

Theory and Application of MCMC Algorithm on Stochastic Volatility Model

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1 Introduction

The variance of returns of assets tends to change over time. The stochastic volatility model (SV) attempts to model this variance by assuming that they follow some latent stochastic process. It appeared first in the literature of theoretical finance on option pricing, though empirical models are usually described in discrete way. There is evidence the SV models offer increased flexibility over the GARCH family (Geweke(1994)).

However, even the simplest SV model is difficult to fit due to various reasons. Naive strategy of maximum likelihood estimation fails due to the fact that the marginal distribution is evaluated in high dimensional space. Monte Carlo Markov Chain (MCMC), on the other hand, faces difficulty in simulating some non-analytical distributions and people needs to resort to indirect approach such as Metroplis-Hastings algorithm or reject-accept sampling. The topic of this essay will be about the MCMC algorithm for SV model. Instead of general theory, we will focus on a simple SV model where the underling stochastic process is a log-normal autoregressive process. As for the MCMC, we will implement some specific algorithms and test their performance on both artificial and real world data.

The outline of this essay is following. We will introduce the model in section 2. In section 3, we derive the Bayesian posterior distribution and review three MCMC algorithms to fit the model. In order to compare them quantitatively, in section 4 we will generate a fake data with the model and test if the model can recover the underlying parameters. In section 5, we will apply the model on the real world data with these methods and check their results and performance. After that we will discuss the problems revealed in the test and possible follow-up work. We gather all the tables and figures at the end.

2 Basic Model and Notation

We will focus on a simple of SV model, which was discussed in many paper, for example Jacquier (1994) and Kim (1998). y_t $t = 1, \dots, N$ is the financial series

we are interested in, prefiltered to have zero mean. The model is

$$\begin{aligned} y_t &= \sqrt{h_t} u_t \\ \ln h_t &= \alpha + \delta \ln h_{t-1} + \sigma_\nu \nu_t \\ u_t, \nu_t &\sim N(0, 1) \end{aligned}$$

h_t is a latent variable measuring the volatility of y , δ is the volatility persistence. σ_ν^2 is the standard deviation of the volatility.

3 Bayesian Analysis

3.1 MCMC

Assume the prior distribution:

$$\begin{aligned} \alpha &\sim N(\alpha_0, \sigma_\alpha^2) \\ \delta &\sim N(\delta_0, \sigma_\delta^2) \\ \sigma_\nu^2 &\sim IG\left(\frac{\nu_0}{2}, \frac{s_0}{2}\right) \end{aligned}$$

We can write the unconditional distribution

$$\begin{aligned} p(y, h, \delta, \alpha, \sigma_\nu^2) &\propto \frac{1}{\sigma_\nu^{2+\nu_0} \sigma_\delta \sigma_\alpha} \exp\left(-\frac{(\delta - \delta_0)^2}{2\sigma_\delta^2} - \frac{(\alpha - \alpha_0)^2}{2\sigma_\alpha^2} - \frac{s_0}{2\sigma_\nu^2}\right) \\ &\times \prod_{t=2}^N \frac{1}{h_t^{\frac{3}{2}} \sigma_\nu} \exp\left(-\frac{y_t^2}{2h_t} - \frac{(\ln h_t - \delta \ln h_{t-1} - \alpha)^2}{2\sigma_\nu^2}\right) \end{aligned} \quad (1)$$

From which we can derive the posterior distributions

$$(\sigma_\nu^2 | h, \alpha, \delta) \sim IG\left(\frac{\nu_0 + N - 1}{2}, \frac{s'}{2}\right)$$

$$s' = s_0 + (N-1)\alpha^2 + (1+\delta^2)S_2 - \delta^2(\ln h_N)^2 - (\ln h_1)^2 - 2\alpha((1-\delta)S_1 - \ln h_1 + \delta \ln h_N) - 2\delta S_3$$

$$(\delta | h, \alpha, \sigma_\nu^2) \sim N\left(\frac{\sigma_\nu^2 \delta_0 + \sigma_\delta^2 (S_3 - \alpha(S_1 - \ln h_N))}{\sigma_\nu^2 + \sigma_\delta^2 (S_2 - (\ln h_N)^2)}, \frac{\sigma_\nu^2 \sigma_\delta^2}{\sigma_\nu^2 + \sigma_\delta^2 (S_2 - (\ln h_N)^2)}\right)$$

$$(\alpha | h, \sigma_\nu^2, \delta) \sim N\left(\frac{\sigma_\alpha^2 ((1-\delta)S_1 - \ln h_1 + \delta \ln h_N) + \sigma_\nu^2 \alpha_0}{\sigma_\nu^2 + (N-1)\sigma_\alpha^2}, \frac{\sigma_\nu^2 \sigma_\alpha^2}{\sigma_\nu^2 + (N-1)\sigma_\alpha^2}\right)$$

