Lecture 17

Coverings.

满射. 原空间可能更大

Def: let $p: E \to B$ be a Continuous surjective map.
an open set U in B is evenly covered by p if
$p^{-1}(U)$ is a disjoint union of open sets V_α in E.
$\bigcup_{\alpha \in J} V_\alpha = p^{-1}(U)$. and each restriction:
$p|V_\alpha : V_\alpha \to U$ is a homeomorphism. $\forall \alpha \in J$

Def: a Continuous surjective map $p: E \to B$ is a covering
if each $b \in B$ has a neighb U which is evenly covered

Example: 53.1 $p: \mathbb{R} \to S^1 \subset \mathbb{C}$. $S^1 = \{z \in \mathbb{C} \; |z| = 1\}$
$p(t) = e^{i2\pi t} = \cos(2\pi t) + i\sin(2\pi t) \in \mathbb{C}$

Remark: $p: E \to B$. then each fiber $p^{-1}(b)$, $b \in B$
has the discrete topology (as a subspace of E)

proof: let $b \in B$. let U be the evenly covered neighb
of b.
$\Rightarrow p^{-1}(U) = \bigcup_{\alpha \in J} V_\alpha$ and $p|V_\alpha \to U$ is a homeo.
in particular. bijective
$\Rightarrow$ each V_α contains exactly one element of the
fiber $p^{-1}(b)$
$V_\alpha \cap p^{-1}(b)$ is a point of the fiber.
V_α is open by definition of covering

$\Rightarrow$ all points of the fiber are open in fiber.

Exercise: $p: E \to B$ covering. with B connected
then each fiber $p^{-1}(b)$ $b \in B$ has the same members
(the same cardinality)

proof: fix a point $b_0 \in B$.
$X = \{ b \in B :$ the fiber $p^{-1}(b)$ has the same cardinality as the fiber $p^{-1}(b_0) \}$
prove that $X = B$
$X \neq \phi$ since $b_0 \in X$.

(1) prove X is open.
let $b \in X$. let U be evenly covered neighb of b.
$\Rightarrow p^{-1}(U) = \bigcup_{\alpha \in J} V_\alpha$ $\quad p|V_\alpha : V_\alpha \to U$ is homeo
each fiber with $b \in U$ has the same cardinality as the cardinality of J.
$\Rightarrow U \subset X \Rightarrow X$ is open

(2) prove X is closed.
prove $B - X$ is open.
let $b \in B - X$ and U is neighb of b.
$p^{-1}(U) = \bigcup_{\alpha \in J} V_\alpha$. the cardinality of each $p^{-1}(b)$ is as same as J's. which is different from $f^{-1}(b_0)$'s.
the $U \subset B - X$. $\Rightarrow B - X$ is open

$\Rightarrow$ $B - X$ is open. 不然是个 seperation

$\Rightarrow$ $X = B$ since B is connected.

Def: let $p: E \to B$ be a covering. If f is a continuous map of some space X to B. a lifting of f is a continuous map $\tilde{f}: X \to E$ s.t. $p \circ \tilde{f} = f$

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & B \end{tikzcd}$$

要求从 e_0 开始
所以唯一

Lemma 54.1 (Path lifting lemma)

let $p: E \to B$ be a covering. $p(e_0) = b_0$ Then any path $f: [0, 1] \to B$ with $f(0) = b_0$ has a unique lifting to a path $\tilde{f}: [0, 1] \to E$ s.t. $\tilde{f}(0) = e_0$

proof: choose a covering $\mathcal{U}$ of B by open sets which are evenly covered. then

$$f^{-1}(\mathcal{U}) = \{ f^{-1}(U), U \in \mathcal{U} \}$$ is an open covering of $[0, 1]$

$[0, 1]$ is a compact metric space

Lebesgue number lemma $\Rightarrow$

there is a subdeversion

$s_0 = 0 < s_1 < \cdots < s_n = 1$ of $[0, 1]$ s.t.

$f([s_i, s_{i+1}])$ is contained in at least one $U \in \mathcal{U}$

let $\tilde{f}(0) = e_0$. suppose that $\tilde{f}$ is defined already on interval $[0, s_i]$. Then we define $\tilde{f}$ on $[s_i, s_{i+1}]$

let $\tilde{f}([s_i\ s_{i+1}]) \subset U \in \mathcal{U}$

$p^{-1}(U) = \bigcup_{\alpha \in J} V_\alpha$

$p|V_\alpha : V_\alpha \to U$ is a homeomorphism

let $\tilde{f}(s_i) \in p^{-1}(U)$

($\tilde{f}$ on s_i is already defined)

let $\tilde{f}(s_i) \in V_{\alpha_0}$. $\alpha_0 \in J$

inverse lemma

for $t \in [s_i, s_{i+1}]$. define $\tilde{f}(t) = ((p|V_{\alpha_0})^{-1} \circ f)(t)$

by the pasting lemma. $\tilde{f}$ is defined now on $[0, s_{i+1}]$

and also continuous

$\Rightarrow$ in finitely many steps. $\tilde{f}$ is defined on $[0,1]$

then prove the uniqueness :

$\tilde{f}(0) \equiv e_0$. suppose $\tilde{f}$ is unique on $[0, s_i]$

prove it is unique on $[s_i, s_{i+1}]$

let $\tilde{f}(s_i) \in V_{\alpha_0}$. $\tilde{f}'(s_i) = \tilde{f}(s_i) \in V_{\alpha_0}$

since $\tilde{f}'([s_i, s_{i+1}])$ is connected

$\tilde{f}'([s_i, s_{i+1}]) \subset V_{\alpha_0}$ since $V_{\alpha_i} \cap V_{\alpha_j} = \phi \ \forall i \neq j$.

