Concrete Mathematics ch. 4

1. what is the smallest positive integer that has exactly k divisors, for $1 \le k \le 6$.

$k=1.$ 1.

$k=2$: $2^{(a)} \Rightarrow (a+1)=2 \Rightarrow a=1 \Rightarrow \boxed{2^1}$

$k=3$: $2^{a} \Rightarrow (a+1)=3 \Rightarrow a=2$

$\Rightarrow 2^2 = \boxed{4}$

$k=4$: $2^a \Rightarrow (a+1)=4 \Rightarrow a=3 \Rightarrow$

or $2^a 3^b$ where $a=b=1 \Rightarrow \boxed{6}$

$k=5$: $2^a \Rightarrow (a+1)=5 \Rightarrow a=4 \Rightarrow 2^4 \boxed{16}$

$k=6$: $2^a 3^b \Rightarrow a=\cancel{1}2,\ b=1 \Rightarrow \boxed{12}$

2. Prove $\text{lcm}(mn)\gcd(mn)=mn$.

~~use it to express~~ $\text{lcm}(mn)$ as $\text{lcm}(n \bmod m, m)$

$\Rightarrow$ use 4.12, 4.14, 4.15

define $m = \prod_p p^{m_p}$ and $n = \prod_p p^{n_p}$.

Then $mn = \prod_p p^{n_p+m_p}$.

Notice that $\gcd(m,n) \Rightarrow k_p \min(m_p, n_p)\ \forall p$

and $\text{lcm}(m,n) \Rightarrow k_p = \max(m_p, n_p)\ \forall p$.

Then if we multiply $\gcd(m,n)\,\text{lcm}(m,n) = k$

Then $k_p = \min(m_p, n_p) + \max(m_p, n_p)\ \forall p$.

It suffices to show: $\gcd(m,n)\,\text{lcm}(m,n) = k = mn$

That is, $\underbrace{\gcd(m,n)\,\mathrm{lcm}(m,n)} = K = \underbrace{mn}$, iff $\forall p$,

$$K_p = \min(m_p, n_p) + \max(m_p, n_p) = n_p + m_p$$

where $\quad mn = \prod_p p^{n_p + m_p} = K$

So each $K_p = n_p + m_p$.

The result is trivial: $\min(m_p, n_p) =$

$m_p[m_p < n_p] + n_p[n_p < m_p]$

$K_p = \min(m_p, n_p) + \max(m_p, n_p)$

$= m_p[m_p < n_p] + n_p[n_p < m_p]$
$+ m_p[m_p > n_p] + n_p[n_p > m_p]$

$\max(m_p, n_p)$
$= m_p[m_p > n_p] + n_p[n_p > m_p]$

$= [m_p < n_p](m_p + n_p) + [m_p > n_p](m_p + n_p)$.

But for any $m_p, n_p \in \mathbb{R}$,

either $m_p < n_p$ or $m_p > n_p$.

$m_p < n_p \Rightarrow !(m_p > n_p)$.

So $\left([m_p < n_p] + [m_p > n_p]\right)(m_p + n_p)$

$= m_p + n_p$ because $[m_p < n_p] + [m_p > n_p]$
is always 1.
$\Rightarrow [m_p \neq n_p]$
when $m_p \neq n_p$.

So we have proved $\gcd(m,n)\,\mathrm{lcm}(m,n) = mn$
with $m_p \neq n_p$.

to prove when $m_p = n_p$: $\min(m_p, m_p) + \max(m_p, m_p) = 2m_p$
$\Rightarrow 2m_p = 2m_p$ ✓

QED.

3. $\pi(x) = \#$ primes not exceeding x.

define $x = \prod_p p^{x_p}$. Then $\pi(x)$

$$= \sum_p [x_p \geq 1]$$

$\Downarrow$

'proved' easily. Assume $x-1$ is not prime and x is not prime

1. Then $\pi(x) = k$ and $\pi(x-1) = k$.

so $k - k = [\underbrace{x \text{ is prime}}_{\text{assum } x \text{ is not prime.}}] = 0$

2. assume x is prime and $x-1$ is not prime.

$\pi(x) = k$, $\pi(x-1) = k-1$

Then: $k-(k-1) = 1 = [x \text{ is prime}]$.

3. Assume x is prime and $x-1$ is prime

$\to$ wait: $x = 3$ and $x - 1 = 2$!

$\to$ $\pi(3) - \pi(2) = 2 - 1 = 1$

$= [x \text{ is prime!}]$.

QED.