$$S_1 = \sum_{t=1}^N \ln h_t \quad S_2 = \sum_{t=1}^N (\ln h_t)^2 \quad S_3 = \sum_{t=2}^N \ln h_t \ln h_{t-1}$$

$$p(h_t | h_{t+1}, h_{t-1}, \delta, \alpha, \sigma_\nu^2) \propto \frac{1}{\sqrt{h_t}} \exp\left(-\frac{y_t^2}{2h_t}\right) \frac{1}{h_t} \exp\left(-\frac{(\ln h_t - \mu_t)^2}{2\sigma^2}\right) \quad (2)$$

$$\mu_t = \frac{\delta \ln h_{t+1} + \delta \ln h_{t-1} + (1 - \delta)\alpha}{1 + \delta^2}$$

$$\sigma^2 = \frac{\sigma_\nu^2}{1 + \delta^2}$$

In addition, following Jacquier (1994) and Geweke (1994a), we will not update h_1 and h_N with (2), we will update them by directly drawing from autoregressive model of $\ln h$.

In summary, the outline of the algorithm is

1. Initialize $h, \alpha, \delta, \sigma_\nu^2$
2. For $t = 2, 3, \dots, N - 1$, draw h_t from $p(h_t | h_{t+1}, h_{t-1}, \delta, \alpha, \sigma_\nu^2)$
3. Draw $\ln h_1$ from $N(\alpha + \delta \ln h_2, \sigma_\nu^2)$, $\ln h_N$ from $N(\alpha + \delta \ln h_{N-1}, \sigma_\nu^2)$
4. Draw σ_ν^2 from $(\sigma_\nu^2 | h, \alpha, \delta)$
5. Draw δ from $(\delta | h, \alpha, \sigma_\nu^2)$
6. Draw α from $(\alpha | h, \delta, \sigma_\nu^2)$
7. Go to step 2

It is easy to simulate the posterior distribution of σ_ν^2 . α and δ , so the only nontrivial part of the MCMC is step 2. Below we will introduce three sampling methods. The comparison of them will be the focus of this project.

3.2 Sampling Method 1: Metroplis-Hastings with Random Walk

Write (2) as a distribution of $\ln h$ rather than h

$$p(\ln h_t | h_{t+1}, h_{t-1}, \delta, \alpha, \sigma_\nu^2) \propto \frac{1}{\sqrt{h_t}} \exp\left(-\frac{y_t^2}{2h_t}\right) \exp\left(-\frac{(\ln h_t - \mu_t)^2}{2\sigma_t^2}\right)$$

Given $\ln h_t^{i-1}$. Each time we simply propose a $\ln h_t^*$ by drawing

$$N(\ln h_t^{i-1}, e^2)$$

where e^2 is preset. And accept it with probability

$$\text{Min}(1, \frac{p(\ln h_t^*)}{p(\ln h_t^{i-1})})$$

The algorithm for updating i th h_t , with $i - 1$ th h_t and all other parameters given, is:

1. Draw $\ln h_t^*$ from $N(\ln h_t^{i-1}, e^2)$
2. Accept this value with probability $\text{Min}(1, \frac{p(\ln h_t^{i*})}{p(\ln h_t^{i-1})})$
3. If accepted, $h_t^i = h_t^*$, else $h_t^i = h_t^{i-1}$

3.3 Sampling Method 2: Metroplis-Hastings with Accept-Reject Sampling

This is the method proposed in Jacquier(1994). The idea is to refine the process of the proposing update in MH. We can "approximate" (2) by an inverse gamma distribution:

$$q(h_t) = \frac{\lambda^\phi}{\Gamma(\phi)} h^{-(\phi+1)} e^{-\frac{\lambda}{h_t}}$$

where

$$\lambda = \frac{1 - 2e^{\sigma^2}}{1 - e^{\sigma^2}} + \frac{1}{2}$$

$$\phi = (\lambda - 1)e^{\mu_t + \frac{\sigma^2}{2}} + \frac{y_t^2}{2}$$

and σ^2 and μ_t are defined under (2). The reason of this choice is that we can choose a inverse gamma distribution which have same first and second moment with the lognormal part of (2), this then combines with the inverse gamma part of (2) to give the above inverse gamma distribution. Define

$$c = 1.1 \left(\frac{p(h)}{q(h)} \right)_{h=\text{mode of } q}$$

A plot of $p(x)$ and $cq(x)$ with selected parameters is given in Figure 1. We will propose candidate h_t^{i*} from $IG(\lambda, \phi)$ and accept it with $\text{Min}(1, \frac{p(h^*)}{cq(h^*)})$, if rejected, repropose until accepted. The winner of accept-reject process will be the candidate of MH process with transition kernel $f(h_t^*) = \text{Min}(p(h_t^*), cq(h_t^*))$. The actual algorithm will be

