Concrete Mathematics
Ch. 1

warmups.

1. n horses 1 to n, horses 1 through n−1 same color, and 2 through n are same color, but middle: 2 through n−1 have to be same color (can't be 2 colors at same time).

∴ horse 1→n−1 have color "a"; color horses 2→n have color "b" but horses 2→n−1 have color "a" ... or "b" .. iff "a" = "b". Thus, horse 1→n−1 have same color and horse n has same color so 1→n has same color.

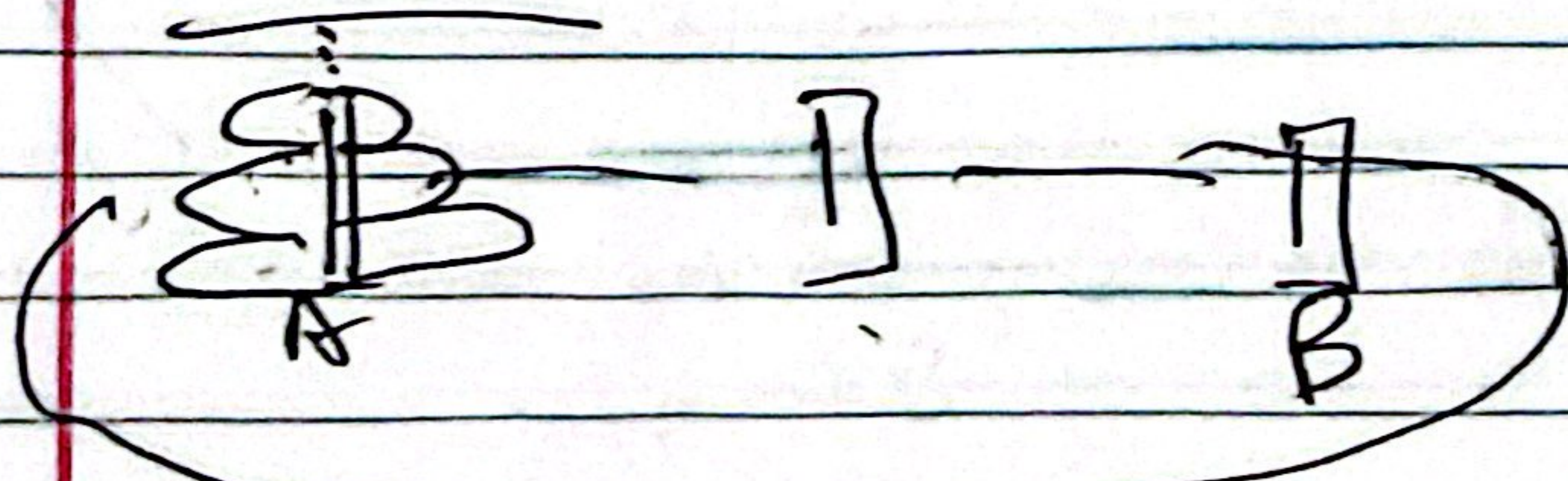
What's wrong here?

~~· horse change color over time.~~

· Proof holds ... until n = 2. {1→1} same color; {2→2} same color. here, there is no "middle group" that allows us to generalize. they could be diff. colors.

2. Tower n disks from peg A to B. direct move disallowed (move to or from middle peg. larger disk must never appear smaller)

n disks



Start with smallest disk → middle → B
next smallest can only go → middle

⋮

Let X_n = shortest # moves to transfer tower of n disks from A to B with rules.

