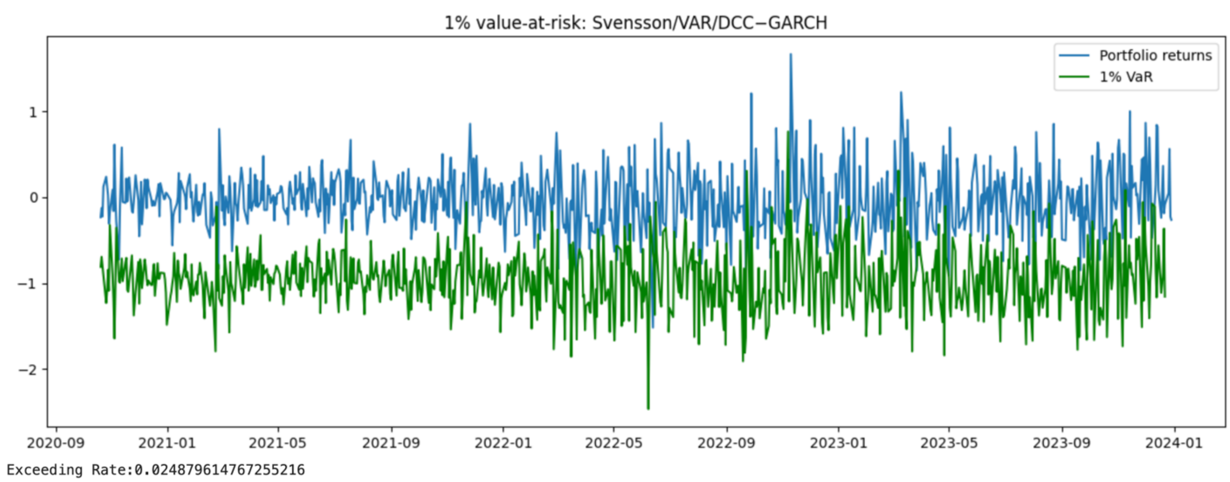

Figure 4.5 1% value−at−risk: Svensson/VAR/DCC−GARCH

This configuration uses the Svensson model, VAR, and DCC-GARCH, with an exceeding rate also at 2.49%. This suggests that whether using AR or VAR, the Svensson model combined with DCC-GARCH performs well in these tests for predicting risk.

In summary, regardless of whether paired with AR or VAR, the Svensson model combined with DCC-GARCH demonstrates better risk prediction performance than the corresponding combinations with the Nelson-Siegel model. This may indicate that the Svensson model is more suitable or accurate in handling such risk estimations.

5. Conclusion

Accurate risk assessment is crucial in risk management. In this context, the use of Value at Risk (VaR) is integral for monitoring market risk exposure and for determining the regulatory capital requirements. While most research on VaR focuses on equity portfolios, this paper expands the discussion to include bond portfolios by introducing a new approaches that leverages well-established term structure factor models, including the dynamic version of the Nelson-Siegel model and Svensson model.

We demonstrate our method empirically using a dataset of constant-maturity U.S. Treasury bills and bonds. By analyzing the expected returns and conditional covariance matrix of these fixed-income assets, we derive out-of-sample VaR estimates for an equally-weighted bond portfolio and conduct a thorough backtesting analysis.

6. References

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Appendix

Model configuration	*VaR*	*hit rate*	*Independence Test p-value*	*Unconditional Coverage Test p-value*
NS - AR - DCCGARCH	0.01	12.46	0.000321252	3.36E-06
	0.025	31.15	0.000254462	0
	0.05	62.3	0.000224592	0
NS - AR - CCCGARCH	0.01	12.46	0.000321252	3.36E-06
	0.025	31.15	0.000254462	0
	0.05	62.3	0.000224592	0
NS - VAR - DCCGARCH	0.01	12.46	0.000594685	1.22E-06
	0.025	31.15	0.000254462	0
	0.05	62.3	0.000955456	0
NS - VAR - CCCGARCH	0.01	12.46	0.000594685	1.22E-06
	0.025	31.15	0.000254462	0

	0.05	62.3	0.000955456	0
Svensson - AR - DCCGARCH	0.01	12.46	0.003467614	0.671453983
	0.025	31.15	1.13E-07	0.003563681
	0.05	62.3	0.000163502	9.01E-06
Svensson - AR - CCCGARCH	0.01	12.46	0.008623027	0.895155835
Svensson - VAR - DCCGARCH	0.01	12.46	0.008623027	0.895155835
	0.025	31.15	5.66E-05	0.007262103
	0.05	62.3	0.000163502	9.01E-06
Svensson - VAR - CCCGARCH	0.01	12.46	0.008623027	0.895155835
	0.025	31.15	5.66E-05	0.007262103
	0.05	62.3	0.000163502	9.01E-06