Robust and Stochastic Portfolio Optimization (Fall 2021)

Final Project Topic:
On Markowitz Portfolio Optimization with Black-Litterman Approach
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1 Abstract

In this project, we consider a problem of portfolio construction. Specifically, we use the optimal portfolio theory proposed by Harry Markowitz, an American financial economist, in 1952. In addition, we also discuss how to construct portfolios that minimize the Conditional Value-at-Risk (CVaR). To lower the risks and elevate the portfolio return in the mean-variance framework. We then incorporate the market equilibrium information with the well-known Black-Litterman approach via inverse optimization. To support our work, we also plan to conduct some empirical studies.

2 Introduction and Motivations

Portfolio optimization plays an important role in determining the strategies of investors. However, in the traditional views, the investors usually predict the future trend by qualitative research since the quantitative research and the optimization problem is difficult. Hence, I am eager to use the different well-known models to do the empirical studies by quantitative method.

The most popular portfolio theory to adjust the preferences of the investor's risk and return is Markowitz's mean-variance model. Markowitz developed his ideas for portfolio optimization in the book "Portfolio Selection: Efficient Diversification of Investments" in 1959. As a result, this project will demonstrate how to apply the Markowitz mean-variance model to find the optimal portfolio construction in specified datasets.

In addition, I found that the investor's view plays an important role in the performance of the portfolio since only relying on the past data(e.g. closed price of stocks) cannot effectively predict the future trend of stocks. Therefore, a motivation for analyzing the inverse optimization problems comes from the Black-Litterman asset allocation model. To analyze, I will use the experiment method which is applying the Black-Litterman approach to the Markowitz mean-variance model and Conditional Value-at-Risk minimization model.

3 Problem Formulation and Preliminaries

In this section, I will show the composition of the Markowitz mean-variance model, Conditional Value-at-Risk(CVaR), and the Black-Litterman approach. The detail of the parameters in each model will be explained in the following section.

3.1 Markowitz mean-vaiance model

3.1.1 Rate of Return

The rate of return is the net gain or loss of an investment over a specified time period, expressed as a percentage of the investment's initial cost.

Let the r_t be the rate of return, P_t is the price of time t, and $P_t - 1$ is the price of time t-1. So the rate of return on market can be written as

$$r_t = \frac{P_t - P_{t-1}}{P_{t-1}}$$

3.1.2 Expected Return

Let n be the number of our assets, and μ_i is the expected return on asset i, $i = 1, ..., n$.

$$\mu_i = (r^t) = \frac{\sum_{t=1}^{m} r_t^i}{m}$$

where r_t^i is the return on asset i between periods t - 1and $t, t = 1, ..., m$ and m is the number of periods.

3.1.3 Variance and Standard Deviation

Risk-averse investors usually consider risk as to the most important factor to their portfolio. The standard deviation is used to measure the risk.

The σ_i is the standard deviation of assets i, and the variance of assets i can be calculated by the following formula

$$\sigma_i^2 = Var(r^i) = \mathbb{E}[(r^i - \mathbb{E}[r^i])^2]$$

3.1.4 Covariance

The risk on the assets i are modeled with the covariance matrix, which is denoted by $\Sigma_{n\times n}$. The matrix of the assets contains the variance of asset i and the covariance between all pairs of assets, i.e.

$$\Sigma_{n\times n} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \dots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \dots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \dots & \sigma_n^n \end{bmatrix}$$

3.1.5 Optimization of Markowitz mean-variance model

The portfolio theory was released by Harry Markowitz in 1990. The parameters of the Markowitz model have already been discussed in the previous section. In addition, to maximize the portfolio mean return, I will use the Markowitz mean-variance model to consider the max return. The form of the model is the following mathematical expression

$$\begin{aligned} &\max_w \mu^T w - \rho w^T \Sigma w \\ &\text{s.t.} \sum_{i=1}^{n} |w_i| = 1 \\ &|w_i| \geq 0 \end{aligned}$$

where μ is the corresponding data mean of the data sets, $\mathbf{w} := [w_1, ..., w_n]^T$ is the weights of the portfolio and $\rho \in (0, 2)$ is the so-called $risk - aversion\ constant$.

3.2 Conditional Value-at-Risk

In this section, I discuss how to obtain risk- minimizing portfolios. I consider the famous Conditional Value-at-Risk(CVaR) method. The CVaR method is defined with respect to the left-tail of the return distribution. The confidence level α level $CVaR$ of a random variable Z is defined as

$$CVaR_\alpha(Z) = [Z|Z \geq VaR_\alpha(Z)]$$

where $\alpha \in (0, 1)$ and VaR_α is the Value-at-Risk.

