Concrete Math

Chapter 2 Sums*

warmups.

1. $\sum_{k=4}^{0} q_k = \sum_{4\le k\le 0} q_k = \sum_{k\in\{\}} q_k = 0$

another way: $\sum_{k=4}^{0} q_k = \boxed{q_4+q_3+q_2+q_1+q_0}$

also: $\sum_{k=m}^{n} q_k = \sum_{k\le n} q_k - \sum_{k<m} q_k = \boxed{-q_1-q_2-q_3}$

2. $x\cdot([x>0]-[x<0])$

$= \begin{cases} x(1-0)=x & \text{if } x>0 \\ x(0-1)=-x & \text{if } x<0 \\ 0 & \text{if } x=0 \end{cases}$

$= \boxed{|x|}$

3. $\sum_{0\le k\le 5} a_k = a_0+a_1+a_2+a_3+a_4+a_5$

$\sum_{0\le k^2\le 5} a_{k^2} = ?$

$\hookrightarrow k^2 \in \{0,1,2,3,4,5\}$

$\rightarrow k \in \{0, \pm 1, \cancel{\pm\sqrt{2}}, \cancel{\pm\sqrt{3}}, \pm\sqrt{4}, \cancel{\pm\sqrt{5}}\}$

but k_{index} must be an integer $\therefore$

$\Rightarrow \boxed{\sum_{-2\le k\le 2} a_{k^2} = a_4+a_1+a_0+a_1+a_4}$

4. $\sum_{1\le i<j<k\le 4} a_{ijk}$

a. summing first on k, then j, then i.

$$\sum_{1\le i\le 4}\ \sum_{i<j\le 4}\ \sum_{j<k\le 4} a_{ijk}$$

How? well, look first at inner

2: want $\Delta i\,\Delta j\,\Delta k$.

so here: $i<j<k\le 4$

- start k: $j<k\le 4$
- remove k: $1\le i<j\le 4$
- start j: $1\le i<j\le 4$
- remove j: $1\le i\le 4$
- start i: $1\le i\le 4$

$$\boxed{\sum_{i=1}^{4}\ \sum_{j=i+1}^{4}\ \sum_{k=j+1}^{4} a_{ijk}}$$

b. summing first on i, j, k:

$$\Rightarrow \sum_{1\le k\le 4}\ \sum_{1\le j<k}\ \sum_{1\le i<j} a_{ijk}$$

$$= \boxed{\sum_{k=1}^{4}\ \sum_{j=1}^{k-1}\ \sum_{i=1}^{j-1} a_{ijk}}$$

- start i: $1\le i<j$
- remove i: $1\le j<k\le 4$
- j: $1\le j<k$ 4
- remove j: $1\le k\le 4$
- write k: $1\le k\le 4$

c.

$$\Rightarrow \sum_{k=1}^{4}\ \sum_{j=1}^{k-1}\left(\sum_{i=1}^{j-1} a_{ijk}\right)$$

sums → 0

$$= \sum_{k=3}^{4}\ \sum_{j=1}^{k-1}\ \sum_{i=1}^{j-1} a_{ijk} + \sum_{j=1}^{0}\ \sum_{i=1}^{j-1} \cdots + \cancel{\sum_{j=1}^{1}\ \sum_{i=1}^{j-1}}$$

$$= \sum_{j=1}^{2}\ \sum_{i=1}^{j-1} a_{ij3} + \sum_{j=1}^{3}\ \sum_{i=1}^{j-1} a_{ij4}$$

$$= 0 + a_{123} + 0 + a_{124} + a_{134} + a_{234}$$

$$= \boxed{a_{123} + a_{124} + a_{134} + a_{234}}$$

$= \sum \left(\sum_{j=1}^{n} a_j\right)\left(\sum_{k=1}^{n} \frac{1}{a_k}\right)$

$= \left(\sum_{j=1}^{n} a_j\right)\left(\sum_{j=1}^{n} \frac{1}{a_j}\right) = (a_1+a_2+\dots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_n}\right)$

$= \boxed{\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{a_j}{a_k}} \to$ can't say: $\sum_{j=1}^{n}\sum_{j=1}^{n} \frac{a_j}{a_j}$.

diff. variables indexing.

This is only correct iff $a_j = a_k$.

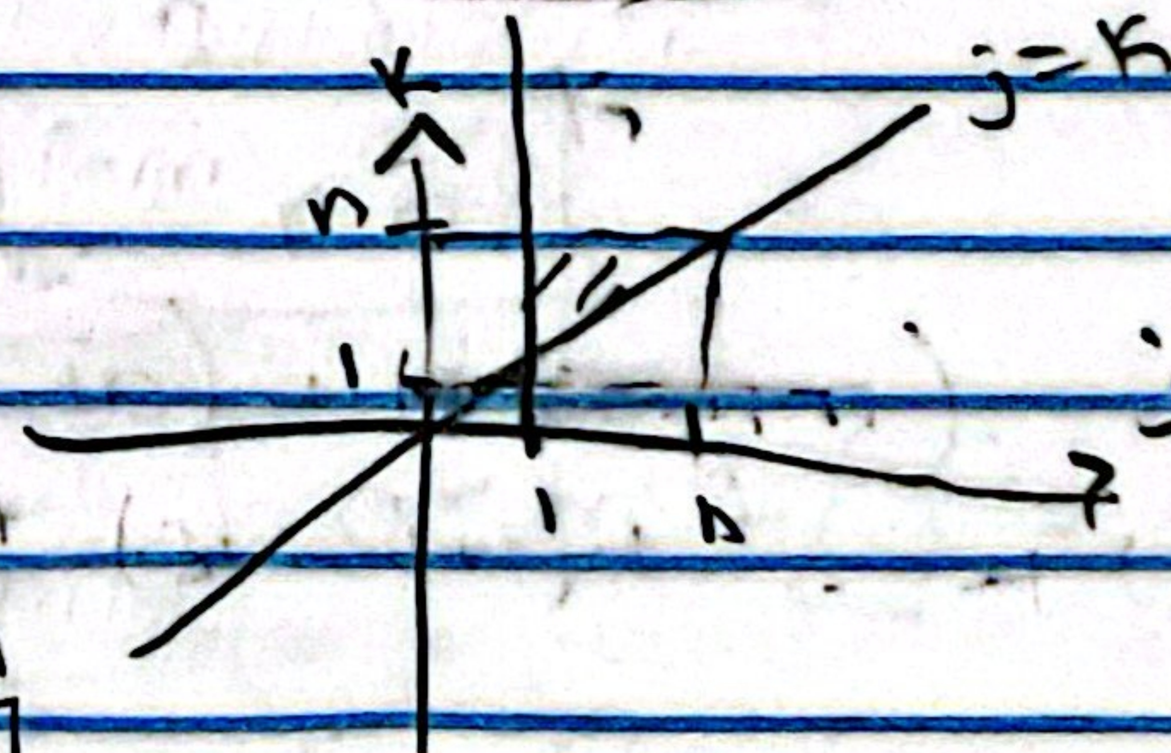
$(a_1+a_2+\dots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_n}\right) \neq n(n)$.

unless $\left(\frac{a_1}{a_1}+\frac{a_1}{a_2}+\frac{a_1}{a_3}+\dots\right)$ all $= 1$.

6.

