

## Theory

Suppose there are  $n$  monomers species. Let

$$M = (M_1, M_2, \dots, M_n) \in \mathbb{R}^d$$

denote the vector of monomer concentrations, and

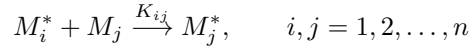
$$M^* = (M_1^*, M_2^*, \dots, M_n^*) \in \mathbb{R}^d$$

denote the vector of concentrations of chains terminating in monomer  $(M_1, M_2, \dots, M_n)$  respectively. Let us also write the rate constant matrix as

$$K \in \mathbb{R}^{n \times n}$$

We shall assume that  $K$  is in general position, i.e.  $K$  is invertible and diagonalizable (which happens almost surely if this is measured with some random error)

Then, the generalized  $n$ -monomer Mayo-Lewis reaction is given by



The rate equations are the following system of ODEs (written in vector form)

$$\begin{aligned} \frac{dM}{dt} &= -(K^T M^*) \circ M \\ \frac{dM^*}{dt} &= (K^T M^*) \circ M - (KM) \circ M^* \end{aligned}$$

where  $K^T$  is the transpose of the matrix  $K$  and  $\circ$  denotes vector Hadamard product, i.e. for vectors  $x$  and  $y$ ,  $x \circ y$  is a vector with components  $x \circ y = (x_1 y_1, x_2 y_2, \dots, x_n y_n)$ .

Now, let us denote by  $\mathbf{1}$  the vector of all 1's, i.e.  $\mathbf{1} = (1, 1, \dots, 1)$ . Then, the monomer mole fraction is a vector

$$f = \frac{M}{\mathbf{1}^T M}$$

Similarly, the fraction of chains terminating in monomers  $M$  is

$$f^* = \frac{M^*}{\mathbf{1}^T M^*}$$

Finally, under the steady-state condition the mole fraction of incorporated monomer is a vector

$$F = \frac{dM/dt}{\mathbf{1}^T dM/dt}$$

We shall assume that  $M$ ,  $M^*$  and  $dM/dt$  are not identically 0 so that the above expressions for  $f$ ,  $f^*$ ,  $F$  are well-defined.

Using the steady-state approximation, we may set  $dM^*/dt = 0$ . This allows us to write, using the ODE system, the following equations

$$F = \frac{f \circ K^T f^*}{f^T K^T f^*}$$

$$0 = (K^T f^*) \circ f - (K f) \circ f^*$$

The last equation can be rewritten

$$(D_f K^T - D_{Kf}) f^* = 0$$

where for any vector  $x$ ,  $D_x$  denotes the diagonal matrix formed by having  $x$  on the diagonal and 0 everywhere else. Since  $K$  is in general position, the matrix  $A(K, f) \equiv (D_f K^T - D_{Kf})$  has a one-dimensional null-space, whose basis we denote by the vector  $v_0(K, f)$ . Then, we immediately have

$$f^* = \frac{v_0(K, f)}{\mathbf{1}^T v_0(K, f)}$$

Therefore, we have the relationship between  $F$  and  $f$

$$F = \frac{f \circ K^T v_0(K, f)}{f^T K^T v_0(K, f)}$$

Conversely, given  $F$  we can also determine  $f$ , but this is in general no longer a one-to-one mapping, hence we may resort to solving the optimization problem

$$f = \arg \min_{x: x_i \geq 0, \sum_i x_i = 1} \left\| F - \frac{x \circ K^T v_0(K, x)}{x^T K^T v_0(K, x)} \right\|^2$$

which can be solved numerically by the L-BFGS-B method.