Timer Option Pricing and Hedging

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1 Introduction

In April 2007, Societe Generale Corporate and Investment Banking (SG CIB) launched a novel financial product termed the timer option. This innovative option allows investors to set the level of volatility used in its pricing. Differing from standard options, the expiration of timer options depends on the accumulated variance of an underlying asset rather than a pre-set date [Bernard and Cui, 2011]. This mechanism allows for a pricing model that accounts for actual realized volatility rather than projected implied volatility. This paper delves into effective simulation techniques to price timer options within stochastic volatility models, building substantially on the foundational work by [Carr and Lee, 2009] and [Saunders, 2010]. Utilizing robust replication strategies and model-free control variates, introducing a streamlined yet accurate approach, particularly within the frameworks of the Heston and Hull and White models, as further analyzed by [Li, 2009].

2 Preliminaries

This section introduces fundamental financial metrics and concepts essential to understand our research's framework and further analysis. First, choose a specific value as our target volatility σ_{target}, and the variance budget based on the execution time T, $VB^{\text{target}} = \sigma^2_{\text{target}} \frac{T_{\text{trade}}}{252}$, where T_{trade} is the number of trading days before the maturity date T. The realized volatility for the observation period D: $\sigma_D^{\text{realized}} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n} u_i^2}$, where daily volatility $u_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$, S_i is the underlying stock price at time t_i, and $t_n = D$. For a one-year observation, the annualized volatility as $\sigma_{\text{realized}} = \frac{\sigma_D^{\text{realized}}}{\sqrt{D}}$. The cumulative volatility of the underlying asset is $VB^{\text{realized}} = \sigma^2_{\text{realized}} \frac{d}{252}$, where d is the number of days since the inception date. The stopping time $\tau = \inf\left\{u > 0 \;\middle|\; \int_0^u V_s\, ds = B\right\}$, is defined as the first hitting time of the realized variance to our chosen variance budget B.

3 Timer Call Options

According to realized variance consumption formula, when the cumulative volatility budget VB^{realized} of the underlying asset hits our variance budget VB^{target}, the time option will automatically execute.

The payoff is

$$\max(S_t - K, 0) \tag{1}$$

To be more specific, our project takes the mandatory execution day into consideration, the option price at time t is as follows

$$C_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(\min(T,\tau_B)-t)} \max(S_{\min(T,\tau)} - K, 0)\right], 0 \leq t < T \tag{2}$$

, where $\mathbb{Q}$ is risk-neutral probability.

3.1 Pricing the Timer Option under the General Stochastic Volatility Model

For timer option, there are two equations to framework the variance and underlying asset by

$$\begin{aligned} dV_t &= \alpha(V_t)\,dt + \beta(V_t)\,dW_t^2 \\ dS_t &= rS_t\,dt + S_t\sqrt{V_t}\left(\sqrt{1-\rho^2}\,dW_t^1 + \rho\,dW_t^2\right) \end{aligned} \tag{3}$$

After modification, the formula becomes

$$\begin{aligned} dV_t &= \alpha(V_t)\,dt + \beta(V_t)\,dW_t^2 \\ \frac{dS_t}{S_t} &= r\,dt + \sqrt{V_t}\left(\sqrt{1-\rho^2}\,dW_t^1 + \rho\,dW_t^2\right) \end{aligned} \tag{4}$$

The Brownian motion W_t^2 is uncorrelated with the Brownian motion W_t^1, which limits the movement of the underlying stock price. ρ is the correlation coefficient between the two general stochastic processes of the underlying asset price and the volatility process. For dividend paying options, the interest rate r is replaced by $r - q$, where q is the dividend rate. In addition, $\alpha(V_t)$ measure the drift of the volatility and $\beta(V_t)$ measure the volatility of the volatility. Obviously, $\alpha(V_t)$ and $\beta(V_t)$ are the measurable functions with respect to n.