$\sum_{k} [1 \le j \le k \le n]$

$= \sum_{j=1}^{n}\sum_{k=j}^{n} 1 \to$



$= \sum_{k=1}^{n}\sum_{j=1}^{k} 1 = \sum_{k=1}^{n} k$

$= \boxed{\frac{n(n+1)}{2}}$

or: $\sum_{j=1}^{n}\sum_{k=j}^{n} 1 = \sum_{j=1}^{n} n-j+1$

$= n+n-1+n-2+\dots+n-n+1 = n+n-1+\dots+1$

$= \boxed{\frac{n(n+1)}{2}}$

(on back)

if want function in j, k:

$$\sum_{j=1}^{n}\sum_{k=j}^{n} 1 \qquad \boxed{[1\le j\le n]\ (n-j+1)}$$

(is in range?)

if yes → multiply by $n-j+1$ terms.

7. $\nabla f(x) = f(x) - f(x-1)$

$$\nabla(x^{\overline{m}}) = x^{\overline{m}} - (x-1)^{\overline{m}}$$

$= x(x+1)\ldots(x+m-1) - (x-1)(x)(x+1)\ldots(x+m-2)$

$= x(x+1)(x+2)\ldots(x+m-2)(x+m-1) - (x-1)(x)(x+1)(x+2)\ldots(x+m-2)$

$= x(x+1)(x+2)\ldots(x+m-2)\left(\cancel{x+m-1-(x-1)}\right)$

$= m\,x(x+1)(x+2)\ldots(x+m-2)$

$= \boxed{m\,x^{\overline{m-1}}}$

8. $0^{\overline{m}} = \boxed{0}$ if $m \ge 0$ $\quad\boxed{\frac{1}{|m|!}\text{ for } m<0}$

9. $x^{\overline{3}} = x(x+1)(x+2)$ $\quad \downarrow x+2$

$x^{\overline{2}} = x(x+1)$ $\quad \downarrow x+1$

$x^{\overline{1}} = x$

$x^{\overline{0}} = 1$ $\quad \downarrow x$

$x^{\overline{-1}} = \frac{1}{x-1}$

$x^{\overline{-2}} = \frac{1}{(x-1)(x-2)}$

$\therefore \boxed{x^{\overline{-n}} = \frac{1}{(x-1)(x-2)\ldots(x-n)} \text{ for } n>0}$

$x^{\overline{m+n}} = x(x+1)\ldots(x+m+n-1)$

$= x(x+1)\ldots(x+m)(x+m+1)+\ldots(x+m+n-1)$

$= \underbrace{x(x+1)\ldots(x+m-1)}_{x^{\overline{m}}} \ \underbrace{(x+m)(x+m+1)\ldots(x+m+n-1)}_{(x+m)^{\overline{n}}}$

$= \boxed{x^{\overline{m}}(x+m)^{\overline{n}}} = x^{\overline{m+n}}$

10. $\Delta(uv) = u\Delta v + Ev\Delta u$

notice LHS: is symmetrical $\Delta(uv) = u(x+1)v(x+1) - u(x)v(x)$
$= v(x+1)u(x+1) - v(x)u(x)$

so $\Delta(vu) = u\Delta v + Ev\Delta u$

but RHS is not... why, how?

~~well $\Delta(vu) = v\Delta u + Eu\Delta v$.~~

$v(x)(u(x+1) - u(x)) + u(x+1)(v(x+1) - v(x))$

$\stackrel{?}{=} u\Delta v + Ev\Delta u$

$\Rightarrow v(x)(u(x+1) - u(x)) + u(x+1)(v(x+1) - v(x)) = u(v(x+1) - v(x)) + v(x+1)(u(x+1) - u(x))$

$\Rightarrow u(x+1)v(x) - \cancel{uv} + \cancel{u(x+1)v(x+1)} - u(x+1)v(x)$

$\stackrel{?}{=} u(x)v(x+1) - \cancel{uv} + \cancel{u(x+1)v(x+1)} - u(x)v(x+1)$

we can remove terms from both sides ONLY here, we know they are equivalent

$\Rightarrow u(x+1)v(x) - u(x+1)v(x) \stackrel{?}{=} u(x)v(x+1) - u(x)v(x+1)$

$LHS = 0 \stackrel{?}{=} 0 = RHS.$ ✓

11. $\sum_{0 \le k < n} (a_{k+1} - a_k)b_k = a_n b_n - a_0 b_0 - \sum_{0 \le k < n} a_{k+1}(b_{k+1} - b_k)$

Consider $\sum_{0\le k<n} (a_{k+1}-a_k)b_k$

$$= \sum_{0\le k<n} a_{k+1}b_k - \sum_{0\le k<n} a_k b_k$$

$$= \sum_{0\le k<n} a_{k+1}b_k - \left(\sum_{1\le k<n} a_k b_k + a_0 b_0\right)$$

$$= \sum_{0\le k<n} a_{k+1}b_k - \left(\sum_{1\le k+1<n} a_{k+1} b_{k+1} + a_0 b_0\right)$$

$$= \sum_{0\le k<n-1} a_{k+1}b_k + a_n b_{n-1} - \left(\sum_{0\le k<n-1} a_{k+1} b_{k+1} + a_0 b_0\right)$$

$$= \sum_{0\le k<n-1} a_{k+1}(b_k - b_{k+1}) + a_n b_{n-1} - a_0 b_0$$

$$= a_n b_{n-1} - a_0 b_0 - \sum_{0\le k<n-1} a_{k+1}(b_{k+1} - b_k)$$

($n-1$th term $\Rightarrow$: $a_n(b_n - b_{n-1})$
$= a_n b_n - a_n b_{n-1}$)

$$= (a_n b_{n-1} - a_0 b_0) - \left(\sum_{0\le k<n} a_{k+1}(b_{k+1} - b_k) - a_n b_n + a_n b_{n-1}\right)$$

$$= \cancel{a_n b_{n-1}} - a_0 b_0 + a_n b_n - \cancel{a_n b_{n-1}} - \sum_{0\le k<n} a_{k+1}(b_{k+1} - b_k)$$

$$= \boxed{a_n b_n - a_0 b_0 - \sum_{0\le k<n} a_{k+1}(b_{k+1} - b_k)}$$

12. $p(k) = k + (-1)^k c$

$k + (-1)^k c = n$

$\Rightarrow \quad n + c = k + \left[(-1)^k + 1\right] c$

$\Rightarrow \quad (-1)^{n+c} = (-1)^{k + \underbrace{[(-1)^k + 1]}_{2 \text{ or } 0} c}$

$\Rightarrow \quad (-1)^{n+c} \equiv (-1)^{k + (0 \bmod 2) \text{ so}}$ same as $(0 \bmod 2$

$\Rightarrow \quad n = k + (-1)^k c \Rightarrow \quad k = n - (-1)^k c$

$\Rightarrow \quad k = n - (-1)^{n+c} c$

which: $p(n - (-1)^{n+c} c) = n - \cancel{(-1)^{n+c}} c + \cancel{(-1)^{n+c}} c$

$= n.$

13. $\sum_{k=0}^{n} (-1)^k k^2 = R_n \quad \Rightarrow \quad R_0 = \alpha$

$R_n = R_{n-1} + (-1)^n (\beta + \gamma n + \delta n^2)$

$R_n = A(n)\alpha + B(n)\beta + C(n)\gamma + D(n)\delta$

$R_n = 1 \Rightarrow \alpha = 1, \quad 1 = 1 + (-1)(\beta + (1)\gamma + \delta(1))$

