Lecture 13

S_Ω: well-ordered uncountable

every section is countable

the minimal uncountable well-ordered set

general property of well-ordered set:

- each element has an immediate successor (or it's the max)
- least upper bound property.

Lemma: If $A \subset S_\Omega$ is countable, then A has an upper bound in S_Ω

proof: $\bigcup_{a\in A} S_a$. S_a: the section of a. $\{x\in S_\Omega \mid x<a\}$

it is a countable union of countable set. it is countable

$\bigcup_{a\in A} S_a \subsetneq S_\Omega$

every element of $S_\Omega - \bigcup_{a\in A} S_a$ is an upper bound of A, $S_\Omega - \bigcup_{a\in A} S_a$ has a least element

Proposition: S_Ω is a Hausdorff space which is not compact but limit point compact.

proof: if $a<b<c$. $(-\infty, c)$ and $(c, +\infty)$ disjoint

if $a<b$. $(-\infty, b)$ and $(a, +\infty)$ disjoint

$\Rightarrow$ Hausdorff

"not compact" $\{S_x. \ x\in S_\Omega\}$ open covering.

$S_x = (-\infty, x) = [\alpha, x)$. α is the least element

There is no finite subcovering since otherwise it would be an open section s.t. $S_a = S_\Omega$

but $a \notin S_\Omega$. $a \in S_\Omega$ contradiction

S_Ω is limit point compact

let A be an infinite subset of S_Ω.

we can assume that A is countable

lemma $\Rightarrow$ A has an upper bound $b \in S_\Omega$

$\Rightarrow$ $A \subset [\alpha, b]$

S_Ω has the least upper bound property.

so $[a,b]$ has the largest element.

so $[a,b]$ is compact $\Rightarrow$ $[a,b]$ limit point compact.

$A \subset [a,b]$. A has a limit point.

so S_Ω limit point compact

(metrizable $\Rightarrow$ first countable)

Corollary S_Ω is not metrizable.

$\bar{S}_\Omega = [\alpha, \Omega]$ is closed interval.

it is compact.

$\bar{S}_\Omega$ is not first countable.

otherwise. at $\Omega \in \bar{S}_\Omega$

there is countable collection of neighb of Ω $\{(x_n, \Omega]\}$

$\{x_n, n \in \mathbb{N}\}$ is countable. $x_n \in S_\Omega$

$\{x_n, n \in \mathbb{N}\} \subset S_\Omega$. so it has a upper bound x in S_Ω

$x > x_n \ \forall n \in \mathbb{N}$.

$(x, \Omega]$ is a neighb of Ω. but it is not contained in $(x_n, \Omega]$ 这个证明似乎有问题?

$\bar{S}_\Omega$ not first countable $\Rightarrow$ not metrizable $\Rightarrow$ S_Ω ....

因为我只要构建一个 countable collection
不要求每一个？
但 S_Ω 到底是否第一可数？
拉流是 Metrizable.
$\Rightarrow$ compact 和 limit point compact 等价.
但 S_Ω 不满足
则 不 Metrizable

Proposition: S_Ω is first countable but not second countable
let $a \in S_\Omega$. a' is the immediate successor of a.
$\{(x, a') . \ x < a\}$ is countable collection of neighbs of a
(我们可以找到 任一 x 的直接后继.
不代表我们可以找到 Ω 的前继？)
(assume a is not the least).
every neighbour U of has a lower bound x' (well-ordered)
$(x', a) \subset U$. so it is first countable.
"not second countable"
let $\mathcal{B}$ be any basis of S_Ω
for each $x \in S_\Omega$ $x \in [\alpha, x')$ which is open
where x' is the immediate successor
$\Rightarrow$ there is $B_x \in \mathcal{B}$ s.t. $x \in B_x \subset [\alpha, x')$
map: $S_\Omega \longrightarrow \mathcal{B}$
$x \longrightarrow B_x$ is injective
S_Ω is not countable. $\Rightarrow \mathcal{B}$ is uncountable.

