Partitions

Lewis Reed

March 2025

1 What is a partition?

Definition 1. A **partition** $p(n)$ of a natural number n is a way of writing n as a sum of positive integer parts.

For example, the partitions of 4 look like this:

$$\begin{aligned}&4\\&3+1\\&2+2\\&2+1+1\\&1+1+1+1\end{aligned}$$

Each number in a partition is called a part. The function $p(n)$ counts the total number of partitions of n. There is no formula for $p(n)$, but Hardy and Ramanujan got very close with their asymptotic formula:

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

$p(n)$ can be given conditions to produce more specific results, where different results can be compared and bijections can be drawn.

Theorem 1. $p(n \mid \text{each part is odd}) = p(n \mid \text{each part is distinct})$

Proof. Starting with an odd-partition part, “pair up” repeated parts and replace them with their double until there are no repeated parts. Once all repeated parts are paired up, the parts of the partition are all distinct.
For example, consider this partition of 10 with only odd parts:

$$5+3+1+1$$

Pair up the two 1s and replace them with 2 to get this new partition:

$$5+3+2$$

There are no more repeated parts, so we are now left with a partition of distinct parts.
Starting with a distinct-part partition, “split” any even parts and replace them with their two halves.

$$5+3+2$$

Split up the 2 and replace it with two 1s to get this new partition:

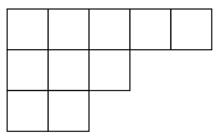
$$5 + 3 + 1 + 1$$

Every odd-part partition uniquely maps to a distinct-part partition and vice versa. This establishes a bijection. ☐

2 Representing partitions with Young diagrams

Definition 2. A **Young diagram** is a diagram used to represent a partition.

It is made up of squares, where each square represents the value of 1. Each part is represented by a row of squares, with the rows going down in decreasing order.
Here is the Young diagram for the partition of 10, 5 + 3 + 2:

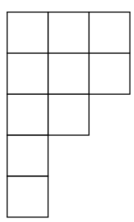


First row: 5 squares
Second row: 3 squares
Third row: 2 squares

These diagrams can be used to further analyse bijections between different types of partitions.

Theorem 2. $p(n \mid \text{largest part has size } k) = p(n \mid \text{has } k \text{ parts})$

Proof. Consider the **conjugate** of the previous Young diagram, where the conjugate is the result of flipping the diagram on its diagonal:



This represents the partition 3 + 3 + 2 + 1 + 1 which is also a partition of 10.

When flipping the diagram, the first row, which is the largest part, becomes the first column, which is the number of parts. In the previous example, the largest part of the original partition has size 5, and the number of parts of the conjugate is 5.
Every Young diagram uniquely maps to its conjugate and vice versa. This establishes a bijection. ☐

3 Representing partitions with generating functions

Definition 3. A **generating function** is a way to represent a sequence of numbers as a power series.

Since the geometric series identity gives:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$$

and similarly,

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + x^8 + \dots,$$

each term represents including that power of x any number of times. Extending this pattern, we obtain the partition generating function:

$$\prod_{n=1}^{\infty} \frac{1}{1-x^n}$$

where each factor $\frac{1}{1-x^n}$ accounts for the possibility of including the number n any number of times in a partition.
Analysing the first six terms of the partition generating function:

$$\prod_{n=1}^{\infty} \frac{1}{1-x^n} = \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^4} \cdot \frac{1}{1-x^5} \cdot \frac{1}{1-x^6} \cdots$$

Each factor can be expanded as follows:

$$\begin{aligned}
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \dots \\
\frac{1}{1-x^2} &= 1 + x^2 + x^4 + x^6 + \dots \\
\frac{1}{1-x^3} &= 1 + x^3 + x^6 + \dots \\
\frac{1}{1-x^4} &= 1 + x^4 + \dots \\
\frac{1}{1-x^5} &= 1 + x^5 + \dots \\
\frac{1}{1-x^6} &= 1 + x^6 + \dots
\end{aligned}$$