$\Rightarrow \tilde{f}' = ((p|V_{\alpha_0})^{-1} \circ f)(t). = \tilde{f}$

it is unique

Theorem 54.2 (homotopy lifting)

$p: E \to B$ covering with $p(e_0) = b_0$

Let $F: I \times I \to B$ be continuous with $F(0 \times 0) = b_0$

Then there is unique lifting $\tilde{F}: I \times I \to E$ s.t

$\tilde{F}(0 \times 0) = e_0$. if F is a path homotopy. (constant on $\{0\} \times I$ and $\{1\} \times I$). then $\tilde{F}$ also proof: Lebesgue number $\Rightarrow$

there are 2 subdiversion:

$s_0 = 0 < s_1 \cdots < s_n = 1$

$t_0 = 0 < t_1 \cdots < t_n = 1$ s.t.

$F([s_i, s_{i+1}] \times [t_j, t_{j+1}])$ is connected in at least one $U \in \mathcal{U}$.

$\tilde{F}(0 \times 0) = e_0$. suppose that $\tilde{F}$ is already defined well on $(I \times [0, t_j]) \cup ([0, s_i] \times [t_j, t_{j+1}])$

then define $\tilde{F}$ on $[s_i, s_{i+1}] \times [t_j, t_{j+1}] = I_i \times J_j$

$\tilde{F}$ is already define on $Y = \{s_i\} \times [t_j, t_{j+1}] \cup [s_i, s_{i+1}] \cup \{t_j\}$

$p^{-1}(U) = \bigcup_{\alpha \in J} V_\alpha \Rightarrow \tilde{F}(Y) \subset V_{\alpha_0}$, $\alpha_0 \in J$

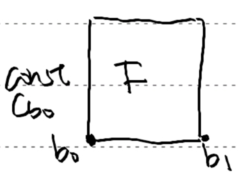
$\Rightarrow \tilde{F}(I_i \times J_j) \subset V_{\alpha_0}$

define $\tilde{F} = (p|V_{\alpha_0})^{-1} \circ F$

$\Rightarrow$ finitely many steps. $\tilde{F}$ is defined on $I \times I$. "existence"

"unique" similar as 54.1

Then. prove $\tilde{F}$ is path-homotopy: if F is



$\tilde{F}|(\{0\} \times I)$ is a lifting of the const path c_{b_0} in $b_0 \in B$

a lifting of c_{b_0} remains in the fiber $p^{-1}(b_0)$ which is discrete.

any path in a discrete space is constant. (since I is connected)

$\Rightarrow \tilde{F}|(\{0\} \times I)$ is constand. similar as $\tilde{F}|(\{1\} \times I)$

$\Rightarrow \tilde{F}$ is path-homotopy

Lecture 18

Theorem 5.4.3 (找 $\tilde{F}$ 对应同伦的两条路径)

$p: E \to B$ covering. $p(e_0) = b_0$. Let f and g be paths in B from b_0 to b_1 which are path-homotopic. And let $\tilde{f}$ and $\tilde{g}$ be liftings to E with initial point e_0. Then also $\tilde{f}$ and $\tilde{g}$ are path-homotopic and in particular have same end point.

proof: $f \underset{H}{\simeq} g$. By the homotopy lifting lemma, H lifts to a path-homotopy $\tilde{H}$

$$\begin{tikzcd}
\cdot \arrow[r, "g"] \arrow[dr, phantom, "H"] & \cdot & & \cdot \arrow[r, "\tilde{g}"] \arrow[dr, phantom, "\tilde{H}"] & \cdot \\
\cdot \arrow[u, "c_{b_0}"] \arrow[r, "f"'] & \cdot \arrow[u, "c_{b_1}"'] & \to & \cdot \arrow[u, "c_{e_0}"] \arrow[r, "\tilde{f}"'] & \cdot \arrow[u, "c_{e_1}"']
\end{tikzcd}$$

by the uniqueness of path-homotopy lifting with a given initial point e_0, $\tilde{f} \underset{\tilde{H}}{\simeq} \tilde{g}$

Theorem 5.4.5: $\pi_1(S^1, 1) \cong \mathbb{Z}$

proof: $\varphi: \pi_1(S^1, 1) \to \mathbb{Z}$

$[w] \in \pi_1(S^1, 1)$

$w: I \to S^1$ path with $w(0) = w(1) = 1 \in S^1$

$p: \mathbb{R} \to S^1$ covering $p(t) = e^{i2\pi t}$

let $\tilde{w}: I \to \mathbb{R}$ be the lifting of w with initial point $\tilde{w}(0) = 0 \in \mathbb{R}$

then $\tilde{w}(1) \in p^{-1}(1) = \mathbb{Z}$

(the rotation number of w)

