Literature Review of Weapon Target Assignment problem and a Branch-Bound-Cut algorithm on it

Guanda Li joint work with Liang Chen and Yu-hong Dai

Institute of Computational Mathematics and Scientific/Engineering Computing,
Academy of Mathematics and Systems Science,
Chinese Academy of Sciences

December 20, 2022

Contents

Background

derivation

- **Weapon Target Assignment problem**
- m n .
- **NP-C**
- 80 80 16.2
-

history

basic formulation

- $I = \{1, \cdots, m\}$, .
- $J = \{1, \cdots, n\}$, .
- $p_{ij} \in [0,1]$, i j .
- a_j, j .
- x_{ij}, i j.

$$\min \quad \sum_{j=1}^{n} a_j \left(\prod_{i=1}^{m} (1 - p_{ij})^{x_{ij}} \right) \tag{1}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} x_{ij} \leq 1 \quad \forall\, i \in I, \tag{2}$$

$$x_{ij} \in \{0, 1\} \quad \forall\, j \in J,\ i \in I. \tag{3}$$

Literature Review

static WTA model 1

dynamic WTA model 1

Semsor WTA model

useful algorithms

basic formulation

- $I = \{1, \cdots, m\}$, .
- $J = \{1, \cdots, n\}$, .
- $p_{ij} \in [0, 1]$, i j .

- a_j, j .
- x_{ij}, i j.

$$\min \quad \sum_{j=1}^{n} a_j \left(\prod_{i=1}^{m} (1 - p_{ij})^{x_{ij}} \right) \tag{4}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} x_{ij} \le 1 \quad \forall\, i \in I, \tag{5}$$

$$x_{ij} \in \{0, 1\} \quad \forall\, j \in J,\ i \in I. \tag{6}$$

formulation

$$\prod_{i=1}^{m}(1-p_{ij}x_{ij}) \iff \prod_{i=1}^{m}(1-p_{ij})^{x_{ij}}, x \in \{0,1\}$$

- x $0,1$
-

$$f_j(x) = \prod_{i=1}^{m} (1 - p_{ij})^{x_{ij}} \quad \forall\, j \in J$$

Hessian H

$$\frac{\partial f_j(x)}{\partial x_{aj} \partial x_{bj}} = \ln(1 - p_{aj}) \ln(1 - p_{bj}) f_j(x)$$

$$H = f(x) \begin{bmatrix} \ln(1-p_{1j})\ln(1-p_{1j}) & \ln(1-p_{1j})\ln(1-p_{2j}) & \cdots & \ln(1-p_{1j})\ln(1-p_{mj}) \\ \ln(1-p_{2j})\ln(1-p_{1j}) & \ln(1-p_{2j})\ln(1-p_{2j}) & \cdots & \ln(1-p_{2j})\ln(1-p_{mj}) \\ \vdots & \vdots & \ddots & \vdots \\ \ln(1-p_{mj})\ln(1-p_{1j}) & \ln(1-p_{mj})\ln(1-p_{2j}) & \cdots & \ln(1-p_{mj})\ln(1-p_{mj}) \end{bmatrix}$$

$$l = \begin{bmatrix} \ln(1-p_{1j}) & \ln(1-p_{2j}) & \cdots & \ln(1-p_{mj}) \end{bmatrix}$$

Hassien

$$H = f(x) l \cdot l^T$$

0

Basic Idea

How to use column generation in WTA Problem

- S
- m j j j 2^m
- 8 1 3 6 $s_{[1,0,1,0,0,1,0,0]}$ $|S| = 2^8 = 256$
- n_{si} 0-1 s i. $n_{s1} = 1,\ n_{s2} = 0$
- $q_{js} = a_j \prod_{i=1}^{m}(1 - n_{si} \cdot p_{ij})$ s j q_{3s} 1,3,6 3 j

transformed formulation

$$
\begin{aligned}
\min \quad & \sum_{j=1}^{n} \sum_{s=0}^{2^m-1} q_{js} y_{js} & \\
\text{s.t.} \quad & \sum_{j=1}^{n} \sum_{s=0}^{2^m-1} n_{si} y_{js} \leq 1 & \forall\, i \in I \\
& \sum_{s=0}^{2^m-1} y_{js} = 1 & \forall\, j \in J \\
& y_{js} \in \{0,1\} & \forall\, j \in J,\; s \in S
\end{aligned}
$$