$\Rightarrow A(n) = 1 \quad \Rightarrow \quad 0 = -\beta - \gamma - \delta$

$0 = \beta + 2\gamma + 4\delta \quad \} \beta, \gamma, \delta = 0$

$0 = -\beta - 3\gamma - \delta(9)$

$R_n = n \Rightarrow$ doesn't work

$R_n = (-1)^n \Rightarrow \quad \alpha = R_0 = (-1)^0 = 1$

$R_n = R_{n-1} + (-1)^n (\beta + \gamma n + \delta n^2)$

$(-1)^n = 1 + B(n)2 + 0 + 0 \quad \Rightarrow \quad (-1)^n = (-1)^{n-1} + (-1)^n(\beta + \gamma n + \delta n^2)$

$\Rightarrow \boxed{B(n) = \frac{(-1)^n - 1}{2}} \quad \Leftarrow \quad -1 = 1 + (-1)(\beta + \gamma n + \delta n^2)$

$\Rightarrow \beta - 2 + \gamma n + \delta n^2 = 0$

$\Rightarrow \beta = 2, \gamma = 0, \delta = 0$

$R_n = (-1)^n n \Rightarrow \quad R_0 = \alpha \Rightarrow \alpha = 0$

$R_n = R_{n-1} + (-1)^n(\beta + \gamma n + \delta n^2)$

$(-1)^n n = (-1)^{n-1}(n-1) + (-1)^n(\beta + \gamma n + \delta n^2)$

$\Rightarrow (-1)^n n = 0 + B(n)(-1) + C(n)(2) \quad \Rightarrow (-1)n = (n-1) + (-1)(\beta + \gamma n + \delta n^2)$

$\Rightarrow -n = n - 1 - \beta - \gamma n - \delta n^2$

$\Rightarrow C(n) = \dfrac{(-1)^n n + B(n)}{2} \quad \Rightarrow (\beta+1) + n(\gamma - 2) + \delta n^2 = 0$

$\Rightarrow \beta = -1, \ \gamma = 2, \ \delta = 0$

$\Rightarrow C(n) = \dfrac{(-1)^n n + \frac{(-1)^n - 1}{2}}{2}$

$\Rightarrow C(n) = \dfrac{2(-1)^n n + (-1)^n + 1}{4} = \dfrac{(-1)^n(2n+1) - 1}{4}$

$R_n = (-1)^n n^2 \quad R_0 = \alpha = 0$

$\Rightarrow R_n = R_{n-1} + (-1)^n(\beta + \gamma n + \delta n^2)$

$\Rightarrow (-1)^n n^2 = 0 + B(n) + (-2)C(n) + 2D(n) \quad \Rightarrow (-1)^n n^2 = (-1)^{n-1}(n-1)^2 + (-1)^n(\beta + \gamma n + \delta n^2)$

$\rightarrow -n^2 = (n-1)^2 + (-1)(\beta + \gamma n + \delta n^2)$

$\Rightarrow (-1)^n n^2 = -(-B(n) + 2C(n)) + 2D(n) \quad \rightarrow 0 = 2n^2 - 2n + 1 - \beta - \gamma n - \delta n^2$

$\Rightarrow (-1)^n n^2 = -(-1)^n n + 2D(n) \quad \Rightarrow 0 = 1 - \beta + n(-\gamma - 2) + n^2(-\delta + 2)$

$\Rightarrow D(n) = \dfrac{(-1)^n(n^2+n)}{2} \quad \Rightarrow \gamma = -2, \ \delta = 2, \ \beta = 1$

Thus $\sum_{k=0}^{n} (-1)^k k^2 \Rightarrow \delta = 1, \ \alpha = \gamma = \beta = 0$

$\rightarrow \boxed{\dfrac{(-1)^n(n^2+n)}{2}}$

14. $\sum_{k=1}^{n} k2^k = \sum_{1\le j\le k\le n} 2^k = \sum_{1\le k\le n}\sum_{1\le j\le k} 2^k = \sum_{1\le k\le n} k2^k$

$\Rightarrow \sum_{1\le j\le k\le n} 2^k = \sum_{1\le j\le n}\sum_{j\le k\le n} 2^k$

$\sum_{j\le k\le n} 2^k = \sum_{0\le k\le n-j} 2^{k+j}$

let $S_n = \sum_{0\le k\le n-j} 2^{k+j}$

$S_{n+1} = S_n + 2^{n+1} = \sum_{0\le k\le n-j+1} 2^{k+j}$

$= 2^j + \sum_{1\le k\le n-j+1} 2^{k+j} = 2^j + \sum_{0\le k\le n-j} 2^{k+j+1}$

$= 2^j + \sum_{0\le k\le n-j} 2^{k+j}\cdot 2 = 2^j + 2S_n = S_n + 2^{n+1}$

$\Rightarrow S_n = 2^{n+1} - 2^j$

$\Rightarrow \sum_{1\le j\le n} 2^{n+1} - 2^j = \sum_{1\le j\le n} 2^{n+1} - \sum_{1\le j\le n} 2^j = n2^{n+1} - \sum_{1\le j\le n} 2^j$

$\Rightarrow \sum_{1\le j\le n} 2^j = R_n \Rightarrow R_{n+1} = R_n + 2^{n+1} = \sum_{1\le j\le n+1} 2^j$

$= 2^1 + \sum_{2\le j\le n+1} 2^j = 2^1 + \sum_{1\le j\le n} 2^{j+1} = 2 + 2R_n$

$\Rightarrow R_n + 2^{n+1} = 2 + 2R_n \Rightarrow R_n = 2^{n+1} - 2$

$\Rightarrow n2^{n+1} - 2^{n+1} + 2 = \boxed{2^{n+1}(n-1)+2}$

15. $\sum_{k=1}^{n} k^3$.

$\boxed{\square}_n + \square_n = 2\sum_{1\le j\le k\le n} jk$.

2.33: $\sum_{1\le j\le k\le n} a_j a_k = \frac{1}{2}\left(\left(\sum_{k=1}^{n} a_k\right)^2 + \sum_{k=1}^{n} a_k^2\right)$

$\Rightarrow \boxed{\square}_n + \square_n = \cancel{2}\left(\cancel{\tfrac{1}{2}}\left(\left(\sum_{k=1}^{n} k\right)^2 + \sum_{k=1}^{n} k^2\right)\right)$

$= \left(\sum_{k=1}^{n} k\right)^2 + \square_n$

$\Rightarrow \boxed{\square}_n = \left(\sum_{k=1}^{n} k\right)^2 = \left(\frac{n(n+1)}{2}\right)^2$

16. $\frac{x^{\underline{m}}}{(x-n)^{\underline{m}}} = \frac{x^{\underline{n}}}{(x-m)^{\underline{n}}}$ unless $x=m$, or $x=n$.

