Supplement on the presence of designed parameters in sensors.

A detailed solution is provided on the relevant problems of introducing design parameters λ into smart sensors in "Low-Complexity Quantized Prescribed Performance Control for Constrained Steer-by-Wire Systems with Input Nonlinearity and Bandwidth Limitations".

Solution:

(i) With the development of electrical automation, smart sensors have emerged. Smart sensors are sensors with information processing capabilities. Smart sensors are equipped with microprocessors, which have the ability to collect, process, and exchange information. They are the product of the integration of sensors and microprocessors. Smart sensors have three advantages over general sensors: high-precision information collection can be achieved through software technology, and the cost is low; they have specific programming automation capabilities and diversified functions. Therefore, we can see from the above that the parameters (i.e., λ and χ) in smart sensors are possible to be designed.

(ii) In the paper "Low-Complexity Quantized Prescribed Performance Control for Constrained Steer-by-Wire Systems with Input Nonlinearity and Bandwidth Limitations," for λ, it can be inferred from Eq.(10) and Eq.(15) ($S = tan(\frac{\pi}{2}\frac{z}{\rho(t)})$ (10), $V = \frac{1}{2}S^2$ (15)) that if V is bounded, then S is bounded and $-\rho(t) < z < \rho(t)$. According to the definition of z (i.e., $z = \lambda x_1 + x_2 - \lambda y_d$), it is known that x_1 tracks y_d with an error of $(-\rho(t) - x_2)/\lambda < x_1 - y_d < (\rho(t) - x_2)/\lambda$, and it can be seen from this that the error is affected by two parameters, i.e., $\rho(t)$ and λ. Therefore, in practical engineering applications, we can first determine the range of parameter λ in advance through simulation and then select a reasonable sensor. Maintain consistency between parameter λ of the controller and λ of the sensor in the design, and adjust the tracking performance of the controller by adjusting $\rho(t)$.

(iii) Due to the quantification of $\chi(t) = \lambda x_1 + x_2$ in the paper, a parameter λ that needs to be designed is introduced. In fact, we can make minor changes to the designed controller to avoid the parameter λ appearing in the sensor. The specific changes and impacts are as follows:

- ***Modified content***

We fine-tune the controller to avoid the presence of parameters λ that need to be designed in the sensor. The key parts are highlighted in italics, and the main changes are as follows

We need to quantify the states x_1 and x_2 separately, i.e.,

$$Q_x(x_i) = \begin{cases} L_i, & Case\ I{:}\, L_i - \psi/2 \leq x_i(t) < L_i + \psi/2 \\ 0, & Case\ II{:} -\psi/2 \leq x_i(t) < \psi/2 \\ -L_i, & Case\ III{:} -L_i - \psi/2 \leq x_i(t) < -L_i + \psi/2 \end{cases} \tag{1}$$

where x_i, $i = 1, 2$, x_1 and x_2 refer to the front wheel steering angle θ_f and angular velocity $\dot{\theta}_f$, respectively.

In addition, we also need to make minor modifications to the error transformation z,

$$z = x_1 + \frac{1}{\lambda}x_2 - y_d \tag{2}$$

where $\lambda > 0$ is the positive constant to be designed.

Remark: From the two equations above, it can be seen that this change is a minor modification to the controller, and the actual content and principle have not changed. For z in the controller, this change only multiplies the coefficient $\frac{1}{\lambda}$ on the original basis ($z = \lambda x_1 + x_2 - \lambda y_d$) and does not affect the performance of the controller's full-state constraints. Below, we present the modified system stability analysis.

- ***Stability analysis***

Provide necessary equations and designed controller

The mathematical model of SbW systems with actuator fault and dead-zone phenomenon can be expressed as

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f(x) + g(t)\ell(t)u + d(t) \\ y = x_1 \end{cases} \tag{3}$$

State quantization using a uniform quantizer:

$$Q_x(x_i) = \begin{cases} L_i, & Case\ I\colon L_i - \psi/2 \le x_i(t) < L_i + \psi/2 \\ 0, & Case\ II\colon -\psi/2 \le x_i(t) < \psi/2 \\ -L_i, & Case\ III\colon -L_i - \psi/2 \le x_i(t) < -L_i + \psi/2 \end{cases} \tag{4}$$