$\varphi([w]) = \tilde{w}(1) \in \mathbb{Z}$. φ is well defined.

if $[w] = [w']$ $\quad w \underset{H}{\simeq} w'$

let $\tilde{w}$ and $\tilde{w}'$ be the liftings of w and w'.

with the initial point $0 \in \mathbb{R}$

by the before theorem. $\tilde{w} \simeq \tilde{w}'$. $\Rightarrow \tilde{w}(1) = \tilde{w}'(1)$

$\Rightarrow \varphi$ is well-defined $\quad$ $\tilde{w}$ 和 w 是一一对应，给 w 找一个和 $\mathbb{Z}$ 有关的函数

then prove that φ is a group homomorphism:

$$\varphi([w] * [v]) \overset{?}{=} \varphi([w]) + \varphi([v])$$

$$\varphi([w * v]) = \tilde{w}(1) + \tilde{v}(1)$$

$$(\widetilde{w * v})(1) = ?$$

从 $\tilde{w}(1)$ 起点的 $\tilde{v}$. $(\tilde{v} + \tilde{w}(1))(1) = \tilde{v}(1)$

$$(\widetilde{w * v}) = \tilde{w} * (\tilde{v} + \tilde{w}(1))$$

$$(\widetilde{w * v})(1) = \tilde{w}(1) + \tilde{v}(1)$$

check the "injective"

let $[w] \in \ker(\varphi)$.

$\Rightarrow \tilde{w}(1) = 0 \Rightarrow \tilde{w}$ is a closed path

$[\tilde{w}] \in \pi_1(\mathbb{R}, 0)$ is trivial group

$\Rightarrow [\tilde{w}] = [c_0] \quad \tilde{w} \underset{H}{\simeq} c_0$

$\Rightarrow w = p \circ \tilde{w} \simeq p \circ c_0 = c_1$

$\Rightarrow [w] = [c_1]$. the $\ker(\varphi)$ is trivial.

φ is injective

check the "surjective"

$\forall n \in \mathbb{Z}$. $u: I \to \mathbb{R}$ be any path with

$u(0) = 0$. $u(1) = n$.

let $w = p \circ u$ a closed path. $[w] \in \pi_1(S^1, 1)$

$\Rightarrow \quad \varphi([\omega]) = \tilde{\omega}(1) = u(1) = n$

$\Rightarrow \quad \varphi$ is surjective

Def: Let $A \subset X$ be a subspace

a continous map: $r: X \to A$ is retraction

s.t. $r|A = id_A$ (↪ 嵌入)

$\Leftrightarrow \quad r \circ i = id_A$. where $i: A \hookrightarrow X$ is the inclusion

Example: $r: \mathbb{R}^n - \{0\} \to S^{n-1}$

$r(x) = \frac{x}{\|x\|}$

Lemma 55.1 if $r: X \to A$ is a retraction

then r_* $\pi_1(X, a_0) \to \pi_1(A, a_0)$ is surjective

and i_* $\pi_1(A, a_0) \to \pi_1(X, a_0)$ is injective

$(a_0 \in A \subset X)$

proof: $r \circ i = id_A \quad \Rightarrow \quad (r \circ i)_* = (id_A)_*$

$\Rightarrow \quad r_* \circ i_* = id_{\pi_1(A, x_0)}$ is bijective

then r_* is surjective. (因为定义域. r_* 没必要单射)

i_* is injective

Theorem 55.2. there is no retraction $i: B^2 \to S^1$

$(B^2 = \{ x \in \mathbb{R}^2 \mid \|x\| \leq 1 \}.\ S^1 = \partial B^2)$

proof: if $r: B^2 \to S^1$ is a retraction

then r_* $\pi_1(B^2, 1) \to \pi_1(S^1, 1)$ is surjective

$\pi_1(B^2,1)$ is trivial group (straight line homotopy)
$\pi_1(S^1,1) \cong \mathbb{Z}$ not trivial.
contradiction.

Remark: there is a retraction
$$r: B^2 - \{0\} \to S^1$$
$$r(x) = \frac{x}{\|x\|}$$

Theorem: 55.6 (Brouwer fixed-point theorem in $d=2$)
Every continuous map: $f: B^2 - B^2$ has a fixed point
$f(x_0) = x_0$
proof suppose $f: B^2 \to B^2$ has no fixed point
$\forall x \in B^2$. $f(x) \neq x$.
$r: B^2 \to S^1$
we can construct a vector $v(x) = f(x) - x$ crossing S^1 at a point. and $r|_{S^1} = id_{S^1}$
r is continuous since f is continuous
r is retraction. contradiction.