$\hookrightarrow$ This implies the following:

$\underbrace{x^{\underline{m}}(x-m)^{\underline{n}}}_{} = \underbrace{x^{\underline{n}}(x-n)^{\underline{m}}}_{}$ (if this holds, the previous holds)

$x^{\underline{m}+\underline{n}} \qquad x^{\underline{n}+\underline{m}}$

meaning

$x^{\underline{m}+\underline{n}} = x^{\underline{m}+\underline{n}}$ ✓

17. $x^{\overline{m}} = (-1)^m(-x)^{\underline{m}} = \cancel{(x+m-1)^{\underline{m}}} = \frac{1}{(x-1)^{-\underline{m}}}$

$\rightarrow x^{\overline{m}} = x(x-1)(x-2)\ldots(x-m+1) = (-1)(-x)(x-1)(x-2)(x-m+1)$

$= (-1)^2(-x)(-x+1)(x-2)(x-3)\ldots(x-m+1)$

$= \ldots = (-1)^m(-x)(-x+1)(-x+2)\ldots(-x+m-1)$

19. $T_0 = 5$

$$2T_n = nT_{n-1} + 3n! \quad , \quad n>0$$

we wish to choose an s_n so that: $s_n b_n = s_{n-1} a_{n-1}$

$$\Rightarrow \quad n s_n = s_{n-1} \cdot 2 = \frac{s_{n-1} \cdot 2}{n} = \frac{s_{n-2} \cdot 2 \cdot 2}{n(n-1)} = \ldots$$

$$= \frac{2^{n-1}}{2 \cdot 3 \cdot 4 \cdots (n-1)(n)} = \frac{2^{n-1}}{n!}$$

Then: $\underbrace{\frac{2^{n-1}}{n!}}_{s_n} \cdot \underbrace{2}_{a_n} \underbrace{T_n}_{T_n} = \underbrace{\frac{2^{n-1}}{n!}}_{\frac{2^{n-2}}{(n-1)!} \cdot 2 \cdot T_{n-1}} \cdot n T_{n-1} + 3n! \cdot \frac{2^{n-1}}{n!}$

$s_{n-1}, \; a_{n-1}, \; T_{n-1}$

Let $S_n = s_n a_n T_n$.

Then $S_n = \frac{2^{n-1}}{n!} \cdot 2 \cdot T_n$

and $S_n = S_{n-1} + 3\cancel{n!} \cdot \frac{2^{n-1}}{\cancel{n!}}$

$$\Rightarrow \quad S_n = S_{n-2} + 3 \cdot 2^{n-2} + 3 \cdot 2^{n-1} \ldots$$

$$\Rightarrow \quad S_n = S_0 + 3 \cdot 2^{0-1} + 3 \cdot 2^{1-1} \ldots + 3 \cdot 2^{n-2} + 3 \cdot 2^{n-1}$$

and $S_n = \frac{2^{0-1}}{0!} \cdot 2 \cdot T_0 + \sum_{k=1}^{n} 3 \cdot 2^{k-1}$

and $S_n = T_0 + \sum_{k=1}^{n} 3 \cdot 2^{k-1}$

$$S_n = 5 + \sum_{k=1}^{n} 3 \cdot 2^{k-1}$$

So $T_n = \frac{S_n}{a_n \cdot s_n} = \frac{S_n}{\left(\frac{2^n}{n!}\right)} = \frac{n!}{2^n}\left(5 + \sum_{k=1}^{n} 3 \cdot 2^{k-1}\right)$

$$= \frac{n!}{2^n}\left(5 + 3 \underbrace{\sum_{k=1}^{n} 2^{k-1}}_{*}\right) = 1$$

$*$: $\sum_{k=1}^{n} 2^{k-1} = R_n$

if $R_n = \sum_{k=1}^{n} 2^{k-1} \Rightarrow R_n + 2^n = R_{n+1} = \sum_{k=1}^{n+1} 2^{k-1}$

$\Rightarrow \quad R_n + 2^n = 1 + \sum_{2 \le k \le n+1} 2^{k-1} \Rightarrow R_n + 2^n = 1 + \sum_{1 \le k \le n}^{n+2} 2^{(k)}$

$\Rightarrow$

$R_n + 2^n = 1 + \sum_{1 \le k \le n} 2^k = 1 + 2 \sum_{1 \le k \le n} 2^{k-1}$

$\rightarrow \quad R_n + 2^n = 1 + 2R_n \Rightarrow R_n = 2^n - 1$

①: $T_n = \frac{n!}{2^n}\left(5 + 3\sum_{k=1}^{n} 2^{k-1}\right) = \frac{n!}{2^n}\left(5 + 3\cdot 2^n - 3\right)$

$= \frac{n!}{2^n}\left(2 + 3\cdot 2^n\right) = \boxed{n!\left(2^{1-n} + 3\right)}$

$1 \le k \le n+1$

20. $\sum_{k=0}^{n} kH_k = S_n$

$S_{n+1} = S_n + (n+1)H_{n+1} = \sum_{k=0}^{n+1} kH_k = 0 + \sum_{k=1}^{n+1} kH_k$

$= 0 + \sum_{0 \le k \le n} (k+1)H_{k+1}$

$\Rightarrow$

$S_n + (n+1)H_{n+1} = \sum_{0 \le k \le n} (k+1)\left(H_k + \frac{1}{k+1}\right)$

$\rightarrow \quad S_n + (n+1)H_{n+1} = \sum_{0 \le k \le n} kH_k + \sum_{0 \le k \le n} H_k + \sum_{0 \le k \le n} 1$

$\Rightarrow$

$\cancel{S_n} + (n+1)H_{n+1} = \cancel{S_n} + \sum_{0 \le k \le n} H_k + n+1$

and $\sum H_k = (n+1)H_n + \cancel{1} - n - \cancel{1}$

so $\boxed{\sum_{0 \le k \le n} H_k = (n+1)H_n - n}$

21. $S_n = \sum_{k=0}^{n} (-1)^{n-k} \Rightarrow S_{n+1} = S_n + (-1)^{n-n-1} = \sum_{0 \le k \le n+1} (-1)^{n-k}$

$$\Rightarrow S_n + (-1)^{-1+n+1} = (-1)^n + \sum_{1 \le k \le n+1} (-1)^{n-k}$$

$$\Rightarrow S_n + (-1)^{-1+n+1} = (-1)^n + \sum_{0 \le k \le n} (-1)^{n-k-1} \quad \hookrightarrow (-1)^{n-k}(-1)^{-1}$$

$$= (-1)^n + (-1)^{-1} \sum_{0 \le k \le n} (-1)^{n-k}$$

$$\to S_n + (-1)^{n+1} = (-1)^n + (-1)^{-1} S_n$$

$$2S_n = (-1)^n + 1$$

$$\to S_n = \frac{1+(-1)^n}{2} = \begin{cases} 1 & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd} \end{cases}$$

$$= \boxed{[n \text{ even}]}$$

→ return 1, if true,
return 0, if false.

22.

$$T_n = \sum_{k=0}^{n} (-1)^{n-k} k$$

$$\to T_{n+1} = T_n + (-1)^{-1}(n+1) = \sum_{k=0}^{n+1} (-1)^{n-k} k$$

$$= (-1)^n (0) + \sum_{k=1}^{n+1} (-1)^{n-k} k = 0 + \sum_{k=0}^{n} (-1)^{n-k-1} (k+1)$$

$$\to T_n + (-1)(n+1) = \sum_{k=0}^{n} (-1)^{n-k-1} k + \sum_{k=0}^{n} (-1)^{n-k-1}$$

$$\to T_n + (-1)(n+1) = (-1) \sum_{k=0}^{n} (-1)^{n-k} k + (-1)[n \text{ even}]$$