Then, update the previous stopping time to the new one, as the known integrated variance $V_{\text{acc}} = \int_0^t V_s\,ds$.

$$\tau_B = \inf\left\{u > t \,\middle|\, \int_t^u V_s\,ds = B - V_{\text{acc}}\right\} \tag{5}$$

The call option before (10) satisfy the following three-dimensional partial differential equation.

$$\frac{\partial c}{\partial t} + rs\frac{\partial c}{\partial S} + \frac{S^2 V}{2}\frac{\partial^2 c}{\partial S^2} + V\frac{\partial c}{\partial I} + \alpha(V)\frac{\partial c}{\partial V} + \frac{\beta^2(V)}{2}\frac{\partial^2 c}{\partial V^2} + \rho s\sqrt{V}\beta(v)\frac{\partial^2 c}{\partial S \partial V} - rc = 0 \tag{6}$$

Under the assumption of perpetuity, the option price is independent of the time t, so the time option is dependent on the stock price (underlying asset price), volatility, and the variance budget. So, the option price has become:

$$C_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-rt}\max(S_t - K, 0)\right] \tag{7}$$

More than that, the initial price of the timer option is equal to:

$$C_0 = \mathbb{E}^{\mathbb{Q}}\left[\max\left(S_0 e^{\left(W_{\tilde{B}} - \frac{B}{2}\right)} - K e^{-r\tau_B}, 0\right)\right] \tag{8}$$

, where the Brownian motion follows:

$$W_{\tilde{u}} = \int_0^u \sqrt{V_t}\left(\sqrt{1-\rho^2}\, dW_T^1 + \rho\, dW_t^2\right) \tag{9}$$

Assume that the variance process is:

$$dV_t = \alpha(V_t)\, dt + \beta(V_t)\, dW_t^2 \tag{10}$$

, the risky asset is modeled under the risk neutral measure $\mathbb{Q}$ as

$$dS_t = rS_t\, dt + S_t\sqrt{V_t}\left(\sqrt{1-\rho^2}\, dW_t^1 + \rho\, dW_t^2\right) \tag{11}$$

where r is the risk-free rate, W_t^1, W_t^2 is the standard Brownian motion. In a general stochastic volatility model given by (29) and (30), the value at time t_x of the option has the following representation

$$S_{t_x} = S_0 \exp\left\{rT + a_{t_x} + \sqrt{b_{t_x}} Z\right\} \tag{12}$$

where a_{t_x} and b_{t_x} are defined by

$$a_{t_x} = \rho(f(V_{t_x}) - f(V_0)) - \rho H_{t_x} - \frac{1}{2}V_{acct_x}, \quad b_{t_x} = (1-\rho^2)V_{acct_x} \tag{13}$$

Given that

$$H_{t_x} = \int_0^{t_x} h(V_{t_x})\, dt \tag{14}$$

$$V_{acct_x} = \int_0^{t_x} V_{t_x}\, dt \tag{15}$$

and where $Z \sim \mathcal{N}(0,1)$ independent of V_{t_x}, H_{t_x} and V_{acct_x}, then f and h are defined by

$$f(v) = \int_0^v \frac{\sqrt{z}}{\beta(z)}\, dz \tag{16}$$

$$h(v) = \alpha(v)f'(v) + \frac{1}{2}\beta^2(v)f''(v) \tag{17}$$

In a general stochastic volatility model given by (19) and (20), the price of a timer call option can be calculated as

$$C_0 = \mathbb{E}\left[S_0 e^{a_\tau + \frac{(1-\rho^2)}{2}V} N(d_1) - Ke^{-r\tau} N(d_2)\right] \tag{18}$$

where $a_\tau = \rho(f(V_\tau) - f(V_0)) - \rho H_\tau - \frac{1}{2}V$ and

$$d_1 = \frac{\log\left(\frac{S_0}{K}\right) + r\tau + a_\tau + \frac{(1-\rho^2)}{2}V}{\sqrt{(1-\rho^2)V}}, \tag{19}$$

$$d_2 = d_1 - \sqrt{(1-\rho^2)V}. \tag{20}$$

3.2 Joint Distribution

Given the Black-Scholes formula for the timer option, then need to simulate the $(V_T, V_{\text{acc}}, H_T)$ to find the exact equation for pricing the timer option.

According to the definition of the timer option, simulate the stochastic volatility process V_t of the timer option through the general stochastic volatility model. In this case, then

$$\tau \triangleq \tau(B) = \inf\left\{t; \int_0^t V_s\, ds \in (0,\infty)\right\}, \tag{21}$$

This formula is the first passage time of the integrated functional of V_s to the fixed level $B \in (0,\infty)$, then the law of the $(V_\tau, V_{\text{acc}}, H_\tau)$, that is

$$(\tau_B, V_{\tau(B)}, H_\tau) \overset{\text{law}}{\sim} \left(\int_0^B \frac{1}{X_s}\, ds, X_B, \int_0^B \frac{h(X_s)}{X_s}\, ds\right) \tag{22}$$

, where X_t is governed by the SDE.

$$\begin{cases} df(X_t) = \frac{h(X_t)}{X_t}\, dt + dB_t, \\ X_0 = V_0, \end{cases} \tag{23}$$