Exercise: $B^1 = [-1,1] \subset \mathbb{R}^1$
every continuous map $f: B^1 \to B^1$ has fixed point
Similar proof.

$$\left.\begin{aligned}&\pi_1(\mathbb{R}^n, 0)\\ &\pi_1(B^n, 0)\\ &\pi_1(\text{convex subset of } \mathbb{R}^n,\ x_0)\end{aligned}\right\} \to \text{trivial}$$

Lemma: if a continuous map $f: S^1 \to X$ extends to a continuous map $F: B^2 \to X$ with $F|S^1 = f$. then $f_*: \pi_1(S^1, 1) \to \pi_1(X, f(1))$ is the trivial homomorphism

proof: let $i: S^1 \hookrightarrow B^2$ be the inclusion

$\Rightarrow f = F \circ i$

$\Rightarrow f_* = (F \circ i)_* = F_* \circ i_*$

$$\underbrace{\pi_1(S^1, 1)}_{\mathbb{Z}} \xrightarrow{i_*} \underbrace{\pi_1(B^2, 1)}_{\text{trivial group}} \xrightarrow{F_*} \pi_1(X, F(1))$$

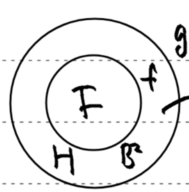
$\Rightarrow f_*$ is trivial

Lemma: If $f \underset{H}{\simeq} g \quad f, g: S^1 \to X$

If f extends to B^2. then also g extends to B^2.

(f_* trivial and g_* is trivial)

proof: $H: S^1 \times I \to X$ be a homotopy between f and g.



$\to S^1 \times I \to X$

ball of $r=2$

a big B^2. F define the small ball. H define the ring

$B^2 \xrightarrow{G} X$. by the pasting Lemma. G is continuous.

Theorem 56.1 (Fundamental theorem of algebra)

Every non-constant complex polynomial

$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has a root in $\mathbb{C}$

$(a_{n-1} \cdots a_0 \in \mathbb{C};\ n \geq 1)$

proof: case 1: suppose $|a_{n-1}| + |a_{n-2}| + \cdots + |a_0| < 1$.

suppose $p(x)$ has no root. $p(x) \neq 0$

let $g: B^2 \to \mathbb{C} - \{0\}$ $\quad B^2 \subset \mathbb{C}$

$g(x) = p(x)$

and $f: S^1 \to \mathbb{C} - \{0\}$ $\quad f(x) = p(x)$ $\quad f = g|S^1$

so f extends to B^2 (g)

privious Lemma $\Rightarrow$ f_* is trivial

$f_* \quad \pi_1(S^1, 1) \to \pi_1(\mathbb{C} - \{0\}, f(1))$ 没证 不用

$\pi_1(S^1, 1) \cong \mathbb{Z}$. $\quad \pi_1(\mathbb{C} - \{0\}, f(1)) \cong \mathbb{Z}$

so f_* is not trivial. contradiction

let $k: S^1 \to \mathbb{C} - \{0\}$

$x \to x^n$

$\Rightarrow f \underset{H}{\simeq} k \quad : S^1 \to \mathbb{C} - \{0\}$ $\quad H: S^1 \times I \to \mathbb{C} - \{0\}$

$H(x,t) = x^n + t(a_{n-1}x^{n-1} + \cdots + a_0)$

$|H(x,t)| \geq x^n - |t(a_{n-1}x^{n-1} + \cdots + a_0)|$

$\geq 1 - |t(|a_{n-1}| + \cdots + |a_0|)| > 0$

$\Rightarrow H(x,t) \neq 0$

f extends to a map $g: B^2 \to \mathbb{C} - \{0\}$

$f \simeq k \Rightarrow k$ extends to a map $B^2 \to \mathbb{C} - \{0\}$

privious lemma: $k_* : \pi_1(S^1, 1) \to \pi_1(\mathbb{C} - \{0\}, 1)$ is trivial

$$\begin{tikzcd}
\pi_1(S^1,1) \cong \mathbb{Z} \arrow[r, "k_*"] \arrow[dr, "\cdot n"'] & \pi_1(\mathbb{C}-\{0\},1) \arrow[d, "r_*"] \\
 & \pi_1(S^1,1) \cong \mathbb{Z}
\end{tikzcd}$$

$$r(x) = \frac{x}{\|x\|}$$

since: $k: x \to x^n$.

$1 \in \mathbb{Z} \to n \in \mathbb{Z}$

$n > 0$. $\Rightarrow$ k_* is not trivial. Contradiction

case 2 : let $x = cy$. $c \in \mathbb{R}^+$

$\Leftrightarrow \quad c^n y^n + a_{n-1} c^{n-1} y^{n-1} + \cdots + a_0 = 0$

$\Leftrightarrow \quad y^n + \frac{a_{n-1}}{c} y^{n-1} + \cdots + \frac{a_0}{c^n} = 0$

we choose c large enough.

$$\left|\frac{a_{n-1}}{c}\right| + \cdots + \left|\frac{a_0}{c^n}\right| < 1$$

$\Rightarrow$ case 1. $p'(y)$ has root $\Rightarrow$ $p(x)$ has root.

Exercise: $p: E \to B$ covering $p(e_0) = b_0$. B connected.

Let $f: X \to B$ continuous $f(x_0) = b_0$

Then. if exist a lift $\tilde{f}$ $X \to E$ with $\tilde{f}(x_0) = e_0$ it is unique.

