Topology

ICTP-MATH.

Lecture 01

可以定义 0 个集合的并为空集.
可以定义 0 个集合的交为全集.
拓扑中的空集和全集是通过交、并操作得到的.

Basis:

Def: X set. a basis for topology on X is a collection $\mathcal{B}$ of subsets of X. s.t.

① for each $x \in X$. there is $B \in \mathcal{B}$ s.t. $x \in B$

$$\bigcup \{ B : B \in \mathcal{B} \} = X$$

$$\bigcup_{B \in \mathcal{B}} B = X$$

② given $B_1, B_2 \in \mathcal{B}$ and $x \in B_2 \cap B_2$. $(x \in X)$
then $x \in B_3 \subset B_1 \cap B_2$

Def Topology $\mathcal{T}$ generated by the basis $\mathcal{B}$:

$U \in \mathcal{T}$: if for every $x \in U$. there is $B_x \in \mathcal{B}$.

s.t. $x \in B_x \subset U$

Claim: $\mathcal{T}$ is a topology: remark: $B \in \mathcal{B}$ is also in $\mathcal{T}$

proof: ① $\phi \in \mathcal{T}$. $X \in \mathcal{T}$ by the first condition of basis

② "arbitary unions".

let $x \in \bigcup_{\alpha \in I} U_\alpha \Rightarrow x \in U_{\alpha_0}$. $\alpha_0 \in I$

$\Rightarrow \exists B_x$. $x \in B_x \subset U_{\alpha_0} \subset \bigcup_{\alpha \in I} U_\alpha$

$\Rightarrow \bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$ is open

③ "finite intersections"

by induction. it suffices to proove two elements.

$U_1, U_2 \in \mathcal{T}$. let $x \in U_1 \cap U_2$. than $x \in U_1$, $x \in U_2$

$\exists B_1, B_2 \in \mathcal{B}$. $x \in B_1 \subset U_1$. $x \in B_2 \subset U_2$

then $x \in B_1 \cap B_2 \subset U_1 \cap U_2$. By the second condition for basis
$\exists B_3 \subset B_1 \cap B_2 \subset U_1 \cap U_2 \Rightarrow U_1 \cap U_2 \in \mathcal{T}$
(ϕ自动满足，因为ϕ中不存在一个点不满足要求)

Lemma 13.1: Let $\mathcal{B}$ a basis for a topology $\mathcal{T}$ on X.
Then $\mathcal{T}$ is equal to the collection of all unions of elements of $\mathcal{B}$

proof: $B \in \mathcal{B}$ is also in $\mathcal{T}$. $\mathcal{T} = \{\cup B\}$

$\mathcal{T}$ is a topo. the unions of B is in $\mathcal{T} \Rightarrow \{\cup B\} \subset \mathcal{T}$

Conversely, given $\forall U \in \mathcal{T}$ for $\forall x \in U$. $\exists B_x \in \mathcal{B}$
s.t. $x \in B_x \Rightarrow U = \bigcup_{x \in U} B_x$. $\Rightarrow \mathcal{T} \subset \{\cup B\}$

So $\mathcal{T}$ equals to the collection of unions of $B \in \mathcal{B}$

Lemma 13.2: Let X be a topology space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X, and each $x \in U$, $\exists C \in \mathcal{C}$ s.t. $x \in C \in U$.

Then $\mathcal{C}$ is a basis for the topo $\mathcal{T}$ on X.

proof: 1. prove that $\mathcal{C}$ is basis

2. prove that the topo $\mathcal{T}'$ generated by $\mathcal{C}$ is equal to $\mathcal{T}$

Lemma 13.3: Let $\mathcal{B}$, $\mathcal{B}'$ be bases for topologies $\mathcal{T}$, $\mathcal{T}'$.
then following are equivalent.