$$T_n - (n+1) = (-)T_n - S_n$$

$$2T_n = (n+1) - S_n$$

$$\boxed{T_n = \frac{n+1}{2} - \frac{1}{2}[n \text{ even}]}$$

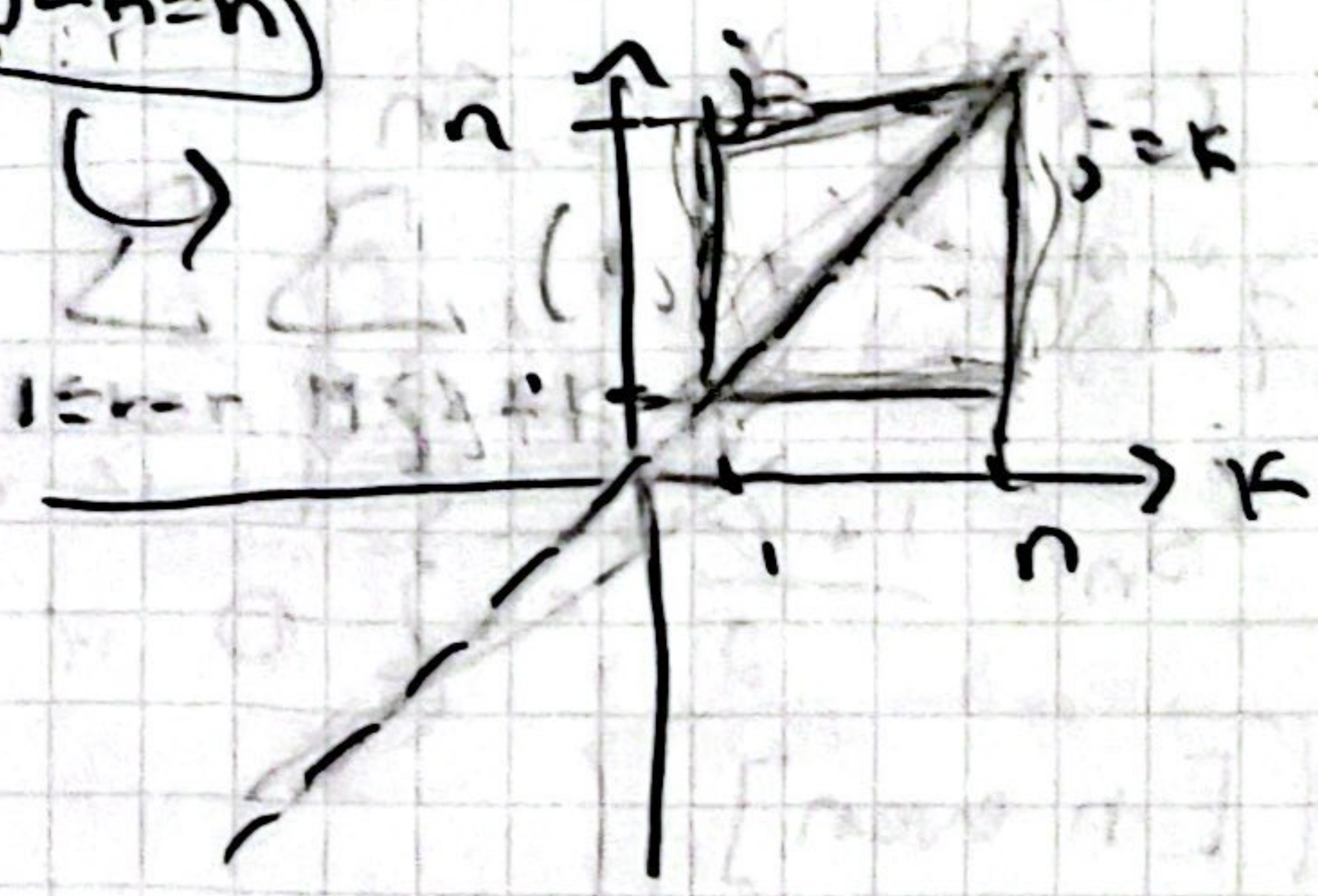
$$u_{n+1} = (n+1)^2 - u_n = u_n + 2T_n + S_n$$
$$= u_n + n + [n \text{ odd}] + [n \text{ even}]$$
$$= u_n + n + 1$$

so $\boxed{u_n = \frac{1}{2} n(n+1)}$

22.

$$S_1 = \sum_{1 \le j < k \le n} (a_j b_k - a_k b_j)^2 = \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \left(\sum_{k=1}^{n} a_k b_k\right)^2$$

$\Rightarrow$



consider : $\{ 1 \le k < j \le n \}$

Check symmetry

now we must verify: $\sum_{1 \le k < j \le n} (a_j b_k - a_k b_j)^2$

$$= \sum_{1 \le k < j \le n} \left((-1)[a_k b_j - a_j b_k]\right)^2 \text{, replace } k \text{ by } j \text{ and } j \text{ by } k.$$

$$= \sum_{1 \le j < k \le n} (a_j b_k - a_k b_j)^2 = S_1$$

$$\Rightarrow 2 S_1 = \sum_{1 \le k, j \le n} (a_j b_k - a_k b_j)^2 - \sum_{1 \le k = j \le n} (a_j b_k - a_k b_j)$$

$$= \sum_{1 \le j \le n} \sum_{1 \le k \le n} (a_j b_k - a_k b_j)^2 - \cancelto{0}{\sum_{1 \le k \le n} (a_k b_k - a_k b_k)}$$

and $2S_{①} = \sum_{1\le j\le n}\sum_{1\le k\le n} (a_j b_k - a_k b_j)^2 =$

$$\sum_{1\le j\le n}\sum_{1\le k\le n} (a_j b_k)^2 - 2a_j a_k b_k b_j + (a_k b_j)^2 .$$

$$= 2\sum_{1\le j\le n}\sum_{1\le k\le n}(a_j b_k)^2 - 2\sum_{1\le j\le n}\sum_{1\le k\le n} 2a_j a_k b_k b_j$$

$$= 2\sum_{1\le j\le n}(a_j)^2 \sum_{1\le k\le n}(b_k)^2 - 2\sum_{1\le j\le n} a_j b_j \sum_{1\le k\le n} a_k b_k$$

$$= 2\sum_{1\le j\le n} a_j^2 \sum_{1\le k\le n} b_k^2 - 2\left(\sum_{1\le k\le n} a_k b_k\right)^2$$

$$\Rightarrow \boxed{S_{①} = \sum_{1\le j<k\le n} (a_j b_k - a_k b_j)^2 = \left(\sum_{1\le j\le n} a_j^2\right)\left(\sum_{1\le k\le n} b_k^2\right) - \left(\sum_{1\le k\le n} a_k b_k\right)^2}$$

23. $\sum_{1\le k\le n} \frac{(2k+1)}{k(k+1)}$

a. $\sum_{1\le k\le n}\left[(2k+1)\left(\frac{1}{k}-\frac{1}{k+1}\right)\right] = \sum_{1\le k\le n} 2+\frac{1}{k}-\frac{2k+1}{k+1}$

$$= 2n + H_n - \sum_{1\le k\le n}\frac{2k+1}{k+1} = 2n + H_n - \sum_{2\le k\le n+1}\frac{2(k-1)+1}{k}$$

$$= 2n + H_n - \sum_{2\le k\le n+1}\frac{2k-2+1}{k} = 2n + H_n - \sum_{2\le k\le n+1}\left(2-\frac{1}{k}\right)$$

$$= 2n + H_n - 2n + \sum_{2\le k\le n+1}\frac{1}{k} = 2n + H_n - 2n - \frac{1}{1} + \frac{1}{1} + \sum_{2\le k\le n+1}\frac{1}{k}$$

$$= H_n - 1 + \sum_{1\le k\le n+1}\frac{1}{k} = H_n - 1 + \frac{1}{n+1} + H_n = \boxed{2H_n - 1 + \frac{1}{n+1}}$$

b. $\sum_{k=1}^{n} \frac{2k+1}{k(k+1)} = \sum_{x=1}^{n+1} \frac{2x+1}{x(x+1)} \delta x$