4. Result of $\left(\frac{0}{1}, \frac{1}{0}, \frac{0}{-1}, \frac{-1}{0}, \frac{0}{1}\right)$

using Stern Brocot generation is ALL

fractions m/n with $m \pm n$.

5. $L^K = \begin{pmatrix} 1 & K \\ 0 & 1 \end{pmatrix}$ $R^K = \begin{pmatrix} 1 & 0 \\ K & 1 \end{pmatrix}$

6. $a \equiv b \mod 0 \iff \boxed{a = b.}$

7.

assume there are n people.

Then we execute all $i \in \{n\}$

where $i \equiv n \mod m$.

The only way for 10 to be executed is if $m = 2, 5, 10$.

if $m=2$, we execute: $(\cancel{2}, \cancel{4}, \cancel{6}, 8, 10 \ldots)$

so 10 is not the 3rd to go.

if $m=5$: we execute $(5, 10, \ldots)$

wait. we need $m \bmod 10 \equiv 0$, $\underbrace{m \bmod K \equiv 0,\ m \bmod K+1 \equiv 0}$

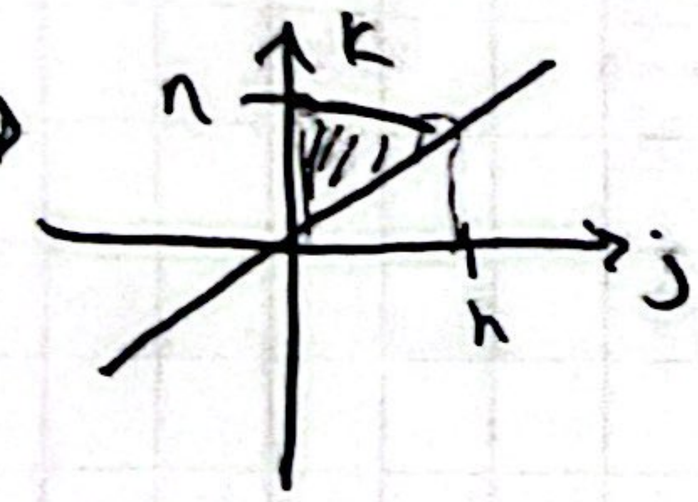
No solution.

11. $g(n) = \sum_{0 \le k \le n} f(k) \iff f(n) = \sum_{0 \le k \le n} \sigma(k)\, g(n-k)$

$\sum_{0 \le k \le n} \sigma(k)\, g(n-k) \Rightarrow$ take $k = n-k$. Then:

$$\sum_{0 \le n-k \le n} \sigma(n-k)\, g(n-(n-k)) = \sum_{0 \le k \le n} \sigma(n-k)\, g(k)$$

$$= \sum_{0 \le k \le n} \sigma(n-k) \sum_{0 \le j \le k} f(j) = \sum_{0 \le k \le n} \sum_{0 \le j \le k} f(j)\, \sigma(n-k)$$

$\Rightarrow$ 

$\Rightarrow \sum_{0 \le j \le n} \sum_{j \le k \le n} f(j)\, \sigma(n-k)$

$= \sum_{0 \le j \le n} f(j) \sum_{j \le k \le n} \sigma(n-k)$ Let's look at this:

$\to f(0) \sum_{0 \le k \le n} \sigma(n-k) + f(1) \sum_{1 \le k \le n} \sigma(n-k) + \dots + f(n) \sum_{n \le k \le n} \sigma(n-k).$

Now consider our original $\sum_{0 \le k \le n} \left[\sigma(k) \sum_{0 \le k \le n-k} f(k) \right]$

$= \sigma(0) \sum_{0 \le k \le n} f(k) + \sigma(1) \sum_{0 \le k \le n-1} f(k) + \dots + \sigma(n) \sum_{0 \le k \le 0} f(k)$

Nothing quite interesting arises. Though: $\sum_{j,k} f(j)[0 \le j \le n]\, \sigma(n-k)[j \le k \le n]$