(1) $\mathcal{T}'$ is finer than $\mathcal{T}$

(2) For each $B \in \mathcal{B}$ and $x \in B$, there is $B' \in \mathcal{B}'$, s.t. $x \in B' \subset B$.

proof: (2) $\Rightarrow$ (1)

Given $U \in \mathcal{T}$. $x \in U$. Since $\mathcal{B}$ generates $\mathcal{T}$.

s.t. $x \in B \subset U$. there is $B' \in \mathcal{B}$. $x \in B' \subset B$.

s.t. $x \in B' \subset B \subset U$. therefore: $U \subset \mathcal{T}'$

(1) $\Rightarrow$ (2)

Given $B \in \mathcal{B}$. $x \in X$ and $x \in B$. B itself is in $\mathcal{T}$

since $\mathcal{T}'$ is finer than $\mathcal{T}$. so $B \in \mathcal{T}'$ generated by $\mathcal{B}'$

there is $B' \in \mathcal{B}'$ s.t. $x \in B' \subset B$

Def: standard topo on $\mathbb{R}$.

topo $\mathbb{R}_l$

topo $\mathbb{R}_K$

lemma 13.4. (1). $\mathbb{R}_l$ & $\mathbb{R}_K$ are strictly finer than standard topo.

(2) $\mathbb{R}_l$ & $\mathbb{R}_K$ are not comparable.

proof (1) obviously

(2) $[1, 2) \in \mathbb{R}_l$

there is no set. U_K in $\mathcal{B}_K$ s.t. $1 \in U_K \subset [1, 2)$

$(-1, 1) - K \in \mathbb{R}_K$

there is no set U_l in $\mathcal{B}_l$ s.t. $0 \in U_l \subset (-1, 1) - K$

if $[a, b)$ satisfies this requirement.

$a \leq 0$. $b > 0$. so $\exists\ 0 < \frac{1}{n} < b$. $\frac{1}{n} \in [a, b)$. $\frac{1}{n} \notin (-1, 1) - K$.

contradiction.

Lecture 02.

Def: A subbasis $\mathcal{S}$ for a topo on X is a collection of subsets of X whose union equals X. (for each $x \in X$, $\exists\ S \in \mathcal{S}$ .s.t. $x \in S$)

(Remarks. not require the intersection condition of basis.)

the topo generated by $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of elements in $\mathcal{S}$

($S \in \mathcal{S}$. then $S \cap S = S \in \mathcal{T}$.)

claim: $\mathcal{T}$ is a topo.

proof: the collection of finite intersections of $S \in \mathcal{S}$ is a basis $\mathcal{B}$, then by lemma 13.1. all unions of $B \in \mathcal{B}$ is a topo generated by $\mathcal{B}$.

Given $x \in X$. $x \in S \in \mathcal{S}$. then $x \in S = S \cap S \in \mathcal{B}$.

it is the first condition of basis.

let $B_1 = S_1 \cap \cdots \cap S_n$, $B_2 = S_1' \cap \cdots \cap S_m' \in \mathcal{B}$.

then $B_1 \cap B_2 = B_3 \in \mathcal{B}$ $x \in B_1$. $x \in B_2$. $x \in B_3$.

$B_3 \subset B_1 \cap B_2$

therefore $\mathcal{B}$ is a basis

The Order Topology:

let X a simple order set. which means that: 1. If $x, y \in X$ and $x \neq y$. then $x < y$ or $y < x$.

2. if $x < y$. $y < z$. then $x < z$

3. $x < x$ is wrong (存在 $\leq$)

Def: let X be a set with simple order.

let $\mathcal{B}$ is a collection of following types of sets:

$$\mathcal{B} = \text{all } \{ (x, y), \ \forall x < y, \ x, y \in X \}$$
$$\cup \text{ all } \{ [x_0, y), \ \forall y > x_0 \in X, \text{ if } x_0 \text{ is the smallest in } X \}$$
$$\cup \text{ all } \{ (x, y_0], \ \forall x < y_0 \in X. \text{ if } y_0 \text{ is the biggest in } X \}.$$

$\mathcal{B}$ is a basis of topo on X. which is called "order topo"

Example: dictionary order.