Let us consider the definition of the CVaR. We let $Z \in L^1(\Omega, F, P)$ The quantity of CVaR at level α can be expressed as

$$CVaR_\alpha(Z) = \inf_{t \in \mathbb{R}} t + \frac{1}{1-\alpha} E[(Z-t)_+]$$

In order to incorporate the Black-Litterman model in the following section, I add the mean return term in my objective function. The goal is to maximize the return and consider the risk measure. The problem can be expressed as:

$$\begin{aligned} \min_{w,y} \; & y + \frac{1}{1-\alpha} \sum_{s=1}^{S} p_s z_s - \mu^T w \\ \text{s.t.} \;\; & z_s \geq Y(\mathbf{w}, \mathbf{r}^s) - y, \quad s = 1, 2, ..., S \\ & z_s \geq 0, \quad s = 1, 2, ..., S \\ & \sum_{i=1}^{n} |w_i| = 1, \quad |w_i| \geq 0 \\ & y \in \mathbb{R}. \end{aligned}$$

where $\mathbf{w} := [w_1, ..., w_n]^T$ is the weights of the portfolio, $Y(\mathbf{w}, \mathbf{r}^s) := -\mathbf{w}^T \mathbf{r}$ represents the loss function, S are the possible scenarios ; e.g., the daily returns in one- year yields , and the z_s is the random variable.

3.3 Black-Litterman Model

In order to make the portfolio more robust, I will incorporate the Black-Litterman methodology into the data perturbations. Black-Litterman Model has been proposed in 1999 in Goldman Sachs. The classical Black-Litterman is driven by two factors: the market equilibrium and the investor's views. Below I will discuss how to build the Black-Litterman(BL) Model and estimate the parameters among it.

3.3.1 Market Equilibrium

First, we should discuss the previous MV optimization problem.

$$\max_{w} \mu^T w - \rho w^T \Sigma w$$

The implied return in the markets is

$$\pi = \mu + \varepsilon_\pi, \quad \varepsilon_\pi \sim \mathcal{N}(0, \tau\Sigma)$$

where μ is the true expected return unknown to the trader, $\tau\Sigma$ is the confidence how we estimate the equilibrium expected returns.

3.3.2 Investor's Views

The Second part is about the investor views. The BL model incorporate the investor views with the market equilibrium in the following procedure. In BL model, the investor's views can be expressed with the linear equation i.e.

$$\mathbf{q} = P\mu + \varepsilon_q, \;\; \varepsilon_q \sim \mathcal{N}(0, \Omega)$$

where $P \in \mathbb{R}^{\mathbb{K} \times \mathbb{N}}$, ε_q represents the degree of confidence in the views, and $\Omega \in \mathbb{R}^{\mathbb{K} \times \mathbb{K}}$ expressing the confidence in the views.

3.3.3 Merging Investor's Views with Market Equilibrium

In order to apply The Black Litterman model, let us combine the investor's view with the market equilibrium.

$$\pi = \mu + \varepsilon_\pi, \;\; \varepsilon_\pi \sim \mathcal{N}(0, \tau\Sigma)$$

$$\mathbf{q} = P\mu + \varepsilon_q, \;\; \varepsilon_q \sim \mathcal{N}(0, \Omega)$$

The result will be the following formula i.e.

$$\mathbf{y} := M\mu + \varepsilon, \;\; \varepsilon \sim \mathcal{N}(0, V)$$

where $\mathbf{y} := \begin{bmatrix}\pi & \mathbf{q}\end{bmatrix}_T, M := \begin{bmatrix} I_{NxN} \\ P \end{bmatrix}$ and $V := \begin{bmatrix} \tau\Sigma & 0 \\ 0 & \Omega \end{bmatrix}$.

3.3.4 Estimate Expected Return on WLS Approach

We merged the market equilibrium and the investor's views. However, we still need to use the weighted least-squares(WLS)approach to estimate the $\hat{\mu}$ i.e.

$$\min_{\mu} (\mathbf{y} - M\mu)^T V^{-1} (\mathbf{y} - M\mu)$$

where $\mathbf{y}$, M, μ, V have already been defined in the previous section.

4 Empirical Studies

In this section, I present the results of my computational experiments on the Dow Jones Industrial Average(DIJA) Index datasets. In section 4.1, I explain the experimental setting which includes the proper definition of market-based portfolios. In section 4.2, I provide a comparison of three portfolios. One of which is the Market-Based Portfolio, another one is Markowitz Portfolio(Markowitz mean-variance model without BL approach), and the last one is CVaR Minimizing Portfolio(CVaR method without BL approach). In section 4.3, I incorporate the Black-Litterman Approach to compare with the previous three models. At the end of this section, I will address the perspectives of the results and give the proper explanation.

