Lecture 05

Theorem 18.1. X, Y topo spaces.

$f: X \to Y$ function. the followings are equivalent

i) f is continous

ii) for every subset A of X, one has $f(\bar{A}) \subset \overline{f(A)}$

iii) for every closed subset B of Y the set $f^{-1}(B)$ is closed in X

iv) for each $x \in X$ and each neighbor V of $f(x)$ there is a neighbor U of x s.t. $f(U) \subset V$ (即在点上连续)

proof: i) $\Rightarrow$ ii)

f is continous. let $x \in \bar{A}$ and let V be a neighbour of $f(x)$. then $f^{-1}(V)$ is open and $x \in f^{-1}(V)$

$f^{-1}(V)$ is a neighbour of x.

$x \in \bar{A}$. $(f^{-1}(V) - \{x\}) \cap A \neq \emptyset$

$f((f^{-1}(V) - \{x\}) \cap A) \neq \emptyset$

$(V - \{f(x)\}) \cap f(A) \neq \emptyset \Rightarrow f(x) \in \overline{f(A)}$

ii) $\Rightarrow$ iii) B is closed in Y and $A = f^{-1}(B)$

we want to show that A is closed $\Leftrightarrow A = \bar{A}$

$f(\bar{A}) \subset \overline{f(A)} = \bar{B} = B$ since B is closed

for every $x \in \bar{A}$. $f(x) \in f(\bar{A}) \subset B$

so $x \in f^{-1}(B) = B$ then $\bar{A} \subset A$. $\bar{A} = A$.

iii) $\Rightarrow$ i) Let V be open in Y. See $B = Y - V$ is closed.

so $f^{-1}(B) = f^{-1}(Y - V) = X - f^{-1}(V)$ is closed in X.

then $f^{-1}(V) = X - (X - f^{-1}(V))$ is open in X. f continous.

Theorem 18.2

i) constant function are continous. (把X全映到Y中一个点 y_0)

ii) let $A \subset X$ be a subspace then the inclusion

$$i: A \hookrightarrow X \quad \text{is continous}$$

proof: $i^{-1}(U) = A \cap U$. is open in A if U is open in X

under the subspace topo

$\Rightarrow$ i is continous.

remarks: the subspace topo the coarsest topo on A

s.t. the inclusion is continous.

(最小的让 inclusion 连续的 topo)

iii) a composition $g \circ f$ of continous func. is continous.

proof: U is open in Z.

$g^{-1}(U)$ is open in Y.

$f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is open in X.

iv) the restriction $f|A: A \to Y$ of a continous func

if $f: X \to Y$ is continous.

proof: the inclusion: $i: A \to X$ is continous

$f|A = f \circ i$ is continous

v) let $f: X \to Y$ be continous. and $f(X) \subset Z \subset Y$

then $g: X \to Z$ $g(x) = f(x)$, is continous

$f(X) \subset Y \subset Z$ then $h: X \to Z$ $h(x) = f(x)$, is continous

proof: 1°. let B is open in Z. then $B = Z \cap U$,

U is open in Y. because $f(X) \subset Z$.

$f^{-1}(U) = f^{-1}(U \cap Z) = f^{-1}(B) = g^{-1}(B)$ is open in X.

2° h is the composition of $f: X \to Y$ and inclusion: $i: Y \to Z$.

(vi) $f: X \to Y$ is continous if $X = \bigcup_\alpha U_\alpha$ s.t. $f|U_\alpha$ is continous $\forall \alpha$.

proof: let V be open Y. then:

$$f^{-1}(V) \cap U_\alpha = (f|U_\alpha)^{-1}(V)$$

$f|U_\alpha$ is continous so $(f|U_\alpha)^{-1}(V)$ is open in U_α and is also open in X. since $\bigcup_\alpha U_\alpha = X$

$$f^{-1}(V) = f^{-1}(V) \cap (\bigcup_\alpha U_\alpha) = \bigcup_\alpha (f|U_\alpha)^{-1}(V)$$ is open. ……

Theorem 18.3 (pasting lemma): (对于有限覆盖同样成立)

let $X = A \cup B$ where A. B closed in X.

let $f: A \to Y$ and $g: B \to Y$ continous.

if $f(x) = g(x)$ for $x \in A \cap B$.

let $h: X \to Y$: $h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$

is well defined and continous.