1. Draw h_t^* from $IG(\lambda, \phi)$, note that both λ and ϕ are functions of h_{t+1}, h_{t-1} and other parameters
2. Accept h_t^* with probability $\text{Min}(1, \frac{p(h_t^*)}{cq(h_t^*)})$
3. If rejected, go to step 1.
4. If $p(h_t^*) < cq(h_t^*)$, $h_t^i = h_t^*$. The algorithm ends.
5. Accept h_t^* with probability $\text{Min}(1, \frac{p(h_t^*)/q(h_t^*)}{p(h_t^{i-1})/q(h_t^{i-1})})$
6. If accepted, $h_t^i = h_t^*$, else $h_t^i = h_t^{i-1}$

3.4 Sampling Method 3: Pure Accept-Reject Sampling

This method was used in Kim(1998). Observe that $h_t^{-1} = \exp(-\ln h_t)$ is a convex function in $\ln h_t$ and can be bounded by a function linear in $\ln h_t$, we can show that:

$$\begin{aligned}
\ln p(\ln h_t | \dots) &= -\frac{1}{2} \ln h_t - \frac{y_t^2}{2h_t} - \frac{(\ln h_t - \mu_t)^2}{2\sigma^2} + \text{constants} \\
&\leq -\frac{1}{2} \ln h_t - \frac{y_t^2}{2} (\exp(-\mu_t)(1 + \mu_t - \ln h_t)) - \frac{(\ln h_t - \mu_t)^2}{2\sigma^2} + \text{constants} \\
&= -\frac{(\ln h_t - \mu'_t)^2}{2\sigma^2} + \text{constants}
\end{aligned} \tag{3}$$

Where

$$\mu'_t = \mu_t + \frac{\sigma^2}{2} (y_t^2 \exp(-\mu_t) - 1)$$

This observation leads to a standard reject-accept sampling:

1. Draw $\ln h_t^*$ from $N(\mu'_t, \sigma^2)$
2. Accept h_t^* with probability $\text{Min}(1, g(h_t^*))$, where

$$g(h_t) = \exp\left(\frac{y_t^2}{2} (\exp(-\mu_t))(1 + \mu_t - \ln h_t) - \frac{1}{h_t}\right)$$

3. If rejected, go to step 1.
4. $h_t^i = h_t^*$

4 Test of Three Sampling Method: Artificial Data

4.1 Shorthand

Before we start, we want give some short names to the algorithms which will be frequently mentioned.

Metroplis-Hastings with random walk: MH+RW, $e = \dots$.

Metroplis-Hastings with reject-accept sampling: MH+RA.

Reject-accept sampling: RA.

4.2 Data

We will set an initial value for α , δ , σ_ν^2 , $\ln h_1$ and generate a sequence of 1000 $\ln h_t$ and y_t following the model.

$$y_t \sim N(0, h_t)$$

$$\ln h_t \sim N(\alpha + \delta \ln h_{t-1}, \sigma_\nu^2)$$

4.3 Result

We generate two artificial data. Both of them have $\alpha = 0$, $\delta = 0.95$, but one has $\sigma_\nu^2 = 0.1$, and the other with $\sigma_\nu^2 = 0.06$. We fit the model with five methods: MH random walk with three different proposed standard deviation e for random walk, and the other two methods.

We run the MCMC cycle for 12000 iterations, taking the posterior mean and standard deviation from samples generated after 4000th iteration. For prior we take $\delta \sim N(0, 10)$, $\alpha \sim N(0, 10)$ and $\sigma_\nu^2 \sim IG(\frac{1}{2}, \frac{1}{2})$. The results are shown in Table 1 and 2. The numbers outside and inside bracket are sample mean and standard deviations respectively. We also present the RMS of h 's error $\sqrt{\sum (h_t - \bar{h}_t)^2 / 1000}$. The estimation of h_t is the mean of the sample.

We observe that MH + RA always have the smallest standard deviation, whether it's the parameters' standard deviation or σ_ν^2 , and gives the greatest volatility persistence. For smaller σ_ν^2 , the improving performance of RW seems very obvious, while the MH+RA seems to insist on its own "inherent" values.

There is no result for $\sigma_\nu^2 = 0.1$ for pure RA method, because with this setup, it sometimes yields an acceptance rate so poor that the MCMC will never finish. Luckily we didn't encounter this problem for the real world data we use in later sections.

5 Test of Three Sampling Method: S&P500

5.1 Data

For this section, we use the S&P500 from 1/1/2007 to 12/31/2010. As in Jacquier(1994) and Gallant(1992), we study the change of log of closing price:

$$y_t = \log(price_{t+1}/price_t)$$

y are plotted in Figure 2. There are 1008 data points in the time series.

5.2 Result

With the same number of iteration and burn-in, and also same prior from previous section. We list the result in Table 3. From Figure 3 to 7 we give all histograms of sample.