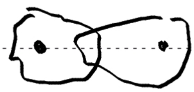
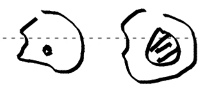
since: let $x \neq y$. $x, y \in S_\Omega$

suppose $x < y$.

$x \in B_x \subset [\alpha, x')$. $y \in B_y \subset [\alpha, y')$

$y \notin [\alpha, x')$ $y \notin B_x$. $B_x \neq B_y$.

The seperation axiom:

T_1:  T_3: 

T_2:  T_4: 

T_3 不能推出T_2吗? closed subset: 

$T_4 \not\Rightarrow T_3 \not\Rightarrow T_2$? $T_2 \Rightarrow T_1$

(书上定义)

regular: T_3 and T_1 假设单点集是闭的

normal: T_4 and T_1 和 T_3 或 T_4

normal $\Rightarrow$ regular $\Rightarrow$ hausdorff $\Rightarrow$ T_1

Theorem X metrizable $\Rightarrow$ X normal.

proof: Hausdorff is satified

T_4. let A, B be disjoint closed subset of X.

for each $a \in A$. there is open ball

$B_a(a, \varepsilon_a)$ disjoint with B.

Same for $b \in B$.

$U = \bigcup_{a\in A} B_d(a, \frac{\varepsilon_a}{2})$ neighbour of A

$V = \bigcup_{b\in B} B_d(b, \frac{\varepsilon_b}{2})$ neighbour of B

if $x \in U \cap V$.

$x \in B_d(a, \frac{\varepsilon_a}{2})$ and $x \in B_d(b, \frac{\varepsilon_b}{2})$. let $\varepsilon_a \geq \varepsilon_b$

$d(a,b) \leq d(x,a) + d(x,b) < \frac{\varepsilon_a}{2} + \frac{\varepsilon_b}{2} \leq \varepsilon_a$

but $b \notin B_d(a, \varepsilon_a) \Rightarrow d(a,b) > \varepsilon_a$.

contradiction

T_1 等价于 单点集是闭集

$\Rightarrow$ $\forall x \in X$

$\exists\, y \in X/\{x\}$ s.t. $y \in U_y$ and $x \notin U_y$

$U = X/\{x\} = \bigcup_{y\in X/\{x\}} U_y$ is open

$\{x\} = X - U$ is closed.

"$\Leftarrow$" $\forall x, y \in X$.

$x \in X/\{y\}$ $y \in X/\{x\}$. T_1 disjoint

"T_3" / "T_4" / "T_2" $\Rightarrow$ "T_1"

Lecture 14

Theorem: X is compact and hausdorff $\Rightarrow X$ is normal

proof: first we prove that X is T_3. ($\Leftrightarrow$ regular)

let $x \in X$ and $A \subset X$ be closed.

$x \in U_x$ is open in X.

$x_1 \in A$. V_{x_1} is open in X.

since X is hausdorff. $\exists$ U_{x_1} and V_{x_1} is disjoint

A is closed. X is compact. $\Rightarrow A$ is compact

$\{V_{x_i}\}$ $x_i \in A$ is an open covering of A.

$\exists$ $i = 1 \cdots n$ $\{V_{x_i}\}$ is finite subcovering

$A \subset V_{x_1} \cup \cdots \cup V_{x_n} = V$

$x \in U_{x_1} \cap \cdots \cap U_{x_n} = U$. $U \cap V = \phi$. $\Rightarrow T_3$

let $A, B \subset X$ be closed

A, B are compact.

$x \in A$. $x \in V_x$ is open in X.

there is U_x which is neighb of B s.t.

U_x. V_x disjoint. since X is T_3

compact: $V_{x_1} \cdots V_{x_n}$ finite cover A.

$A \subset V_{x_1} \cup \cdots \cup V_{x_n} = V$

$B \subset U_{x_1} \cap \cdots \cap U_{x_n} = U$.

V and U disjoint.