Proof: let C be closed in Y.

$$h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$$

$f^{-1}(C)$ is closed in A and $g^{-1}(C)$ is closed in B.

since A. B is closed. $f^{-1}(C)$. $g^{-1}(C)$ is closed in X

$h^{-1}(C)$ is closed in X. ……

(注: $U \subset X$ is open. U is closed in U. but open in X)

Product topo again:

Theorem 18.4 let $f: A \to X \times Y$ $f(a) = (f_1(a), f_2(a))$

Then f is continuous if and only if

$f_1: A \to X$. $f_2: A \to Y$ are both continuous.

proof: $\exists ! \lambda$ $\pi_1: X \times Y \to X$ $\pi_2: X \times Y \to Y$

$\pi_1^{-1}(U) = U \times Y$. $\pi_2^{-1}(V) = X \times V$

$\pi_1(f(a)) = f_1(a)$ $\pi_2^{-1}(f(a)) = f_2(a)$

... ...

Def: $\prod_{\alpha \in I} X_\alpha$. cartesian product $\{(x_\alpha).\ \alpha \in I, x_\alpha \in X_\alpha\}$

for $\beta \in I$: $\pi_\beta: \prod_{\alpha \in I} X_\alpha \longmapsto X_\beta$

be the projection onto the β'th coordinate.

$\pi_\beta((x_\alpha)_{\alpha \in I}) = x_\beta$. is continuous.

Define $S_\beta = \{\pi_\beta^{-1}(U_\beta),\ U_\beta \text{ is open in } X_\beta\}$

Consists of open sets. and also $S = \bigcup_{\beta \in I} S_\beta$.

Def: The topo on $\prod_{\alpha \in I} X_\alpha$ is the topo generated by the subbasis S. this topo is the coarsest topo s.t. all projections Π_β are continous

$$S \xrightarrow[\text{of } \pi_\beta^{-1}(U_\beta)]{\text{finite intersections}} B \text{ basis.}$$

Suppose $\beta_1, \dots, \beta_n \in J$ are different.

$\pi_{\beta_1}^{-1}(U_{\beta_1}) \cap \cdots \cap \pi_{\beta_n}^{-1}(U_{\beta_n})$

$= \prod_{\alpha \in J} U_\alpha$. if $\alpha = \beta_1 \cdots \beta_n$. $U_\alpha = U_{\beta_i}$

$\alpha \neq \beta_1 \cdots \beta_n$. $U_\alpha = X_\alpha$

Theorem 19.2:

$\mathcal{B} = \{ \prod_{\alpha \in J} U_\alpha$: U_α open in X_α for finitely many indices $\alpha = \beta_i$

$U_\alpha = X_\alpha$ for others $\}$

product topology 之后不会说明一般都是 product topo

$\mathcal{B}' = \{ \prod_{\alpha \in J} U_\alpha$. U_α open in $X_\alpha \}$ on $\prod_{\alpha \in J} X_\alpha$

box topology

The box topology is finer than product topology

finite : they are equal

infinite: The box topology is strictly finer.

考虑 product topo. 对有限叉乘成立的条件一般也适用于 任意叉乘

但考虑 box topo 时则不一定.

Theorem 19.6:

Let $f : A \to \prod_{\alpha \in J} X_\alpha$. with product topo be given by

$f(a) = (f_\alpha(a))_{\alpha \in J}$. f_α the component func.

so $f_\alpha = \pi_\alpha \cdot f$ then:

f is continuous $\Longleftrightarrow$ f_α is continuous $\forall \alpha$.

proof: let $\pi_\beta^{-1}(U_\beta) \in S$ subbasis for the product topo.

($\Rightarrow$): $f_\beta = \pi_\beta \cdot f$ is composite of π_β and f two continous func. f_β is continous

($\Leftarrow$) $f^{-1}(\pi_\beta^{-1}(U_\beta)) = (\pi_\beta \cdot f)^{-1}(U_\beta)$

$= f_\beta^{-1}(U_\beta)$ is open in A,

since f_β is continous

so f is continous

注：开集是子基的有限交的并.

开集的逆像是子基的逆像的有限交的并.

子基的逆像是开. 则开集的逆像也是.

($\Rightarrow$) is also right for box topo.

($\Leftarrow$) not always hold for box topo

Example: $\mathbb{R}^\omega = \prod_{n\in\mathbb{Z}^+} \mathbb{R}$, $\mathbb{R}$ is standard topo.

$\mathbb{R}^\omega = \{(x_1, x_2 \cdots) \quad x_i \in \mathbb{R}\}$

$f: \mathbb{R} \to \mathbb{R}^\omega$: $f(t) = (t, t, \cdots)$

if $\mathbb{R}^\omega$ has product topo.

f is continous since all componet func is continous

suppose $\mathbb{R}^\omega$ has box topo

then $\prod_{n\in\mathbb{Z}^+}(-\frac{1}{n}, \frac{1}{n})$ is open in $\mathbb{R}^\omega$.