$\to u(x) = 2x+1$, $\Delta u(x) = \cancel{2x}+3-\cancel{2x+1}$ $= 2$ $\quad \frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$

$\to \sum_{x=1}^{n+1} (2x+1)\,\Delta\left(-\frac{1}{x}\right)\delta x$ $\quad$ let $\Delta v(x) = \frac{1}{x(x+1)}$,

$= \sum_{1}^{n+1} u(x)\,\Delta v(x)\,\delta x$ $\quad$ then $\Delta v(x) = \frac{1}{x} - \frac{1}{x+1}$

$\Rightarrow \Delta v(x) = -\left(\frac{1}{x+1} - \frac{1}{x}\right)$

$= -\Delta\left(\frac{1}{x}\right)$

so $v(x) = -\frac{1}{x}$

$= uv - \sum Ev\,\Delta u$

$= (2x+1)\left(-\frac{1}{x}\right) - \sum_{1}^{n+1}\left(-\frac{1}{x+1}\right)(2)$

$= -2 - \frac{1}{x} + \sum_{1}^{n+1} \frac{2}{x+1}\,\delta x$

$= \left[-2 - \frac{1}{x} + \sum_{1} 2x^{\underline{-1}}\,\delta x\right]_1^{n+1} = \left[-2 - \frac{1}{x} + 2(H_x)\right]_1^{n+1}$

$= \cancel{-2} - \frac{1}{n+1} \cancel{+2} + \frac{1}{1} + 2H_{n+1} - 2H_1$

$= 1 - \frac{1}{n+1} + 2\left(H_n + \frac{1}{n+1}\right) - 2$

$= \boxed{-1 + \frac{1}{n+1} + 2H_n}$

24. $\sum_{0 \le k < n} \frac{H_k}{(k+1)(k+2)}$ $\to$ consider $\frac{1}{(x+1)(x+2)}$.

$\to \sum_{0}^{n} \frac{H_x}{(x+1)(x+2)}\,\delta x$ $\quad x^{\underline{-1}} = \frac{1}{x+1}$

$x^{\underline{-2}} = \frac{1}{(x+1)(x+2)}$

$= \sum_{0}^{n} x^{\underline{-2}} H_x\,\delta x$

$\rightarrow \sum_0^n x^{\underline{-2}} H_x \, \delta x$

let $\Delta v(x) = x^{\underline{-2}}$, $v(x) = \frac{x^{\underline{-1}}}{-1}$

$u(x) = H_x$

$\Delta u(x) = x^{\underline{-1}}$

$\rightarrow \sum_0^{n-1} \Delta v(x)\, u(x)\, \delta x = \frac{x^{\underline{-1}}}{-1} H_x - \sum \frac{(x+1)^{\underline{-1}}}{-1} x^{\underline{-1}} \delta x$

$= -x^{\underline{-1}} H_x + \sum x^{\underline{-1-1}} \delta x = -x^{\underline{-1}} H_x + \sum x^{\underline{-2}} \delta x$

$= -\frac{1}{x+1} H_x + \frac{x^{\underline{-1}}}{-1} = \left[-\frac{1}{x+1} H_x - \frac{1}{x+1}\right]_0^n$

$= \left[\left(\frac{-1}{x+1}\right)(H_x + 1)\right]_0^n = -\frac{1}{n+1}(H_n + 1) + 1$

$\rightarrow \boxed{1 - \frac{H_{n+1}}{n+1}}$

Now consider: $\sum_0^n x^{\underline{m}} H_x \, \delta x$ $\left(\int x^m \ln x \, dx\right)$

$u(x) = H_x$, $v(x) = \frac{x^{\underline{m+1}}}{m+1}$

$\Delta u(x) = x^{\underline{-1}}$, $\Delta v(x) = x^{\underline{m}}$

$\rightarrow \sum_0^n \Delta v(x)\, u(x)\, \delta x$

$= H_x \cdot \frac{x^{\underline{m+1}}}{m+1} - \sum \frac{(x+1)^{\underline{m+1}}}{m+1} x^{\underline{-1}} \delta x$

$= H_x \cdot \frac{x^{\underline{m+1}}}{m+1} - \sum \frac{x^{\underline{-1+m+1}}}{m+1} \delta x$

$= H_x \cdot \frac{x^{\underline{m+1}}}{} - \sum \frac{x^{\underline{m}}}{m+1} \delta x$

$$\Rightarrow \left[H_x \cdot \frac{x^{m+1}}{m+1} - \frac{x^{m+1}}{(m+1)^2} \right]_0^n$$

$$\Rightarrow H_n \cdot \frac{n^{m+1}}{m+1} - \frac{n^{m+1}}{(m+1)^2} - 0 + 0$$

$$\Rightarrow \boxed{\frac{H_n \cdot n^{m+1}}{m+1} - \frac{n^{m+1}}{(m+1)^2}}$$

25. $\prod_{k \in K} a_k \rightarrow$ product of a_k for all $k \in K$

$\Rightarrow$

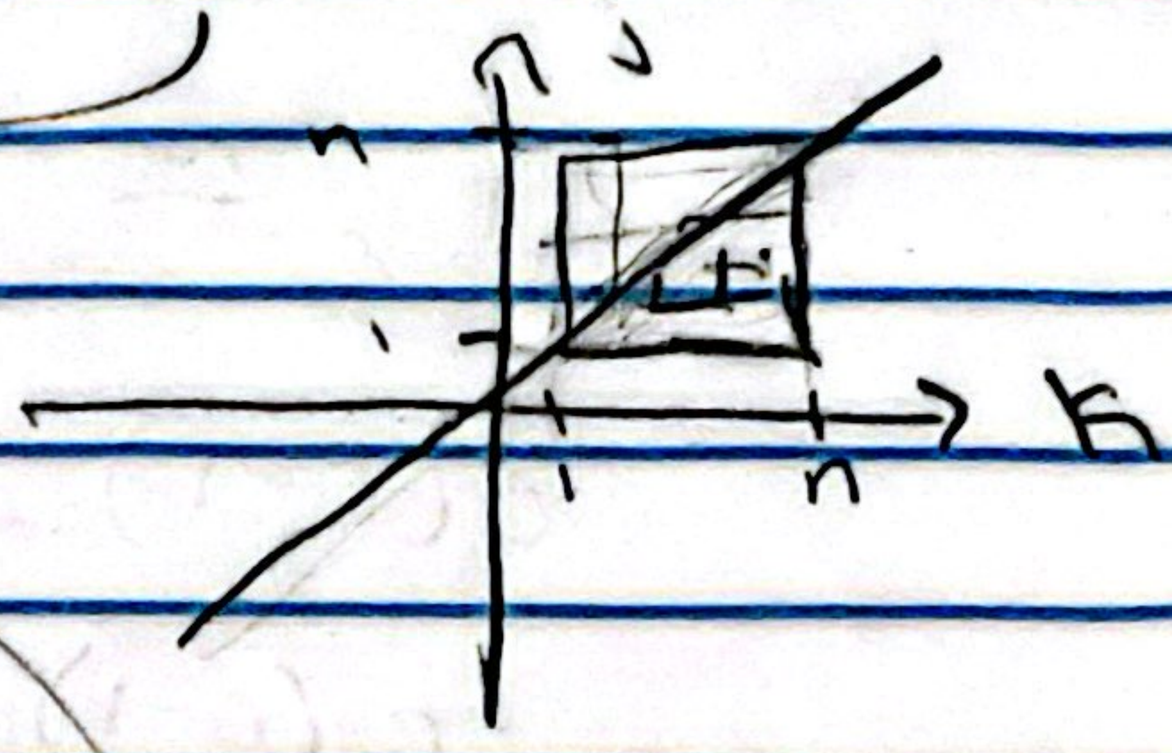
- $\prod_{k \in K} a_k^c = \left(\prod_{k \in K} a_k\right)^c$
- $\prod_{k \in K} a_k b_k = \left(\prod_{k \in K} a_k\right)\left(\prod_{k \in K} b_k\right)$
- $\prod_{k \in K} a_k = \prod_{p(k) \in K} a_{p(k)}$