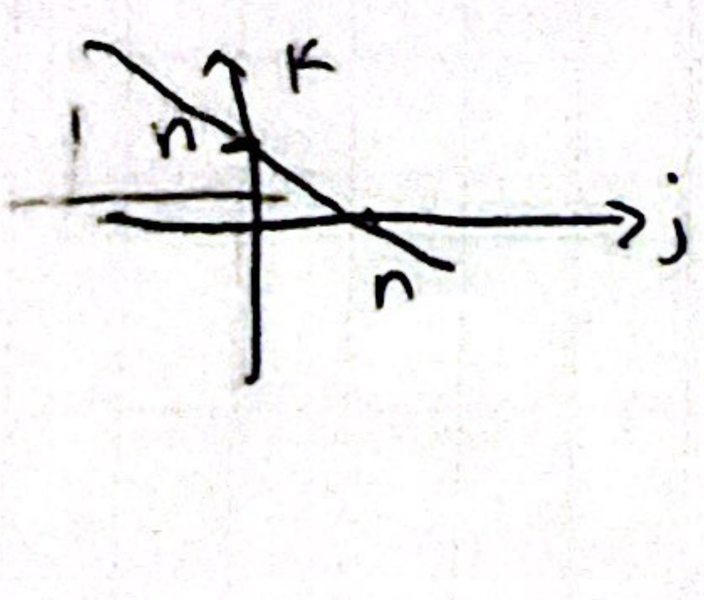
$= \sum_{j,k} f(j)[0 \le j \le n]\, \sigma(k)[0 \le k \le n-j]$

$n-k = k$

$j \le n-k \le n$

$0 \le k \le n-j$

$= \sum_{0 \le j \le n} \sum_{0 \le k \le n-j} \sigma(k)\, f(j)$



This is great news.

we want $\sum_{0 \le j \le n} \sum_{0 \le k \le n-j} \sigma(k) f(j) = f(n)$.

let's expand this once again:

$$f(0) \sum_{0 \le k \le n} \sigma(k) + f(1) \sum_{0 \le k \le n-1} \sigma(k) + \dots f(n-1) \sum_{0 \le k \le 1} \sigma(k)$$

$$+ f(n) \sum_{0 \le k \le 0} \sigma(k) = f(n).$$

realizing that, in order for this to hold ...

The first n terms (0 to $n-1$) must sum to 0.

And, $\sum_{k=0} \sigma(k) = 1 \Rightarrow \{ \sigma(0) = 1.$

This is now something σ must satisfy

This has the following result:

$$\left(f(0) + f(0) \sum_{1 \le k \le n} \sigma(k) \right) + \left(f(1) + f(1) \sum_{1 \le k \le n-1} \sigma(k) \right)$$

$$+ \dots + \left(f(n-1) \overset{f(n-1)}{\sum_{k=1}} \sigma(k) \right) + f(n) = f(n).$$

we notice almost a recurrence like relation.

Note that $f(n-1) + f(n-1) \sum_{k=1} \sigma(k) = 0$

so $\sum_{k=1} \sigma(k) = -1$.

$\Rightarrow \boxed{\sigma(1) = -1.}$

we continue:

$$\left[f(0) + f(0)(-1) + f(0) \sum_{2 \le k \le n} \sigma(k) \right] + \left[f(1) + f(1)(-1) + f(1) \sum_{2 \le k \le n-1} \sigma(k) \right]$$

$$+ \dots + \left(f(n-2) + f(n-2)(-1) + f(n-2) \sum_{k=2} \sigma(k) \right) + 0 + f(n)$$

$$= f(n).$$

we see immediately this collapses into:

$$f(0) \sum_{2 \le k \le n} \sigma(k) + f(1) \sum_{2 \le k \le n-1} \sigma(k) + \dots + f(n-2) \sum_{2 \le k \le 2} \sigma(k) + f(n) = f(n)$$

In order for this to hold for all functions f and all values $n \geq 0$, we need this to identically be zero:

$$f(0)\sum_{2\le k\le n}\sigma(k) + f(1)\sum_{2\le k\le n-1}\sigma(k) + \dots + f(n-2)\sum_{2\le k\le 2}\sigma(k) \equiv 0.$$

which can only be possible if

$$\sigma(k) = 0 \text{ for all } k \geq 2.$$

And we have our function:

$$\boxed{\sigma(n) = \begin{cases} 1 & n=0 \\ -1 & n=1 \\ 0 & n\geq 2. \end{cases} \Rightarrow \sigma(n) = [n=0]-[n=1]}$$

12. Simplify $\sum_{d\backslash m}\sum_{k\backslash d} \mu(k)\, g(d/k)$.

$$\sum_{d,k} \mu(k)\, g(d/k)\, [d\backslash m][k\backslash d]$$

$$= \sum_{d,k} \mu(k)\, g(d/k)\, [\tfrac{m}{d}=N_1][\tfrac{d}{k}=N_2]$$

$$= \sum_{d,k} \mu(k)\, g(d/k)\, [m=N_1 d][d=N_2 k]$$

$$= \sum_{k,N_2} \mu(k)\, g(N_2)\, [m=N_1 N_2 k]$$

$$: \sum_{k}\sum_{N_1,N_2>0} \mu(k)\, g(N_2)\, [m=N_1 N_2 k] \quad [m/k = d N_2]$$

$$\Rightarrow \sum_{k}\sum_{N_1,N_2>0} \mu(k)\, g(N_2)\, [m=N_1 k][\ldots]$$

14. $\gcd(km, kn) = k\gcd(m,n)$

Let $m = \prod_p p^{m_p}$, $n = \prod_p p^{n_p}$, $k = \prod_p p^{k_p}$.