$f^{-1}(\prod_{n\in\mathbb{Z}^+}(-\frac{1}{n}, \frac{1}{n})) = \bigcap_{n\in\mathbb{Z}^+}(-\frac{1}{n}, \frac{1}{n}) = \{0\}$

因为在 product topo 中 无法定义一个 $\prod_{n\in\mathbb{Z}^+}(-\frac{1}{n}, \frac{1}{n})$ 的无限叉乘开集.

避开了无穷交导致不开出现的问题

on $\mathbb{R}^{\omega}$: the box topo is strictly finer than product

Observation. $\prod_{n\in\mathbb{Z}^+}(-1,1)$ is open in box topo

but not contain any basis of product topo

$\mathrm{Int}(\prod(-1,1)) = \phi$ in product topo.

(注：有限交的并集反而会把有一些已经缩小的空间又放大，没法得到一个 $\prod U_\alpha$ 的形式)

Lecture 06

The metric topology:

Def: A metric on X is a func.

$$d: X \times X \to \mathbb{R}$$

having following properties:

(1) $d(x,y) \geq 0$

(2) $d(x,y) = d(y,x)$

and $d(x,y) = 0$ if and only if $x = y$

(3) $d(x,y) + d(y,z) \geq d(x,z)$

for $\varepsilon > 0$. $B_d(x,\varepsilon) = \{y \in X,\ d(x,y) < \varepsilon\}$

is open ball centred x with radius ε.

Def: $\mathcal{B} = \{B_d(x,\varepsilon) : x \in X,\ \varepsilon > 0\}$ is a basis for a topo on X, called metric topology.

Lemma: If $y \in B_d(x,\varepsilon)$ then there is $\delta > 0$ s.t. $B_d(y,\delta) \subset B_d(x,\varepsilon)$

Claim: $\mathcal{B}$ is a basis.

easy to prove

Def: a topo space X is metrizable if there is a metric d on X s.t. the metric topo associated with d is equal to the given topo $\mathcal{T}$ on X.

Problem: Find necessary and sufficient topological conditions s.t. a topo. space is metrizable

Example: X with discrete topo.

$$d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y. \end{cases}$$

topology basis: $\mathcal{B} = \{ B_d(x, 1) \text{ open} \}$

X with indiscrete topo

Lemma: X is metrizable $\Rightarrow$ X is hausdorff

proof: $\varepsilon = \frac{1}{2} d(x,y)$

$B_d(x, \varepsilon) \cap B_d(y, \varepsilon) = \phi$

suppose: $z \in B_d(x, \varepsilon) \cap B_d(y, \varepsilon) \neq \phi$.

$d(x,y) \leq d(x,z) + d(y,z) < 2\varepsilon = d(x,y)$ Contradiction

Def: norm of x: $\|x\| = (x_1^2 + \cdots + x_n^2)^{1/2}$

euclide metric: $d(x,y) = \|x - y\|$

Square metric: $d(x,y) = \max\{ |x_i - y_i| \}$

Theorem 20.2. 一个开球中包含另一个, 则另一个 topo is finer

Theorem 20.3 The standard topo on $\mathbb{R}^n$ is the topo generated by euclidean metric and square metric, which is equal to the product topo on $\mathbb{R} \times \cdots \times \mathbb{R}$, $\mathbb{R}$ with standard topo.

proof: $\rho(x, y) \le d(x, y) \le \sqrt{n}\,\rho(x, y)$

therefore $B_d(x, \varepsilon) \subset B_\rho(x, \varepsilon)$

$$B_\rho(x, \varepsilon/\sqrt{n}) \subset B_d(x, \varepsilon)$$

s.t. the topos generated by those 2 metric are the same

let $B = (a_1, b_1) \times \cdots \times (a_n, b_n)$ be an basis element and let $x \in B$. for each i. there is ε_i s.t.

$(x_i - \varepsilon_i, x_i + \varepsilon_i) \subset (a_i, b_i)$ 反复用 lemma 13.3

choose $\varepsilon = \min\{\varepsilon_i\}$. then $B_\rho(x, \varepsilon) \subset B$. so the ρ-topo is finer than the product topo

conversely. let $B_\rho(x, \varepsilon)$ be a basis element and let $y \in B_d(x, \varepsilon)$. it is trivial that

$B = B_\rho(x, \varepsilon) = \prod_i (x_i - \varepsilon, x_i + \varepsilon)$ is a basis element of product topo. $y \in B \subset B_d$.

product topo is finer than ρ-topo

Question: $\mathbb{R}^{\omega} = \prod_{n\in\mathbb{Z}^+} \mathbb{R}$ is metrizable ?