Lemma 31.1

(1) X is T_3 (regular)

$\Leftrightarrow$ for every $x \in X$. and neighb U of x.

$\exists$ neighb V of x s.t. $\bar{V} \subset U$.

(2) X is T_4 (normal)

for $\forall$ closed subset $A \subset X$ and neighb U of A:

$\exists$ neighb V of A s.t. $\bar{V} \subset U$

proof: "(1) $\Rightarrow$" Let $x \in X$ and $x \in U$ is open in X.

so $X - U$ is closed in X.

since X is T_3. $\exists$ neighb V of x and neighb W of $X-U$. s.t. $V \cap W = \phi$.

since $X-U \subset W$. $V \cap X-U = \phi$.

$V \subset U$

if $y \in X-U$. $y \in W$ and $W \cap V = \phi$.

y is not limit point of V.

so $\bar{V} \cap X-U = \phi$. $\bar{V} \subset U$.

"$\Leftarrow$" let $x \subset X$

Let $A \subset X - \{x\}$ be closed

$U = X-A$ is open and $x \in U$.

$\exists$ $x \in V \subset U$. s.t. $\bar{V} \subset U$

Since $\bar{V} \subset U$. $X-\bar{V} \supset A$. Is open

and $X-\bar{V} \cap V = \phi$

$\Rightarrow$ X is T_3

(2) similarly. replace point by another closed subset

Theorem 33.1 (Urysohn theorem)

let X be T_4 (normal) and A, B disjoint closed subsets of X. there exists a continuous map $f: X \to [0,1]$ s.t. $f(A) = \{0\}$ and $f(B) = \{1\}$

(which means that A, B can be sepereted by a continuous real function)

Observation X is T_4

$\Longleftrightarrow$ $\forall$ 2 closed subsets in X is sperated

("$\Leftarrow$" can be proved by choosing 2 neighbs $f^{-1}([0,\frac{1}{2}))$ and $f^{-1}((\frac{1}{2},1])$).

proof "$\Rightarrow$" let A, B disjoint closed in X

let $P = \mathbb{Q} \cap [0,1]$ (all rational in $[0,1]$)

P is countable. we define

for each $p \in P$ an open set U_p in X. s.t.

whenever $p < q$, $q \in P$ then $\overline{U_p} \subset U_q$

arrange P in a sequence starting with $\{1, 0, \underbrace{\cdots\cdots}_{\text{arbitrary}}\}$

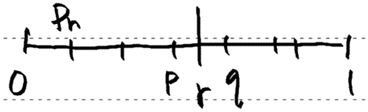
we define U_p recursively following this sequence:

let $U_1 = X - B$ be neighb of A.

X is T_4 $\Rightarrow$ there is neighb $\underline{U_0 \text{ of } A}$. s.t. $\overline{U_0} \subset U_1$

Suppose inductively that U_p is already defined for the finite n rational number sequence P_n

let r be the next rational number in sequence



let p be the immediate predecessor of r in P_n in the natural order in $[0,1]$.

let q be the ... successor ...... , $U_q \supset \bar{U}_p$

U_q is a neighb of $\bar{U}_p$. $\bar{U}_p$ is closed.

X is T_4 $\Rightarrow$ a neighb U_r of $\bar{U}_p$ s.t. $\bar{U}_r \subset U_q$.

by induction. U_p is defined for all $p \in P$

(此时可以排成 $\{1, 0, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \dots\dots\}$)

extend the definition to all $p \in Q$

setting : $U_p = \phi$ if $p < 0$

$U_p = X$ if $p > 1$

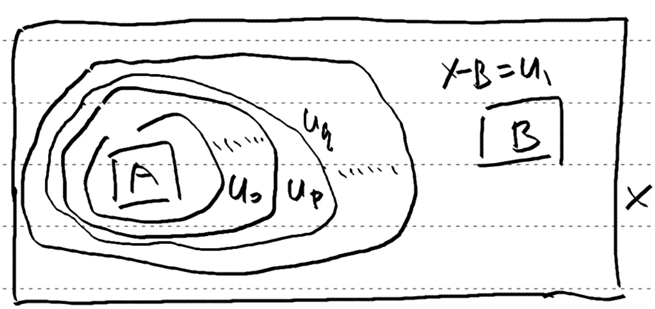
definition of f :