Then $\gcd(km, kn) = \prod_p p^{\min(m_p + k_p,\, n_p + k_p)}$

$\Rightarrow$ we can rewrite $\min(m_p+k_p,\, n_p+k_p) = (m_p+k_p)[m_p+k_p < n_p+k_p] + (n_p+k_p)[n_p+k_p < m_p+k_p]$

$$= k_p([m_p+k_p < n_p+k_p] + [m_p+k_p > n_p+k_p]) + m_p[m_p+\cancel{k_p} < n_p+\cancel{k_p}] + n_p[n_p+\cancel{k_p} < m_p+\cancel{k_p}]$$

$$= k_p([m_p+k_p < n_p+k_p] + [m_p+k_p > n_p+k_p]) + \underbrace{m_p[m_p<n_p] + n_p[n_p<m_p]}_{\text{describes exactly } \min(m_p,n_p).}$$

$$= \underbrace{k_p([m_p+\cancel{k_p} < n_p+\cancel{k_p}] + [m_p+\cancel{k_p} > n_p+\cancel{k_p}])}_{} + \min(m_p,n_p).$$

$$k_p(\underbrace{[m_p<n_p] + [m_p>n_p]}_{\text{we would love for this to include } m_p=n_p.}) + \min(m_p,n_p)$$

we note that when $m_p = n_p$:

$$\min(m_p+k_p,\, m_p+k_p) = m_p+k_p = k_p + \min(m_p, m_p)$$

So: $k_p + \min(m_p,n_p)$.

$\Rightarrow \gcd(km,kn) = \prod_p p^{k_p + \min(m_p,n_p)} = \prod_p p^{k_p} \cdot \prod_p p^{\min(m_p,n_p)}$

$= k\gcd(m,n)$. QED.

17. $f_n = 2^{2^n}+1$

prove $f_m \perp f_n$ if $m<n$.

well, $f_m \perp f_n \Leftrightarrow f_n \bmod f_m \not\equiv 0$

so it suffices to show $f_n \bmod f_m$ is not zero for any m, n with $m<n$.

we have $2^{2^n}+1 \bmod (2^{2^m}+1)$

$$\equiv \left(2^{2^m \cdot 2^{n-m}}+1\right) \bmod (2^{2^m}+1)$$

$$\equiv \left((2^{2^m})^{2^{n-m}}+1\right) \bmod (2^{2^m}+1)$$

$$\equiv \left((-1)^{2^{n-m}}+1\right) \bmod (2^{2^m}+1)$$

$\Rightarrow$ if $m<n$, $2^{n-m}>1$.

and: $(1+1) \bmod (2^{2^m}+1)$

$$\equiv 2 \bmod (2^{2^m}+1)$$

Thus $\gcd(f_n, f_m) = \gcd(2, f_m) = 1$.

18. 2^n+1 is prime if n is a power of 2.

if $2^n+1 = p$, then

suppose $\exists\, p \mid n$. Then $n = p \cdot k$

So $2^{pk}+1 = (2^k+1)(2^{k(p-1)} - 2^{k(p-2)} + \dots - 2^k + 1)$

$\Rightarrow 2^8+1 = (2^4+1)(2^{4(1)} - 2^0 + 1)$

19. $\sum_{1 \le k < n} \left\lfloor \frac{\varphi(k+1)}{k} \right\rfloor$

$\Rightarrow$ consider $\left\lfloor \frac{\varphi(k+1)}{k} \right\rfloor$

$\Rightarrow$ returns 1 when $k+1$ is prime because $\varphi(p) = p-1$.

$\Rightarrow$ $[p \text{ prime}]$

and $\varphi(k+1) \le k$.

$\Downarrow$

max or sup $\varphi(k+1) = k$.

Thus

$0 \le \left\lfloor \frac{\varphi(k+1)}{k} \right\rfloor \le 1$.

So: $\boxed{\pi(n)}$