Theorem 20.1 if $d: X \times X \to \mathbb{R}$ is a metric on X. then $\bar{d}$ $X \times X \to \mathbb{R}$. $\bar{d}(x, y) = \min\{d(x, y), 1\}$ is also a metric on X.

This bounded metric induces the same topo on X as d.

Def: Let J be any index set consider

$$\mathbb{R}^J = \prod_J \mathbb{R} = \{(x_\alpha)_{\alpha\in J} \quad x_\alpha \in \mathbb{R}_\alpha\}$$

define $\bar{\rho} : \mathbb{R}^J \times \mathbb{R}^J \to \mathbb{R}$. $\bar{\rho}(x, y) = \sup\{\bar{d}(x_\alpha, y_\alpha)\}$ ≤ 1

where $\bar{d}$ is the standard bounded metric

It is called the uniform metric on $\mathbb{R}^J$

The topo. it induces is called uniform topo.

Theorem: 20.4 the uniform topo on $\mathbb{R}^J$ is finer than the produce topo. and coarser than box topo. strictly when J is infinite.

(when J is finite. they are the same)

proof: give $x = (x_\alpha)_{\alpha\in J}$ and a product basis element $\prod U_\alpha$ Let $\alpha_1 \cdots \alpha_n$ be indices for wich $U_\alpha \neq \mathbb{R}$, $x \in \prod U_\alpha$

Then for each i. choose $\varepsilon_i > 0$ s.t. the ε_i-ball centred at x_i in $\bar{d}$ metric is contained in U_{α_i}. this we can do because U_{α_i} is open in $\mathbb{R}$.

Let $\varepsilon = \min\{\varepsilon_i\}$.

Then $x \in B_{\bar\rho}(x, \varepsilon) \subset \prod U_\alpha$. so the uniform topo is finer than the product topo

On the other hand let $B = B_{\bar\rho}(x, \varepsilon)$ and the neighbour $U = \prod (x_\alpha - \frac{1}{2}\varepsilon, x_\alpha + \frac{1}{2}\varepsilon)$ of x is contained in B. Then $x \in U \subset B_{\bar\rho}(x, \varepsilon)$. the box topo is finer than the uniform topo

When J is infinite: prove uniform ~ is strictly finer than product ~.

$B_{\bar\rho}(0, 1)$ is open in the uniform topo but not open in the product topo

$U = \prod_{n \in \mathbb{Z}^+} (x_n - \varepsilon, x_n + \varepsilon)$ is open in box topo. $0 < \varepsilon < 1$

$y = (x_n + \frac{n}{n+1}\varepsilon)_{n \in \mathbb{Z}^+}$ is in U.

but not in $B_{\bar\rho}(x, \varepsilon)$ since

$\bar\rho = \sup\{\bar d(x, y)\} = \sup\{\frac{n\varepsilon}{n+1}\} = \varepsilon$ not less than ε.

For the point y there is no basis element of uniform topo ($\bar\rho$-ball) centered at it and contained in U.

U is not open in uniform topo. ......

即使取 $x' = (x_n + \frac{n}{n+1}\varepsilon)_{n \leq N} \ldots (x_n + \frac{N}{N+1}\varepsilon)_{n > N}$.

$\bar\rho = \frac{1}{N+1}\varepsilon$. 无穷后 $\lim \bar d = \frac{\varepsilon}{N+1} = \bar\rho$.

但要求是 $\lim \bar d < \bar\rho$ 才行. 但 $\bar\rho > \frac{1}{N+1}\varepsilon$

会导到 $\bar\rho$-ball 在 U 外去了.

极限时取上确界导致小于变等于.

但 我要求 小于 !!!!!

Lecture 07.

in $\mathbb{R}^J$ $\quad \bar{\rho}(x, y) = \sup\{\bar{d}(x_\alpha, y_\alpha) \;\; \alpha \in J\}$.

where $\bar{d}(x_\alpha, y_\alpha) = \min\{|x_\alpha - y_\alpha|, 1\}$

uniform metric induces uniform topo

Theorem 20.5: Let $x = (x_i)_{i \in \mathbb{Z}^+}$ $y = (y_i)_{i \in \mathbb{Z}^+} \in \mathbb{R}^\omega$.

Let $D(x, y) = \sup\{\frac{\bar{d}(x_i, y_i)}{i}\}$. then D is a metric on $\mathbb{R}^\omega$, which induces the product topo.

proof: the triangle Inequality: for all i:

$$\frac{\bar{d}(x_i, z_i)}{i} \le \frac{\bar{d}(x_i, y_i)}{i} + \frac{\bar{d}(y_i, z_i)}{i} \le D(x, y) + D(y, z)$$

s.t. $D(x, z) = \sup\{\frac{\bar{d}(x_i, z_i)}{i}\} \le D(x, y) + D(y, z)$

Let U be open in the metric topo.

choose an ε-ball $B_D(x, \varepsilon) \subset U$.