So $x_n = 3x_n + 2$

$x_0 = 0$

$\Rightarrow x_0 + 1 = 1$

$x_n + 1 = 3x_{n-1} + 3$

let $u_n = x_n + 1$

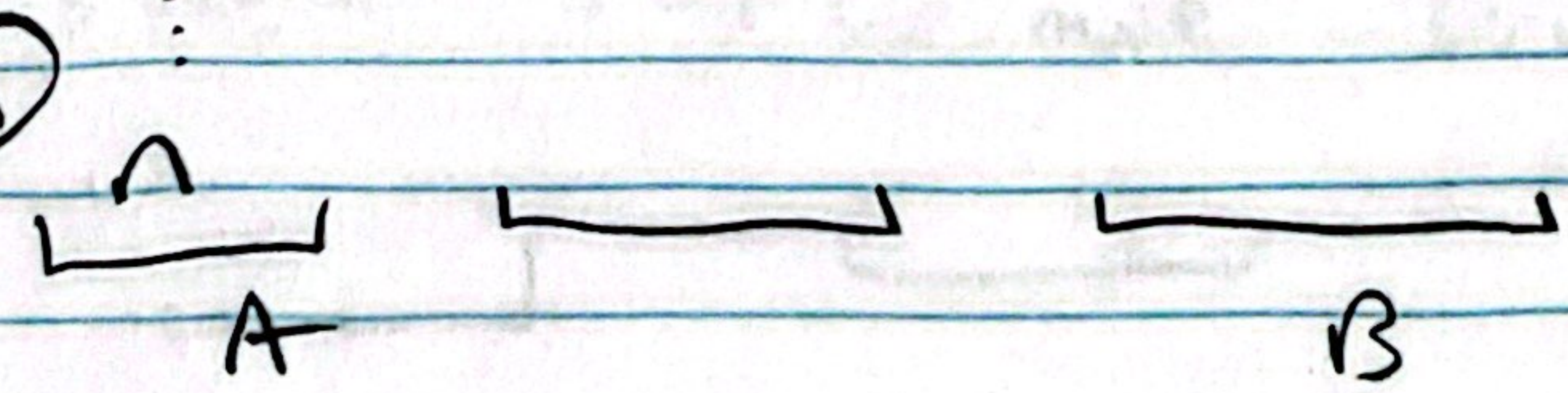
$\Rightarrow u_0 = 1$

$u_n = 3u_{n-1} \Rightarrow u_n = 3^n$

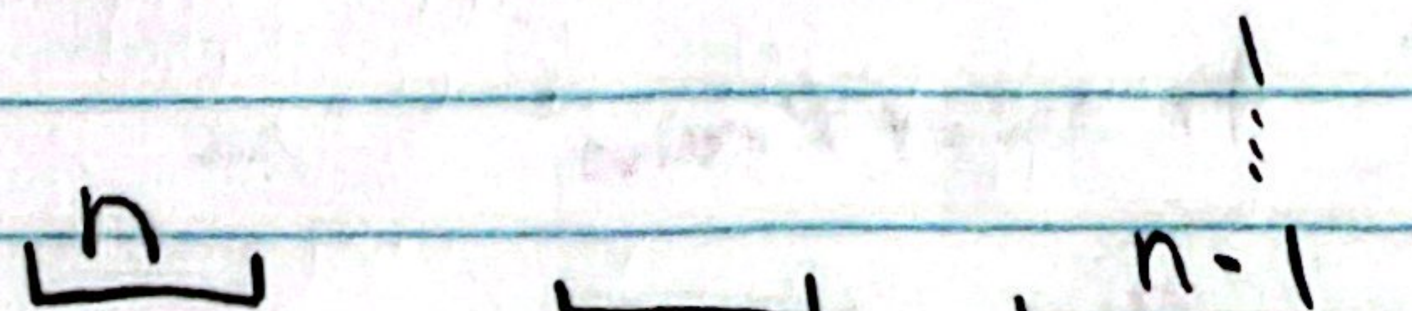
(question 2 omitted for clarity reasons)

$\Rightarrow 3^n = x_n + 1 \Rightarrow \boxed{x_n = 3^n - 1}$

3. ①

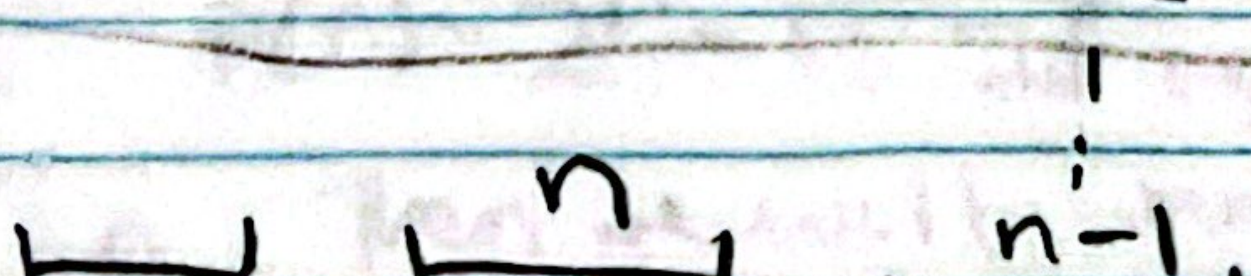


to get to B, we must have



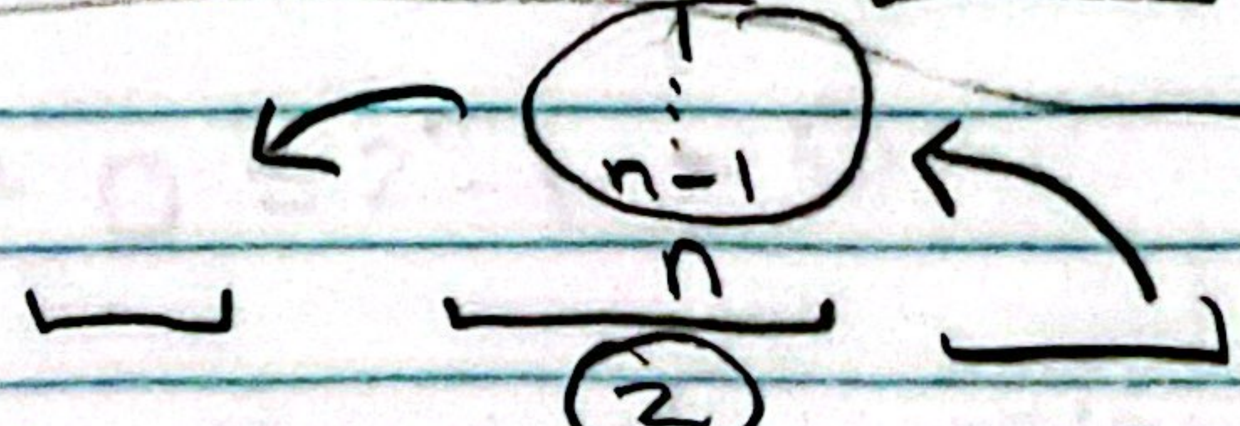
at some point.

Then:



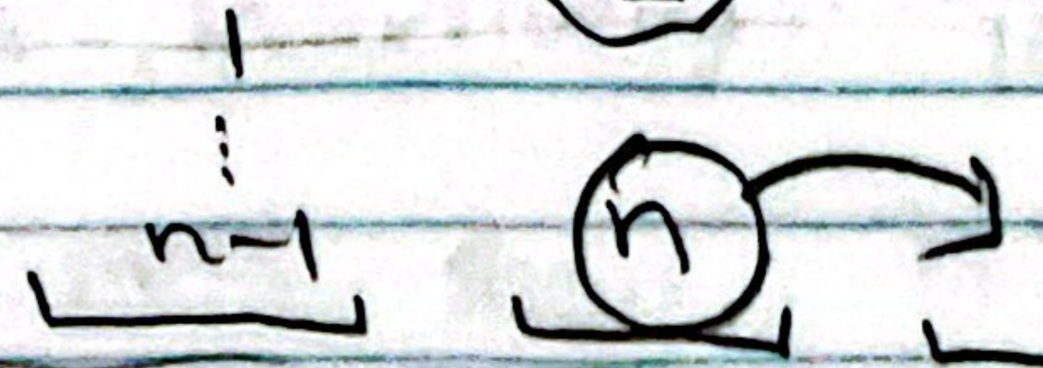
is the only possible next step.