When we multiply these expansions, the coefficient of x^6 in the product will correspond to $p(6)$. To find this coefficient, we need to identify all combinations of terms whose powers sum to 6:

$$\begin{aligned}
&x^6 \text{ from } \frac{1}{1-x} : \text{ represents the partition } 6 = 1+1+1+1+1+1 \\
&x^4 \cdot x^2 \text{ from } \frac{1}{1-x} \text{ and } \frac{1}{1-x^2} : \text{ represents } 6 = 1+1+1+1+2 \\
&x^2 \cdot x^4 \text{ from } \frac{1}{1-x} \text{ and } \frac{1}{1-x^4} : \text{ represents } 6 = 1+1+4 \\
&x^2 \cdot x^2 \cdot x^2 \text{ from } \frac{1}{1-x^2} : \text{ represents } 6 = 2+2+2 \\
&x^3 \cdot x^3 \text{ from } \frac{1}{1-x^3} : \text{ represents } 6 = 3+3 \\
&x^3 \cdot x^2 \cdot x \text{ from } \frac{1}{1-x^3}, \frac{1}{1-x^2}, \text{ and } \frac{1}{1-x} : \text{ represents } 6 = 3+2+1 \\
&x^3 \cdot x \cdot x \cdot x \text{ from } \frac{1}{1-x^3} \text{ and } \frac{1}{1-x} : \text{ represents } 6 = 3+1+1+1 \\
&x^2 \cdot x^2 \cdot x \cdot x \text{ from } \frac{1}{1-x^2} \text{ and } \frac{1}{1-x} : \text{ represents } 6 = 2+2+1+1 \\
&x^4 \cdot x \cdot x \text{ from } \frac{1}{1-x^4} \text{ and } \frac{1}{1-x} : \text{ represents } 6 = 4+1+1 \\
&x^5 \cdot x \text{ from } \frac{1}{1-x^5} \text{ and } \frac{1}{1-x} : \text{ represents } 6 = 5+1 \\
&x^6 \text{ from } \frac{1}{1-x^6} : \text{ represents the partition } 6 = 6
\end{aligned}$$

Counting these combinations, we find that $p(6) = 11$.

More generally, if you want to find $p(k)$, you can set the upper limit of the product to k and ignore any terms in the series with an exponent greater than k. So, when $k = 6$, we can expand $\prod_{n=1}^{6} \frac{1}{1-x^n}$, ignoring the terms where the exponent is greater than 6. We end up with:

$$(1 + x + x^2 + x^3 + x^4 + x^5 + x^6) \cdot (1 + x^2 + x^4 + x^6) \cdot (1 + x^3 + x^6) \cdot (1 + x^4) \cdot (1 + x^5) \cdot (1 + x^6)$$

which, when expanded out, gives us:

$$1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6$$

The coefficient of x^6 is 11, which is equal to the number of partitions of 6.

Theorem 3. Let $p_2(k)$ be the number of partitions of k where each part is of size at most 2. The generating function for $p_2(k)$ is given by the product $\prod_{n=1}^{2} \frac{1}{1-x^n}$.

Proof. To find $p_2(k)$, the number of partitions of k with parts of size at most 2, we only need to expand the first two terms of the generating function:

$$\frac{1}{1-x} \cdot \frac{1}{1-x^2}$$

Which is equivalent to:

$$(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + \dots) \cdot (1 + x^2 + x^4 + x^6 + x^8 + \dots)$$

When we expand this out, the coefficient of x^k in the product will correspond to $p_2(k)$. For example, if we want to find $p_2(8)$, we look at the coefficient of x^8 in:

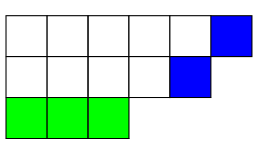
$$1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + 4x^6 + 4x^7 + 5x^8 + \dots$$

Therefore, $p_2(8) = 5$.

$\square$

4 Franklin's involution

Definition 4. Consider a Young diagram of a partition of distinct parts. We can name two regions of the diagram:



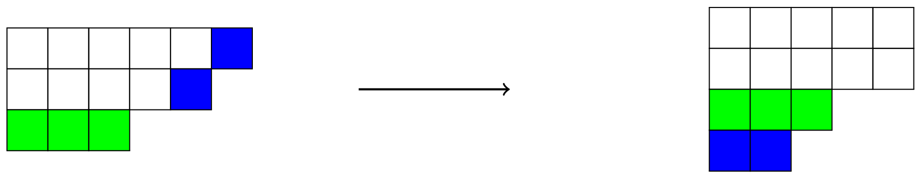
Blue - We call this the **slope**.
Green - This is the **base**, which is simply the smallest part in the partition.