- $I = \{1, \cdots, m\}$, .
- $J = \{1, \cdots, n\}$, .
- $S = \{1, \cdots, 2^m\}$, .
- n_{si} s i
- y_{js} 0-1 j s
- q_{js} s j

transformed formulation continuous

$$
\begin{aligned}
\min \quad & \sum_{j=1}^{n} \sum_{s=0}^{2^m-1} q_{js} y_{js} && \\
\text{s.t.} \quad & \sum_{j=1}^{n} \sum_{s=0}^{2^m-1} n_{si} y_{js} \le 1 && \forall\ i \in I \\
& \sum_{s=0}^{2^m-1} y_{js} = 1 && \forall\ j \in J \\
& y_{js} \in \{0,1\} && \forall\ j \in J,\ s \in S
\end{aligned}
$$

-
-
-

LP Relaxition

$$
\begin{aligned}
\min \quad & \sum_{j=1}^{n} \sum_{s=0}^{2^m-1} q_{js} y_{js} & \\
\text{s.t.} \quad & \sum_{j=1}^{n} \sum_{s=0}^{2^m-1} n_{si} y_{js} \leq 1 & \forall\, i \in I \\
& \sum_{s=0}^{2^m-1} y_{js} = 1 & \forall\, j \in J \\
& y_{js} \geq 0 & \forall\, j \in J,\; s \in S
\end{aligned}
$$

- $y_{js} \geq 0$
- $n \times 2^m$
-

Dual Problem

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{m} u_i + \sum_{j=1}^{n} v_j \\
\text{s.t.} \quad & \sum_{i=1}^{m} x_{is} u_i + v_j \le q_{js} \quad (P) \\
& u_i \le 0, \quad v_j \ \text{free}
\end{aligned}
$$

-
- u^*, v^*
- $\sum_{i=1}^{m} x_{is} u_i + v_j > q_{js}$

Restricted Dual Problem

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{m} u_i + \sum_{j=1}^{n} v_j \\
\text{s.t.} \quad & \sum_{i=1}^{m} x_{is} u_i + v_j \leq q_{js} \quad (P) \\
& u_i \leq 0, \quad v_j \text{ free}
\end{aligned}
$$

subproblem

$$
\begin{aligned}
\min \quad & a_j \prod_{i=1}^{m} (1 - p_{ij} x_{is}) - \sum_{i=1}^{m} x_{is} u_i - v_j \\
\text{s.t.} \quad & j \in J, \quad s \in S
\end{aligned}
$$

- $\sum_{i=1}^{m} x_{is} u_i + v_j > q_{js}$
- q_{js}
- j

transformed model

:

$$\sum_{j=1}^{n} a_j \left(\prod_{i=1}^{m} (1 - p_{ij})^{x_{ij}} \right)$$

η_j $\prod_{i=1}^{m} (1 - p_{ij})^{x_{ij}}$

$$\min \quad \sum_{j=1}^{n} a_j \eta_j \tag{7}$$

$$\text{s.t.} \quad \eta_j \geq \prod_{i=1}^{m} (1 - p_{ij})^{x_{ij}}, \quad \forall\, j \in J \tag{8}$$

$$\sum_{j=1}^{n} x_{ij} \leq 1 \quad \forall\, i \in I, \tag{9}$$

$$x_{ij} \in \{0, 1\} \quad \forall\, j \in J \quad i \in I. \tag{10}$$

Basic idea of outer approximation

$f(x)$ x^*

$$f(x) \geq f(x^*) + \nabla f(x^*)(x - x^*)$$

$\eta \geq f(x)$

$$\eta \geq f(x) \geq f(x^*) + \nabla f(x^*)(x - x^*)$$

constraint formulation of outer approximation

$f_j(x) = \prod_{i=1}^{m}(1-p_{ij})^{x_{ij}} \quad j \in J \qquad \bar{x}$

$$\nabla f(\bar{x})(x-\bar{x}) = f(\bar{x})\sum_{i=1}^{m}\ln(1-p_{ij})(x_{ij}-\bar{x_{ij}})$$
$$= f(\bar{x})\sum_{i=1}^{m}\ln(1-p_{ij})x_{ij} - f(\bar{x})\sum_{i=1}^{m}\ln(1-p_{ij})\bar{x_{ij}}$$