for $x \in X$. let $Q(x) = \{p \text{ rational} : x \in U_p\}$

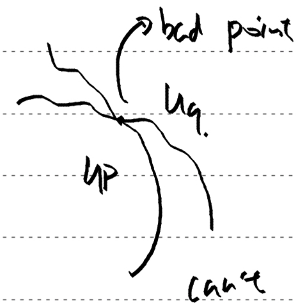
$f : X \to [0,1]$: $f(x) \equiv \inf(Q(x))$

$\Rightarrow$ $0 \le f(x) \le 1$.

$f(A) = \{0\}$ $f(B) = \{1\}$



$q > p$



$\Rightarrow \bar{U}_p \subset U_q$

claim: f is continuous

two facts: (1) $x \in \bar{U}_r \iff f(x) \leq r$

(2) $x \notin U_r \iff f(x) \geq r$.

(1) if $x \in \bar{U}_r$. $x \in U_s$ for all $s > r$.

then $f(x) = \inf Q(x) \leq r$

(2) if $x \notin U_r$ $x \notin U_s$ for all $s < r$

then $f(x) = \inf Q(x) \geq r$

let $x_0 \in X$. let $f(x_0) \in (c, d) \subset \mathbb{R}$

let $c < p < f(x_0) < q < d$.

let $U = U_q - \bar{U}_p$ open

then $x_0 \in U_q$ $x_0 \notin \bar{U}_p$ U is neighb of x_0

so $f(U) \subset [p, q] \subset (c, d)$ is open

$\Rightarrow$ f is continuous.

Lecture 15

Theorem 34.1 (Urysohn metrization theorem)

Every regular space X with a countable basis is metrizable.

normal 时也成立. (可数基 → 有穷交集 → T_4?)

Lemma: There is a countable collection of continuous functions: $f_n: X \to [0,1]$, $n \in \mathbb{N}$ s.t. $\forall x_0 \in X$ and neighb U of x_0, $\exists\ n$ s.t. $f_n(x_0) > 0$ and $f_n(X-U) = \{0\}$

proof: let $\mathcal{B} = \{B_n,\ n \in \mathbb{N}\}$ countable basis of X

for each pair of index n, m. $\bar{B}_n \subset B_m$ by the Urysohn lemma. Choose a continuous func. s.t

$g_{n,m}(\bar{B}_n) = \{1\}$ and $g_{n,m}(X - B_m) = \{0\}$

Consider $x_0 \in U$. choose a basis element B_m s.t.

$x_0 \in B_m \subset U$. since X is regular.

$\forall$. x_0 and closed subset $X - B_m$:

$\exists$ $x_0 \in \bar{B}_n$ $\quad X - B_m \subset X - \bar{B}_n$. $\quad \bar{B}_n \subset B_m$

$\Rightarrow$ $g_{n,m}(x_0) = 1$. $\quad g_{n,m}(X-U) = \{0\}$

(ω : 可数)

proof of theorem: define $F: X \to \mathbb{R}^\omega$ with product topo

$F(x) = (f_1(x), f_2(x) \ \ldots\ ) = (f_n(x))$ $\quad n \in \mathbb{N}$

claim: F is an imbeding (homeomorphical)

$\Rightarrow$ $X \cong F(X) \subset \mathbb{R}^\omega$ is metrizable.

$\Rightarrow$ X is metrizable

proof of the claim:

1) F is injective

Let $x, y \in X$. $x \neq y$

X is $T_1 \Rightarrow \exists$ a neighb U of x s.t. $y \notin U$.

as the lemma before $\exists B_n \in \mathcal{B}$. $x \in \bar{B}_n \subset B_m \subset U$

$\Rightarrow g_{n,m}(x) = 1$, $g_{n,m}(y) = 0$

choose these $g_{n,m}$ as f_n.