Then:

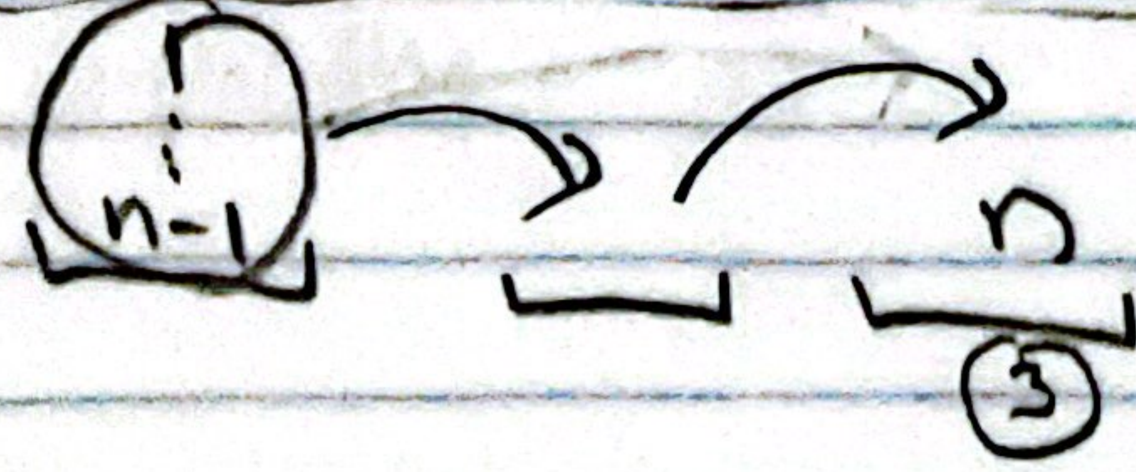


must occur over a series of steps (x_{n-1} steps precisely)

It follows:



And:



is the only quickest next step.

So from ①, ②, ③, we see that n our stack must traverse from each peg.

= book ans: 3^n possible arrangements (disk can be on any peg!) Must hit them all ; shortest solution takes $3^n - 1$. This construction is like a ternary Gray code running through all nums $(0\ldots0)_3$ to $(2\ldots2)_3$. Changing 1 digit a time.

4. Any starting & ending configurations of n disks on 3 pegs that are more than $2^n - 1$ moves apart

T_n = necessary moves

$\Rightarrow$ want to see if $T_n \nleq 2^n - 1$.

Assume $P(n) = 2^n - 1 \geq T_n$. If we show this is true for T_n, then we will have solved problem.

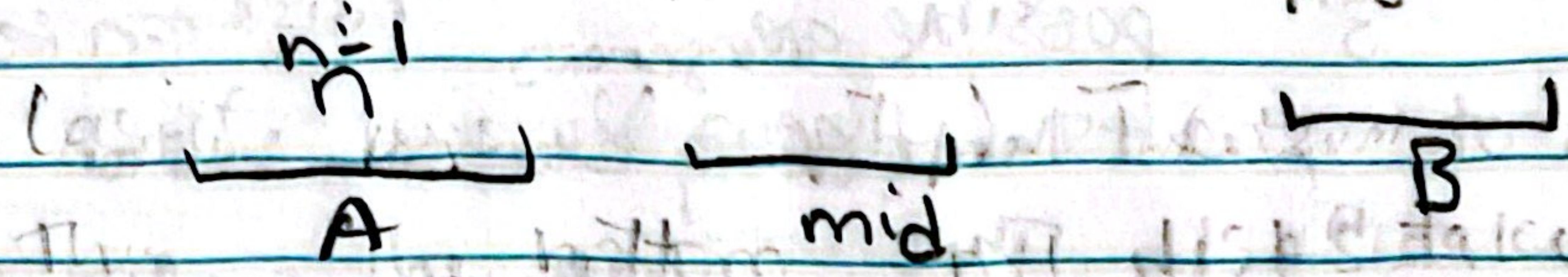
$T(0) = 0 \leq 2^0 - 1 = 0$

Assume, inductively: $T(n-1) \leq 2^{n-1} - 1$.

We must show $T(n) \leq 2^n - 1$ follows implicitly.

suppose i have $n-1$ disks $\rightarrow T(n-1) \le 2^{n-1}-1$

adding an n^{th} disk as the largest disk, Then:



where can n be? either A, mid, or B.

① Let n be at mid: takes $n-1$ to mid will by definition need $T(n-1)$ steps.

② let n be at B: $n \rightarrow$ mid $= 1$, $n-1 \rightarrow$ mid $= T(n-1)$

So this takes $T(n-1)+1$ steps.

③ let n be at A: $\underbrace{n-1 \rightarrow B}_{T(n-1)}$, $\underbrace{n \rightarrow mid}_{1}$, $\underbrace{n-1 \rightarrow mid}_{T(n-1)}$ $\Rightarrow 2T(n-1)+1$.

So we see:

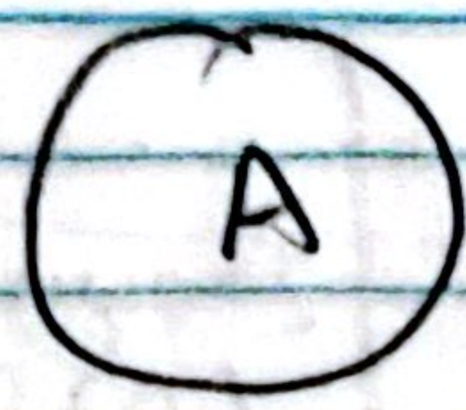
①: $T(n-1)$

②: $T(n-1)+1$

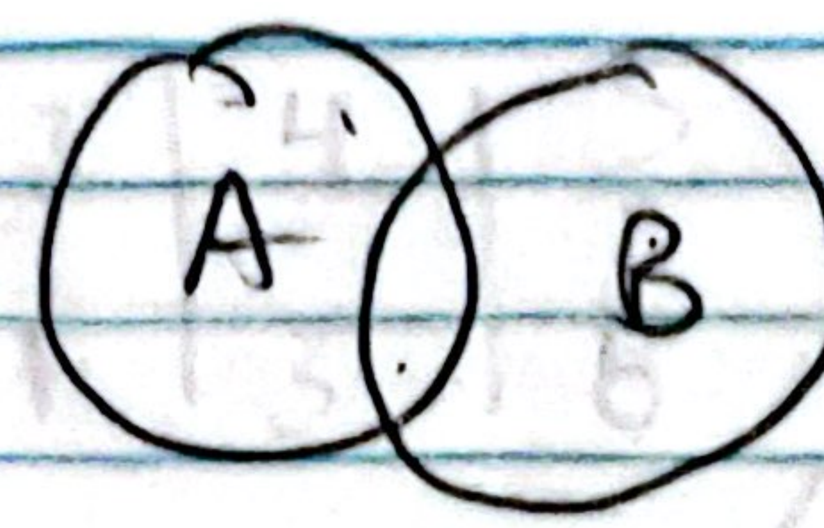
③ $2T(n-1)+1$

So $T(n-1) \le T(n) \le 2T(n-1)+1$

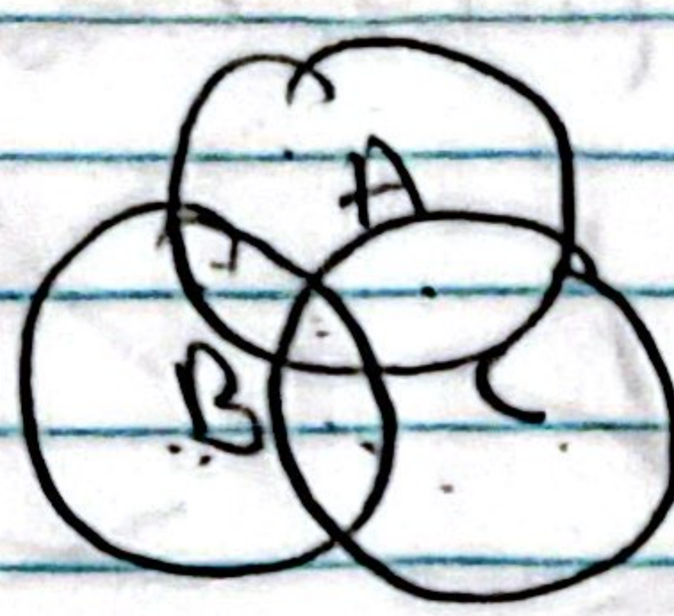
and by induction $T(n) \le 2(2^{n-1}-1)+1 = 2^n-1$

5. 

$\Rightarrow 2$



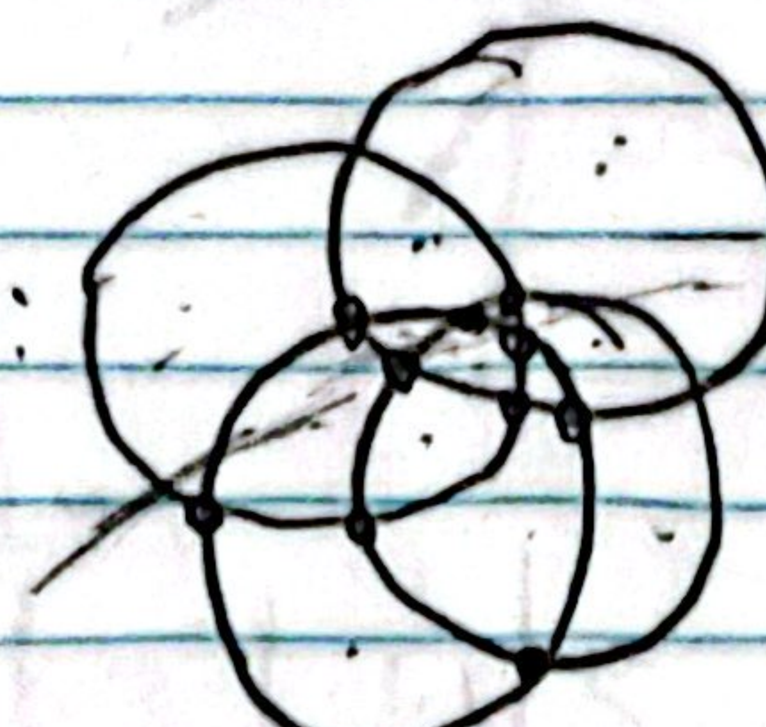
$\Rightarrow 4$



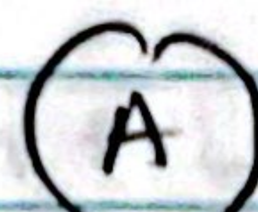
$\Rightarrow 8$ ... so is 4 sets 16?

Ans: No. Let's see:

We see why immediately: each circle can only intersect another at at most 2 points. Therefore, our growth is not exponential: 2^n simply.

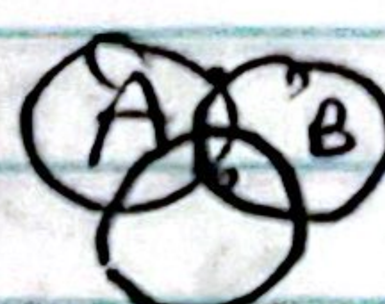

$\Rightarrow$ 14 regions

Instead, let's ignore the outer region



$\Rightarrow 1$,

2 points create another region for total of $1+1+1=3$.



$\Rightarrow$ 2 intersections on A: 1 region
2 intersections on mid: 1 region
2 intersections on B: 1 region
itself: 1 region

add to previous.

