Partitions

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1 What is a partition?

Definition 1. A **partition** $p(n)$ of a natural number n is a way of writing n as a sum of positive integer parts.

For example, the partitions of 4 look like this:

$$\begin{aligned}
&4\\
&3+1\\
&2+2\\
&2+1+1\\
&1+1+1+1
\end{aligned}$$

Each number in a partition is called a part. The function $p(n)$ counts the total number of partitions of n. There is no formula for $p(n)$, but Hardy and Ramanujan got very close with their asymptotic formula:

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

$p(n)$ can be given conditions to produce more specific results, where different results can be compared and bijections can be drawn.

Theorem 1. $p(n \mid \text{each part is odd}) = p(n \mid \text{each part is distinct})$

Proof. Starting with an odd-partition part, 'pair up' repeated parts and replace them with their double until there are no repeated parts. Once all repeated parts are paired up, the parts of the partition are all distinct. For example, consider this partiton of 10 with only odd parts:

$$5+3+1+1$$

Pair up the two 1s and replace them with 2 to get this new partition:

$$5+3+2$$

There are no more repeated parts, so we are now left with a partiton of distinct parts.
Starting with a distinct-part partition, 'split' any even parts and replace them with their two halves.

$$5+3+2$$

Split up the 2 and replace it with two 1s to get this new partition:

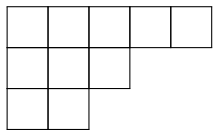
$$5+3+1+1$$

Every odd-part partition uniquely maps to a distinct-part partition and vice versa. This establishes a bijection. □

2 Representing partitions with Young diagrams

Definition 2. A **Young diagram** is a diagram used to represent a partition.

It is made up of squares, where each square represents the value of 1. Each part is represented by a row of squares, with the rows going down in decreasing order.
Here is the Young diagram for the partition of 10, 5 + 3 + 2:

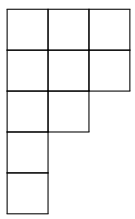


First row: 5 squares
Second row: 3 squares
Third row: 2 squares

These diagrams can be used to further analyse bijections between different types of partitions.

Theorem 2. $p(n \mid \text{largest part has size } k) = p(n \mid \text{has } k \text{ parts})$

Proof. Consider the **conjugate** of the previous Young diagram, where the conjugate is the result of flipping the diagram on its diagonal:



This represents the partition 3 + 3 + 2 + 1 + 1 which is also a partiton of 10.

When flipping the diagram, the first row, which is the largest part, becomes the first column, which is the number of parts. In the previous example, the largest part of the original partition has size 5, and the number of parts of the conjugate is 5.
Every Young diagram uniquely maps to its conjugate and vice versa. This establishes a bijection. □

3 Representing partitions with generating functions

Definition 3. A **generating function** is a way to represent a sequence of numbers as a power series.

Since the geometric series identity gives:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$$

and similarly,

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + x^8 + \dots,$$

each term represents including that power of x any number of times. Extending this pattern, we obtain the partition generating function:

$$\prod_{n=1}^{\infty} \frac{1}{1-x^n}$$

where each factor $\frac{1}{1-x^n}$ accounts for the possibility of including the number n any number of times in a partition.

Analysing the first six terms of the partition generating function:

$$\prod_{n=1}^{\infty} \frac{1}{1-x^n} = \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^4} \cdot \frac{1}{1-x^5} \cdot \frac{1}{1-x^6} \cdot \ldots$$

Each factor can be expanded as follows:

$$\begin{aligned}
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \ldots \\
\frac{1}{1-x^2} &= 1 + x^2 + x^4 + x^6 + \ldots \\
\frac{1}{1-x^3} &= 1 + x^3 + x^6 + \ldots \\
\frac{1}{1-x^4} &= 1 + x^4 + \ldots \\
\frac{1}{1-x^5} &= 1 + x^5 + \ldots \\
\frac{1}{1-x^6} &= 1 + x^6 + \ldots
\end{aligned}$$

When we multiply these expansions, the coefficient of x^6 in the product will correspond to $p(6)$. To find this coefficient, we need to identify all combinations of terms whose powers sum to 6:

$$\begin{aligned}
&x^6 \text{ from } \frac{1}{1-x} : \text{ represents the partition } 6 = 1+1+1+1+1+1 \\
&x^4 \cdot x^2 \text{ from } \frac{1}{1-x} \text{ and } \frac{1}{1-x^2} : \text{ represents } 6 = 1+1+1+1+2 \\
&x^2 \cdot x^4 \text{ from } \frac{1}{1-x} \text{ and } \frac{1}{1-x^4} : \text{ represents } 6 = 1+1+4 \\
&x^2 \cdot x^2 \cdot x^2 \text{ from } \frac{1}{1-x^2} : \text{ represents } 6 = 2+2+2 \\
&x^3 \cdot x^3 \text{ from } \frac{1}{1-x^3} : \text{ represents } 6 = 3+3 \\
&x^3 \cdot x^2 \cdot x \text{ from } \frac{1}{1-x^3}, \frac{1}{1-x^2}, \text{ and } \frac{1}{1-x} : \text{ represents } 6 = 3+2+1 \\
&x^3 \cdot x \cdot x \cdot x \text{ from } \frac{1}{1-x^3} \text{ and } \frac{1}{1-x} : \text{ represents } 6 = 3+1+1+1 \\
&x^2 \cdot x^2 \cdot x \cdot x \text{ from } \frac{1}{1-x^2} \text{ and } \frac{1}{1-x} : \text{ represents } 6 = 2+2+1+1 \\
&x^4 \cdot x \cdot x \text{ from } \frac{1}{1-x^4} \text{ and } \frac{1}{1-x} : \text{ represents } 6 = 4+1+1 \\
&x^5 \cdot x \text{ from } \frac{1}{1-x^5} \text{ and } \frac{1}{1-x} : \text{ represents } 6 = 5+1 \\
&x^6 \text{ from } \frac{1}{1-x^6} : \text{ represents the partition } 6 = 6
\end{aligned}$$

Counting these combinations, we find that $p(6) = 11$.
More generally, if you want to find $p(k)$, you can set the upper limit of the product to k and ignore any terms in the series with an exponent greater than k. So, when $k = 6$, we can expand $\prod_{n=1}^{6} \frac{1}{1-x^n}$, ignoring the terms where the exponent is greater than 6. We end up with:

$$(1 + x + x^2 + x^3 + x^4 + x^5 + x^6) \cdot (1 + x^2 + x^4 + x^6) \cdot (1 + x^3 + x^6) \cdot (1 + x^4) \cdot (1 + x^5) \cdot (1 + x^6)$$

which, when expanded out, gives us:

$$1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6$$

The coefficient of x^6 is 11, which is equal to the number of partitions of 6.

Theorem 3. Let $p_2(k)$ be the number of partitions of k where each part is of size at most 2. The generating function for $p_2(k)$ is given by the product $\prod_{n=1}^{2} \frac{1}{1-x^n}$.

Proof. To find $p_2(k)$, the number of partitions of k with parts of size at most 2, we only need to expand the first two terms of the generating function:

$$\frac{1}{1-x} \cdot \frac{1}{1-x^2}$$

Which is equivalent to:

$$(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + \dots) \cdot (1 + x^2 + x^4 + x^6 + x^8 + \dots)$$

When we expand this out, the coefficient of x^k in the product will correspond to $p_2(k)$. For example, if we want to find $p_2(8)$, we look at the coefficient of x^8 in:

$$1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + 4x^6 + 4x^7 + 5x^8 + \dots$$

Therefore, $p_2(8) = 5$.

$\square$