$\Rightarrow F(x) \neq F(y)$

2) F is continous.

Since all component function $f_n \in \{g_{n,m}\}$

is continous $\Rightarrow F$ is continous

3) $X \to F(X)$ is surjective trivial.

4) F^{-1} is continous:

let $Z = F(X) \subset \mathbb{R}^{\omega}$. prove that

$\forall$ open subset U in X. $F(U)$ is open in Z.

let $z_0 \in F(U)$. $x_0 \in U$: $F(x_0) = z_0$

Choose n. s.t. $f_n(x_0) = g_{n,m}(x_0) > 0$. and

$f_n(X - U) = \{0\}$ by the lemma.

let $V \equiv \pi_n^{-1}((0, +\infty))$ open in $\mathbb{R}^{\omega}$.

$W = V \cap Z$. open in Z.

prove that $z_0 \in W \subset F(U)$ ($\Rightarrow F(U)$ is open in Z)

$\pi_n(z_0) = \pi_n(F(x_0)) = f_n(x_0) > 0 \in (0, \infty)$

So $z_0 \in V = \pi_n^{-1}((0, \infty))$

$\forall z \in W$. $z = F(x)$ so $\pi_n(z) = f_n(x) > 0$, for some $x \in X$.

since $f_n(X - U) = \{0\}$. f_n vanish outside of U.

$\Rightarrow x \notin X - U$. $x \in U$. $\Rightarrow z = F(x) \in F(U)$, $W \subset F(U)$

Theorem: X compact. Hausdorff

X is metrizable $\Leftrightarrow$ X has countable basis

proof: "$\Leftarrow$" X is compact. Hausdorff. $\Rightarrow$ X is normal

X normal. second countable $\Rightarrow$ X is metrizable

"$\Rightarrow$" $\mathcal{B}_n$ is a finite open covering consisting of open balls. with $\frac{1}{n}$ radius. since X is compact.

let $\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$ $\quad$ $\mathcal{B}$ is countable.

let $x \in U \subset X$.

$\exists$ $n \in \mathbb{N}$, $B_d(x, \frac{1}{n}) \subset U$.

choose $x \in B_d(y, \frac{1}{2n}) \in \mathcal{B}_{2n} \subset \mathcal{B}$

$B_d(y, \frac{1}{2n}) \subset B_d(x, \frac{1}{n}) \subset U$. ......

Homotopy and fundamental group

Def: two continuous maps $f, f' \; X \to Y$
are homotopic ($f \simeq f'$) if there exists a continuous map
$F: X \times I \to Y$ s.t.
$F(x,0) = f(x)$ $\quad F(x,1) = f'(x)$
f, f': $f \underset{F}{\simeq} f'$.
(homotopy is a continuous difference between f and f')

Lemma 51.1: homotopy is an equivalence relation on the set of continuous maps from X to Y.

- $f \simeq f$ by the constant homotopy:
  $F(x,t) = f(x)$
- $f \underset{F}{\simeq} g \Rightarrow g \underset{G}{\simeq} f$.
  $G(x,t) = F(x, 1-t)$
- $f \underset{F}{\simeq} g$. $\quad g \underset{G}{\simeq} h$.
  $\Rightarrow f \underset{H}{\simeq} h$ $\quad H(x,t) = \begin{cases} F(X, 2t) & 0 \le X \le \frac{1}{2} \\ G(x, 2t-1) & \frac{1}{2} < X \le 1 \end{cases}$

$H(x,1) = h = G(x,1)$

$t = \frac{1}{2}$: $\quad H(X, \frac{1}{2}) = F(x,1) = g = G(x,0)$ $\quad H(x,1) = h = G(x,1)$

$F(x,0) = f = H(x,0)$ $\quad H(x,0) = f = F(x,0)$

$H(X, \frac{1}{2}) = F(X,1) = G(X,0) = g$

$H(x,0) = F(x,0) = f$

$H(x,1) = G(x,1) = g$

Example:

1) every continous map $f: X \to \mathbb{R}^n$.
is homotopic to a constant map C_{z_0}.
$C_{z_0}: X \to \mathbb{R}^n$ $\quad C_{z_0}(x) = z_0$
$f \underset{H}{\simeq} C_{z_0}$ $\quad H(x,t) = (1-t)f(x) + tz_0$
("straight line homotopy" only in $\mathbb{R}^n$)

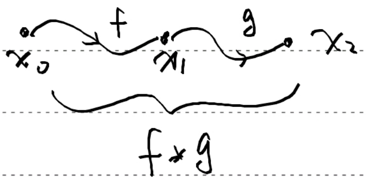
2) path: $f: [0,1] \to X$.
then f is homotopic to a constant path: $C_{f(0)}$
$H(s,t) = f((1-t)s)$
$f \underset{H}{\simeq} C_{f(0)}$

Def: path homotopy:
Let: $f, g: I \to X$ be paths with same initial point $f(0)=g(0)=x_1$ and same end point $f(1)=g(1)=x_2$
then f and g are (path) homotopy if
$F: I \times I \to X$ s.t.
$F(s,0) = f(s)$ $\quad F(s,1) = g(s)$
and $F(0,t) = x_1$ $\quad F(1,t) = x_2$.

Remark: path-homotopy is an equivalence relation on the set of the paths from one fixed point to another fixed point on X.

Def: Let f be a path in X from x_0 to x_1.
and g a path from x_1 to x_2
then $f * g : I \to X$.

$$(f * g)(s) = \begin{cases} f(2s) & 0 \le s \le \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \le s \le 1 \end{cases}$$



$f * g$ is a path by (pasting lemma) which is called the product of f and g (defined only $f(1) = g(0)$)

Def: for a path $f : I \to X$. let $[f]$ be the equivalence class of f w.r.t path-homotopy
let $f * g$ be path with $f(1) = g(0)$.
then: $[f] * [g] = [f * g]$. which is well-defined.

Check: $[f] = [f']$ if $f \underset{F}{\simeq} f'$
$[g] = [g']$ if $g \underset{G}{\simeq} g'$.

then $(f * g) \underset{H}{\simeq} (f' * g')$

$$H(s,t) = \begin{cases} F(2s, t) & 0 \le s \le \frac{1}{2} \\ G(2s-1, t) & \frac{1}{2} < s \le 1. \end{cases}$$

$H(s, 0) = (f * g)(s)$

$H(s, 1) = (f' * g')(s)$ by the definition of H and $f * g$

H is continuous by pasting Lemma.

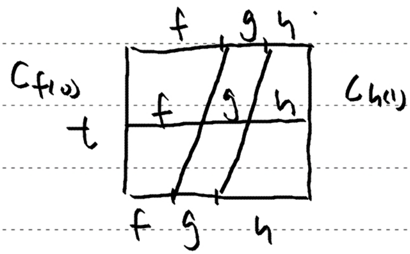
Lecture 16

Theorem 51.2

(1) $([f]*[g])*[h] = [f]*([g]*[h])$

proof: we have to prove

$$(f*g)*h \underset{H}{\simeq} f*(g*h)$$



$\xrightarrow{H} X$

Exercise 1 $f: I \to X$ is a path and $r: I \to I$ is a continuous map s.t. $r(0)=0$, and $r(1)=1$ then f and $f \circ r$ are path-homotopic

proof: $F(s,t) = t f(s) + (1-t)(f \circ r)(s)$

$F(s,1) = f(s) \qquad F(s,0) = (f \circ r)(s)$

$F(0,t) = t f(0) + (1-t)(f \circ r)(0) = f(0) = x_0$

$F(1,t) = t f(1) + (1-t)(f \circ r)(1) = f(1) = x_1$

remark: for reparametrize of f.