We can manipulate these regions using a certain method, called **Franklin's involution**:

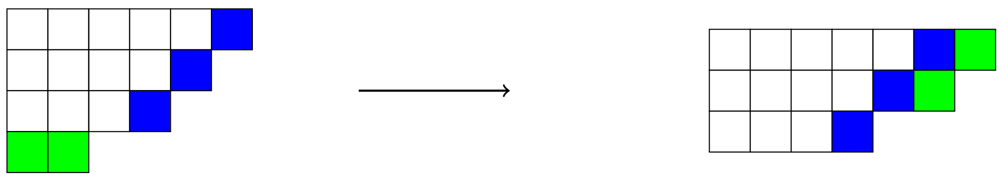
- If the slope is less than the base, we can "move" the slope down below the base to become the new base.
- If the slope is greater than or equal to the base, we can "move" the base up to become the new slope.

Theorem 4. If Franklin's involution is successful, it will map a partition with either an even number of distinct parts to one with an odd number of distinct parts.

Proof. For our previous example of the partition of 14, 6 + 5 + 3, which maps a partition with an odd number of parts to one with an even number of parts:



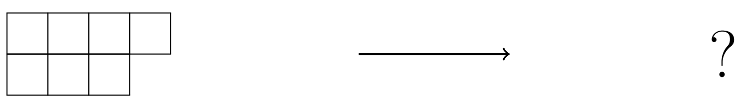
For this example of the partition of 17, 6 + 5 + 4 + 2, which maps a partition with an even number of parts to one with an odd number of parts:



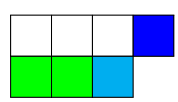
The transformation uniquely maps a partition with an even number of parts to one with an odd number of distinct parts, and vice versa. This establishes a bijection.

□

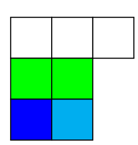
For all the partitions of numbers like 10, 9 and 8, every partition with an even number of distinct parts pairs up exactly with a partition with an odd number of distinct parts. However, for the partitions of a number like 7, there is a "leftover" partition which does not map to any counterpart. This particular example is the partition of 3 + 4, which looks like this:



The problem with this particular Young diagram is that the slope and base overlap:

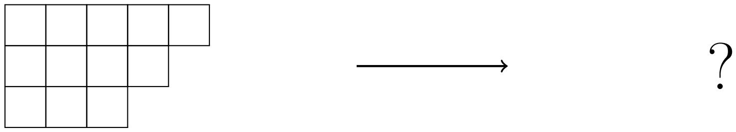


If you try and perform the involution anyway, you would want to move the slope down to become a new base, since the slope is less than the base. This would look like this:

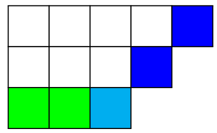


Since this involves moving a part of the old base, this is creates the partition 3 + 2 + 2, which is no longer a partition of distinct parts. This means the involution fails.

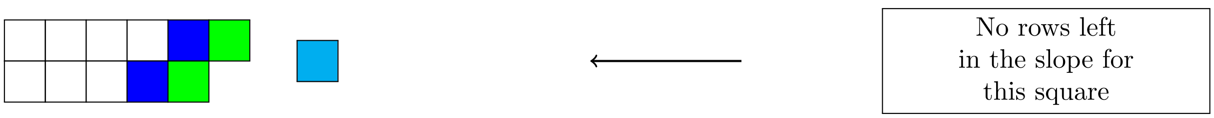
Another example of where there is a leftover partition which does not map to any counterpart is the partitions of 12. The particular diagram where the involution fails is for the partition 5 + 4 + 3, which looks like this:



In this case the slope and base also overlap:



Trying to perform the involution anyway, means you would want to move the base up to become the new slope, since the slope is equal to the base. This would look like this:



Since this involves moving a part of the old slope, there are less rows than the new slope, leaving one square free. This means the involution fails.

There is a pattern for which numbers have partitions where Franklin's involution fails. The numbers 7 and 12 are examples of generalised pentagonal numbers.

5 Pentagonal number theorem

Definition 5. The set of **generalised pentagonal numbers** is an extended version of the set of pentagonal numbers:

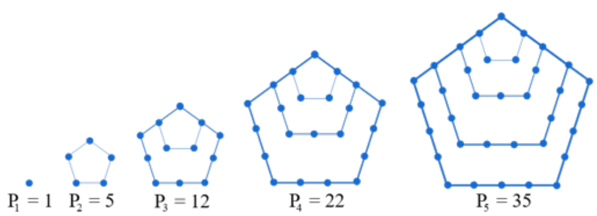


Figure 1: Pentagonal numbers. Source: `https://www.math.net/pentagonal-number`, March 2025.