Input quantization using hysteresis quantizer:

$$Q_v(v) = \begin{cases} v_i, & Case\ I\colon \dfrac{v_i}{1+\varpi} < v \le v_i, Q_v^- \ge v_i \\ & or\ v_i \le v < \dfrac{v_i}{1-\varpi}, Q_v^- \le v_i \\ v_i(1+\varpi), & Case\ II\colon v_i \le v \le \dfrac{v_i}{1-\varpi}, Q_v^- \ge v_i(1+ \\ & \varpi) or \dfrac{v_i}{1-\varpi} \le v < v_i,\ Q_v^- \le v_i(1+\varpi) \\ 0, & Case\ III\colon 0 \le v \le \dfrac{v_i}{1+\varpi}, Q_v^- \ge v_i(1+ \\ & \varpi)\ or\ \dfrac{v_i}{1+\varpi} < v < v_1, Q_v^- = 0 \\ -Q_v(-v), & Case\ IV\colon v < 0. \end{cases} \tag{5}$$

Please refer to the manuscript for detailed parameter definitions on page 3.

The following error transformation is given as

$$z = x_1 + \frac{1}{\lambda} x_2 - y_d \tag{6}$$

The control law $v(t)$ is designed as

$$v(t) = -\eta S(z(t)), \quad S(z(t)) = \tan\left(\frac{\pi}{2}\frac{z(t)}{\rho(t)}\right) \tag{7}$$

The prescribed performance function $\rho(t)$ is

$$\rho(t) = \begin{cases} \xi_1 + (\xi_0 - \xi_1) e^{\left(-\frac{t}{t_\xi - t}\right)}, & Case\ I\colon 0 \le t < t_\xi \\ \xi_1, & Case\ II\colon otherwise \end{cases} \tag{8}$$

The ETM is constructed as

$$u(t) = Q_v(v(t_k)) \quad \forall t \in [t_k, t_{k+1}),\ t_1 = 0 \tag{9}$$

$$t_{k+1} = \begin{cases} inf\left\{t > t_k : |e_v(t)| \ge \varrho |v(t)| + m\right\}, \\ \qquad Case\ I\colon if\ |v(t)| \le \kappa \\ inf\{t > t_k \colon |e_v(t)| \ge m\}, Case\ II\colon otherwise \end{cases} \tag{10}$$

Please refer to the manuscript for detailed parameter definitions on pages 4-5.

Remark: From (4) to (10), it can be seen that the changes to the controller are only two parts: state quantization and error transformation.

Proof of stability.

Based on the quantification (4)-(5) and event-triggering mechanism (9)-(10), the time derivative of S can be expressed as

$$\dot{S} = \frac{\pi}{2\rho^2 cos^2\left(\frac{\pi}{2}\frac{z(t)}{\rho(t)}\right)}(\rho(t)(\frac{1}{\lambda}(f(x) + g\ell(t)u(t) + d(t)) + x_2 - \dot{y}_d(t)) - \dot{\rho}(t)z(t)). \tag{11}$$

For the SbW system (3) with the developed control schemes (6)-(10) and state and input quantizers (4)-(5), the main results can be summarized in the following theorem.

Theorem 1: Consider the SbW system (3) with model uncertainty, input nonlinearity, the quantizers (4)-(5), and ETM in the controller-to-actuator channel (9)-(10). Under Assumptions 1-2 hold, the control objective can be guaranteed by the developed control schemes (6)-(7) with the condition $|z(0)| < \rho(0)$, and all signals in the closed-loop system are guaranteed to be bounded.

Proof: Consider the barrier Lyapunov function is

$$V = \frac{1}{2}S^2(t). \tag{12}$$

Remark: The manuscript introduces the barrier Lyapunov function technique to prove the stability of the system. From (12), it can be concluded that if V is bounded, then S is bounded, and $-\rho(t) < z < \rho(t)$. According to the definition of z (6), it is known that x_1 tracks y_d with an error of $-\rho(t) - \frac{1}{\lambda}x_2 < x_1 - y_d < \rho(t) - \frac{1}{\lambda}x_2$.