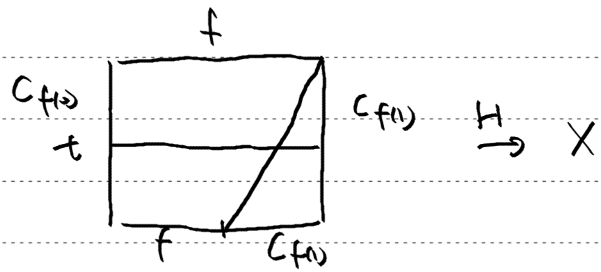
Exercise 2: f, g $X \to \mathbb{R}^n$ are continuous map then f, g are homotopic

(2) $f: I \to X$ is a path.

$\Rightarrow$ $[f] * [C_{f(1)}] = [f]$

$[C_{f(0)}] * [f] = [f]$

proof of (2) have to prove $f * C_{f(1)} \underset{H}{\simeq} f$



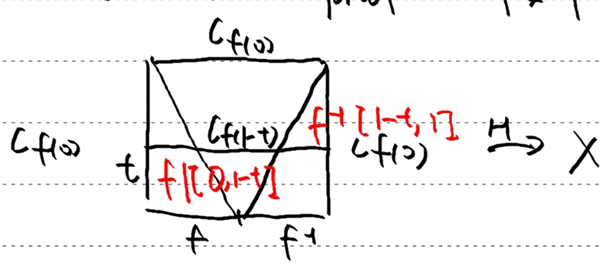
(3) $f: I \to X$ is a path.

define $f^{-1}: I \to X$ $f^{-1}(s) = f(1-s)$

which called the reverse of f. then

$[f] * [f^{-1}] = [C_{f(0)}]$ and $[f^{-1}] * [f] = [C_{f(1)}]$

proof: we have to proof $f * f^{-1} \underset{H}{\simeq} C_{f(0)}$



the formal proof:

two facts: (a) $k: X \to Y$ is a continous map. and if F is a path homotopy in X between f and g. then $k \circ F$ is a path homotopy in Y between $k \circ f$ and $k \circ g$

(b) $k: X \to Y$ is a continous map. and if f and g are paths in X with $f(1) = g(0)$. then

$$k \circ (f * g) = (k \circ f) * (k \circ g)$$

first prove (2):

let $i: I \to I$ be the identity map. which is the path in I from 0 to 1. then $e_0 * i$ is also a path from 0 to 1 in I

since I is convex. there is always a path-homotopy G in I between i and $e_0 * i$. Then ..

Def: X top. space. $x_0 \in X$ fixed. a closed path

$\pi_1(X, x_0) = \{[f] : f : I \to X : f(0) = f(1) = x_0$ is a path$\}$

is a fundamental group of X with base point x_0

product: $[f] * [g] = [f * g]$ is well defined

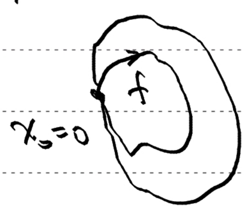
(by 5.1.2 (1)) the product is associative

identy: $[C_{x_0}]$ is the unit element. (by 5.1.2 (2))

inverse: $[f]^{-1} = [f^{-1}]$ (by 5.1.2 (3))

$\Rightarrow \pi_1(X, x_0)$ is a group

Example: $\pi_1(\mathbb{R}^n, 0) = \{[C_0]\}$ is the trivial group



$[f] \in \pi_1(\mathbb{R}^n, x_0)$

$f \underset{H}{\simeq} C_0$

$H(s,t) = f((1-t)s)$; $[f] = [C_0]$

路径的同伦等价类才是群元素.