Def: X an ordered set. $a \in X$. rays determined by a:

$(a, +\infty)$ $(-\infty, a)$ $[a, +\infty)$ $(-\infty, a]$

Product Topology:

X and Y are topological spaces, there is a simple way to define on the cartesian product $X \times Y$

(证明 $B_1 \cap B_2 \in \mathcal{B}$ 就行)

Def: X, Y: topo spaces:

the product topo is generated by basis $\mathcal{B}$:

$$\mathcal{B} = \{ U \times V. \quad U \text{ is open in } X. \quad V \text{ is open in } Y. \}$$

claim: $\mathcal{B}$ is a basis.

proof: Since $X \times Y$ is an element of $\mathcal{B}$. the first condition is trivial

$U_1 \times V_1$, $U_2 \times V_2$ $\in \mathcal{B}$. $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$

$U_1 \cap U_2$ is open in X. $V_1 \cap V_2$ is open in Y. $(U_1 \times V_1) \cap (U_2 \times V_2) \in \mathcal{B}$.

Theorem 15.1: X. Y. topo spaces. if $\mathcal{B}_X$ is basis of X. $\mathcal{B}_Y$ is basis of Y. then collection

$\mathcal{B}' = \{ U' \times V' \mid U' \in \mathcal{B}_X.\ V' \in \mathcal{B}_Y \}$ is the basis $\mathcal{B}$ of product topo on $X \times Y$.

proof: apply lemma 13.2.

an open set $W \subset X \times Y$. point $x \times y \in W$.

there is a basis element of product topo = $U \times V$ s.t.

$x \times y \in U \times V \subset W$. $\quad U \times V \in \mathcal{B}$

U and V is open in X and Y.

there is $U' \in \mathcal{B}_X$ $V' \in \mathcal{B}_Y$ s.t. $x \in U' \subset U$. $y \in V' \subset V$

s.t. $x \times y \in U' \times V' \subset U \times V \subset W$. and $U' \times V' \in \mathcal{B}'$

by the lemma 13.2, $\mathcal{B}'$ is the basis of the same topo. (product topo)

Example: dictionary topo on $\mathbb{R}^2$ is strictly finer than the standard product topo.

$\mathbb{R}^2_{\text{dictionary}} = \mathbb{R}_d \times \mathbb{R}$

($\mathbb{R}$: real number with standard topo.)

($\mathbb{R}_d$ real number with discrete topo)

$(a \times b,\ c \times d)_{\text{dictionary}}$

$= \{a\} \times (b, +\infty) + (a, b) \times (c, d) + \{c\} \times (-\infty, d)$

Lecture 3

16. Subspace topology.

Def X topo space. $Y \subset X$. subset

$$\tau_Y = \{Y \cap U.\ U \in \tau\} \quad \tau: \text{topo of } X.$$

this is a topo on Y.

claim: τ_Y is a topo on Y

proof: $\phi = Y \cap \phi$. $Y = Y \cap X$.

$$(U_1 \cap Y) \cap \cdots \cap (U_n \cap Y) = (U_1 \cap \cdots \cap U_n) \cap Y$$

$$\bigcup_{\alpha \in J} (U_\alpha \cap Y) = (\bigcup_{\alpha \in J} U_\alpha) \cap Y$$

Lemma 16.1. if $\mathcal{B}$ is a basis for topo on X. then collection $\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for the subspace topo. on Y.

proof: Given U is open in X. and $y \in U \cap Y$. so $y \in U$

so there is $B \in \mathcal{B}$ s.t. $y \in B \subset U$.

s.t. $y \in B \cap Y \subset U \cap Y$. it follows lemma 13.2.

s.t. $\mathcal{B}_Y$ is a basis

Lemma 16.2. Let Y be a subset of X. If U is open in Y and Y is open in X. then U is open in X

proof: U is open in Y. $U = Y \cap V$, V is open in X. Since Y and V are open in X. $U = Y \cap V$ is open in X.

