1. 交点

   Problem.

   (input)

   射线方程: $\boldsymbol{x}(t) = \boldsymbol{x}_0 + t\hat{\boldsymbol{d}}$

   曲面方程: $f(\boldsymbol{x}) = 0$

   - Answer

     交点: $f\left(\boldsymbol{x}_0 + t\hat{\boldsymbol{d}}\right) = 0$

     求解 t 即可. $t = dis$ 即射线出发点与交点的距离.

   Include.

   (a) 射线、平面交点

   (input) 平面方程: $\boldsymbol{a}^T\boldsymbol{x} = b$

$$
\begin{aligned}
dis &= \frac{\boldsymbol{a}^T\boldsymbol{x}_0 - b}{\boldsymbol{a}^T\hat{\boldsymbol{d}}} \\
&= \frac{\left(\sum\limits_{i=1}^{\dim} a_i o_i\right) - b}{\sum\limits_{i=1}^{\dim} a_i d_i} \qquad \text{(分量形式)}
\end{aligned}
$$

   Proof.

$$
\begin{aligned}
\Rightarrow \boldsymbol{a}^T\left(\boldsymbol{x}_0 + t\hat{\boldsymbol{d}}\right) - b = 0 \\
t\boldsymbol{a}^T\hat{\boldsymbol{d}} + \left(\boldsymbol{a}^T\boldsymbol{x}_0 - b\right) = 0 \\
t &= \frac{\boldsymbol{a}^T\boldsymbol{x}_0 - b}{\boldsymbol{a}^T\hat{\boldsymbol{d}}} \\
&= \frac{\left(\sum\limits_{i=1}^{\dim} a_i o_i\right) - b}{\sum\limits_{i=1}^{\dim} a_i d_i} \qquad \text{(分量形式)}
\end{aligned}
$$

   i. 射线、平面图形交点

   A. 射线、三角形交点

   (input) 三角形顶点 $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$

$$
\begin{aligned}
dis &= \frac{(\boldsymbol{x}_0 - \boldsymbol{v}_1) \times \boldsymbol{e}_1 \cdot \boldsymbol{e}_2}{\hat{\boldsymbol{d}} \times \boldsymbol{e}_2 \cdot \boldsymbol{e}_1} \\
u &= \frac{\hat{\boldsymbol{d}} \times \boldsymbol{e}_2 \cdot (\boldsymbol{x}_0 - \boldsymbol{v}_1)}{\hat{\boldsymbol{d}} \times \boldsymbol{e}_2 \cdot \boldsymbol{e}_1} \\
v &= \frac{(\boldsymbol{x}_0 - \boldsymbol{v}_1) \times \boldsymbol{e}_1 \cdot \hat{\boldsymbol{d}}}{\hat{\boldsymbol{d}} \times \boldsymbol{e}_2 \cdot \boldsymbol{e}_1} \\
\boldsymbol{e}_1 &= \boldsymbol{v}_2 - \boldsymbol{v}_1 \\
\boldsymbol{e}_2 &= \boldsymbol{v}_3 - \boldsymbol{v}_1
\end{aligned}
$$

   有交点条件: $u \geq 0, v \geq 0, u + v \leq 1$

   Proof.

   射线三角形交点:

$$
\begin{aligned}
\boldsymbol{x}_0 + t\hat{\boldsymbol{d}} &= (1-u-v)\,\boldsymbol{v}_1 + u\boldsymbol{v}_2 + v\boldsymbol{v}_3 \\
\Rightarrow \boldsymbol{x}_0 - \boldsymbol{v}_1 &= \begin{pmatrix} -\hat{\boldsymbol{d}} & \boldsymbol{v}_2 - \boldsymbol{v}_1 & \boldsymbol{v}_3 - \boldsymbol{v}_1 \end{pmatrix} \begin{pmatrix} t \\ u \\ v \end{pmatrix} \\
&= \begin{pmatrix} -\hat{\boldsymbol{d}} & \boldsymbol{e}_1 & \boldsymbol{e}_2 \end{pmatrix} \begin{pmatrix} t \\ u \\ v \end{pmatrix} \qquad (\boldsymbol{e}_1 = \boldsymbol{v}_2 - \boldsymbol{v}_1, \boldsymbol{e}_2 = \boldsymbol{v}_3 - \boldsymbol{v}_1)
\end{aligned}
$$