Lecture 19

Fundamental group of some 2-manifolds

Theorem 59.1 (special case of Van Kampen theorem)

$X = A \cup B$. A, B are open. $A \cap B$ path-connected

let $x_0 \in A \cap B$

If $\pi_1(A, x_0)$ and $\pi_1(B, x_0)$ are trivial, then also $\pi_1(X, x_0)$ is trivial.

Proof: Let $[w] \in \pi_1(X, x_0)$ $w: I \to X$ path with $w(0) = w(1) = x_0$

have to prove that w is path-homotopic to the constant path c_{x_0}

by Lebesgue number Lemma. apply it to the open covering $\{w^{-1}(A), w^{-1}(B)\}$ of $I = [0,1]$

there is $t_0 = 0 < t_1 < \cdots < t_l = 1 \in [0,1]$

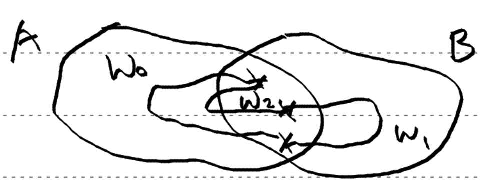
s.t. $w([t_{i-1}, t_i]) \subset A$ or B

alternatily in A and B (互轮流在A和B中)

$\Rightarrow w(t_i), w(t_{i+1}) \in A \cap B$

let $w_i = w|[t_i, t_{i+1}]$ reparanized to $[0,1]$

so $w_i | [0,1] \to X$



$\Rightarrow \quad w \simeq \underbrace{w_0 * w_1 * \cdots w_{l-1}}_{\text{reparamization of } w}$ (节点都在$A\cap B$)

let $k_i : I \to A\cap B$ be a path from x_0 to $w_i(1)$

$\Rightarrow \quad w \simeq \big(w_0 * (k_0^{-1})\big) * \big((k_0) * w_1 * (k_1^{-1})\big) * k_1) * \cdots * (k_{l-2}^{-1} * \big((k_{l-2}) * w_{l-1}\big)$

constant   constant

all closed path in A or B.

$\Rightarrow$ all homotopic to c_{x_0} since $\pi_1(A, x_0)$, $\pi_1(B, x_0)$ trivial.

$\Rightarrow \quad w \simeq c_{x_0} \quad [w] = [c_{x_0}]$

(书上还有更一般的形式)

Theorem: 59.3. For $n \geq 2$. $\pi_1(S^n, x_0)$ is trivial

(path-connected . trivial fundamental group $\to$ simply connected)

proof: let $S^n_+ = S^n - \{(0,0,\cdots,-1)\}$

$S^n_- = S^n - \{(0,0,\cdots,1)\}$ $S = S^n_+ \cup S^n_-$

then $S^n_+ \cong \mathbb{R}^n$. $S^n_- \cong \mathbb{R}^n$

$S^n_\pm \to \mathbb{R}^n$: $f(x) = f(x_1 \cdots x_{n+1}) = \frac{1}{1 \mp x_{n+1}}(x_1, \cdots x_n)$ homeomorphism

$\Rightarrow \pi_1(S^n_+, x_0)$ $\pi_1(S^n_-, x_0)$ are trivial.

$S^n_+ \cap S^n_- = S^n - \{(0, \cdots 0, \pm 1)\} \cong \mathbb{R}^n - \{0\}$

which is path connected since $n \geq 2$.

by van-Kampen theorem $\Rightarrow \pi_1(S^n, x_0)$ is trivial.

Example: $\mathbb{R}^n - \{0\} \cong S^{n-1} \times \mathbb{R}^+$

$x \to \frac{x}{\|x\|} \times \|x\|$

Proposition: $\pi_1(X \times Y, x_0 \times y_0) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$

proof: $\pi_1(X \times Y, x_0 \times y_0) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$

$[w] \to [P_1 \circ w] \times [P_2 \circ w]$

$P_1 : X \times Y \to X$

$P_2 : X \times Y \to Y$ projection

is well-defined homomorphism.

inverse: $\varphi : \pi_1(X, x_0) \times \pi_1(Y, y_0) \to \pi_1(X \times Y, x_0 \times y_0)$

$[w_1]$ $[w_2] \to [w_1 \times w_2]$

$(w_1 \times w_2)t = w_1(t) \times w_2(t)$

well-defined.

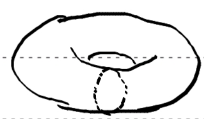
Corollary: $\pi_1(\mathbb{R}^3-\{0\}, x_0) \cong \pi_1(S^1, x_0) \times \underbrace{\pi_1(\mathbb{R}^+, 1)}_{\text{trivial}}$

$$\cong \pi_1(S^1, x_0)$$
$$\cong \mathbb{Z}$$

(isomorphism given by the rotation number)

Corollary: $S^1 \times S^1$

$S^1 \times S^1 \subset \mathbb{C} \times \mathbb{C} \subset \mathbb{R}^4$

 $\subset \mathbb{R}^3$

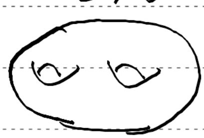
$\pi_1(S^1 \times S^1, 1 \times 1) \cong \pi_1(S^1, 1) \times \pi_1(S^1, 1)$

$\cong \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$

Corollary: S^2 and $S^1 \times S^1$ are not homeomorphic.