Choose N s.t. $\frac{1}{N} < \varepsilon$.

let $V = (x_1 - \varepsilon, x_1 + \varepsilon) \times \cdots \times (x_N - \varepsilon, x_N + \varepsilon) \times \mathbb{R} \times \mathbb{R} \cdots$

$\forall y \in \mathbb{R}^\omega$ $\quad \frac{\bar{d}(x_i, y_i)}{i} \le \frac{1}{N}$ for $i \ge N$

Therefore $D(x, y) \le \max\{\frac{\bar{d}(x_1, y_1)}{1}, \cdots \frac{\bar{d}(x_N, y_N)}{N}, \frac{1}{N}\}$

if $y \in V$. $D(x, y) \le \varepsilon$. s.t. $x \in V \subset B_D(x, \varepsilon)$

so metric topo is finer than product topo.

Conversely. Consider $U = \prod_{1 \le \alpha_n} U_i \prod_{i > \alpha_n} \mathbb{R}$. basis element of product topo.

and $x \in U$.

choose $(x_i - \varepsilon_i, x_i + \varepsilon_i) \subset U_i$ for $i \le \alpha_n$

define $\varepsilon = \min\{\frac{\varepsilon_i}{i}; i \le \alpha_n\}$

The $x \in B_D(x, \varepsilon) \subset U$

metric topo is finer than product topo

so they are equal.

Lemma (Sequence Lemma): 21.2

X topo space. $A \subset X$.

(1) If there is a sequence in A which converges to $x \in X$ then $x \in \bar{A}$

(2) Conversely suppose that X is metrizable. then if $x \in \bar{A}$, there is a sequence in A that converges to x.

proof: (1) $\forall U$ is neighbour of x. $\exists N$. when $i > N$. $a_i \in U$.

since $a_i \in A$. $\Rightarrow U \cap A \neq \phi$. $\Rightarrow x \in \bar{A}$

(2) X is metrizable with metric d.

let $x \in \bar{A}$ and $\{a_n\} = \{a_n \in B_d(x, \frac{1}{n}) \cap A \neq \phi\}$

let U be a neighbour of x there is $\varepsilon > 0$.

$B_d(x, \varepsilon) \subset U$.

let $N \in \mathbb{N}$: $\frac{1}{N} < \varepsilon \Rightarrow B_d(x, \frac{1}{N}) \subset U$.

for $n > N$. $B_d(x, \frac{1}{n}) \subset B_d(x, \frac{1}{N}) \subset U$

$\Rightarrow$ for $n > N$. $a_n \in U$.

$\Rightarrow$ there is a sequence $a_n \to x$

Theorem 21.1 : $f: X \to Y$.

X and Y are metrizable topo space with metric d_X. d_Y.

f is continuous $\Leftrightarrow$

given $x \in X$ and $\varepsilon > 0$. $\exists\ \delta > 0$ s.t.

$d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$

proof: $\Rightarrow$. $B_Y(f(x), \varepsilon)$ is open.

$f^{-1}(B_Y(f(x), \varepsilon))$ is open in X.

$\Rightarrow \exists\ B_X(x, \delta) \subset f^{-1}(B_Y(f(x), \varepsilon))$

$d_X(x, y) < \delta$. $y \in B_X(x, \delta)$

$\Rightarrow f(y) \in f(B_X(x, \delta)) \subset B_Y(f(x), \varepsilon) \Rightarrow d_Y(f(x), f(y)) < \varepsilon$

"$\Leftarrow$" let V be open in Y and $f(x) \in V$

$\exists\ B_Y(f(x), \varepsilon) \subset V$

for $f(y) \in B_Y(f(x), \varepsilon) \subset V$.

$x, y \in f^{-1}(V)$ and $d_X(x, y) < \delta$

$\Rightarrow y \in B_X(x, \delta) \subset f^{-1}(V)$.

$f^{-1}(V)$ is open

......