$\bar{x}$

outer approximation model

X

$$\min \quad \sum_{j=1}^{n} a_j \eta_j \tag{11}$$

$$\text{s.t.} \quad \eta_j \geq f(\bar{x}) \sum_{i=1}^{m} \ln(1 - p_{ij})(x_{ij} - \bar{x_{ij}}) + f(\bar{x}), \quad \forall\, j \in J,\ \bar{x} \in X \tag{12}$$

$$\sum_{i=1}^{m} x_{ij} \leq 1 \quad \forall\, j \in J, \tag{13}$$

$$x_{ij} \in \{0, 1\} \quad \forall\, j \in J \quad i \in I. \tag{14}$$

Idea of outer approximation method

$$\min \quad \sum_{j=1}^{n} a_j \eta_j$$

$$\text{s.t.} \quad \eta_j \geq f(\bar{x}) \sum_{i=1}^{m} \ln(1-p_{ij})(x_{ij} - \bar{x_{ij}}) + f(\bar{x}), \quad j \in J, \ \bar{x} \in X$$

$$\sum_{i=1}^{m} x_{ij} \leq 1 \quad j \in J, \quad x_{ij} \in \{0,1\} \quad j \in J \quad i \in I$$

12

Idea

1. 12 x, η
2. 12
3. 12 2

$$
\begin{aligned}
\min \quad & \sum_{j=1}^{n} a_j \eta_j \\
\text{s.t.} \quad & \eta_j \geq f(\bar{x}) \sum_{i=1}^{m} \ln(1 - p_{ij})(x_{ij} - \bar{x_{ij}}) + f(\bar{x}), \quad j \in J,\ \bar{x} \in X \\
& \sum_{i=1}^{m} x_{ij} \leq 1 \quad j \in J, \quad x_{ij} \in \{0, 1\} \quad j \in J \quad i \in I
\end{aligned}
$$

-
- η_j
- $\eta < f_j(x)$

Branch-Bound-Cut Algorithm

$$\min \quad \sum_{j=1}^{n} a_j \eta_j \tag{15}$$

$$\text{s.t.} \quad \eta_j \geq \prod_{i=1}^{m} (1 - p_{ij})^{x_{ij}}, \quad \forall\, j \in J \tag{16}$$

$$\sum_{j=1}^{n} x_{ij} \leq 1 \quad \forall\, i \in I, \tag{17}$$

$$x_{ij} \in \{0, 1\} \quad \forall\, j \in J \quad i \in I. \tag{18}$$

-

Basic Idea

$$\min \quad \sum_{j=1}^{n} a_j \eta_j \tag{19}$$

$$\text{s.t.} \quad \eta_j \geq f(\bar{x}) \sum_{i=1}^{m} \ln(1 - p_{ij})(x_{ij} - \bar{x_{ij}}) + f(\bar{x}), \quad \forall\, j \in J,\ \bar{x} \in X \tag{20}$$

$$\sum_{i=1}^{m} x_{ij} \leq 1 \quad \forall\, j \in J, \tag{21}$$

$$x_{ij} \in \{0, 1\} \quad \forall\, j \in J \quad i \in I. \tag{22}$$

-
-
-

a potential improvement

$$\min \quad \sum_{j=1}^{n} a_j \eta_j \tag{23}$$

$$\text{s.t.} \quad \eta_j \geq f(\bar{x}) \sum_{i=1}^{m} \ln(1-p_{ij})(x_{ij} - \bar{x_{ij}}) + f(\bar{x}), \quad \forall\, j \in J,\ \bar{x} \in X \tag{24}$$

$$\sum_{i=1}^{m} x_{ij} \leq 1 \quad \forall\, j \in J, \tag{25}$$

$$x_{ij} \in \{0, 1\} \quad \forall\, j \in J \quad i \in I. \tag{26}$$

-
- LP
-
-

computational result of branch-bound-cut algorithm

Numerical Experiment Environment

only use the feasible solution cut

also use the fractional silution cut

camparition of two situations

Future Research Plan

Acknowledgement and Reference

Thank you!