$f(v)$ and $h(v)$ are defined as

$$f(v) = \int_0^v \frac{\sqrt{z}}{\beta(z)}\, dz, \tag{24}$$

$$h(v) = \alpha(v) f'(v) + \frac{1}{2}\beta^2(v) f''(v), \tag{25}$$

3.3 Heston Model

In this model,

$$\begin{aligned} \alpha(V_t) &= \kappa(\theta - V_t) \\ \beta(V_t) &= \gamma\sqrt{V_t} \end{aligned} \tag{26}$$

With substitution, S_t and V_t are assumed to satisfy the following stochastic differential equations:

$$dS_t = rS_t\,dt + \sqrt{V_t}S_t\left(\sqrt{1-\rho^2}\,dW_t^1 + \rho\,dW_t^2\right) \tag{27}$$

$$dV_t = \kappa(\theta - V_t)\,dt + \gamma\sqrt{V_t}\,dW_t^2 \tag{28}$$

where r is the risk-free rate, where W_t^1, W_t^2 is the standard Brownian motion with correlation coefficient ρ.

Then substitute $\alpha(s)$, $\beta(s)$ into $f(s)$ and $h(s)$. Therefore, get

$$f(v) = \int_0^v \frac{\sqrt{V_t}}{\gamma\sqrt{V_t}}\,dV_t = \frac{v}{\gamma} \tag{29}$$

$$h(v) = \kappa(\theta - V_t)f'(v) + \frac{1}{2}\gamma^2 V_t \cdot 0 = \frac{\kappa(\theta - V_t)}{\gamma} \tag{30}$$

So, the Y_t is governed by these two formulas

$$\begin{cases} dY_t = \left(\frac{\kappa\theta}{Y_t} - \kappa\right)dt + \gamma d\hat{W}_t \\ Y_0 = V_0, \end{cases} \tag{31}$$

The joint law in the Heston model has become

$$(\tau_B, V_{\tau(B)}, H_\tau) \overset{\text{law}}{\sim} \left(\int_0^B \frac{1}{Y_s}\,ds, Y_B, \frac{\kappa\theta\tau - \kappa B}{\gamma}\right) \tag{32}$$

3.4 Hull-White Model

In this model,

$$\begin{aligned} \alpha(V_t) &= \alpha V_t \\ \beta(V_t) &= \gamma V_t \end{aligned} \tag{33}$$

Under the Hull–White model, S_t and V_t are assumed to satisfy the following stochastic differential equations:

$$dS_t = rS_t\,dt + \sqrt{V_t}S_t\left(\sqrt{1-\rho^2}\,dW_t^1 + \rho\,dW_t^2\right) \tag{34}$$

$$dV_t = \alpha V_t\,dt + \gamma V_t\,dW_t^2 \tag{35}$$

Then substitute the $\alpha(s)$, $\beta(s)$ to $f(s)$ and $h(s)$. So,

$$f(v) = \int_0^v \frac{\sqrt{V_t}}{\gamma V_t}\,dV_t = \frac{1}{\beta}\int_0^v \frac{1}{\sqrt{V_t}}\,dV_t = \frac{2\sqrt{v}}{\gamma} \tag{36}$$

$$h(v) = \alpha(v)f'(v) + \frac{1}{2}\beta^2(v)f''(v) = \frac{av}{\gamma\sqrt{v}} + \frac{1}{2}\gamma^2 v^2\left(\frac{1}{2\gamma}v^{-\frac{3}{2}}\right) = \left(\frac{\alpha}{\gamma} - \frac{\gamma}{4}\right)\sqrt{v}. \tag{37}$$

Therefore,

$$\begin{cases} dY_t = \left(\frac{2\alpha}{\gamma^2} - \frac{1}{2}\right) \frac{1}{Y_t}\, dt + d\hat{W}_t, \\ Y_0 = \frac{2}{\gamma}\sqrt{V_0}, \end{cases} \tag{38}$$

And the joint law in the Hull-White model has become

$$(\tau_B, V_{\tau(B)}, H_\tau) \overset{\text{law}}{\sim} \left(\frac{4}{\gamma^2} \int_0^B \frac{1}{Y_s^2}\, ds, \frac{\gamma^2}{4} Y_B^2, \left(\frac{2\alpha}{\gamma^2} - \frac{1}{2}\right) \int_0^B \frac{1}{Y_s}\, ds \right) \tag{39}$$

In the Hull-White Model, the price of a timer call option can be simplified as

$$C_0 = \mathbb{E}\left[S_0 e^{\left(a_T + \frac{(1-\rho^2)}{2} B\right)} N(d_1) - K e^{-rT} N(d_2) \right] \tag{40}$$

where N is the cumulative distribution function of a standard normal distribution.