The results are quite consistent with the artificial data test. It is observed that MH + RA gives a highest persistence and smallest standard deviation on all parameters, and pure RA always gives the highest standard deviation and covariance. There are very little covariance between parameters.

For each h_t , we compute its sample mean and plot the log of the mean in Figure 8. It is a relief to observe that all five pictures reflects the abrupt change in the middle of Figure 2 (caused by the financial crisis) by showing a peak at the same position. Their trend are also very similar, but curiously MH + RA gives a smoother plot (the fourth one) .

5.3 Time

In Table 4 we listed the time and rejection/repeat rate of each method for 12000 iterations. Repeat rate applies for methods involving MH algorithm, while rejection rate applies for methods involving RA sampling. Note that only the HM+RA method have both. The rates for MH+RA is in agreement with that argued in Jacquier (2004). The slow speed for MH+RA is due to the fact it simulates inverse-gamma distribution (possibly for multiple times) in every update.

5.4 Autocorrelation

We give the autocorrelation function of parameters fit by all methods in Figure 9 to 13. The autocorrelation function are computed and plotted using `acf()` in stats library of R.

6 Discussion

In general, all methods we studied are usable in practice and give reasonable estimations of volatility of real world financial series. However, there are some

problems lurking around. The RA method is threatened by poor acceptance rate, and actually in Kim (1998) the author already talked about improvement on it. The MH + RA method seems to have bias in estimation. Also we haven't study for detail the convergence rate of each method which is important for MCMC. All these issues worth a closer investigations. They might be due to theoretical reasons or just implementation issues.

MCMC has its advantage but other methods is not to be forget. In fact one can use methods of moments or quasi-likelihood Kalman filtering to fit the SV model. It would be interesting to compare them to MCMC.

The model we studied is a very basic one. By extending it in various way, one can hope to uncover more features of the data. One such example is the fat-tailed model mentioned Jacquier (2004) which assumes that

$$y_t = \sqrt{h_t} \sqrt{\lambda_t} u_t$$

$$\lambda_t \sim \text{InverseGamma}$$

and thus breaks the normality of variance of data. A more complicated topic would be to extend the model to include the evolution of multiple assets together and try to find correlations. We hope to address these questions in the future.

While SV models are deep in themselves, there is a wide range of topics one can study by reaching out to other models. For example, the comparison between GARCH family and SV model has been a popular topic for a long time. As in Kim (1998) one can compute the likelihood ratio statistics and bayes factor to compare these models.

7 Acknowledgement

The author would like to thank Prof. James Scott for many helpful discussions, especially for suggesting the random walk algorithm for MCMC which is an important part of this project.

8 Tables

Method	δ	α	σ_ν^2	RMS of error in h
MH + RW, $e = 0.05$	0.926(0.029)	0.012(0.011)	0.08(0.016)	0.988
MH + RW, $e = 0.1$	0.949(0.013)	0.007(0.009)	0.077(0.019)	0.9327
MH + RW, $e = 0.3$	0.949(0.014)	0.008(0.009)	0.074(0.018)	0.9087
MH + RA	0.977(0.008)	0.013(0.007)	0.027(0.004)	1.287
RA	0.949(0.016)	0.008(0.009)	0.071(0.017)	0.88

Table 1: Results of fitting SV model on artificial data generated from $\alpha = 0$, $\delta = 0.95$, $\sigma_\nu^2 = 0.06$

Method	δ	α	σ_ν^2	RMS of error in h
MH +RW, $e = 0.05$	0.929(0.019)	0.015(0.012)	0.104(0.017)	1.0721
MH +RW, $e = 0.1$	0.893(0.031)	0.012(0.015)	0.193(0.056)	1.1803
MH + RW, $e = 0.3$	0.915(0.022)	0.011(0.013)	0.153(0.039)	1.0417
MH + RA	0.973(0.009)	0.016(0.009)	0.0323(0.005)	1.3961
RA				

Table 2: Results of fitting SV model on artificial data generated from $\alpha = 0$, $\delta = 0.95$, $\sigma_\nu^2 = 0.1$, no result for RA because the accept rate is too poor

Method	δ	α	σ_ν^2	$Cov(\delta, \alpha)$ ($\times 10^{-4}$)	$Cov(\delta, \sigma_\nu^2)$ ($\times 10^{-4}$)	$Cov(\alpha, \sigma_\nu^2)$ ($\times 10^{-4}$)
MH +RW, $e = 0.05$	0.976(0.008)	-0.214(0.073)	0.06(0.012)	5.9	-0.12	-1.18
MH +RW, $e = 0.1$	0.977(0.008)	-0.199(0.067)	0.061(0.009)	5.07	-0.27	-2.3
MH + RW, $e = 0.3$	0.973(0.009)	-0.24(0.083)	0.068(0.015)	7.8	-0.83	-7.3
MH + RA	0.986(0.006)	-0.113(0.051)	0.029(0.004)	3.1	-0.05	-0.44
RA	0.974(0.008)	-0.216(0.071)	0.068(0.015)	6.68	-0.72	-6.36