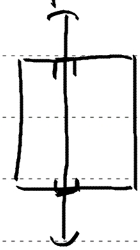
Example: $I = [-1, 1] \subset \mathbb{R}$.

$I^2 \subset \mathbb{R}^2$ $\mathbb{R}^2$ with dictionary order topo.

I^2 with the subspace topo inherits from $\mathbb{R}^2$ with dictionary.

this sub space topo is strictly finer than dictionary order topo on I^2.

illustration:



open in the subspace topo
but not open in the dictionary topo

(to prove that a set is open, we can prove that every point of this set is contained in a basis element contained in the set).

Theorem 16.3 if A is a subspace of X and B is a subspace of Y. then the product topo on $A \times B$ is the same as the topo inherits as a subspace of $X \times Y$

proof: $U \times V$ is a basis element of $X \times Y$.

the $(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$ is a basis element of subspace topo on $A \times B$

$U \cap A$ is open in subspace topo on A.

$V \cap B$ is open in sub space topo on B.

So $(U \cap A) \times (V \cap B)$ is the basis element of product topo on $A \times B$.

(means the topos are the same)

So the bases of subspace topo and the product topo are the same.

Given an order set X. we say that a subset $Y \subset X$ is convex if for each pair of points $a < b$ of Y. the entire interval (a, b) of points of X lies in Y.

（字典序 $\mathbb{R}^2$. $I = (-1, +1)$. I^2 不是 convex 的）

Theorem 16.4 Let X be an order set with order topo. let Y be a subset of X that is convex in X. Then the order topo on Y is the same as the subspace topo inherits X.

proof: $(a, +\infty)$ in X.

if $a \in Y$. $(a, +\infty) \cap Y = \{ x \mid x > a \text{ and } x \in Y \}$

an open ray in Y.

if $a \notin Y$. if $a <$ the lower bound of Y. $(a, +\infty) \cap Y = Y$

if $a >$ the upper bound of Y. $(a, +\infty) \cap Y = \emptyset$

the case $(-\infty, a)$ is similar. they are open in order topo on Y.

the sets $(a, +\infty) \cap Y$ and $(-\infty, a) \cap Y$ form a subbasis for the subspace topo on Y.

the order topo contains the subspace topo.

the reverse: any open ray in Y equals the intersections of open ray in X and Y. and since the open rays in Y form a subbasis of order topo on Y.

the subspace topo contains the order topo

so they are equal.

17: Closed sets & Limit Points

Def: subset $A \subset X$ is closed if $X - A$ is open

theorem 17.1: X and ϕ are closed

finite unions of closed sets are closed

- $\bigcup_i^n (X - U_i) = X - \bigcap_i^n U_i$ $\quad \bigcap_i U_i$ is open

arbitrary intersections of closed sets are closed

- $\bigcap_i (X - U_i) = X - \bigcup_i U_i$ $\quad \bigcup_i U_i$ is open

theorem 17.2. $Y \subset X$ a subspace of X. then $A \subset Y$ is closed if and only if it equals the intersection of a closed set of X with Y.

proof: $(\Leftarrow)$: $A = Y \cap C$ $\quad C$ is closed in X. $C = X - U$. U is open

$A = Y \cap (X - U) = Y \cap X - Y \cap U = Y - Y \cap U$.

$Y \cap U$ is open in Y.

s.t. $A = Y - Y \cap U$ is closed

$(\Rightarrow)$ A is closed in Y. $\quad A = Y - U_Y$. U_Y is open

s.t. $U_Y = U \cap Y$ $\quad U$ is open in X.

s.t. $A = Y - U \cap Y = X \cap Y - U \cap Y = (X - U) \cap Y$.

$X - U$ is closed in X.

Theorem. 17.3 $Y \subset X$ a subspace of X. if A is closed in Y and Y is closed in X. then A is closed in X.