The time derivative of V is computed based on (11) as

$$\dot{V} = \frac{S\pi}{2\rho^2 cos^2\left(\frac{\pi}{2}\frac{z(t)}{\rho(t)}\right)}(\rho(t)(\frac{1}{\lambda}f(x) + \frac{1}{\lambda}g\ell(t)u(t) + \frac{1}{\lambda}d(t) + x_2 - \dot{y}_d(t)) - \dot{\rho}(t)z(t)). \tag{13}$$

Remark: Compared to before modification, since x_2 has changed to $\frac{1}{\lambda}x_2$. From (13), it can be seen that $f(x)$ is the uncertainty term, $g\ell(t)$ is the control gain, and $d(t)$ is the disturbance. After multiplying them by the coefficient $\frac{1}{\lambda} > 0$, there is no substantial change.

From (10), one gets that there exist $|o_i(t)| < 1, i = 1, 2,$ such that $Q_v(v(t)) = (1 + o_1(t)\rho_1)u(t) + o_2(t)m$. The relationship between $Q_v(v(t))$ and $u(t)$ can be expressed as

$$u(t) = \frac{Q_v(v(t))}{1 + o_1(t)\rho_1} - \frac{o_2(t)m}{1 + o_1(t)\rho_1}. \tag{14}$$

Substituting (14) to (13) yields

$$\dot{V} = \frac{S\pi}{2\rho^2 cos^2\left(\frac{\pi}{2}\frac{z(t)}{\rho(t)}\right)}(\rho(t)(\frac{1}{\lambda}f(x) + go^{'}(t)Q_v(v(t)) + d_1^{'}(t) + x_2 - \dot{y}_d(t)) - \dot{\rho}(t)z(t)). \tag{15}$$

with $o^{'}(t) = \frac{\ell(t)}{\lambda(1+o_1(t)\rho_1)}$ and $d_1^{'}(t) = \frac{1}{\lambda}d(t) - go^{'}(t)o_2(t)m$.

From (5) and *Lemma 1*, one gets that $Q_v(v(t)) = \alpha(v)v(t) + \gamma(v)$. Thus, substituting (5) to (15) yields

$$\dot{V} = \frac{S\pi}{2\rho^2 cos^2\left(\frac{\pi}{2}\frac{z(t)}{\rho(t)}\right)}(\rho(t)(\frac{1}{\lambda}f(x) + g\ell^{'}(t)v(t) + d_2^{'}(t) + x_2 - \dot{y}_d(t)) - \dot{\rho}(t)z(t)). \tag{16}$$

with $\ell^{'}(t) = o^{'}(t)\alpha(v)$ and $d_2^{'}(t) = d_1^{'}(t) + go^{'}(t)\gamma(v)$.

Combined with the definition of $z(t)$ and state quantizer (4), one gets that

$$|x_i(t) - Q_x(x_i)| \leq \frac{\psi}{2},\ and\ |z - \tilde{z}| \leq \frac{\psi}{2}(1 + \frac{1}{\lambda}). \tag{17}$$

where z is not state quantized, and $\tilde{z}$ is state quantized.

Remark: Compared to the error $|z + \lambda y_d - Q_x(\chi)| \leq \psi/2$ generated by state quantization before modification, the modified error is multiplied by $(1 + 1/\lambda)$, and when λ is large enough, the two errors are almost the same. Therefore, this change has no theoretical impact on the controller.

Combine (17), it can be concluded that there exists a variable $o_3(t), |o_3(t)| \leq 1$, such that

$$z(t) = \tilde{z} + \frac{\psi}{2} o_3(t)(1 + \frac{1}{\lambda}). \tag{18}$$

The following is an analysis of the worst-case dynamics of $S(t)$, i.e., $\rho(t) - \frac{\psi}{2}(1 + \frac{1}{\lambda}) \leq |\tilde{z}| \leq |z(t)| \leq \rho(t)$, where $\frac{\psi}{2}(1 + \frac{1}{\lambda})$ is the maximum quantization error of the state quantizer, which can be derived as $sign(z) = sign(\tilde{z})$.