S^2: orientable surface of genus 0

$S^1 \times S^1$ orientable surface of genus 1

 " " of genus 2

real projective space $\mathbb{R}P^n$.

$\mathbb{R}^3$ vector space

$\mathbb{R}P^2$ = all lines in $\mathbb{R}^3$ through 0

= all subspace of $\mathbb{R}^3$ of dim-1

each line through 0 intersects $S^2 \subset \mathbb{R}^3$ in 2 antipodal points equivalence relation on S^2:

$\{x, -x\} = [x]$

$x \in S^2$. equivalence class of x

$p: S^2 \to \mathbb{RP}^2 = \{[x] : x \in S^2\}$

$p(x) = [x]$

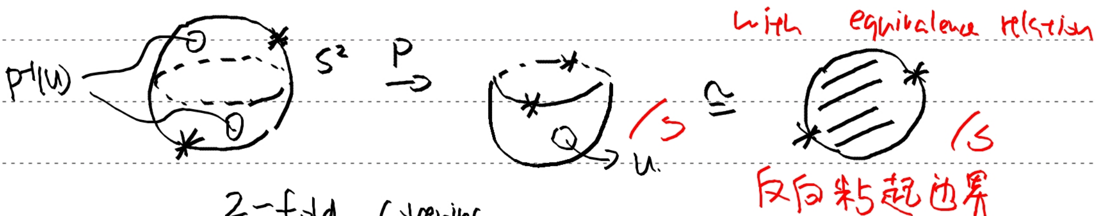
$S^2 \subset \mathbb{R}^3$. subspace topo.

$\mathbb{RP}^2$ has the quotient topo.

$U \subset \mathbb{RP}^2$ is open $\Leftrightarrow$ $p^{-1}(U)$ is open in S^2

the finest topo. on $\mathbb{RP}^2$ s.t. p is continuous

$\Rightarrow$ $p: S^2 \to \mathbb{RP}^2$ covering



2-fold covering.

each fiber has 2 elements: antipodal points

Theorem $\pi_1(\mathbb{RP}^2, [x_0]) \cong \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$

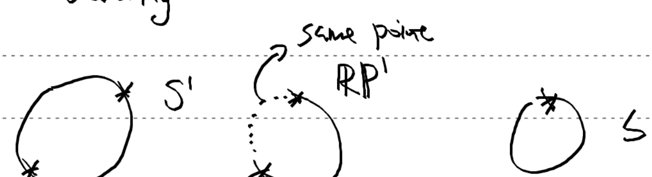
group with 2 element

proof: define $\mathbb{RP}^n = \{[x] = \{x, -x\}, x \in S^n\}$

$p: S^n \to \mathbb{RP}^n$ $\quad p(x) = [x]$

2-fold covering

example: $\mathbb{RP}^1$.



$\mathbb{RP}^1 \cong S^1$

Corollary: $\mathbb{R}P^2$ is not homeomorphic to S^2

$\mathbb{R}P^2$ = non-orientable surface of genus 1

$\mathbb{R}P^2$ 和 $\mathbb{R}P$ 区别和万有覆盖不同有关.

proof (more general case) $\pi_1(\mathbb{R}P^n, [x_0]) \cong \mathbb{Z}_2$, $n>1$

$(\pi_1(\mathbb{R}P, [x_0]) \cong \pi_1(S^1, x_0) \cong \mathbb{Z})$ 一维没法转向

$p: S^2 \to \mathbb{R}P^2$ $\quad p(x) = [x] = \{x, -x\}$.

2-fold covering

$$\underset{\substack{\Downarrow \\ [w]}}{\pi_1(\mathbb{R}P^2, [x_0])} \longmapsto (\{-1, 1\}, \cdot) \cong \mathbb{Z}_2$$

$w: I \to \mathbb{R}P^2$. path $\quad w(0) = w(1) = [x_0]$. $x \in S^2$

let $\tilde{w}: I \to S^2$ be a lifting of w.

with $\tilde{w}(0) = x_0 \in S^2$.

then $\tilde{w}(1) = \varepsilon x_0$. $\varepsilon = \pm 1$

let $\varphi([w]) \equiv \varepsilon \in \{-1, 1\}$.

then 1). φ is well defined by the homotopy lifting Lemma

2) φ is a homomorphism of groups

3) φ is injective:

Let $[w] \in \ker(\varphi) \Rightarrow \tilde{w}(1) = x_0 = \tilde{w}(0)$

$(\varepsilon([w]) = 1)$

$\Rightarrow \tilde{w}$ is a closed path in S^2. $[\tilde{w}] \in \underset{\text{trivial}}{\pi_1(S^2, x_0)}$

$\Rightarrow [\tilde{w}] = [c_{x_0}]$ $\quad \tilde{w} \underset{H}{\simeq} c_{x_0}$

$\Rightarrow w = p \circ \tilde{w} \underset{p \circ H}{\simeq} p \circ c_{x_0} = c_{[x_0]}$. $[w] = [c_{[x_0]}]$

$\Rightarrow \ker(\varphi)$ is trivial.