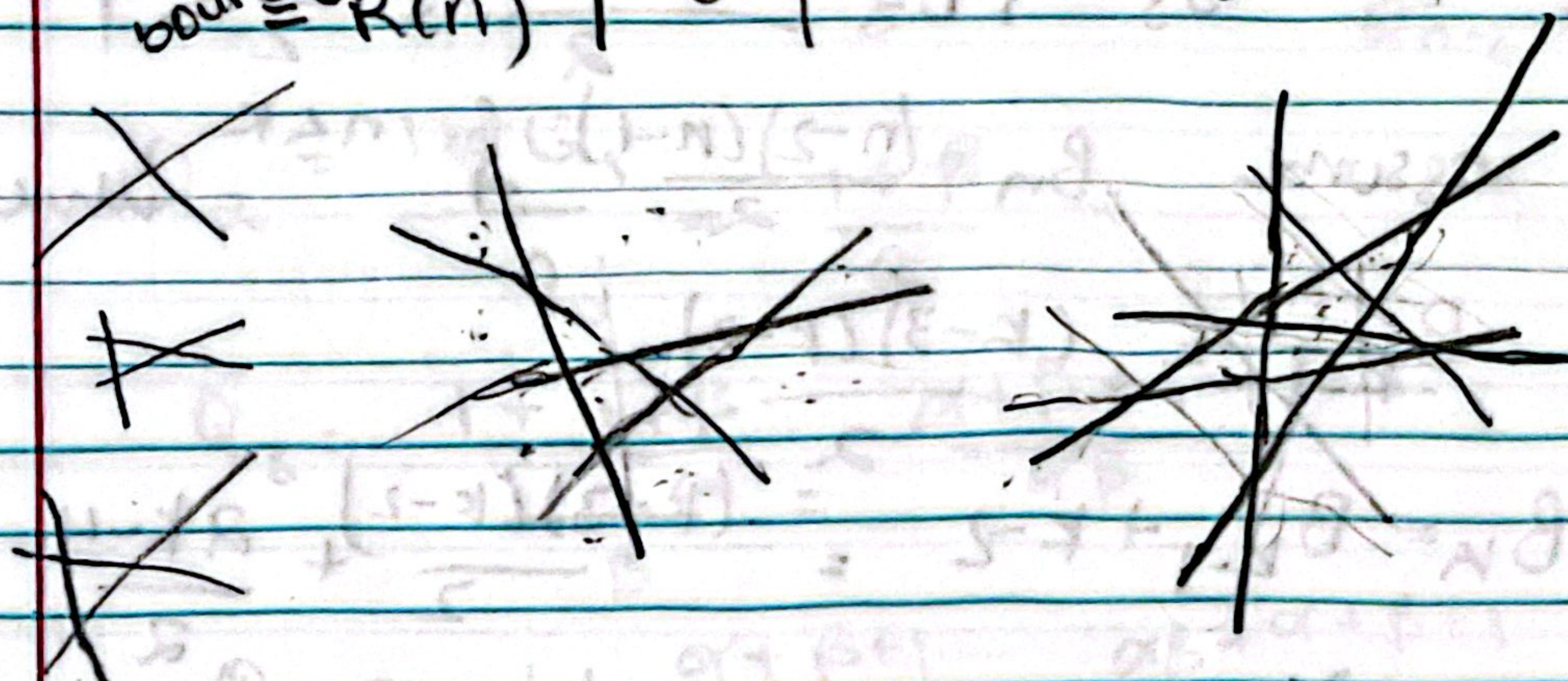
So define C_n as num regions:

$$\underbrace{C_n - 1}_{\text{to account for ignored outer}} = C_{n-1} + \underbrace{1}_{\text{itself}} + C_{n-1} \Rightarrow C_n = 2C_{n-1} + 2 \quad n > 1$$

$$C_1 = 1$$

6.

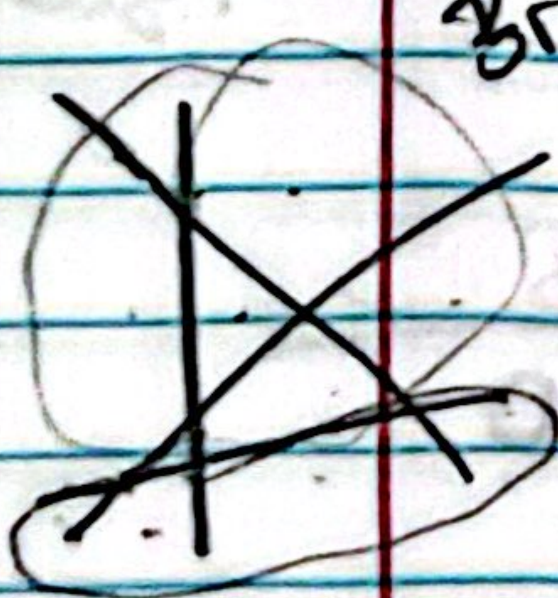
n	1	2	3	4	5
max num regions bounded $= R(n)$	0	0	1	3	6



L_n (max # regions)

n	1	2	3	4	5
$L(n)$	2	4	7	11	16

4 lines



$$\underline{L_n = n + L_{n-1}}$$

4

(1.) when have n new regions, we can cut maximum of 2 unbounded. Thus $n-2$ must be bounded regions.

creating 2 new unbounded regions

$$\Rightarrow R(n) = R_{n-1} + n-2$$

$$= R_{n-2} + n-3 + n-2$$

$$\vdots$$

$$R(n) = R_0 + \dots + n-2 = 0 + 0 + 0 + 1 + 2 + \dots + n-2$$

$$= \boxed{\frac{(n-2)(n-1)}{2}}$$

Induction:

Base case: $B_3 = 1$, $B_3 = \frac{(3-2)(3-1)}{2} = \frac{2}{2} = 1$

assume $B_n = \frac{(n-2)(n-1)}{2}$ for $n < k$ $\Rightarrow$ where

$$B_{k-1} = \frac{(k-3)(k-2)}{2}$$

$$B_k = B_{k-1} + k - 2 = \frac{(k-3)(k-2)}{2} + \frac{2k-4}{2}$$

$$= \frac{k^2 - 3k + 2}{2} = \frac{(k-2)(k-1)}{2}.$$

(2) Notice $L(n) - R(n) =$ num regions $-$ num bounded regions

$=$ num unbounded regions $= u(n)$.

n	1	2	3	4	5
$u(n) = L(n) - R(n)$	2	4	6	8	10

huh. $u(n) = 2n$.

$$\Rightarrow 2n = L(n) - R(n)$$

$$\Rightarrow R(n) = L(n) - 2n$$

$$\rightarrow L(n) = L(n-1) + n$$
$$= L(n-2) + n - 1 + n$$
$$= L(n-3) + n-2 + n-1 + n$$
$$= L(0) + 1 + 2 + \cdots + n-2 + n-1 + n$$
$$= \frac{n(n+1)}{2}$$

$$\Rightarrow \boxed{R(n) = \frac{n(n+1)}{2} - 2n = \frac{(n-2)(n-1)}{2}}$$

HW exercises:

8. $Q_0 = \alpha$, $Q_1 = \beta$, $Q_n = \dfrac{1+Q_{n-1}}{Q_{n-2}}$, $n > 1$

$$Q_2 = \frac{1+Q_1}{Q_0} = \frac{1+\beta}{\alpha}$$

$$Q_3 = \frac{1+\frac{1+\beta}{\alpha}}{\beta} = \frac{\alpha+\beta+1}{\alpha\beta}$$

$$Q_4 = \frac{1+\frac{\alpha+\beta+1}{\alpha\beta}}{\frac{1+\beta}{\alpha}} = \frac{\alpha\beta+\alpha+\beta+1}{\cancel{\alpha}\beta} \cdot \frac{\cancel{\alpha}}{1+\beta}$$

$$= \frac{(\alpha+1)\cancel{(\beta+1)}}{\beta\cancel{(1+\beta)}} = \frac{\alpha+1}{\beta}$$

$$Q_5 = \frac{1+\frac{\alpha+1}{\beta}}{\frac{\alpha+\beta+1}{\alpha\beta}} = \frac{\cancel{\beta+\alpha+1}}{\cancel{\beta}} \cdot \frac{\alpha\cancel{\beta}}{\cancel{\alpha+\beta+1}} = \alpha$$

$$Q_6 = \frac{1+\alpha}{\frac{\alpha+1}{\beta}} = \beta.$$

Therefore cycle $\overbrace{Q_0, Q_1, Q_2, Q_3, Q_4}^{\text{repeats}}, \ldots$

periodic.

so we can define:

$Q_n = \alpha$, $n \equiv 0 \bmod 5$

$Q_n = \beta$, $n \equiv 1 \bmod 5$

$Q_n = \frac{1+\beta}{\alpha}$, $n \equiv 2 \bmod 5$

$\ldots$ $Q_n = \frac{\alpha+1}{\beta}$, $n \equiv 4 \bmod 5$

9. $P(n)$: $x_1 \cdots x_n \le \left(\frac{x_1+\cdots+x_n}{n}\right)^n$, if $x_1, \ldots, x_n \ge 0$

a) $x_n = \frac{(x_1+\cdots+x_{n-1})}{n-1}$

want to show $P(n-1)$: $x_1 \cdots x_{n-1} \le \left(\frac{x_1+\cdots+x_{n-1}}{n-1}\right)^{n-1}$

follows from $P(n)$:

$$x_1 \cdots x_{n-1}\left(\frac{x_1+\cdots x_{n-1}}{n-1}\right) \le \left(\frac{x_1+\cdots+x_{n-1}+x_n}{n}\right)^n$$

$$\to \quad x_1 \cdots x_{n-1}\left(\frac{x_1+\cdots x_{n-1}}{n-1}\right) \le \left(\frac{x_1+\cdots+x_{n-1}+\left(\frac{x_1+\cdots+x_n}{n-1}\right)}{n}\right)^n$$

consider RHS: $\left(\frac{\frac{(n-1)(x_1+\cdots+x_{n-1})+x_1+\cdots+x_{n-1}}{n-1}}{n}\right)^n$

$$\to \left(\frac{(x_1+\cdots+x_{n-1})(\cancel{n-1+1})}{(n-1)\cancel{n}}\right)^n = \left(\frac{x_1+\cdots+x_{n-1}}{n-1}\right)^n$$

$$\to \quad x_1 \cdots x_{n-1}\left(\frac{x_1+\cdots+x_{n-1}}{n-1}\right) \le \left(\frac{x_1+\cdots+x_{n-1}}{n-1}\right)^n$$

$$\Rightarrow \quad x_1 \cdots x_{n-1} \le \left(\frac{x_1+\cdots x_{n-1}}{n-1}\right)^{n-1}$$

$= P(n-1)$ Qed.

b) $P(n)$ and $P(2)$ imply $P(2n)$

$P(2)$: $x_1 x_2 \le \left(\frac{x_1+x_2}{2}\right)^2$

$P(n)$ $x_1 x_2 \ldots x_n \le \left(\frac{x_1+\cdots+x_n}{n}\right)^n$

$P(2n)$: $x_1 x_2 \dots x_n x_{n+1} \dots x_{2n} \leq \left(\frac{x_1 + \dots + x_{2n}}{2n}\right)^{2n}$

we would like to show this inductive hypothesis from $P(2)$ and $P(n)$.