Table 3: Results of fitting SV model on S&P500 daily log change

Method	Time for 12000 iteration (second)	reject rate	repeat rate
MH + RW, $e = 0.05$	213.42		0.1
MH + RW, $e = 0.1$	210.64		0.18
MH + RW, $e = 0.3$	215.87		0.44
MH +RA	451.58	0.09	0.0002
RA	171.39	0.008	

Table 4: Time and reject/repeat rate of each method fitting S&P 500 daily log change

9 Figures

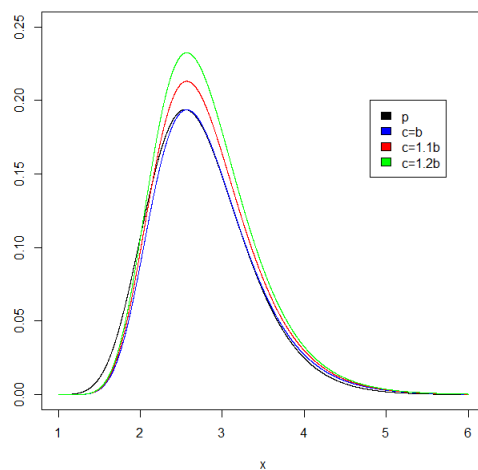


Figure 1: Plot of $p(x)$ and $cq(x)$ with various value of c . b is the value of p/q at the mode of q . $\sigma^2 = 0.05$, $\mu = 1$, $y = 1$