4 Delta Hedging with Profit and Loss

Equity options are subject to the risks associated with both the direction of the asset price and its volatility. The Black-Scholes hedging principle suggests that delta hedging can mitigate the risk from price movements. However, delta hedging often falls short in real-world settings due to numerous deviations from the assumptions of the Black-Scholes model. Challenges include inaccurate volatility estimates, sudden shifts in asset prices, limited liquidity, and the significant costs associated with frequent trading. [Kwok and Zheng, 2022] illustrate that employing delta hedging with a selected time-dependent hedge volatility can produce a profit and loss (P&L) that correlates with both the realized variance and the cash gamma position, which is calculated as the option gamma multiplied by the square of the asset price.

Consider an option trader who sells an option at time zero, priced using the current market's implied volatility σ_0^i. The time-t price of the option is given by $V_t = V(S_t, t; \sigma_t^i)$, where σ_t^i represents the implied volatility derived from traded option prices. For its remaining duration, the option is delta-hedged with a chosen time-dependent hedge volatility σ_t^h.

The instantaneous volatility process is denoted by σ_t. The assumed dynamics of S_t permit no jumps. The time-dependent implied volatility σ_t^i is derived from traded option prices at varying times. To summarize, for $t \in [0, T]$, there are three volatilities: 1. σ_t: the instantaneous volatility of S_t. 2. σ_t^i: the implied volatility from market option prices. 3. σ_t^h: the time-dependent hedge volatility adopted by the hedger.

Upon selling an option, seeks to create a replication portfolio composed of units of the underlying asset and a money market account. Initially, at time t_0, the proceeds from the option sale, V_0^*, are allocated to buy Δ_0^h units of the underlying asset and deposit the remaining funds, $V_0^* - \Delta_0^h S_0$, into the

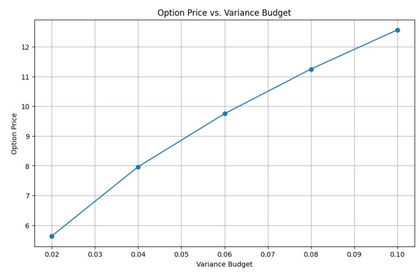

Figure 1: Hull-White Variance Budget

money market account. The hedge ratio is then dynamically adjusted over time, dependent on a selected volatility σ_t^h. This hedge ratio is calculated as follows: $\Delta_t^h = \frac{\partial}{\partial S} V(S_t, t; \sigma_t^h)$. Replication continues with the remaining amount, $V_t^* - \Delta_t^h S_t$, being placed in the market money account M_t, where $M_t = V_t^* - \Delta_t^h S_t$.

The delta-hedged portfolio consists of a short position in one unit of the option and a long position in Δ_t^h units of the underlying asset, along with a money market account holding $V_t^* - \Delta_t^h S_t$. The profit and loss (P&L) of the delta-hedged portfolio over the infinitesimal time interval $[t, t + dt]$ comprises the following three components:

- The change in the option's value: $-dV_t^*$;
- P&L from the dynamic position of the underlying asset and dividend income: $\Delta_t^h (dS_t + qS_t\, dt)$;
- Risk-free interest earned from the money market account: $r(V_t^* - \Delta_t^h S_t)\, dt$.

Consequently, we have the following formula for the total P&L at maturity T ([Carr and Madan, 1998]):

$$\Pi_T = e^{rT} \left[V(S_0, 0; \sigma_0^i) - V(S_0, 0; \sigma_0^h) \right] + \int_0^T e^{r(T-t)} \frac{\Gamma_t^h S_t^2}{2}\, dt$$

Since the factor $\Gamma_t^h S_t^2$ appears as a cash term in Π_T, it is commonly called the *cash gamma* or *dollar gamma*.

5 Sensitivity Analysis

Sensitivity analysis is a tool to determine which factors affect a model's output most, contribute to assessing a model's reliability and its reaction to variable changes.