Now: begin with $x_1 x_2 \dots x_n x_{n+1} \dots x_{2n}$,

$\rightarrow (\underbrace{x_1 x_2 \dots x_n}_{\leq \left(\frac{x_1 + \dots + x_n}{n}\right)^n})(\underbrace{x_{n+1} \dots x_{2n}}_{\leq \left(\frac{x_{n+1} + \dots + x_{2n}}{n}\right)^n})$

so $(x_1 x_2 \dots x_n)(x_{n+1} \dots x_{2n}) \leq \left(\frac{x_1 + \dots + x_n}{n}\right)^n \left(\frac{x_{n+1} + \dots + x_{2n}}{n}\right)^n$

consider RHS: $\left(\frac{x_1 + \dots + x_n}{n}\right)^n \left(\frac{x_{n+1} + \dots + x_n}{n}\right)^n$

if we show $\left(\frac{x_1 + \dots + x_n}{n}\right)^n \left(\frac{x_{n+1} + \dots + x_n}{n}\right)^n \leq \left(\frac{x_1 + \dots + x_{2n}}{2n}\right)^{2n}$

Then we will be done (I will explain why at end).

$\rightarrow$ nth root $\rightarrow \left(\frac{x_1 + \dots + x_n}{n}\right)\left(\frac{x_{n+1} + \dots + x_n}{n}\right) \leq \left(\frac{x_1 + \dots + x_{2n}}{2n}\right)^2$

$\rightarrow (x_1 + \dots + x_n)(x_{n+1} + \dots + x_n) \leq \left(\frac{x_1 + \dots + x_{2n}}{2}\right)^2$

Recall: $x_1 x_2 \leq \left(\frac{x_1 + x_2}{2}\right)^2 \Rightarrow k_1 = x_1 + \dots + x_n$
$k_2 = x_{n+1} + \dots + x_n$

Thus: $k_1 k_2 \leq \left(\frac{k_1 + k_2}{2}\right)^2$ $x_1 + \dots + x_{2n} = k_1 + k_2$

which we know is true (base case).

Therefore, dear reader, look back up on the page for our previous conjecture $\left(\frac{x_1 + \dots + x_n}{n}\right)^n \left(\frac{x_{n+1} + \dots + x_n}{n}\right)^n \leq \left(\frac{x_1 + \dots + x_{2n}}{2n}\right)^{2n}$ has been proved.

So far Then, we have This definitively (by induction)

$$\underbrace{(x_1 x_2 \dots x_n)(x_{n+1} \dots x_{2n})}_{A} \le \underbrace{\left(\frac{x_1+\dots+x_n}{n}\right)^n \left(\frac{x_{n+1}+\dots+x_{2n}}{n}\right)^n}_{B}$$

and proved $\underbrace{\left(\frac{x_1+\dots+x_n}{n}\right)^n \left(\frac{x_{n+1}+\dots+x_{2n}}{n}\right)^n}_{B} \le \underbrace{\left(\frac{x_1+\dots+x_{2n}}{2n}\right)^{2n}}_{C}$

And So: $A \le B$, $B \le C$

so $A \le C$ must follow by transitivity;

That is, $\boxed{x_1 x_2 \dots x_n x_{n+1} \dots x_{2n} \le \left(\frac{x_1+\dots+x_{2n}}{2n}\right)^{2n}}$

is implied through $P(2)$ and $P(n)$. qed.

C. $P(n)$ & $P(2) \to P(2n)$

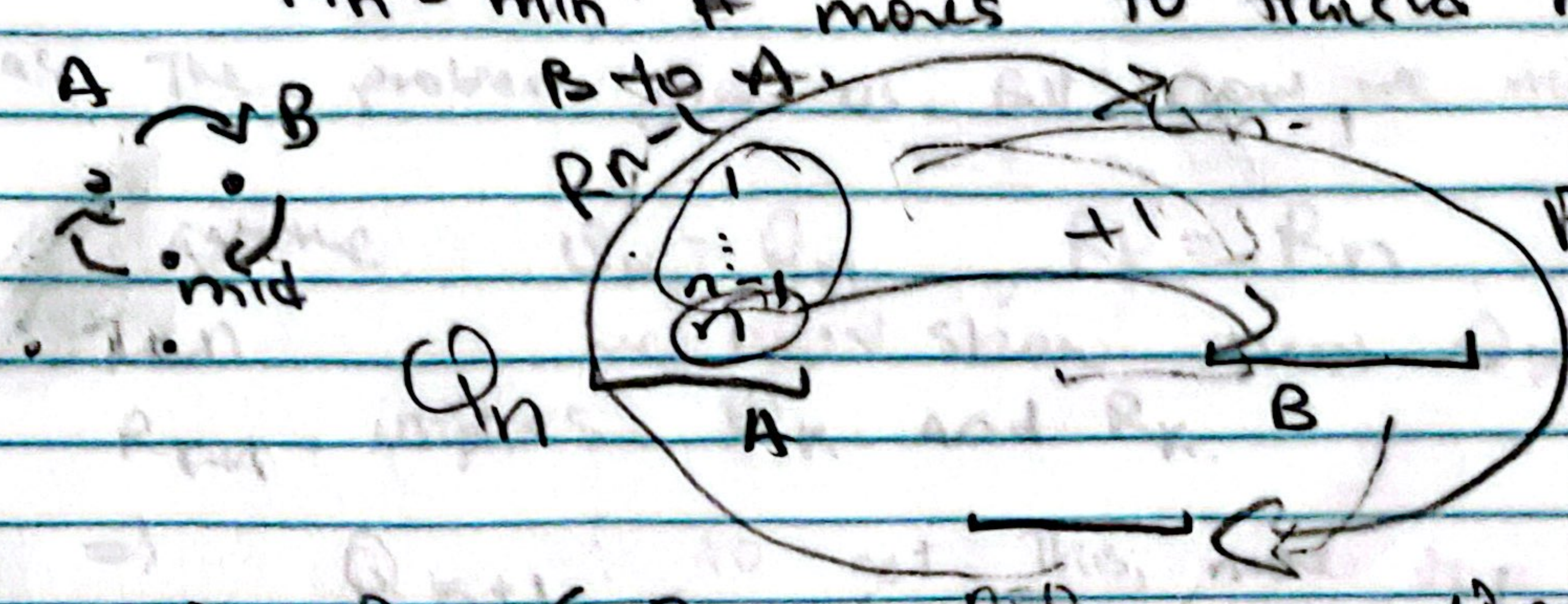
$\Rightarrow$ we cover all $2, 4, 8, 16 \dots$

$P(n-1)$ covers rest.