$$
\begin{aligned}
t &= \frac{\left|(\boldsymbol{x}_0 - \boldsymbol{v}_1) \quad \boldsymbol{e}_1 \quad \boldsymbol{e}_2\right|}{\left|-\hat{\boldsymbol{d}} \quad \boldsymbol{e}_1 \quad \boldsymbol{e}_2\right|} \\
u &= \frac{\left|-\hat{\boldsymbol{d}} \quad (\boldsymbol{x}_0 - \boldsymbol{v}_1) \quad \boldsymbol{e}_2\right|}{\left|-\hat{\boldsymbol{d}} \quad \boldsymbol{e}_1 \quad \boldsymbol{e}_2\right|} \\
v &= \frac{\left|-\hat{\boldsymbol{d}} \quad \boldsymbol{e}_1 \quad (\boldsymbol{x}_0 - \boldsymbol{v}_1)\right|}{\left|-\hat{\boldsymbol{d}} \quad \boldsymbol{e}_1 \quad \boldsymbol{e}_2\right|}
\end{aligned}
$$

$$
\begin{aligned}
\left|\boldsymbol{a} \quad \boldsymbol{b} \quad \boldsymbol{c}\right| &= \boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c} \\
&= -\boldsymbol{a} \times \boldsymbol{c} \cdot \boldsymbol{b} \qquad (\text{混合积公式}) \\
t &= \frac{(\boldsymbol{x}_0 - \boldsymbol{v}_1) \times \boldsymbol{e}_1 \cdot \boldsymbol{e}_2}{\hat{\boldsymbol{d}} \times \boldsymbol{e}_2 \cdot \boldsymbol{e}_1} \\
u &= \frac{\hat{\boldsymbol{d}} \times \boldsymbol{e}_2 \cdot (\boldsymbol{x}_0 - \boldsymbol{v}_1)}{\hat{\boldsymbol{d}} \times \boldsymbol{e}_2 \cdot \boldsymbol{e}_1} \\
v &= \frac{(\boldsymbol{x}_0 - \boldsymbol{v}_1) \times \boldsymbol{e}_1 \cdot \hat{\boldsymbol{d}}}{\hat{\boldsymbol{d}} \times \boldsymbol{e}_2 \cdot \boldsymbol{e}_1}
\end{aligned}
$$

B. 射线、圆交点
Note.
先与平面计算交点, 再看交点与圆心的距离是否小于半径.

C. 射线、任意平面图案交点
Note.
先与平面计算交点, 再看交点是否在图案内部.

(b) 射线、球面交点
(input) 球方程: $\|\boldsymbol{x} - \boldsymbol{c}\|_2 - R = 0$

$$
\begin{aligned}
dis &= \frac{-b \pm \sqrt{\Delta}}{2a} \\
\Delta &= b^2 - 4ac \\
&= \left(2\hat{\boldsymbol{d}}^T (\boldsymbol{x}_0 - \boldsymbol{c})\right)^2 - 4\left(\hat{\boldsymbol{d}}^T \hat{\boldsymbol{d}}\right)\left((\boldsymbol{x}_0 - \boldsymbol{c})^T (\boldsymbol{x}_0 - \boldsymbol{c}) - R^2\right) \\
&= 4\left(\sum_{i=1}^{\dim} d_i (o_i - c_i)\right)^2 - 4\left(\sum_{i=1}^{\dim} d_i^2\right)\left(\left(\sum_{i=1}^{\dim} (o_i - c_i)^2\right) - R^2\right) \qquad (\text{分量形式})
\end{aligned}
$$

若 $\Delta \geq 0$ 有交点; 若 $\Delta < 0$ 无交点.
Proof.