Def: $A \subset X$ is a subset. the interior of A is defined as the union of all open set contained in A

$\text{Int}(A) = \mathring{A} = \bigcup\{U \text{ open in } X.\ U \subset A\}$, the largest open set contained in A, $\mathring{A}$ is open

the closure of A defined as the intersection of all closed sets containing A

$\text{Cl}(A) = \bar{A} = \bigcap\{C \text{ closed in } X,\ A \subset C\}$, the smallest closed set containing A. $\bar{A}$ is closed.

remarks: $\mathring{A} \subset A \subset \bar{A}$

theorem 17.4. $Y \subset X$ is a subspace. let A be a subset of Y. $\bar{A}$ is the closure of A in X. then the closure of A in Y equals $\bar{A} \cap Y$

proof: since $A \subset \bar{A}$ and $A \subset Y$.

then $A \subset \bar{A} \cap Y$ and $\bar{A} \cap Y \subset Y$, $\bar{A} \cap Y$ is closed

since $\bar{A}_Y$ is the intersection of all closed sets in Y containing A. so $\bar{A}_Y \subset \bar{A} \cap Y$

on the other hand. $\bar{A}_Y$ is closed in Y. so $\bar{A}_Y = C \cap Y$. where C is closed in X. since $A \subset \bar{A}_Y = C \cap Y$.

so $A \subset C$. since $\bar{A} = \bigcap\{C.\ C \text{ closed in } X.\ A \subset C\}$.

so $\bar{A} \subset C$. s.t. $\bar{A} \cap Y \subset C \cap Y = \bar{A}_Y$

so $\bar{A}_Y = \bar{A} \cap Y$.

也可以 $\bar{A}_Y = \bigcap\{C_Y.\ C_Y \supset A\}$. $C_Y = C \cap Y$.

$\Rightarrow \bar{A}_Y = (\bigcap\{C.\ C \text{ closed in } X.\ C \supset A\}) \cap Y = \bar{A} \cap Y$.

Theorem 17.5: let A be a subset of topo. space X.
(i) Then $x \in \bar{A}$ if and only if every open set U containing x intersects A.
(ii) Supposing the topo of X is given by a basis. then $x \in \bar{A}$ if and only if every basis element B containing x intersects A

proof: (i) 反证法:

$x \notin \bar{A} \Leftrightarrow$ there is an open set U containing x doesn't intersect A

$\Rightarrow$ if $x \notin \bar{A}$. the set $U = X - \bar{A}$ is open and contains x which doesn't doesn't intersect A.

$\Leftarrow$ if $x \subset U$ and U is open. U doesn't intersect A $X-U$ is closed and $X-U$ contains A. s.t. $\bar{A} \subset X-U$. therefore. x can't be in $\bar{A}$

(ii). Apply (i)

$\Rightarrow$ every basis element containing x must intersect A since every basis element is open

$\Leftarrow$ every basis element B containing x intersects A. so every open set U containing x intersects A because $B \subset U$. so $x \in \bar{A}$ by (i).

Def: neighbourhood of x: the open set containing x.

Def: $A \subset X$, $x \in X$, x is a limit point of A if every neighbourhood U of x intersects A in a point different from x.

Theorem 17.6. Let A be a subset of X. A' be the set of all limit points of A. then: $\bar{A} = A \cup A'$

proof: $\Rightarrow$ by 17.5. $A' \subset \bar{A}$. by definition $A \subset \bar{A}$

s.t. $A' \cup A \subset \bar{A}$

$\Leftarrow$. $x \in \bar{A}$. if $x \in A$. $x \in A \cup A'$ is trivial

if $x \notin A$. by 17.5. neighbourhood of x intersects A. but not at point x. so $x \in A'$

s.t. $\bar{A} \subset A' \cup A$.