φ is injective.

4) φ is surjective

let u be a path in S^2 from x_0 to $-x_0$

and $w = p \circ u$.

$\Rightarrow$ w is a closed path. $[w] \in \pi_1(\mathbb{R}P^2, [x_0])$

$\tilde{w} = u$, $\tilde{w}(1) = u(1) = -x_0$.

$\varphi([w]) = -1$.

$\Rightarrow$ φ is surjective

surface.	fundamental group
S^2	trivial
torus	$\mathbb{Z}^2$
$\mathbb{R}P^2$	$\mathbb{Z}_2$

很多更 general 的定理还要看看.

§53- §60

2). $\varphi([w] * [w']) = \varphi([w * \underbrace{w(1)}_{} \cdot w')])$

$= \varepsilon \cdot \varepsilon' = \varphi([w]) \cdot \varphi([w'])$

homomorphism

Lecture 20

Homotopy equivalence

Def let $f: X \to Y$ be continuous. f is a homotopy equivalence if there is a continuous map $g: Y \to X$ s.t. $g \circ f \simeq id_X$ $f \circ g \simeq id_Y$

Exercise a homeomorphism f:

X and Y are homotopy equivalent if there is a homotopy equivalence $f: X \to Y$: this is an equivalence relation for topo space.

Def: a topo space X is contractible if id_X is homotopic to a constant map c_{x_0} i.e. $id_X \simeq c_{x_0}$. $x_0 \in X$.

Exercise X is contractible

$\iff$ X is homotopy equivalent to a 1-point space

proof: "$\Rightarrow$" $id_X \simeq c_{x_0}$. $x_0 \in X$.

$f: X \to \{x_0\}$ constant.

$g: \{x\} \hookrightarrow X$ inclusion

$f \circ g = id_{\{x_0\}}$. $g \circ f = c_{x_0} \simeq id_X$

"$\Leftarrow$" $f: X \to \{x_0\}$. $g: \{x_0\} \to X$.

$g \circ f = c_{g(x_0)} \simeq id_X$. $g(x_0) \in X$.

X is contractible.

Example. $\mathbb{R}^n$ is contractible

$id_{\mathbb{R}^n} \underset{H}{\simeq} c_0 \qquad 0 \in \mathbb{R}^n$

$H(x,t) = (1-t)x$

Example:

$X \simeq X \times I$

$f : X \times I \to X \qquad f(x \times t) = x$

$g : X \to X \times I \qquad g(x) = x \times 0$

$f \circ g = id_X$

$g \circ f : x \times t \to x \times 0$

$g \circ f \underset{H}{\simeq} id_{X \times I}$

$H((x \times t), s) = x \times st.$

Exercise: If Y is contractible then $X \simeq X \times Y$

Example: $\mathbb{R}^n - \{0\} \cong S^{n-1} \times \mathbb{R}^+ \simeq S^{n-1}$

$r \quad \mathbb{R}^n - \{0\} \to S^{n-1} \qquad x \to \frac{x}{\|x\|}$

$i : S^{n-1} \hookrightarrow \mathbb{R}^n - \{0\}$ inclusion

$r \circ i = id_{S^{n-1}}$ (r is retraction)

$i \circ r \underset{H}{\simeq} id_{\mathbb{R}^n - \{0\}}$

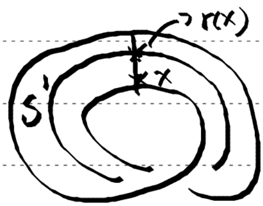
with $H(x,t) = (1-t)x + t\frac{x}{\|x\|}$

straight line homotopy

Def: a retraction $r: X \to A$ ($r \circ i = id_A$)
is a deformation retraction if
$i \circ r \simeq id_X$.

$\Rightarrow$ a deformation retraction is homotopy equivalence

Ex: Mobius band $M \simeq S^1$



$M \subset \mathbb{R}^3$

the central line is S^1

$r: M \to S^1$ is a retraction

$i \circ r \simeq id_M$

$\Rightarrow$ r is a deformation retraction

$\Rightarrow$ $M \simeq S^1 \simeq S^1 \times I$ cylinder.

Proposition: let $H: X \times I \to Y$ be a homotopy

$f_t(x) = H(x,t)$ for a fixed $t \in I$

$f_0 \underset{H}{\simeq} f_1$

fixed $x_0 \in X$. let $\alpha: I \to Y$ be the path

$\alpha(t) = f_t(x_0) = H(x_0, t)$.