4.1 Experimental setting

Let me start explaining the detail of our experimental of our setting. I use the 5 years(2016 2020) DIJA index for my dataset to evaluate the performance of 2021. There are five portfolios, one Market-Based portfolio, one portfolio of Markowitz Portfolio without BL approach, one portfolio of Markowitz Portfolio with BL approach, one CVaR Minimizing Portfolio, and one CVaR Minimizing Portfolio with BL approach. The performance of the portfolio is evaluated with respect to average reward, standard deviation. I also report one risk-adjusted performance measure: the ratio of average reward to standard deviation (also known as the Sharpe ratio).

4.2 Portfolio Comparison Without Black-Litterman Approach

I first compare the performance of the market-based and the other two portfolios in the below table.

Portfolio	Return	Risk	Sharpe Ratio
Market Based Portfolio	26.04%	20.19	1.21%
Markowitz Portfolio	36.14%	20.34	1.70%
CVaR Minimizing Portfolio	3.60%	5.27	0.41%

We can observe that Market Based Portfolio performs worse than the Markowitz Portfolio in return and the Sharpe ratio. Though the risk of Market Based Portfolio is smaller than Markowitz Portfolio, the return of it is greatly less than the Markowitz Portfolio. However, the performance of Market Based Portfolio is better than CVaR Minimizing Portfolio in the Sharpe ratio. I consider that is because CVaR Minimizing Portfolio aims to minimize the risk without any consideration. Hence, we can conclude that the Markowitz Portfolio under the DIJA index in 2021 can have better performance than the Market Based Portfolio. In addition, the CVaR Minimizing Portfolio can truly minimize the risk under the condition, and the risk of CVaR Minimizing Portfolio is only 5.27, which is the smallest compared to other portfolios.

4.3 Portfolio Comparison on Black-Litterman Approach

In this section, I apply the BL approach to the model and incorporate the following investor's views which were collected from the news at the end of 2020.

- Apple perform well in 2021 due to its constant growth.
- Merck has already better performance in sales since the vaccine is important to combat the pandemic.
- Goldman Sachs will perform slightly better than J.P.Morgan due to its better performance in 2020.
- As a result of the pandemic, the sales of Walmart will decrease dramatically.

Among these five investors' views, I am confident with the first and the second views, so I give higher credit in the confidence covariance matrix. The comparison of the portfolios on the BL approach is in the below table.

Portfolio	Return	Risk	Sharpe Ratio
Markowitz Portfolio	36.14%	20.34	1.70%
CVaR minimizing Portfolio	3.60%	5.27	0.41%
Markowitz Portfolio w/ BL approach	38.06%	14.81	2.47%
CVaR minimizing Portfolio w/ BL approach	27.10%	27.25	0.94%

In the table, the results show that when we apply the Black-Litterman approach to adjust our data mean, we can increase return and lower the risk(standard deviation). The Sharpe ratio of each portfolio with BL approach is better than those without. However, after applying the approach to the CVaR minimizing Portfolio, the return and the risk change greatly, which represents that CVaR minimizing Portfolio has no longer only been considered to minimize risk. The CVaR minimizing Portfolio incorporates the investor view to find the optimization of the problem.

4.4 Results

Finally, I have already separately addressed the finding in the previous section. Let us observe the overall empirical results.

Portfolio	Return	Risk	Sharpe Ratio
Market Based Portfolio	26.04%	20.19	1.21%
Markowitz Portfolio	36.14%	20.34	1.70%
CVaR Minimizing Portfolio	3.60%	5.27	0.41%
Markowitz Portfolio w/ BL approach	38.06%	14.81	2.47%
CVaR Minimizing Portfolio w/ BL approach	27.10%	27.25	0.94%

In the above overall table, when we consider the return, Markowitz Portfolio w/ BL approach has the highest return. When we focus on risk, CVaR Minimizing Portfolio has the lowest standard deviation. And the last, the Sharpe ratio of Markowitz Portfolio w/ BL approach is the highest. Most of the portfolios perform better than Market Based Portfolio. However, CVaR Minimizing Portfolio w/ BL approach has very worse performance. It has a lower Sharpe ratio than Market Based Portfolio, so it means CVaR Minimizing Portfolio w/ BL approach should undertake more risk to get a higher return. In the end, among the five portfolios, Markowitz Portfolio w/ BL approach has the best performance under the condition of the 2021 DIJA index.

5 Conclusion

In this paper, I addressed the basic portfolio construction, combing the market-based approaches and analytical researches. At first, I introduce the Markowitz mean-variance model, The framework of the Conditional Value-at-Risk method and the Black-Litterman model are subsequently been explained. We then explore the possibilities of applying the Black-Litterman approach to the different models. My computational experiments in section 4 show that the Black-Litterman approach does improve the performance in most cases, though the Black-Litterman approach does not suitable for to CVaR Minimizing Portfolio. It may be because I put too much confidence in the investor's views, which cause the risk to increase largely. However, the Black-Litterman approach still performs well in Markowitz Portfolio, so this approach may work in Markowitz Portfolio under the condition I set in section 4.1.

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