Logic Coursework

1) i) NOT can be expressed using the connective ¬

A	B	¬(A → ¬B)	¬A → B
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	1

Therefore:

- AND can be expressed using the formula ¬(A -> ¬B)
- OR can be expressed using the formula ¬A -> B

Thus, {¬, ->} is a complete set of connectives.

ii)

A	B	A → 0	(A → (B → 0)) → 0	(A → 0) → B
0	0	1	0	0
0	1	1	0	1
1	0	0	0	1
1	1	0	1	1

Therefore:

- NOT can be expressed using the formulae A -> 0 and B -> 0
- AND can be expressed using the formula (A -> (B -> 0)) -> 0
- OR can be expressed using the formula (A -> 0) -> B

Thus, {->, 0} is a complete set of connectives.

iii)

A	B	A NAND A	(A NAND A) NAND (B NAND B)
0	0	1	0
0	1	1	1
1	0	0	1
1	1	0	1

Therefore:

- NOT can be expressed using the formulae A NAND A and B NAND B
- AND can be expressed using the connective ∧
- OR can be expressed using the formula (A NAND A) NAND (B NAND B)

Thus, {NAND, ∧} is a complete set of connectives.

iv) AND can be expressed using the connective ∧
OR can be expressed using the connective ∨
However, NOT cannot be expressed using either connective.

- A ∧ A does not negate A
- A ∨ A also does not negate A

Thus, {∧, ∨} is not a complete set of connectives.

2) i) $((\neg p \vee q) \to r) \to (\neg s \vee t)$

$(\neg(\neg p \vee q) \vee r) \to (\neg s \vee t)$

$\neg(\neg(\neg p \vee q)) \vee r) \vee \neg s \vee t$

$((\neg p \vee q) \wedge \neg r) \vee \neg s \vee t$

$((\neg s \vee \neg p \vee q) \wedge (\neg s \vee \neg r)) \vee t$

$(t \vee \neg s \vee \neg p \vee q) \wedge (t \vee \neg s \vee \neg r)$ CNF

ii) $((\neg p \vee q) \wedge \neg r) \vee \neg s \vee t$

$(\neg r \wedge \neg p) \vee (\neg r \wedge q) \vee \neg s \vee t$ DNF

3)

Tseitin's Algorithm converts propositional formulae into conjunctive normal form, while avoiding the exponential increase in the number of terms that comes with the standard approach of using De Morgan's law and distributive properties.

$A \longleftrightarrow (x_1 \wedge x_2 \wedge x_3)$

$(A \to (x_1 \wedge x_2 \wedge x_3)) \wedge ((x_1 \wedge x_2 \wedge x_3) \to A)$

$(\neg A \vee (x_1 \wedge x_2 \wedge x_3)) \wedge (\neg(x_1 \wedge x_2 \wedge x_3) \vee A)$

$(\neg A \vee x_1) \wedge (\neg A \vee x_2) \wedge (\neg A \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3 \vee A) = F_{x_1 \wedge x_2 \wedge x_3}$

$B \longleftrightarrow (y_1 \wedge y_2 \wedge y_3)$

$(\neg B \vee y_1) \wedge (\neg B \vee y_2) \wedge (\neg B \vee y_3) \wedge (\neg y_1 \vee \neg y_2 \vee \neg y_3 \vee B) = F_{y_1 \wedge y_2 \wedge y_3}$

$C \longleftrightarrow (A \to B)$

$C \longleftrightarrow (\neg A \vee B)$

$$(C \to (\neg A \lor B)) \land ((\neg A \lor B) \to C)$$

$$(\neg C \lor \neg A \lor B) \land (\neg(\neg A \lor B) \lor C)$$

$$(\neg C \lor \neg A \lor B) \land ((A \land \neg B) \lor C)$$

$$(\neg C \lor \neg A \lor B) \land (C \lor A) \land (C \lor \neg B) = F_{A \to B}$$

$$D \leftrightarrow (C \lor z)$$

$$(D \to (C \lor z)) \land ((C \lor z) \to D)$$

$$(\neg D \lor (C \lor z)) \land (\neg(C \lor z) \lor D)$$

$$(\neg D \lor C \lor z) \land ((\neg C \land \neg z) \lor D)$$

$$(\neg D \lor C \lor z) \land (D \lor \neg C) \land (D \lor \neg z) = F_{C \lor z}$$