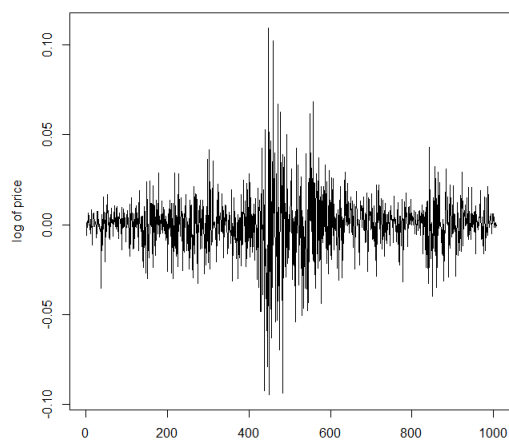


Figure 2: Daily change of S&P500 closing price from 2007-2010

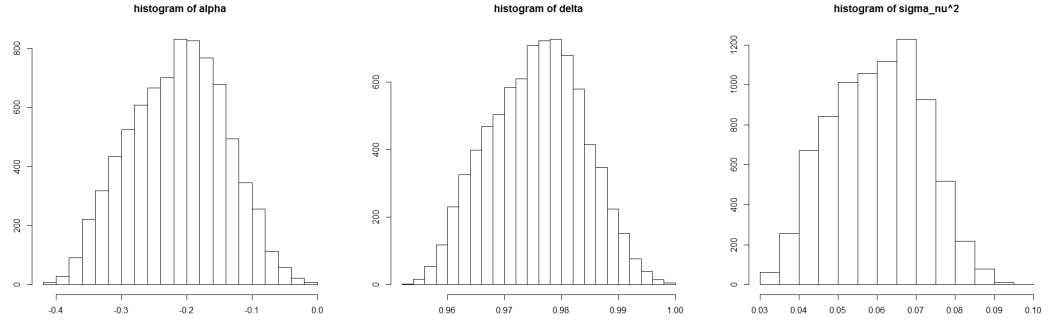


Figure 3: Histograms of MH + RW with $e = 0.05$ on S&P500

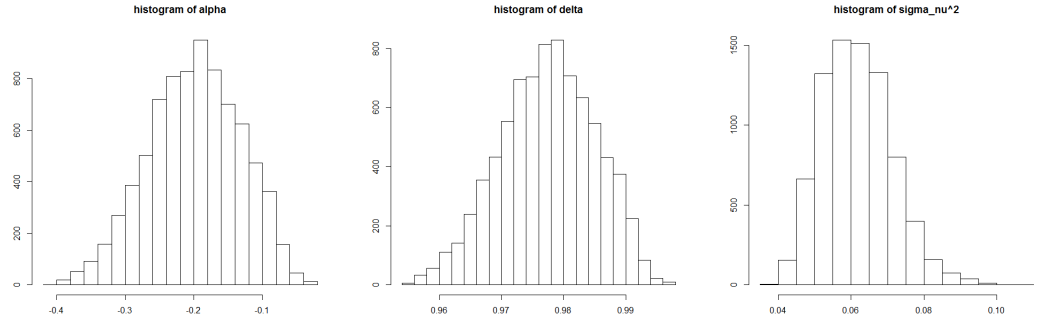


Figure 4: Histograms of MH + RW with $e = 0.1$ on S&P500

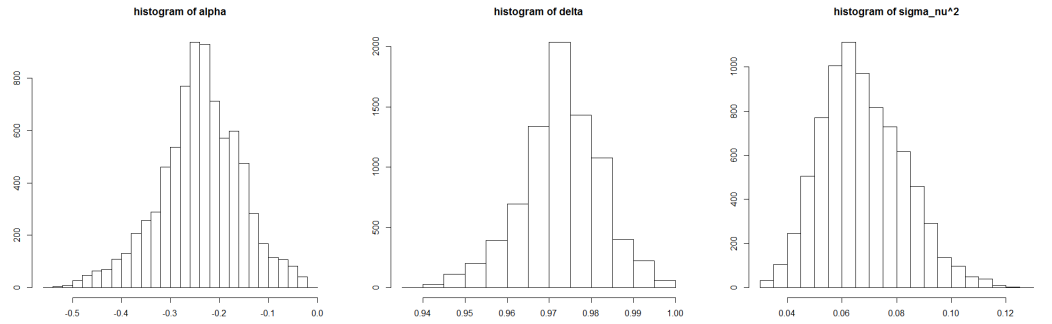