10. Moves must be clockwise.

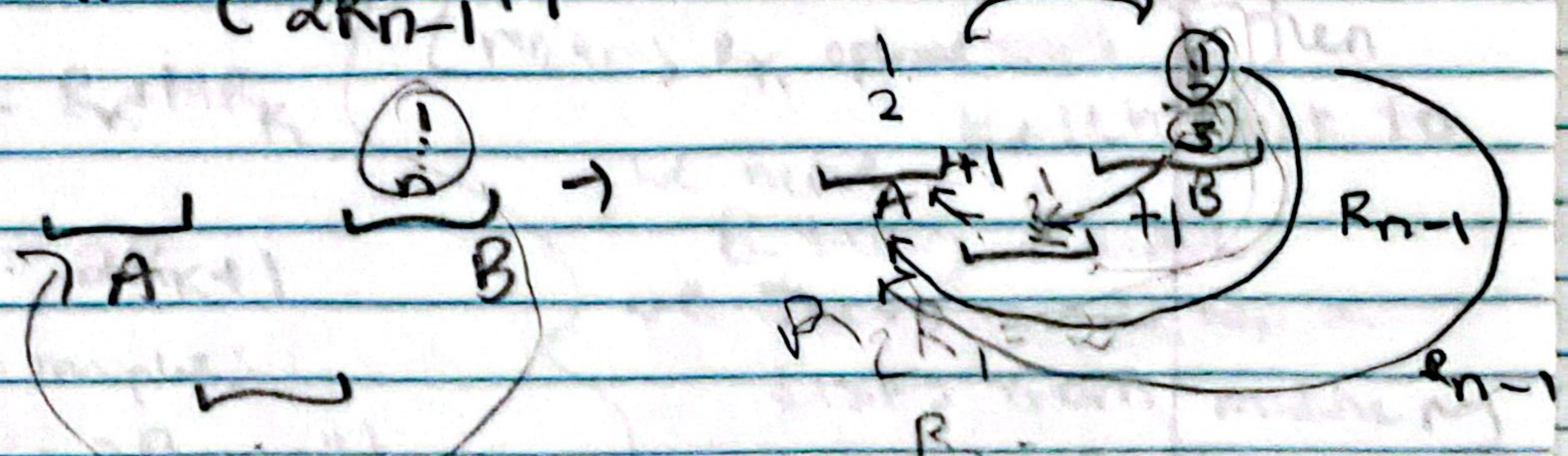
$\therefore$ Q_n = min # moves to transfer n disks from A to B.

R_n = min # moves to transfer n disks from B to A.



$\Rightarrow$ $Q_n = \begin{cases} 0 \\ 2R_{n-1}+1 \end{cases}$

R_n:



$R_1 = 2$

$R_2 \Rightarrow$: $1 \to mid \to A$, $2 \to mid$, $1 \to B$, $2 \to A$, $1 \to mid \to A$
7 R_2

R_3: $1 \to mid \to A$, $2 \to mid$, $1 \to B$, $2 \to A$, $1 \to mid \to A$
$3 \to mid$, $1 \to B \to mid$, $2 \to B$, $1 \to A \to B$,
$3 \to A$, R_2

$\vdots$

$R_n = R_{n-1} + 1 + Q_{n-1} + 1 + R_{n-1} = 2R_{n-1} + 1 + Q_{n-1} + 1$

but $Q_n = 2R_{n-1}+1 \to R_{n-1} = \frac{Q_n - 1}{2}$

$\Rightarrow R_n = 2\left(\frac{Q_n - 1}{2}\right) + 1 + Q_{n-1} + 1 = Q_n + Q_{n-1} + 1$

So, from our steps, we have determined

$$Q_n = \begin{cases} 0 & n=0 \\ 2R_{n-1}+1 & n>0 \end{cases}, \quad R_n = \begin{cases} 0 & n=0 \\ Q_n+Q_{n-1}+1 & n>0 \end{cases}$$

as the problem gave us. But now we must prove it.

assume $Q_K = Q_n$, $R_K = R_n$.

Then we must show that Q_{K+1} and R_{K+1} implies Q_K and R_K.

$\Rightarrow$ Q_{K+1}: to get this, move top K disks from A to middle peg (requiring R_K optimal steps). Then we move K+1th disk to B, taking 1 move. We then move top K disks from middle peg to B (requiring R_K optimal steps).

$$Q_{K+1} = R_K + 1 + R_K$$

$$= 2R_K + 1$$

which implies

$Q_K = 2R_{K-1}+1$.

same thing for R_K.

15. $I(n) =$ # penultimate survivor

n	order	$I(n)$
1	1	1
2	2, 1	2
3	2, 1, 3	1
4	2, 4, 3, 1	3
5	2, 4, 1, 5, 3	5

$2n$ '' 2 ; 4 ; $\Rightarrow$ $2n-1$ ' 3, $2n-3$ 5, … $\Rightarrow I(2n) = 2J(n)+1$

$I(2n+1) = 2J(n)+1$

$I(2) = 2 \quad I(2n) = 2I(n) - 1$

$\Rightarrow I(3) = 1 \quad I(2n+1) = 2I(n)+1$

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$I(n)$	2	1	3	5	1	3	5	7	9	11	1	3	1				

$I(3 \cdot 2^m) = 1$

$\Rightarrow \boxed{I(3 \cdot 2^m + l) = I(3 \cdot 2^m) + 2l \quad \text{where} \quad 0 \le l < 3 \cdot 2^m}$

$\Rightarrow I(3 \cdot 2^m + l) = 1 + 2l, \quad m \ge 0, \quad 0 \le l < 3 \cdot 2^m,$