The sum of the three contributions is equal to the dilatation

$$
\begin{aligned}
-\operatorname{div}\vec{v} &= \frac{\partial^2 \Phi}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \varphi}\left(\frac{1}{r}\frac{\partial \Phi}{\partial \varphi}\right) \\
&= \frac{\partial^2 \Phi}{\partial z^2} + \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \varphi^2},
\end{aligned}
\tag{45}
$$

and the wave equation in cylindrical coordinates is

$$
\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \varphi^2} + \frac{\partial^2 \Phi}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2}. \tag{46}
$$

21.4. The Solution of the Wave Equation — General Cylindrical Coordinates

The differential equation, Eq. (46), like every partial differential equation has not only an infinite number of solutions but also has an infinite number of special groups of solutions. Of these we consider again a group that is particularly suited for our purposes. We consider the group of standing and progressive harmonic waves; this group of solutions is sufficient to construct any physically possible solutions by superimposing wave terms. These solutions are of the form

$$
\Phi = R(r)\, Z(z)\, \Psi(\varphi)\, T(t). \tag{47}
$$

This expression is substituted into the wave equation which then is divided by $R\,Z\,\Psi\,T$ to obtain

$$
\left[\frac{1}{R}\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\cdot\frac{1}{R}\frac{\partial R}{\partial r} + \frac{1}{r^2\Psi}\frac{\partial^2 \Psi}{\partial \varphi^2}\right] + \left[\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}\right] = \frac{1}{c^2}\frac{\partial^2 T}{T\,\partial t^2}. \tag{48}
$$

The first bracketed series of terms depends only on R and φ, the second on z, and the right-hand side depends only on t. This equation can be satisfied for all values of z, φ, t and r only if the terms in the brackets are constant. We must have

$$
\frac{1}{c^2 T}\frac{\partial^2 T}{\partial t^2} = -k^2 \tag{49}
$$

$$
\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = -k_z^2 \tag{50}
$$

$$
\frac{1}{R}\left(\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r}\right) + \frac{1}{r^2\Psi}\frac{\partial^2 \Psi}{\partial \varphi^2} = -k_r^2. \tag{51}
$$

The arbitrary constants $-k^2$, $-k_z^2$ and $-k_r^2$ have been written as negative squares. The solutions thus obtained are particularly simple and of the form best suited for our purposes. The last expression can then be written in the form

$$
\frac{r^2}{R}\left(\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r}\right) + k_r^2 r^2 = -\frac{1}{\Psi}\frac{\partial^2 \Psi}{\partial \varphi^2}. \tag{52}
$$

The left-hand side now depends only on r; the right-hand side only on φ. The two sides must again be equal to a constant. Hence,

$$\frac{1}{\Psi}\frac{\partial^2\Psi}{\partial\varphi^2} = -m^2, \tag{53}$$

$$\frac{1}{R}\left(\frac{\partial^2 R}{\partial r^2}+\frac{1}{r}\frac{\partial R}{\partial r}\right)+k_r^2-\frac{m^2}{r^2}=0, \tag{54}$$

$$m, k_r, k_z = \text{constants}.$$

The constants are not independent of each other. Substitution of Eqs. (49), (50) and (51) into the differential equation yields

$$k_z^2 = k^2 - k_r^2. \tag{55}$$

The solution of Eq. (49) is

$$\bar{T} = A' e^{j\omega t} + [B' e^{-j\omega t}]. \tag{56}$$

The bracketed term is unnecessary because of the double signs of $\pm m$, $\pm k_z$. It leads to the same real solution as the first term. Equation (50) is solved by

$$\bar{Z} = \bar{C} e^{j k_z z} + \bar{D} e^{-j k_z z}, \tag{57}$$

and Eq. (53) by

$$\bar{\Psi} = \bar{E} e^{j m\varphi} + \bar{F} e^{-j m\varphi}. \tag{58}$$

Equation (54) represents the general Bessel's equation; its solutions in terms of progressive waves are the Hankel functions of first and second kind and of order m:

$$R(r) = \bar{A} H_m^{(1)}(k_r r) + \bar{B} H_m^{(2)}(k_r r). \tag{59}$$

The standing wave solutions are the Bessel functions and Neumann functions of order m:

$$R(r) = \bar{A} J_m(k_r r) + \bar{B} N_m(k_r r). \tag{60}$$

The complete solution, then, is given by

$$\bar{P} = \sum_m \{\bar{A}_m \cos(m\varphi+\alpha_m)\, Z_m^{(1)}(k_r r)\, e^{-j k_z z} + \bar{B}_m \cos(m\varphi+\beta_m)\, Z_m^{(2)}(k_r r)\, e^{+j k_z z}\}, \tag{61}$$

where

$$k_z = \sqrt{k^2 - k_r^2} \tag{62}$$

and $Z_m^{(1)}$ and $Z_m^{(2)}$ are either the functions J_m and N_m or $H_m^{(1)}$ and $H_m^{(2)}$. All values of m and k_z are admissible.

21.5. Sound Propagation in Circular Tubes

The solution, Eq. (60), applies for a circular tube if k_r is determined so that it satisfies the boundary conditions. For a tube with rigid walls

$$\left.\frac{\partial}{\partial r} J_m(k_r r)\right|_{r=R} = k_r \cdot J_m'(k_r R) = 0 \tag{63}$$