Figure 5: Histograms of MH + RW with $e = 0.3$ on S&P500

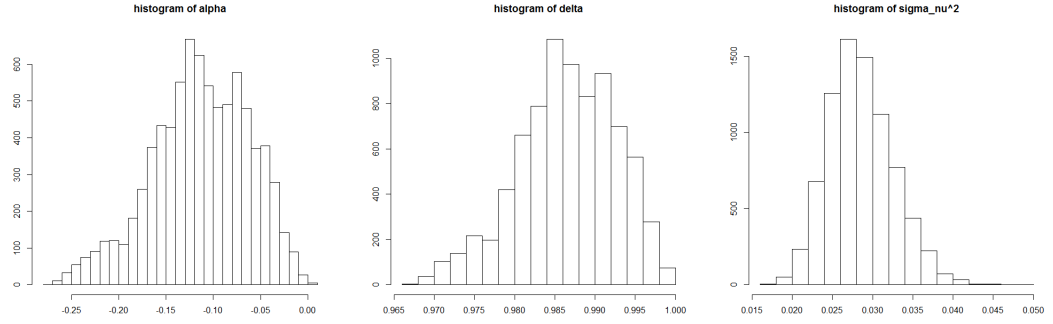


Figure 6: Histograms of MH + RA on S&P500

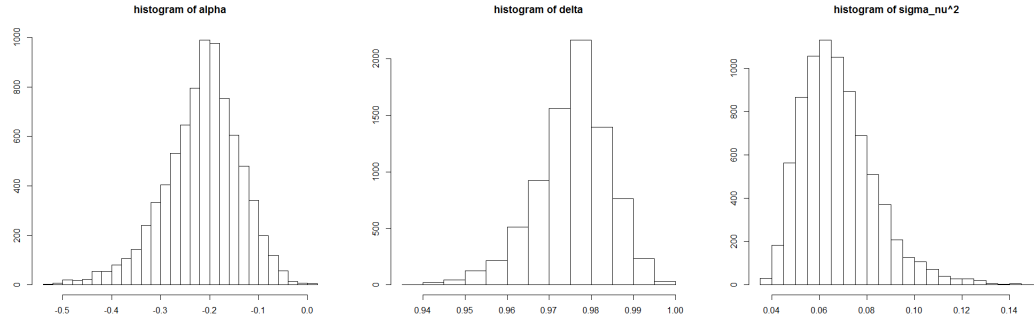


Figure 7: Histograms of RA on S&P500

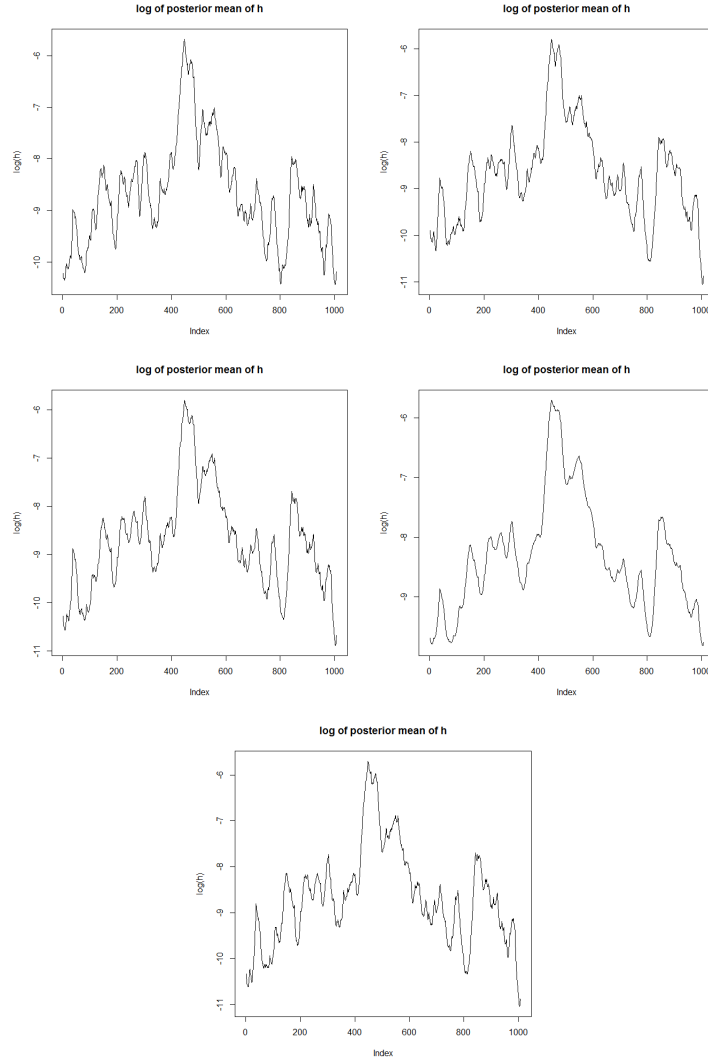


Figure 8: Log of mean of h_t of five methods on S&P500, in the order of appearance in previous tables