Theorem 21.3 let $f: X \to Y$ function X, Y topo space.
f is continous $\Rightarrow$
for every converge sequence $x_n \to x$ in X
the converse "$\Leftarrow$" holds if X is metrizable
the image sequence $f(x_n) \to f(x)$ in Y

proof: "$\Rightarrow$" let V be neighbour of $f(x) \in Y$
then $f^{-1}(V)$ is open in X. and $f^{-1}(V)$ is neighbour of x.
so. $\exists N$. $n \geq N$. $x_n \in f^{-1}(V)$.
s.t. $f(x_n) \in V$ for $n \geq N$. $\Rightarrow f(x_n) \to f(x)$

"$\Leftarrow$" let $A \subset X$. we show that $f(\bar{A}) \subset \overline{f(A)}$
if $x \in \bar{A}$. by lemma 21.2.
$\exists \{x_n\} \in A$ and $x_n \to x$. since X metrizable
since $f(x_n) \to f(x)$ and $f(x_n) \in f(A)$
s.t. $f(x) \in \overline{f(A)}$ by lemma 21.2
Hence $f(\bar{A}) \subset \overline{f(A)}$ as desired

Proposition: $\mathbb{R}^\omega$ with the box topo is not metrizable.
proof: let $A \subset \mathbb{R}^\omega$. $A = \{(x_n)_{n \in \mathbb{Z}^+}.\ x_n \in \mathbb{R}^+\}$
clearly: $0 = (0, 0, \dots) \in \bar{A}$
let U be a neighbour of 0.
$\Rightarrow 0 \in \prod_{n \in \mathbb{Z}^+} U_n \subset U$, U_n is open in $\mathbb{R}$
$\Rightarrow 0 \in \prod_{n \in \mathbb{Z}^+} (-\varepsilon_n, \varepsilon_n) \subset \prod_{n \in \mathbb{Z}^+} U_n \subset U$
then $\left(\frac{\varepsilon_n}{2}\right)_{n \in \mathbb{Z}^+} \in A \cap U$
$\Rightarrow 0 \in \bar{A}$

if we can't find a sequence converge to 0.

by lemma 21.2. $\mathbb{R}^\omega$ with box topo isn't metrizable.

let a_n be any sequence in $A \subset \mathbb{R}^\omega$

$a_n = (x_{1n}, x_{2n} \cdots) = (x_{in}) \in \mathbb{R}^\omega$. $x_{in} > 0$.

let $U = (-x_{11}, x_{11}) \times (-x_{22}, x_{22}) \cdots$

$= \prod_{n\in\mathbb{N}} (-x_{nn}, x_{nn})$ be a neighbour of 0

s.t. $a_n \notin U$ for any $n \in \mathbb{N}$.

Since $a_{nn} \notin (-a_{nn}, a_{nn})$

$\cdots$

可数 $\Leftrightarrow$ 所有数可以排成一个数列

Classical diagonal arguement of Cantor:

consider a sequence a_n of all real numbers in $[0,1]$

$a_1 = 0. x_{11}\ x_{21}\ x_{31} \cdots$ $\qquad x_{ij} \in \{0,1,2\cdots 9\}$

$a_2 = 0. x_{12}\ x_{22}\ x_{32} \cdots$

$a_3 = 0. x_{13}\ x_{23}\ x_{33} \cdots$

let $b = 0.b_1 b_2 b_3 \cdots$ s.t. $b_i = \begin{cases} 0 & \text{if } x_{ij} \neq 0 \\ 1 & \text{if } x_{ii} = 0 \end{cases}$

$\Rightarrow b_i \neq a_n \ \forall n$ contradiction!

$\Rightarrow [0,1]$ is not countable

Proposition. If J is uncountable, then $\mathbb{R}^J$ with the product topo is not metrizable.

proof let $A = \{(x_\alpha)_{\alpha\in J} \in \mathbb{R}^J\}$ where

$x_\alpha = 0$ for finite α. and $x_\alpha = 1$ for others let $0 = (0, 0 \cdots) \in \mathbb{R}^J$. the $0 \in \bar{A}$ since:

let $0 \in \prod_\beta U_\beta$ $\quad U_\beta \neq \mathbb{R}$ only for finite β.

$U_\beta = \mathbb{R}$ for others.

U is open and $0 \in U$. s.t. $0 \in \prod_\beta U_\beta \subset U$.

$(x_\alpha) \in \prod_\beta U_\beta \cap A$ where $x_\alpha = 1$ and $U_\beta = \mathbb{R}$, $\alpha = \beta$

$\prod_\beta U_\beta \cap A \neq \phi$. $\quad U \cap A \neq \phi \Rightarrow 0 \in \bar{A}$

let a_n any sequence in A 定义有限个坐标是零.

$a_n = (x_{in}) \in \mathbb{R}^J$, $i \in J$ and finite i s.t. $x_{in} = 0$

$J_n = \{ i \in J.\ x_{in} = 0 \}$.