Where a pentagonal number P_k follows:

$$P_k = \frac{k(3k-1)}{2} \text{for } k = 0, 1, 2, 3, \ldots$$

And a generalised pentagonal number follows:

$$P_k = \frac{k(3k-1)}{2} \text{for } k = 0, \pm 1, \pm 2, \pm 3, \ldots$$

Theorem 5. The theorem states that:

$$\prod_{n=1}^{\infty}(1-x^n)=\sum_{-\infty}^{\infty}(-1)^k x^{\frac{k(3k-1)}{2}}$$

This means that if we expand the left-hand side, we get a series with plus and minus signs, and the powers of x in the series are pentagonal numbers.

Proof. Considering the cases where Franklin's involution fails, we can use this to derive:

$$p_e(n)=p_o(n)+\epsilon(n)$$

Where $p_e(n)$ and $p_o(n)$ are the number of partitions with an even or odd number of distinct parts, respectively, and:

$$\epsilon(n)=\begin{cases}(-1)^k & \text{if } n=\frac{k(3k-1)}{2}\\ 0 & \text{if anything else}\end{cases}$$

We know this is true because for most n, the number of partitions with an even number of distinct parts is the same as the number of partitions with an odd number of distinct parts. However, when n is a pentagonal number, Franklin's involution fails in exactly one case, leading to an extra even or odd partition. It is exactly one case because for a given n that is a generalised pentagonal number, there is exactly one way to form the Young diagram where the overlap occurs.

Reframing this in the form of a generating function:

$$\sum_{n=0}^{\infty}(p_e(n)-p_o(n))\cdot x^n$$

From the previous identity, we know this is equal to:

$$\sum_{n=0}^{\infty}\epsilon(n)\cdot x^n$$

This can also be written as:

$$\sum_{-\infty}^{\infty}(-1)^k x^{\frac{k(3k-1)}{2}}=(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)\dots$$

We know this because the product $(1-x)(1-x^2)(1-x^3)\cdots$ expands to a sum where almost every term cancels out due to the pairing, except for the special cases. This is why the sum ends up being

$$\sum_{-\infty}^{\infty}(-1)^k x^{\frac{k(3k-1)}{2}},$$

which is exactly what the pentagonal number theorem states.

□

From this, we can see:

$$\prod_{n=1}^{\infty}(1-x^n)=1-x-x^2+x^5+x^7-x^{12}-\dots$$

6 Euler's recurrence relation

Definition 6. Euler's recurrence relation expresses the partition function $p(n)$ in terms of previous values using contributions from generalized pentagonal numbers. It is given by:

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + \dots$$

where the terms correspond to the generalized pentagonal numbers $P_k = \frac{k(3k-1)}{2}$ for $k = \pm1, \pm2, \pm3, \dots$, and the signs alternate based on k.

Theorem 6. For all $n \geq 1$,

$$p(n) = \sum_{k \neq 0} (-1)^{k+1} p(n - P_k)$$

where $P_k = \frac{k(3k-1)}{2}$ are the generalized pentagonal numbers, and the sum runs over all nonzero integers k.

Proof. Recall the generating function for $p(n)$:

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{j=1}^{\infty} \frac{1}{1-x^j}$$

We also now know, from the pentagonal number theorem, that:

$$\prod_{j=1}^{\infty} (1-x^j) = 1 - x - x^2 + x^5 + x^7 - x^{12} - \dots$$

Using what we now know, we can rewrite the partition generating function:

$$\begin{aligned}
\sum_{n=0}^{\infty} p(n)x^n &= \prod_{j=1}^{\infty} \frac{1}{1-x^j} \\
&= \frac{1}{\prod_{j=1}^{\infty}(1-x^j)} \\
&= \frac{1}{1 - x - x^2 + x^5 + x^7 - x^{12} - \dots}
\end{aligned}$$

Therefore, we can write:

$$(\sum_{n=0}^{\infty} p(n)x^n) \cdot (1 - x - x^2 + x^5 + x^7 - x^{12} - \dots) = 1$$

Since this is simply the generating function multiplied by 1 over itself. We can expand this expression to get:

$$\sum_{n=0}^{\infty} p(n)x^n - \sum_{n=1}^{\infty} p(n-1)x^n - \sum_{n=2}^{\infty} p(n-2)x^n - \sum_{n=5}^{\infty} p(n-5)x^n - \sum_{n=7}^{\infty} p(n-7)x^n - \dots = 1$$

Since we know the sum is 1, the coefficients of the x^n terms must equal 0. Therefore:

$$p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - \dots = 0$$

Which we can rearrange to be:

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + \dots$$

This is exactly Euler's recurrence relation.

$\square$