$\prod_{\substack{j \in J \\ k \in K}} a_{j,k} = \prod_{j \in J} \prod_{k \in K} a_{j,k}$

$\prod_{k \in K} a_k = \prod_{k} a_k^{[k \in K]}$

$\prod_{k \in K} c = c^{\# K}$

26. $\prod_{1\le j\le k\le n} a_j a_k = \prod_{1\le k\le n}\prod_{1\le j\le k} a_j a_k$



consider:

$$\prod_{1\le k\le j\le n} a_j a_k = \prod_{1\le k\le j\le n} a_k a_j$$

$$= \prod_{1\le j\le k\le n} a_j a_k$$ ✓ symmetry holds.

$$\prod_{1\le j,k\le n} a_j a_k = \prod_{1\le j\le n}\prod_{1\le k\le n} a_j a_k$$

$$\Rightarrow \left(\prod_{1\le j\le k\le n} a_j a_k\right)^2 = \left(\prod_{1\le j,k\le n} a_j a_k\right)\left(\prod_{1\le j=k\le n} a_j a_k\right)$$

$$\Rightarrow \left(\prod_{1\le j\le k\le n} a_j a_k\right)^2 = \left(\prod_{1\le j,k\le n} a_j a_k\right)\left(\prod_{1\le k\le n} a_k^2\right)$$

$$\prod_{1\le j\le n}\prod_{1\le k\le n} a_j a_k$$

$$= \prod_{1\le j\le n}\left(a_j^n \cdot (a_1 \cdot a_2 \cdot a_3 \cdot a_4 \cdots)\right)$$

$$= \prod_{1\le j\le n}\left(a_j^n \prod_{1\le k\le n} a_k\right)$$

$$= \prod_{1\le j\le n} a_j^n \cdot \prod_{1\le j\le n}\prod_{1\le k\le n} a_k$$

$$= \left(\prod_{1\le k\le n} a_k^n\right)^2$$

$$\Rightarrow \left(\prod_{1\le j\le k\le n} a_j a_k\right)^2 = \left(\prod_{1\le k\le n} a_k^{2n}\right)\left(\prod_{1\le k\le n} a_k^2\right)$$

$$\Rightarrow \boxed{\prod_{1\le j\le k\le n} a_j a_k = \prod_{1\le k\le n} a_k^{n+1}}$$

27. $\Delta(c^{\underline{x}})$ $\Rightarrow$ $\sum_{k=1}^{n} (-2)^{\underline{k}}/k$

$\Rightarrow c^{\underline{x+1}} - c^{\underline{x}}$

$c(c-1)\dots(c-x-1+1) - c(c-1)\dots(c-x+2)(c-x+1)$

$= c(c-1)\dots(c-x+1)(c-x) - c(c-1)\dots(c-x+2)(c-x+1)$

$= c(c-1)\dots(c-x+1)((c-x)-1)$

we would like a $c-x$ here.

$= c(c-1)\dots(c-x+1)\_\_\,(c-x-1)$

$= \dfrac{c(c-1)\dots(c-x+1)(c-x)(c-x-1)}{c-x}$

$= \dfrac{c^{\underline{x+2}}}{c-x}$ so $\Delta(c^{\underline{x}}) = \dfrac{c^{\underline{x+2}}}{c-x}$

$\Delta(-2^{\underline{x}}) = \dfrac{-2^{\underline{x+2}}}{-2-x}$

$\Rightarrow$ $\Delta((-2)^{\underline{x-2}}) = \dfrac{-2^{\underline{x-2+2}}}{-2-(x-2)} = \dfrac{(-2)^{\underline{x}}}{-x}$

$\Rightarrow \sum_{k=1}^{n} \dfrac{(-2)^{\underline{k}}}{k} = -\sum_{k=1}^{n+1} \dfrac{(-2)^{\underline{x}}}{-x}\,\delta x$

$= -\left[(-2)^{\underline{x-2}}\right]^{n+1} = -\left((-2)^{\underline{n-1}} - (-2)^{\underline{-1}}\right)$

$= \boxed{(-2)^{\underline{-1}} - (-2)^{\underline{n-1}}} = \dots \boxed{(-1)^n n! - 1}$

$$k^2 - (k-1)(k+1)$$
$$= k^2 - (k^2 - 1) =$$

28. $$\sum_{k\ge 1} \frac{1}{k(k+1)} = \sum_{k \ge 1} \frac{k}{k+1} - \frac{k-1}{k}.$$

Does not converge absolutely so

$$\sum_{k\ge1}\sum_{j\ge1}\left(\frac{k}{j}[j=k+1] - \frac{j}{k}[j=k-1]\right)$$
$$\ne \sum_{j\ge1}\sum_{k\ge1}\left(\frac{k}{j}[j=k+1] - \frac{j}{k}[j=k-1]\right)$$

29. $$\left(\sum_{k=1}^{n} \frac{(-1)^k k}{4k^2-1}\right) = \sum_{k=1}^{n} \frac{(-1)^k k}{(2k-1)(2k+1)}$$

$$= \sum_{k=1}^{n} (-1)^k \left(\frac{\frac14}{2k-1} + \frac{\frac14}{2k+1}\right)$$

$$= (-1)\left(\frac{\frac14}{1} + \cancel{\frac{\frac14}{3}}\right) + (1)\left(\cancel{\frac{\frac14}{3}} + \frac{\frac14}{5}\right)$$
$$+ (-1)\left(\cancel{\frac{\frac14}{5}} + \cancel{\frac{\frac14}{7}}\right) + \dots + (-1)^{n}$$
$$+ (-1)^{n-1}\left(\cancel{\frac{\frac14}{2n-3}} + \cancel{\frac{\frac14}{2n-1}}\right) + (-1)^n\left(\cancel{\frac{1/4}{2n-1}} + \frac{\frac14}{2n+1}\right)$$

$$= -\frac14 + (-1)^n \frac{\frac14}{2n+1} = \boxed{-\frac14 + \frac{(-1)^n}{8n+4}}$$

30. $1050 \Rightarrow 1$ way

$n+n+1 = 1050 \Rightarrow 2n = 1049 \Rightarrow \times$

$n+n+1+n+2 = 1050 \Rightarrow 3n = 1047 \Rightarrow \checkmark$

$n+n+1+n+2+n+3 = 1050 \Rightarrow 4n+6 = 1050$

$4n = 1044 \Rightarrow \checkmark$

we will have a winner, whenever:

$n+n+1+n+2+\dots+n+k = 1050$

$\Rightarrow (k+1)n + \frac{k(k+1)}{2} = 1050$ and $n = \frac{1050 - \frac{k(k+1)}{2}}{k+1}$

$\Rightarrow (k+1)\left(n+\frac{k}{2}\right) = 1050$ so

$\rightarrow (k+1)(2n+k) = 2100$

$\rightarrow a = k+1,\ b = 2n+k$

(1) $\rightarrow k = a-1$, $n = \frac{b-a+1}{2}$

where $ab = 2100$, $\rightarrow 2100 = 4 \cdot 3 \cdot 5^2 \cdot 7$

we need $b-a+1$ even so b and a must be opposite parity.