Remark: Compared to $\rho(t) - \frac{\psi}{2} \leq |Q_x(\chi) - \lambda y_d| \leq |z(t)| \leq \rho(t)$ before modification (Note that $Q_x(\chi) - \lambda y_d$ represents $\tilde{z}$ before modification), $\rho(t) - \frac{\psi}{2}(1 + \frac{1}{\lambda}) < \rho(t) - \frac{\psi}{2}$, but when λ increases, $\rho(t) - \frac{\psi}{2}(1 + \frac{1}{\lambda}) \to \rho(t) - \frac{\psi}{2}$, and this change has no critical effect on the proof.

The above analysis concludes that there exists a positive time-varying variable $o_4(t), \underline{o}_4 \leq o_4(t) < 1$, with $\underline{o}_4$ being an unknown positive constant, such that

$$\tan\left(\frac{\pi}{2}\frac{\tilde{z}}{\rho(t)}\right) = o_4(t) \tan\left(\frac{\pi}{2}\frac{z(t)}{\rho(t)}\right). \tag{19}$$

Remark: As of (19), the effect of minor changes to the controller has ended. From the stability analysis, it can be seen that this change is not a critical factor for stability. The following analysis is the same as before the modification.

Substituting (19) into (16) yields

$$\begin{aligned}\dot{V} =& \frac{S\pi}{2\rho^2 cos^2\left(\frac{\pi}{2}\frac{z(t)}{\rho(t)}\right)}(\rho(t)(\frac{1}{\lambda}f(x) - g\eta\ell^{'}(t)o_4(t)S(z(t)) \\ &+ d_2^{'}(t) + x_2 - \dot{y}_d(t)) - \dot{\rho}(t)z(t)).\end{aligned} \tag{20}$$

Then, the following conditions are given to analyze the dynamic behavior of V: (i) From *Lemma 2*, it can be concluded that $x_i, i = 1, 2$ are bounded, and $f(x)$ and g are also bounded. (ii) Combined with the Assumption 2 and (14)-(19), one can get that $\ell^{'}(t)$ and $d_2^{'}(t)$ are bounded, and $g\ell^{'}(t)o_4(t) > 0$. (iii) From (8), bounded $\rho(t)$ and $\dot{\rho}(t)$ can be obtained. From the above analysis, it can be seen that there exists an unknown positive constant $\bar{\Gamma}$ such that

$$\bar{\Gamma} \geq |\rho(t)\frac{1}{\lambda}f(x) + \rho(t)d_2^{'}(t) + \rho(t)x_2 - \rho(t)\dot{y}_d(t) - \dot{\rho}(t)z(t))|.$$

Substituting the above inequality into (20), one gets

$$\dot{V} \leq \frac{|S|\pi}{2\rho^2 cos^2\left(\frac{\pi}{2}\frac{z(t)}{\rho}\right)}\left(\bar{\Gamma} - \rho(t)g\eta\ell^{'}(t)o_4(t)|S|\right) \tag{21}$$

$$= \frac{\rho(t)g\eta\ell^{'}(t)o_4(t)|S|\pi}{2\rho^2 cos^2\left(\frac{\pi}{2}\frac{z(t)}{\rho}\right)}\left(\frac{\bar{\Gamma}}{\rho(t)g\eta\ell'(t)o_4(t)} - |S|\right). \tag{22}$$

From (22), one can obtain that there exists an unknown positive constant Θ, $|S| = \Theta$ for $t = t_\Theta$ such that

$$\dot{V}|_{t=t_\Theta} \leq 0, \quad if \quad \Theta \geq max\left\{|S(z(0))|, \frac{\bar{\Gamma}}{\rho(t)g\eta\ell'(t)o_4(t)}\right\}.$$

Based on the above inequality, one gets that V is bounded, i.e., $V \leq \frac{1}{2}\Theta^2$. Taking the inverse of the S-function, one has

$$-\rho(t) < S^{-1}(-\Theta) \leq z(t) \leq S^{-1}(\Theta) < \rho(t). \tag{23}$$

The above result (23), with the conclusion of *Lemma 2*, implies that the tracking error of the SbW system can converge to the prescribed neighborhood of the origin. Besides, the above analysis means that all closed-loop signals remain bounded.The stability analysis of the SbW system is completed.

In summary, from (i)-(iii), we can see that whether the parameter λ in the sensor can be designed by us is not a crucial and inevitable issue for the entire controller design.