$$\begin{aligned} F = {} & F_{x_1 \land x_2 \land x_3} \land F_{y_1 \land y_2 \land y_3} \land F_{A \to B} \land F_{C \lor z} \land D \\ = {} & (\neg A \lor x_1) \land (\neg A \lor x_2) \land (\neg A \lor x_3) \land (\neg x_1 \lor \neg x_2 \lor \neg x_3 \lor A) \\ & \land (\neg B \lor y_1) \land (\neg B \lor y_2) \land (\neg B \lor y_3) \land (\neg y_1 \lor \neg y_2 \lor \neg y_3 \lor B) \\ & \land (\neg C \lor \neg A \lor B) \land (C \lor A) \land (C \lor \neg B) \\ & \land (\neg D \lor C \lor z) \land (D \lor \neg C) \land (D \lor \neg z) \land D \end{aligned}$$

4) i) If the LHS of the implication is true, for each x there is a y_0 where $E(x, y_0)$ and $E(y_0, z)$ are true for all z. For the RHS, for each x we can use the same y_0 from the LHS, where $E(y_0, z)$ is true with the given z, since it is true for all z. Therefore, the statement is logically valid.

ii) If the LHS of the implication is true, for each x there is a y_0 where $E(x, y_0)$ is true and there is a u_0 where $E(u_0, v)$ is true for all v.
For the RHS, we can use the same u_0 from the LHS which means $E(u_0, v)$ is true for all v. Then, for each x, we can use the same y_0 where $E(x, y_0)$ is true. Therefore, the statement is logically valid.

iii) If the LHS of the implication is true, for each x there is a y_0 where $R(x, y_0, z)$ is true for all z.
For the RHS, there must exist an x_0 where for all y, there is a z_0 making $R(x_0, y, z)$ true. The LHS guarantees a suitable y_0 for each x, but it does not guarantee that there is a particular x_0 that works for every y. Therefore, the statement is logically invalid.

iv) If the LHS of the implication is true, then for all x and y, E(x, y) is true.
For the RHS, E(x, y) is also true for all x and y. Since it uses a ∨ connective, the whole RHS is true, regardless of the result of E(y, z). Therefore, the statement is logically valid.

5) i) For x = 0, y = 1 and z = 2:

- If w = 0 then E(x, w) = E(0, 0) does not hold.
- If w = 1 then E(y, w) = E(1, 1) does not hold.
- If w = 2 then E(z, w) = E(2, 2) does not hold.

Therefore, a w does not exist for all x, y and z, and the statement is false.

ii) If x = 0:

- For y = 1 and z = 2:
    - If w = 0 then E(x, w) = E(0, 0) does not hold.
    - If w = 1 then E(y, w) = E(1, 1) does not hold.
    - If w = 2 then E(z, w) = E(2, 2) does not hold.

If x = 1:

- For y = 0 and z = 2:
    - If w = 0 then E(y, w) = E(0, 0) does not hold.
    - If w = 1 then E(x, w) = E(1, 1) does not hold.
    - If w = 2 then E(z, w) = E(2, 2) does not hold.

If x = 2:

- For y = 0 and z = 1:
    - If w = 0 then E(y, w) = E(0, 0) does not hold.
    - If w = 1 then E(z, w) = E(1, 1) does not hold.
    - If w = 2 then E(x, w) = E(2, 2) does not hold.

Since there is no x where for every y, z pair there is a w, the statement is false.

iii) For any y, you can pick the x to be the same as the y. Then for any z, you can pick the w to be the only number not chosen yet, and this will hold for all the E relations, since they only allow for w to not be equal to any of x, y and z. Therefore, the statement is true.

iv) Since E only holds if the two variables are not equal, there is no choice of x, y and z where all possibilities of w do not equal any of them. Thus, it is impossible to find x, y and z such that every w in the domain is different from all three. Therefore, the statement is false.

v) The E relations say that w cannot be equal to any of x_2, y_2, z_2 and z. For when x_1, y_1 and z_1 are all distinct, x_2, y_2 and z_2 can be chosen so there is one 'leftover' value which could be a possible choice for w. However, there has to be a choice for w which holds for all z. So, when z is equal to the 'leftover' value, there is no choice for w. Therefore, the statement is false.

vi) For the statement to be true, you would need two 'leftover' variables after the choice of x_2 and y_2, so that z and/or z_2 could simultaneously take up just one of those leftover variables, leaving a choice for w. However, after the choice of x_2, this does not work for when y_1 is equal to x_2, since then y_2 could not equal x_2. Since for any choice of x_2 it doesn't hold for all y_1, the statement is false.