Def: let α be a path on X from x_0 to x_1

define a map: $\hat{\alpha}$ $\pi_1(X, x_0) \to \pi_1(X, x_1)$

$\hat{\alpha}([f]) = [\alpha^{-1}] * [f] * [\alpha]$

Theorem 5.2.1. the map $\hat{\alpha}$ is group isomorphism.

proof: $\hat{\alpha}([f]) * \hat{\alpha}([g]) = [\alpha^{-1}] * [f] * [\alpha] * [\alpha^{-1}] * [g] * [\alpha]$

$= [\alpha^{-1}] * [f] * [g] * [\alpha]$

$= \hat{\alpha}([f] * [g])$ homomorphism verify that $\hat{\alpha}$ has inverse.

denote $\hat{\beta}([h]) = [\beta^{-1}] * [h] * [\beta]$ where $\beta = \alpha^{-1}$

$$\hat{\alpha}(\hat{\beta}([h])) = [\alpha^{-1}] * ([\alpha] * [h] * [\alpha^{-1}]) * [\alpha]$$
$$= [h]$$

$\hat{\beta}$ is the inverse of $\hat{\alpha}$.

$\hat{\alpha}$ is bijective. $\hat{\alpha}$ is isomorphism

Corollary: If X is path-connected.

for all $x_0, x_1 \in X$ then

$$\pi_1(X, x_0) \cong \pi_1(X, x_1)$$

(in general, the isomorphism is not canonical, but depends on the choice of α)

Def: X is simply connected if X is path-connected and $\pi_1(X, x_0)$ is trivial (does not depend on choice of x_0)

Lemma 52.3: If X is simply connected then any 2 paths with the same initial point and end point are path-homotopic

Theorem: $\pi_1(S^1, x_0) \cong \mathbb{Z}$

Def: let $h: X \to Y$ be continuous with $h(x_0) = y_0$

$h_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$

$$h_*([f]) = [h \circ f]$$

is a homomorphism of groups, called the homomorphism induced by h.

it is well defined since

$f \underset{H}{\simeq} f' \Rightarrow h \circ f \underset{h \circ H}{\simeq} h \circ f'$

then

$$h_*([f]) * h_*([f']) = [h \circ f] * [h \circ f']$$
$$= [(h \circ f) * (h \circ f')] = [h \circ (f * f')]$$
$$= h_*([f] * [f'])$$

Theorem 52.4 (1) if $X \xrightarrow{f} Y \xrightarrow{g} Z$ continuous maps

$f(x_0) = y_0$, $g(y_0) = z_0$

$$\Rightarrow (g \circ f)_* = g_* \circ f_*$$

homomorphism between $\pi_1(X, x_0)$ and $\pi_1(Z, z_0)$

(2) and $(id_X)_* = id_{\pi_1(X, x_0)}$

identity map $\Rightarrow$ identity homomorphism

proof: (1) $(g \circ f)_*([w]) \equiv [(g \circ f) \circ w]$

$$= g_*([f \circ w]) = g_*(f_*([w]))$$
$$= (g_* \circ f_*)([w])$$

$$\Rightarrow (g \circ f)_* = g_* \circ f_*$$

(2) $(id_X)_*([w]) = [id_X \circ w] = [w]$

$$\Rightarrow (id_X)_* = id_{\pi_1(X, x_0)}$$

Corollary 52.5. If $f: X \to Y$ is homeomorphism with $f(x_0) = y_0$

then $f_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is isomorphism

proof let $g: Y \to X$ be the inverse map of f

(also continous since f is homeomorphism

then $(g \circ f)_* = (id_X)_*$ since $g \circ f = f \circ g = id_X$

$= id_{\pi_1(X, x_0)}$

similarly $(f \circ g)_* = id_{\pi_1(Y, y_0)}$

$\Rightarrow f_*$ is isomorphism with inverse g_*

($\exists$ inverse $\Rightarrow$ bijective)

Remark: If X and Y are path connected and $\pi_1(X, x_0)$ is not isomorphic to $\pi_1(Y, y_0)$ then X and Y are not homeomorphism

The Fundemental Group is an example of Topological Invariance of Path-connected Space.

The fundemental group $(\pi_1(X, x_0), h_*)$ is a functor from the category of topo space with base points and continous based point preserving map to the category of groups and group homomorphism:

$h: X \to Y \qquad h(x_0) = y_0$

$\Rightarrow h_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$