The sensitivity analysis conducted under the Hull-White model demonstrates the relationship between option pricing and market parameters such as the variance budget and interest rates. Figure5 indicates a positive correlation

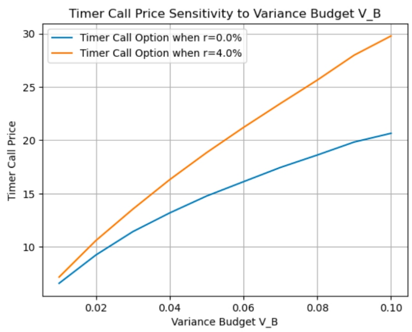

Figure 2: Heston Variance Budget

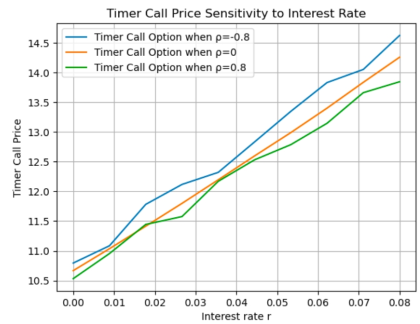

Figure 3: Heston Interest

between the option price and the variance budget: as the variance budget increases, the option price also rises. This suggests that higher uncertainty in the market, as captured by the variance budget, translates into a higher premium for the option.

The sensitivity analysis of option pricing within the Heston model framework depicted in the charts illustrates how option prices react to changes in variance budget and interest rates. For the variance budget, the upward trending graph indicates that an increase in market uncertainty, represented by a larger variance budget, leads to higher option prices, reflecting the higher risk premium required by option sellers. On contrast, the interest rate sensitivity chart exhibits a nonlinear relationship where option prices increase sharply with lower interest rates and plateau as interest rates rise. This pattern suggests a declining sensitivity to rates at higher levels, which could be due to the decreased impact of discounting future cash flows.

6 Experimental Result

In the end, we utilized the dynamic hedging strategy and construct a portfolio consisting of the AAPL and JPM stock and their corresponding timer options. The portfolio initiates with an initial capital of \$1,000,000. Concurrently, a variance path is created for gauging the volatility of the options, utilizing parameters such as kappa, theta, and Delta. With updates are made to tau, we employ the prevailing stock prices along with the Black-Scholes model's parameters, the options' delta δ and intrinsic value are computed.

To perpetuate a dynamic equilibrium within the hedge, the hedge ratio is re-calibrated on a daily basis, leading to a recalculated stock position. The value of the portfolio is revised to factor in the adjustments in stock holdings and the valuation of the options. The cash balance is also consistently updated to reflect the profits or losses stemming from fluctuations in stock prices and the assessment of the options' worth.

Figure 4 shows the fluctuating 'Daily Stock Position' of JPM under Heston

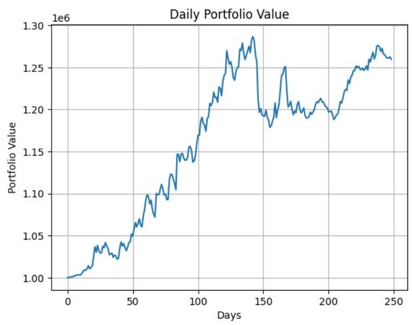

Figure 4: JPM Heston Portfolio Value

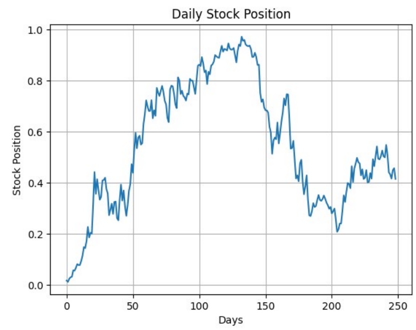

Figure 5: AAPL Heston Portfolio Value

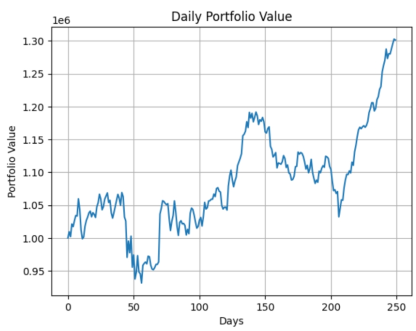

Figure 6: JPM H-W Portfolio Value

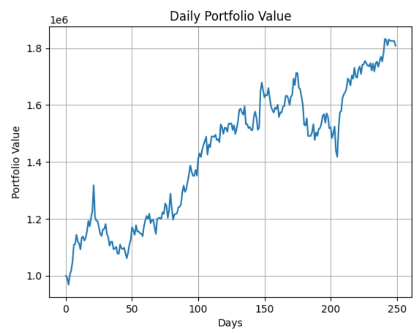

Figure 7: AAPL H-W Portfolio Value

model. It starts just below 0.6, once dips to around 0.3 twice, but eventually increases to approximately 0.9. It reflects the adjustments in the number of shares held as part of the dynamic hedging strategy, reacting to changes in the underlying stock price or option delta. In Figure 5, it depicts the 'Daily Portfolio Value' of AAPL over 252 trading days within one year. It is volatile but ultimately trends upwards, surpassing $1.2 million. The upward trajectory, especially noticeable in the latter half of the period, suggests successful hedging that provided positive returns. The resulting growth in portfolio value, indicates an effective hedging strategy that capitalizes on market movements while mitigating risk.