Corollary 17.7. A subset of X is closed if and only if it contains its all limit points

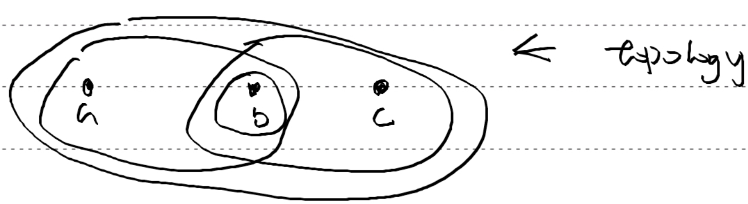
proof: A is closed if and only if $A = \bar{A}$.

$\bar{A} = A \cup A' = A$. s.t. $A' \subset A$

(极限点可以在 A 内)

Hausdorff space.

an important example and motivation:



对于一个数列 $x_n = b$ x 可以收敛于 a. b. c.

因为 $x_n = b$ 总也在 a. c 的邻域内

Def: A topological space is called a "Hausdorff space" if for each pair x_1, x_2 of distinct points of X there exists neighbourhood U_1 and U_2 of x_1 and x_2 respectively that are disjoint.

Theorem 17.8: Every finite point set in a Hausdorff space is closed.

proof it suffices to show every one point set $\{x_0\}$ is closed. If x is any different point from x_0, then x and x_0 have disjoint neighbourhood U and V.

since U doesn't intersect $\{x_0\} \subset V$. so x can not belong to the closure of $\{x_0\}$ so the closure of $\{x_0\}$ is itself. so $\{x_0\}$ is closed.

Hausdorff: T_2 axiom.

space where finite point set is closed: T_1 axiom

T_1 is weaker than T_2

T_1 axiom: $\exists U_1, U_2 \ni x_1 \in U_1 \quad x_2 \in U_2$

$x_1 \notin U_2 \quad x_2 \notin U_1$

(不要求 U_1 U_2 不相交. 只要求不互相包含 x_1, x_2

$T_1 \Leftrightarrow$ finite point set closed:

proof: $\Rightarrow$ trivial. X_0 存在不包含 x 的近邻. 所以 x 不属于闭包. 所以闭包为本身 闭集.

$\Leftarrow$ let x, y $x \neq y$.

$\{x\}$ is closed. $U-\{x\}$ is open.

and $y \in U-\{x\}$. Is a neighbourhood of y. not containing x

similarly: $U-\{y\}$ is a neighbourhood of x. not containing y.

so X is T_1

Lecture 04.

Example: let X be a set. let $\mathcal{T}$ be the collection of all subsets U of X s.t. $X-U$ either is finite or is all of X. Then $\mathcal{T}$ is a topology on X. called "finite complement topo"

Claim: X with finite complement topo is T_2

proof: $x_1, x_2 \in X$.

$X-\{x_2\}$ is open since $X-(X-\{x_2\}) = \{x_2\}$ is finite

$X-\{x_2\}$ is a neighbourhood of x_1 that does not contain x_2.

similar for x_2.

remark: if X is finite. it is discrete topo.

but not T_2 if X is infinite.

proof: $x_1, x_2 \in X$ and $x_1 \neq x_2$.

$\forall\, U_1, U_2$ are open in X and $x_1 \in U_1$, $x_2 \in U_2$

$X-U_1$ is finite $X-U_2$ is finite.

so $(X-U_1) \cup (X-U_2) = X-(U_1 \cap U_2)$ is finite

since U_1, U_2 are open, $U_1 \cap U_2$ is open.

so $X-(U_1 \cap U_2)$ is finite

since X is infinite. $U_1 \cap U_2 \neq \phi$.

so X is not T_2

Theorem 17.9. Let X be a space satisfying T_2 axiom. $A \subset X$. Then x is a limit point of A if and only if every neighbourhood of x contains infinitely many points of A.

proof: ($\Leftarrow$) if every U containing x intersects A in infinitely many points it certainly intersects A in some point other than X itself. so x is a limit point of A

($\Rightarrow$): x is a limit point of A and suppose that a neighbourhood U of x intersects A in only finite points. The U also intersects $A - \{x\}$ in finite points $\{x_1, x_2 \cdots x_m\}$

$\{x_1, x_2 \cdots x_m\}$ is closed. s.t. $X - \{x_1, x_2 \cdots x_m\}$ is open.

so $U \cap (X - \{x_1, x_2 \cdots x_m\})$ is a neighbourhood of x but that intersects the $A - \{x\}$ not at all. contradiction.