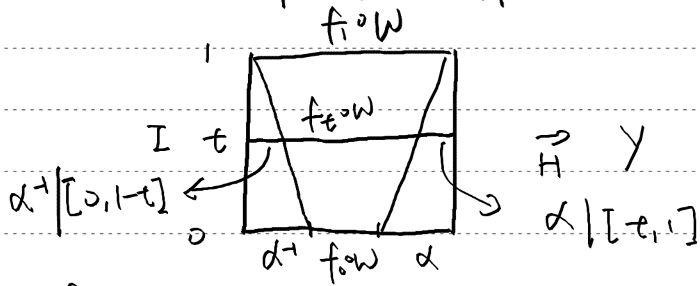
then $\hat{\alpha} \circ (f_0)_* = (f_1)_*$

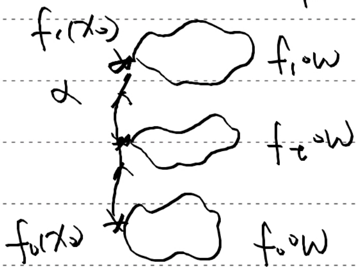
$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "(f_0)_*"] \arrow[dr, "(f_1)_*"'] & \pi_1(Y, f_0(x_0)) \arrow[d, "\hat{\alpha}", "\cong"'] \\
& \pi_1(Y, f_1(x_0))
\end{tikzcd}$$

proof: let $[w] \in \pi_1(X, x_0)$ $\qquad \hat{\alpha} = [\alpha^{-1}] * \cdots * [\alpha]$

have to prove: $[\alpha^{-1}] * [\underbrace{f_0 \circ w}_{(f_0)_*[w]}] * [\alpha] = [\underbrace{f_1 \circ w}_{(f_1)_*[w]}]$

$\Leftrightarrow \quad \alpha^{-1} * (f_0 \circ w) * \alpha \simeq_H f_1 \circ w$





Theorem: if: $f: X \to Y$ is a homotopy equivalence with $f(x_0) = y_0$. then $f_* \, \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.

proof: $\quad f: X \to Y \qquad g: Y \to X.$

$\qquad g \circ f \simeq id_X \qquad f \circ g \simeq id_Y$

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "f_*"] \arrow[d, "\hat{\alpha}"', "\cong"] & \pi_1(Y, y_0) \arrow[dl, "g_*"] \\
\pi_1(X, g(y_0)) &
\end{tikzcd}$$

$$\left.\begin{aligned} (g\circ f)_* &= g_*\circ f_* \\ (id_X)_* &= id_{\pi_1(X,x_0)} \end{aligned}\right\} \quad g_*\circ f_* = \hat{\alpha}\circ id_{\pi_1(X,x_0)} = \hat{\alpha}$$

f_*: injective $\qquad g_*$: surjective

$$\begin{tikzcd}
\pi_1(X,x_0) \arrow[r, "f_*", "\text{injective}"'] \arrow[d, "\alpha"', "\cong"] & \pi_1(Y,y_0) \arrow[d, "\beta", "\cong"'] \arrow[dl, "g_*"', "\text{injective, surjective}"] \\
\pi_1(X,g(y_0)) \arrow[r, "f'_*"', "\text{surjective}"] & \pi_1(Y,f(g(y_0)))
\end{tikzcd}$$

$\Rightarrow$ g_* is also an isomorphism

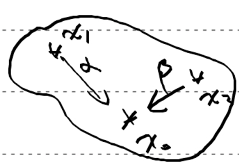
by symmetry. f_* is also an isomorphism
(invert the roles of f and g)

Corollary: if X is contractible, then
$\pi_1(X,x_0)$ is the trivial group

proof: $id_X \simeq C_{x_0}$ constant map $\Leftrightarrow$ $X \simeq \{x_0\}$
$f: X \to \{x_0\}$ homotopy equivalence
$\Rightarrow$ f_* is isomorphism.
$\pi_1(X,x_0)$ is trivial

Exercise: X contractible $\Rightarrow$ X is path connected.

$id_X \underset{H}{\simeq} C_{x_0}$



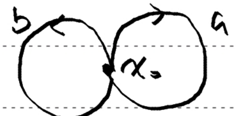
$\Rightarrow \alpha(t) = H(x_1,t) \qquad (\beta(t) = H(x_2,t))$

$\Rightarrow \alpha * \beta^{-1}$ is a path from any x_1 to x_2

Corollary: if X, Y are path connected, homotopy equivalent. then $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$ $\forall x_0 \in X, y_0 \in Y$

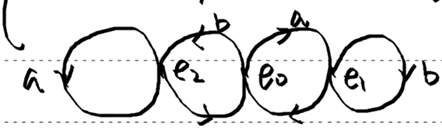
Problem: S^2 is simply connected. Is S^2 contractible? NO!

2 circles with "figure 8"

$B =$  $\subset \mathbb{R}^2$

p

$a * b \not\simeq b * a$ $\quad [a]*[b] \neq [b]*[a]$



p is a 3-fold covering

$p^{-1}(x_0) = \{e_0, e_1, e_2\}$

$\widetilde{a*b}$. initial point e_0. has end point e_1

$\widetilde{b*a}$ initial point e_0. has end point e_2

$\Rightarrow$ $a*b \not\simeq b*a$

(If the are homotopic. the liftings have always the same end point by the homotopy lifting lemma)

$\Rightarrow \pi_1(B, x_0)$ is not ablien.

$\pi_1(B, x_0) \cong \mathbb{Z} * \mathbb{Z}$ free product.

(free non ablien group of order 2)

free ablien group is $\mathbb{Z} \times \mathbb{Z} \cong \pi_1(\text{tours})$