$\underbrace{\bigcup_{n \in N}}_{\text{contable union of finite sets is contable}} J_n \subsetneq J$

Choose $\beta \in J$. $\beta \notin \bigcup J_n$

then $x_{\beta n} = 1 \quad \forall n \in N$.

then $U = \pi_\beta^{-1}((-1, 1))$ is a neighbour of 0.

s.t. $a_n \notin U \quad \forall n \in N$ since $x_{\beta n} = 1 \notin (-1, 1)$

因为有不可数个坐标分量. 总可以在数到无穷个中找到一个坐标为 1. 不在邻域 $(-1, 1)$ 中的分量.

by lemma 21.2 $\mathbb{R}^J$ with product topo is not metrizable. (J is uncountable)

Lecture 08

Def: topo space X is first countable in a point if there is a countable collection $\{U_n\}_{n \in \mathbb{N}}$ of neighbour of x. s.t. any neighbour U of x contains at least one U_n.

We can also assume: $U_1 \supset U_2 \supset U_3 \cdots$

then $U_i \subset U$ for $i \geq N$.

fundamental system of neighbour.

X is first countable if X is first countable in every $x \in X$

Example: X: metrizable $\Rightarrow X$ first countable.

take $U_i = B_d(x, \frac{1}{i})$

first countability is a necessary topological condition for metrizability.

Observation The sequence lemma holds if we replace "metrizable" by "first countable".

proof: replace $B_d(x, \frac{1}{n})$ by U_n: $U_n \supset U_{n+1} \cdots$

$\mathbb{R}^\omega$ with box topo is not first countable in $0 \in \mathbb{R}^\omega$

$\mathbb{R}^J$ J uncountable with product topo is not first countable in $0 \in \mathbb{R}^J$.

proof by sequence lemma.

Def: X second countable : $\mathcal{B}$ is a countable basis for X.

Proposition: Second countable $\Rightarrow$ first countable.

proof: let $\mathcal{B}$ be a countable basis of X.

let $x \in X$. $\mathcal{B}' = \{B \in \mathcal{B} ; x \in B\}$

s.t. $\mathcal{B}'$ is countable $\mathcal{B}' = \{B_1, B_2 \cdots\}$

$U_1 = B_1$, $U_2 = B_1 \cap B_2$, $U_3 = B_1 \cap B_2 \cap B_3 \cdots$

if U is any neighbour of x, then there is $B \in \mathcal{B}$:

s.t. $B \in \mathcal{B}'$ and $B \subset U$.

so $\exists$ $U_n = B \cap \cdots \subset U$.

Def: X, Y topo spaces.

a function $f: X \to Y$ is a "quotient map" if:

f is continuous, surjective and :

U is open in Y if and only if $f^{-1}(U)$ open in X.

(连续性只要求"$\Rightarrow$", 没要求"$\Leftarrow$")

(stronger than continuity)

Def: given the topo in X, then Y has the "finest" topo

s.t. f is continous.

this is the "quotient topo."

(或者说 给定 $f: X \to Y$ surjective. 有且只有一个拓扑

使 f 是 quotient map. 即 quotient topo)

(易证确实只有一个)

Def: $f: X \to Y$

subset $C \subset X$. if $y \in Y$ $C \cap f^{-1}(\{y\}) \neq \phi$.

then $C \supset f^{-1}(\{y\})$

we say C is saturated.

quotient map $\Leftrightarrow$

f is continuous and f maps saturated open sets of X to open sets in Y

Def: X topo space. let X^* be a partition of X into disjoint subsets whose union is X.

let $f: X \to X^*$ $f(x) = X_i$ where $x \in X_i$

in the quotient topo induced by f. the space X^* is called a quotient space.

Theorem: 22.1 let $p: X \to Y$ quotient map. let A be a subspace of X. that is saturated with respect to q. let $q: A \to p(A)$

(1) if A is either open or closed in X. then q is quotient map (存在不开不闭集合的)

(2) if p is either an open map or a closed map. then q is quotient map.

(证明暂略)

Theorem 22.2

corollary 22.3

$f(\{g^{-1}(z) \mid z \in Z\}) = z$

单满

双射

Connected spaces:

Def: A separation of topo space X is a pair U, V subsets of X s.t.

U and V are open. means: U and V are both open and closed.

$X = U \cup V$, $U \cap V = \phi$, $U, V \neq X, \phi$.

X is connected if there is no separation of X.

Obs: X is connected:

$\Leftrightarrow$ the only subsets of X that are both open and closed are X and ϕ.

Example: $\mathbb{R}_l$ is not connected.

$\mathbb{R}_l = (-\infty, a) \cup [a, +\infty)$ is a separation

Lemma 23.2.

If X a topo space. C and D are separation of X. and $Y \subset X$ is a connected subspace. then $Y \subset C$ or $Y \subset D$

$Y = Y \cap X = Y \cap (C \cup D) = (Y \cap C) \cup (Y \cap D)$

$Y \cap C$ and $Y \cap D$ is open and disjoint in Y

$Y \cap C = Y - (Y \cap D)$ is closed.

same as $Y \cap D$.