Similarly, for Hull-White method, we construct the portfolio of one underlying stock and the corresponding timer option for both JPM and AAPL. The performance for both portfolio also show a similar upward trend, generating positive return in 2023.

7 Conclusion

In conclusion, this study presents comprehensive analyses and simulations for pricing timer options within stochastic volatility models, particularly focusing on the Heston and Hull-White frameworks. It integrates advanced replication strategies and employs control variates to enhance pricing accuracy. The study underscores the effectiveness of dynamic delta hedging strategies in managing risks and optimizing portfolio value over time, as evidenced by the positive returns on investment in AAPL and JPM stock portfolios. The findings affirm the significance of sensitivity analysis in modeling and the impactful role of volatility parameters on option pricing and hedging effectiveness.

For the future work, there may also be an interest in extending the models to account for more complex market factors or exploring the influence of other types of stochastic processes. Additionally, considering the computational complexity of the models, efforts could be made to refine the algorithms for greater efficiency and scalability. Finally, the study opens doors for comparative analysis with other volatility models, offering a broader perspective on option pricing and hedging strategies.

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A Appendix

A.1 Proof: Pricing of Timer Option

Proof. Recall that f and h are defined as above, by Ito's lemma to $f(V_t)$, then

$$df(V_t) = f'(V_t)\left[\alpha(V_t)\,dt + \beta(V_t)\,dW_t^2\right] + \frac{1}{2}\beta^2(V_t)f''(V_t)\,dt = h(V_t)\,dt + \sqrt{V_t}\,dW_t^2 \tag{41}$$

Then, do the integration from 0 to T on both sides of the above equation, then

$$\int_0^T \sqrt{V_t}\,dW_t^2 = f(V_T) - f(V_0) - \int_0^T h(V_t)\,dt, \tag{42}$$

Since

$$d(\ln S_t) = \left(r - \frac{V_t}{2}\right)dt + \rho\sqrt{V_t}\,dW_t^2 + \sqrt{1-\rho^2}\sqrt{V_t}\,dW_t^1, \tag{43}$$

Now plug equation (44) into (45), can get

$$\ln\frac{S_T}{S_0} = \int_0^T (r - \frac{V_t}{2})\,dt + \sqrt{1-\rho^2}\int_0^T \sqrt{V_t}\,dW_t^1 + \rho\left[f(V_T) - f(V_0) - \int_0^T h(V_t)\,dt\right], \tag{44}$$

Then can get

$$S_T = S_0 \exp\left(rT + a_T + \sqrt{1-\rho^2}\int_0^T \sqrt{V_t}\,dW_t^1\right), \tag{45}$$

And sort out the formula

$$S_T = S_0 \exp\left(rT + a_T + \sqrt{b_T}Z\right), Z \sim \mathcal{N}(0,1), \tag{46}$$

where a_T and b_T are as defined earlier. □

A.2 Proof: Timer Call Option Pricing

Proof. The proof is very similar to the standard European case. In this case $\xi_\tau = V$, and τ is random. Now condition on (τ, H_τ, V_τ) and the price of the timer option is obtained by

$$C_0 = \mathbb{E}\left[\mathbb{E}\left[e^{-r\tau}(S_\tau - K)^+ \mid \tau, H_\tau, V_\tau\right]\right]. \tag{47}$$

The conditional expectation is the Black Scholes formula for the call option, then

$$C_0 = \mathbb{E}\left[C_{\mathrm{BS}}\left(\tilde{S}_0, K, r, \sqrt{\frac{V}{\tau}}, \tau\right)\right] \tag{48}$$

where C_{BS} is given by (13) with $\tilde{S}_0 = S_0 e^{a_\tau + \frac{b_\tau}{2}}$, where a_τ and b_τ are defined by (22) in Lemma 2.1 and $b_\tau = \frac{(1-\rho^2)}{2}V$. □