$U \cap (X - \{\cdots\}) = U - \{\cdots\}$. T_1 保证也不是 ϕ.

Def: X topo space. a sequence $x_1, x_2 \cdots$ in X converges to $x \in X$ if. for every neighbour, $\exists N \in \mathbb{N}$, s.t. $x_n \in U$ for all $n \geq N$: $x_n \to x$.

Observation: in T_2 space if x_n converges. then the limit x is unique.

如果不是. 存在 x_1 x_2. 有不相交邻域 U_1 U_2, x_n 在 U_1 中 则不可能在 U_2 中

Example: $\mathbb{N}$ with finite complement topo (T_1, not T_2)

1, 2, 3, 4, 5 ⋯⋯ $\to$ converges to 1. 2. 3 4. 5 ⋯⋯

收敛于所有数

Continuous function.

Def: X, Y topo spaces.

i): a func $f: X \to Y$ is continuous if preimages of open sets V in Y are still open in X:

$f^{-1}(V) = \{x \in X \mid f(x) \in V\}$ is open if V is open.

(global property)

ii). $f: X \to Y$ is continuous in a point $x \in X$ if for every neighbour V of $f(x)$, there exists a neighbour U of x s.t. $f(U) \subset V$.

(local property)

Theorem 18.1 $f: X \to Y$ is continuous

$\Leftrightarrow$ (4) f is continuous in every point $x \in X$

proof: "$\Rightarrow$" Let $x \in X$. V neighbour of $f(x)$

$U \equiv f^{-1}(V)$ is open in X

and $f(U) = V \subset V$.

"$\Leftarrow$" Let V be open in Y and $x \in f^{-1}(V)$

then $f(x) \in V$, V is neighbur of $f(x)$.

f is continous in $x \in X$. so there is a neighbour U_x of x s.t. $f(U_x) \subset V$.

then $f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$ is open in X.

Example: $id: \mathbb{R} \to \mathbb{R}_l$. $\mathbb{R}_l$: $\mathbb{R}$ with lower limit topo is not continous

$id^{-1}[a,b) = [a,b)$ is closed in $\mathbb{R}$.

id: $\mathbb{R}_l \to \mathbb{R}$ is continuous.

(2) 一个 $f(\bar{A}) \neq \overline{f(A)}$ 的例子.

$X = (0, 1]$ $Y = \mathbb{R}$ with standard topo.

id $X \to Y$. $A = (0, 1) \subset X$.

$f(A) = (0, 1]$. $\overline{f(A)} = [0, 1]$

Def: $f: X \to Y$ is a homeomorphism "$\cong$"

if f is a continuous bijection and

$f^{-1}: Y \to X$ is also continuous

Remarks. 一个双射 f 连续.

f^{-1} 不一定. 比如 $\mathbb{R}_l$ 和 $\mathbb{R}$ 那例子

所以一个连续双射不一定对应一个同胚

$\mathbb{R}$ 有可数基 (a, b) $a, b \in \mathbb{Q}$

$\mathbb{R}_l$ 没有可数基. ∵ 对于 $[x, x+1)$ 有不可数个

至少对应一个 $[x, y)$ $y > x$ 的基

Def: A topo space X is second countable if X has a countable base.

Corollary: $\mathbb{R}_l$ is not second countable.

TOPOLOGY INVARIENCE.

拓扑不变量.

X Y 同胚 用开集表示 X 的性质 Y 中也有对应属性

拓扑性质