$\Rightarrow Y \cap C = \phi$ or $Y \cap D = \phi$.

$\Rightarrow Y \subset D$ or $Y \subset C$

Theorem: suppose $A_\alpha \subset X$ are connected. $\alpha \in J$.
and $\bigcap_{\alpha \in J} A_\alpha \neq \phi$. then
$\bigcup_{\alpha \in J} A_\alpha$ is connected.

proof: let $p \in \bigcap_\alpha A_\alpha$ suppose that
$Y = \bigcup_\alpha A_\alpha = C \cup D$. $p \in C$ or $p \in D$.
suppose $p \in C$.
A_α is connected. so $A_\alpha \subset C$ or D.
$p \in C$ and $p \in A_\alpha$. s.t. all. $A_\alpha \subset C$.
$\Rightarrow \bigcup_\alpha A_\alpha \subset C$. $D = \phi$.
Y has no separation

Theorem 23.4. Let $A \subset X$ be connected. and
$A \subset B \subset \bar{A}$. then B is connected
in particular. the closure of a connected subset is
connected (in the case $B = \bar{A}$)

proof: suppose $B = C \cup D$.
$\Rightarrow A \subset C$ or $A \subset D$. suppose $A \subset C$
then $\bar{A} \subset \bar{C}$. $B \subset \bar{A} \subset \bar{C}$
C is closed in B. $\bar{C} = C$
$B \subset \bar{C} = C$ since $C \subset B$. $C = B$.
s.t. $C = B$ and $D = \phi$. no separation

Theorem 23.5 : if $f: X \to Y$ is continuous and X is connected, then $f(X)$ is connected.

proof: we can assume $Y = f(X)$. so f is surjective

($g: X \to f(X)$ $g(x) = f(x)$ is continuous $\Leftrightarrow$ f continuous)

suppose $Y = C \cup D$

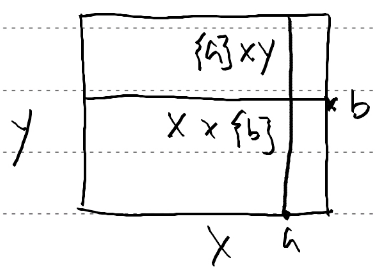
$\Rightarrow$ $X = f^{-1}(C) \cup f^{-1}(D)$

X is connected. $f^{-1}(C) = \phi$ or $f^{-1}(D) = \phi$.

$\Rightarrow$ $C = \phi$ or $D = \phi$. ....

Theorem: An arbitrary cartesian product of connected spaces is connected.

proof: we first prove for case of 2 connected spaces X, Y.



fix $b \in Y$

$X \times \{b\} \cong X$. is connected.

$\{a\} \times Y \cong Y$ is connected.

$T_{ab} = \{a\} \times Y \cup X \times \{b\}$ is connected.

$\bigcap_{a \in X} T_{ab} = X \times \{b\} \neq \phi$

$\Rightarrow$ $X \times Y = \bigcup_a T_{ab}$ is connected

for the case of arbitrary produce.

$X = \prod_{\alpha \in J} X_\alpha$ X_α connected.

构建稠密子集 D $\bar{D} = X$.

D 自通.

$0 = (0, 0, 0, \dots)$ 其他 x_i 全是 0.

$D = \{ x \in X.$ 有限个 $x_i \neq 0 \}$. (可以是 0 个 x_i).

有限子集 $F \subset J$.

$D_F = \{ x \in X, x_i = 0 \quad i \notin F \}$

$D = \bigcup_{F \subset J} D_F$.

$D_F = \prod_{\alpha \in F} X_\alpha \times \prod_{\alpha \notin F} \{0\} \cong \prod_{\alpha \in F} X_\alpha$.

有限积道通. D_F 道通.

$0 \in \bigcap D_F \neq \phi$

$D = \bigcup D_F$ 道通.

D 稠密: $\mathcal{B} = \prod_{i \in J} U_i$ finitely many $U_i \neq X_i$

$\forall B \in \mathcal{B}$. $\exists F$. $\alpha \notin F$. $U_\alpha = X_\alpha$

$x \in B$. $x_\alpha = 0$. $\alpha \notin F$.

$x \in D_F \cap B \neq \phi$.

$\Rightarrow D$ 稠密 $\bar{D} = X$. Connected.

($\forall x \in X$. $\forall U$ containing x.

$U \cap D \neq \phi$. x is limit point of D.

$\bar{D} = D \cup \{x\} = X$.

稠密 $\Leftrightarrow$).