$$
\begin{aligned}
&\Rightarrow (\boldsymbol{x} - \boldsymbol{c})^T (\boldsymbol{x} - \boldsymbol{c}) - R^2 = 0 \\
&\Rightarrow \left(\boldsymbol{x}_0 + t\hat{\boldsymbol{d}} - \boldsymbol{c}\right)^T \left(\boldsymbol{x}_0 + t\hat{\boldsymbol{d}} - \boldsymbol{c}\right) - R^2 = 0 \qquad (\text{相交，代入}) \\
&\Rightarrow t^2 \left(\hat{\boldsymbol{d}}^T \hat{\boldsymbol{d}}\right) + t\left(2\hat{\boldsymbol{d}}^T (\boldsymbol{x}_0 - \boldsymbol{c})\right) + \left((\boldsymbol{x}_0 - \boldsymbol{c})^T (\boldsymbol{x}_0 - \boldsymbol{c}) - R^2\right) = 0
\end{aligned}
$$

$$
\begin{aligned}
\Delta &= b^2 - 4ac \\
&= \left(2\hat{\boldsymbol{d}}^T\left(\boldsymbol{x}_0 - \boldsymbol{c}\right)\right)^2 - 4\left(\hat{\boldsymbol{d}}^T\hat{\boldsymbol{d}}\right)\left(\left(\boldsymbol{x}_0 - \boldsymbol{c}\right)^T\left(\boldsymbol{x}_0 - \boldsymbol{c}\right) - R^2\right) \\
&= 4\left(\sum_{i=1}^{\dim} d_i\left(o_i - c_i\right)\right)^2 - 4\left(\sum_{i=1}^{\dim} d_i^2\right)\left(\left(\sum_{i=1}^{\dim}\left(o_i - c_i\right)^2\right) - R^2\right) \qquad \text{(分量形式)}
\end{aligned}
$$

若 $\Delta \geq 0$ 有交点; 若 $\Delta < 0$ 无交点.

$$
t = \frac{-b \pm \sqrt{\Delta}}{2a}
$$

(c) 射线、椭球面交点

(input) 椭球方程: $\left(\boldsymbol{x} - \boldsymbol{c}\right)^T \boldsymbol{P}^{-1}\left(\boldsymbol{x} - \boldsymbol{c}\right) = 1$

$$
\begin{aligned}
dis &= \frac{-b \pm \sqrt{\Delta}}{2a} \\
\Delta &= b^2 - 4ac \\
&= \left(2\hat{\boldsymbol{d}}^T\boldsymbol{P}^{-1}\left(\boldsymbol{x}_0 - \boldsymbol{c}\right)\right)^2 - 4\left(\hat{\boldsymbol{d}}^T\boldsymbol{P}^{-1}\hat{\boldsymbol{d}}\right)\left(\left(\boldsymbol{x}_0 - \boldsymbol{c}\right)^T\boldsymbol{P}^{-1}\left(\boldsymbol{x}_0 - \boldsymbol{c}\right) - 1\right)
\end{aligned}
$$

若 $\Delta \geq 0$ 有交点; 若 $\Delta < 0$ 无交点.

Proof.

$$
\begin{aligned}
\Rightarrow \left(t\hat{\boldsymbol{d}} + \left(\boldsymbol{x}_0 - \boldsymbol{c}\right)\right)^T\boldsymbol{P}^{-1}\left(t\hat{\boldsymbol{d}} + \left(\boldsymbol{x}_0 - \boldsymbol{c}\right)\right) - 1 = 0 \\
t^2\left(\hat{\boldsymbol{d}}^T\boldsymbol{P}^{-1}\hat{\boldsymbol{d}}\right) + t\left(2\hat{\boldsymbol{d}}^T\boldsymbol{P}^{-1}\left(\boldsymbol{x}_0 - \boldsymbol{c}\right)\right) + \left(\left(\boldsymbol{x}_0 - \boldsymbol{c}\right)^T\boldsymbol{P}^{-1}\left(\boldsymbol{x}_0 - \boldsymbol{c}\right) - 1\right) = 0
\end{aligned}
$$

$$
\begin{aligned}
\Delta &= b^2 - 4ac \\
&= \left(2\hat{\boldsymbol{d}}^T\boldsymbol{P}^{-1}\left(\boldsymbol{x}_0 - \boldsymbol{c}\right)\right)^2 - 4\left(\hat{\boldsymbol{d}}^T\boldsymbol{P}^{-1}\hat{\boldsymbol{d}}\right)\left(\left(\boldsymbol{x}_0 - \boldsymbol{c}\right)^T\boldsymbol{P}^{-1}\left(\boldsymbol{x}_0 - \boldsymbol{c}\right) - 1\right)
\end{aligned}
$$

若 $\Delta \geq 0$ 有交点; 若 $\Delta < 0$ 无交点.

$$
t = \frac{-b \pm \sqrt{\Delta}}{2a}
$$

(d) 射线、矩体交点