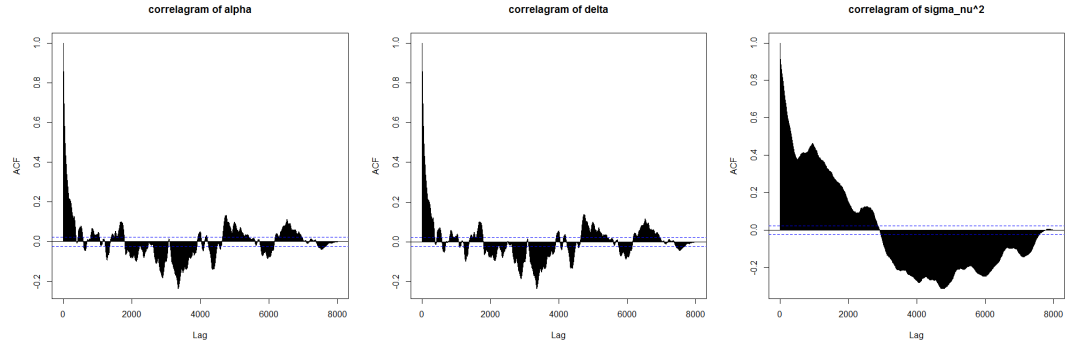


Figure 9: Correlagram of MH + RW with $e = 0.05$ on S&P500

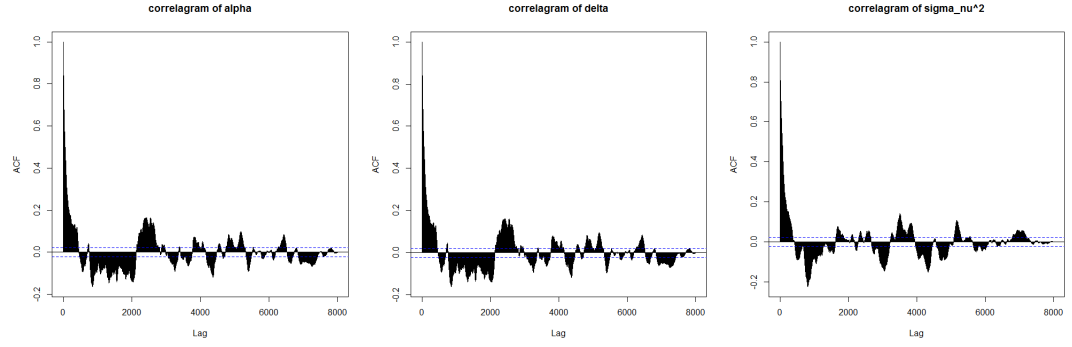


Figure 10: Correlagram of MH + RW with $e = 0.1$ on S&P500

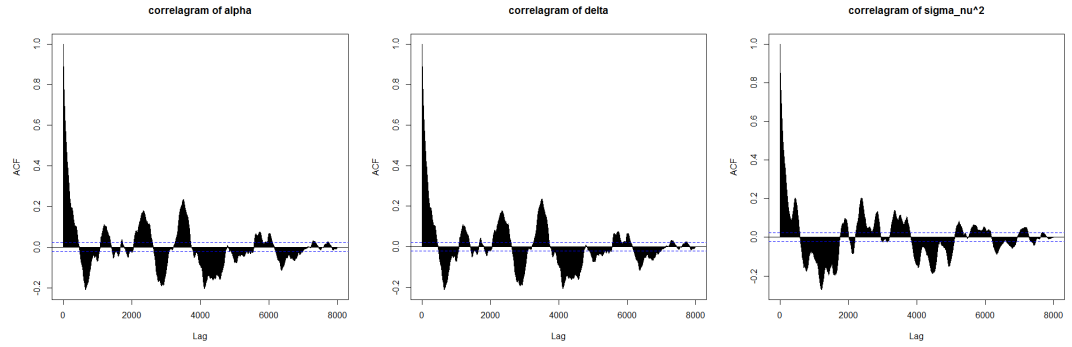


Figure 11: Correlagram of MH + RW with $e = 0.3$ on S&P500

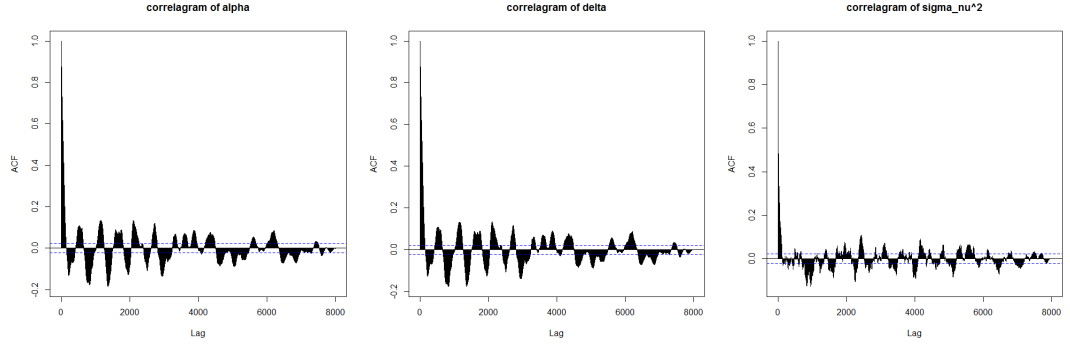


Figure 12: Correlagram of MH + RA on S&P500

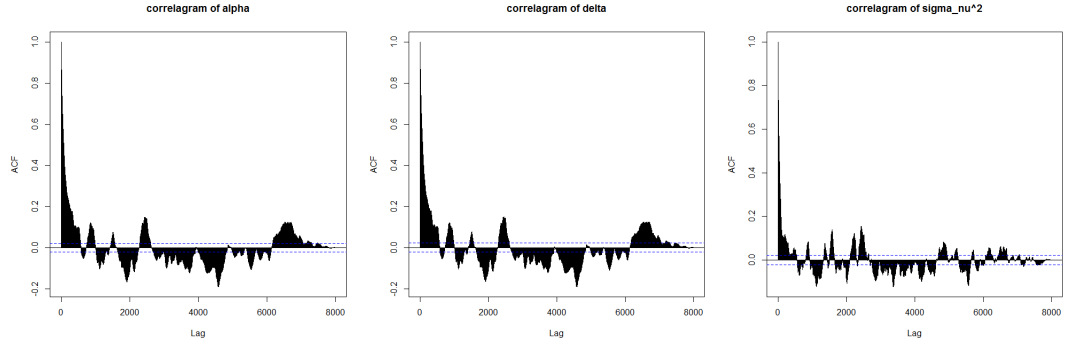


Figure 13: Correlagram of RA on S&P500

10 Codes

All codes of this project, along with reference and .png figures appeared in the essay are available in <https://github.com/yinanzhu12/MCMC-SV>.

11 Reference

1. Eric Jacquier, Nicholas G. Polson and Peter E. Rossi, "Bayesian Analysis of Stochastic Volatility Models", Journal of Business & Economic Statistics, 1994, Vol. 12, No. 4
2. Eric Jacquier, Nicholas G. Polson and Peter E. Rossi, "Bayesian analysis of stochastic volatility models with fat-tails and correlated errors", Journal of Econometrics, 122(2004) 185-212

3. John Geweke, "Prior for Macroeconomic Time Series and Their Application", *Econometric Theory*, Vol. 10, 609-632 (1994a)
4. John Geweke, "Bayesian Comparison of Econometric Models", working paper, Federal Reserve Bank of Minneapolis Research Department, (1994b)
5. Siddhartha Chib and Edward Greenberg, "Understanding the Metropolis-Hastings Algorithm", *The American Statistician*, Vol. 49, No. 4. (Nov., 1995), pp.327-335
6. A. Ronald Gallant, Peter E. Rossi, George Tauchen, "Stock Prices and Volume", *The Review of Financial Studies*, Vol. 5, No. 2. (1992), pp. 199-242
7. Sangjoon Kim, Neil Shephard, Siddhartha Chib, "Stochastic Volatility: Likelihood Inference and Comparison with ARCH Model", *Review of Economic Studies* (1998) 65, 361-393