2100 has: $3 \cdot 5^2 \cdot 7 \Rightarrow (2)(3)(2) = 12$ odd factors

and $2^2 \cdot 3 \cdot 5^2 \cdot 7 \Rightarrow (3)(2)(3)(2) = 36$ total

so 24 even factors.

So, choosing b to be 1 of 12 odd factors, we have surplus of evens. This implies that, for a and b, we have 12 pairs of factors that satisfy (1).

$\boxed{\text{So the answer is 12 ways to write 1050 consecutive sum.}}$

Another solution: $\sum_{i=a}^{b-1} x$ gives some # of consecutive integers added together. Then: $\sum_{i=a}^{b-1} x = 1050$.

$$\rightarrow \sum_a^b x \,\delta x = 1050 \Rightarrow \sum_a^b x^{\underline{1}} \,\delta x = \left[\frac{x^{\underline{2}}}{2}\right]_a^b$$

$$= \frac{b^{\underline{2}}}{2} - \frac{a^{\underline{2}}}{2} = \frac{1}{2}(b-a)(b+a-1) = 1050$$

and $(b-a)(b+a-1) = 1050 \rightarrow$ Solution follows, similar to the first.

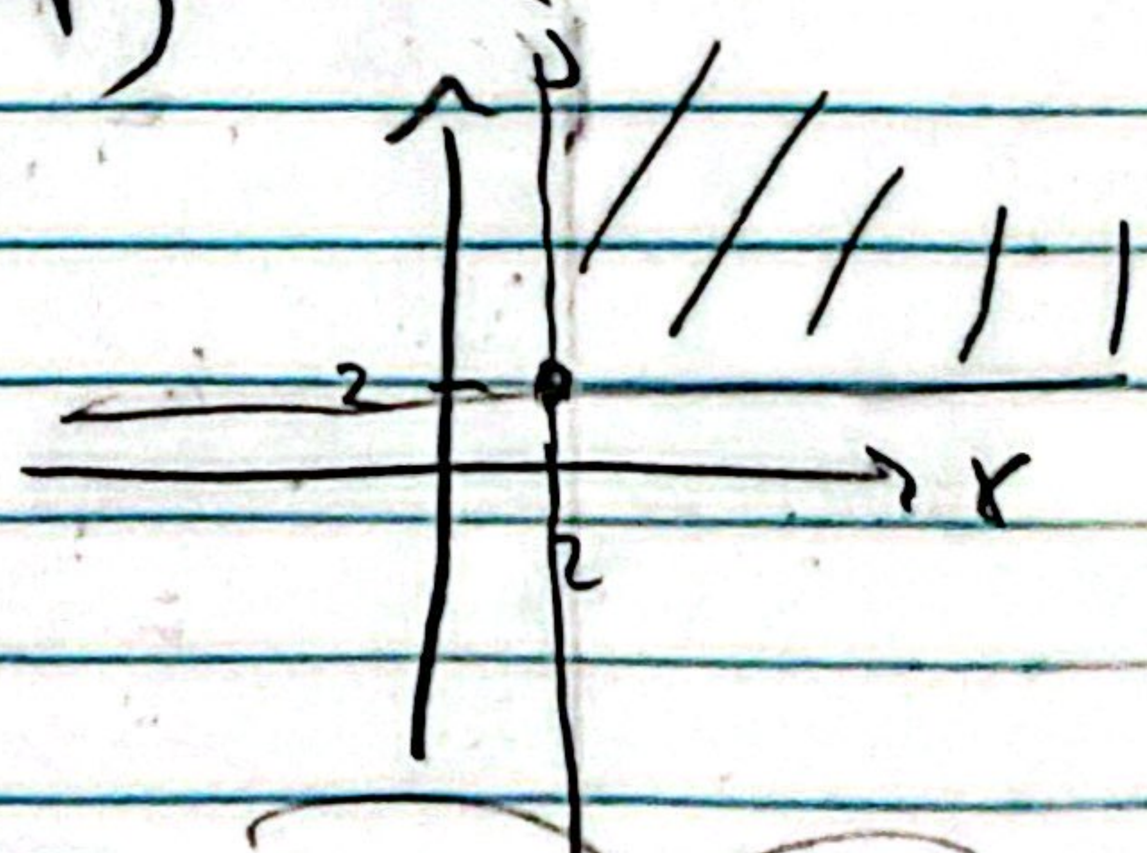
$$\boxed{\prod_{p\geq 2} (n_p+1) \text{ ways to represent } \prod_p p^{n_p}.}$$

31. $1 + \frac{1}{2^k} + \frac{1}{3^k} + \dots = \sum_{j\geq 1} \frac{1}{j^k} = \zeta(k)$

$$\Rightarrow \sum_{k\geq 2} (\zeta(k) - 1) = 1$$

$$\Rightarrow \sum_{k\geq 2} \left(\left[1 + \frac{1}{2^k} + \frac{1}{3^k} + \dots\right] - 1\right)$$

$$\Rightarrow \sum_{k\geq 2} \sum_{j\geq 2} \frac{1}{j^k} \Rightarrow$$



$$\Rightarrow \sum_{j\geq 2} \sum_{k\geq 2} \frac{1}{j^k}$$

$\hookrightarrow$ geometric series $\frac{1}{j^2} + \frac{1}{j^3} + \frac{1}{j^4} + \dots$

$\Rightarrow a = \frac{1}{j^2},\ r = \frac{1}{j}$

$\Rightarrow \dfrac{\frac{1}{j^2}}{1 - \frac{1}{j}}$

$$\Rightarrow \sum_{j\geq 2} \frac{\frac{1}{j^2}}{1 - \frac{1}{j}}$$

$$= \sum_{j\geq 2} \frac{1}{j^2}\left(\frac{j}{j-1}\right)$$

$$\Rightarrow \sum_{j\geq 2}\left(\frac{\frac{1}{2}}{j-1}+\frac{-\frac{1}{2}}{j+1}\right)=\frac{1}{2}+\cancel{\frac{-\frac{1}{2}}{3}}+\frac{\frac{1}{2}}{2}+\cancel{\frac{-\frac{1}{2}}{4}}+\cancel{\frac{\frac{1}{2}}{3}}$$

$$+\cancel{\frac{-\frac{1}{2}}{5}}+\cancel{\frac{\frac{1}{2}}{4}}+\cancel{\frac{-\frac{1}{2}}{6}}+\cancel{\frac{\frac{1}{2}}{5}}+\cancel{\frac{-\frac{1}{2}}{7}}+\cancel{\frac{\frac{1}{2}}{6}}+\cancel{\frac{-\frac{1}{2}}{8}}+\dots$$

$\Rightarrow$ telescopes, converges abs.

to: $\frac{1}{2}+\frac{\frac{1}{2}}{2}=\boxed{\frac{3}{4}}$