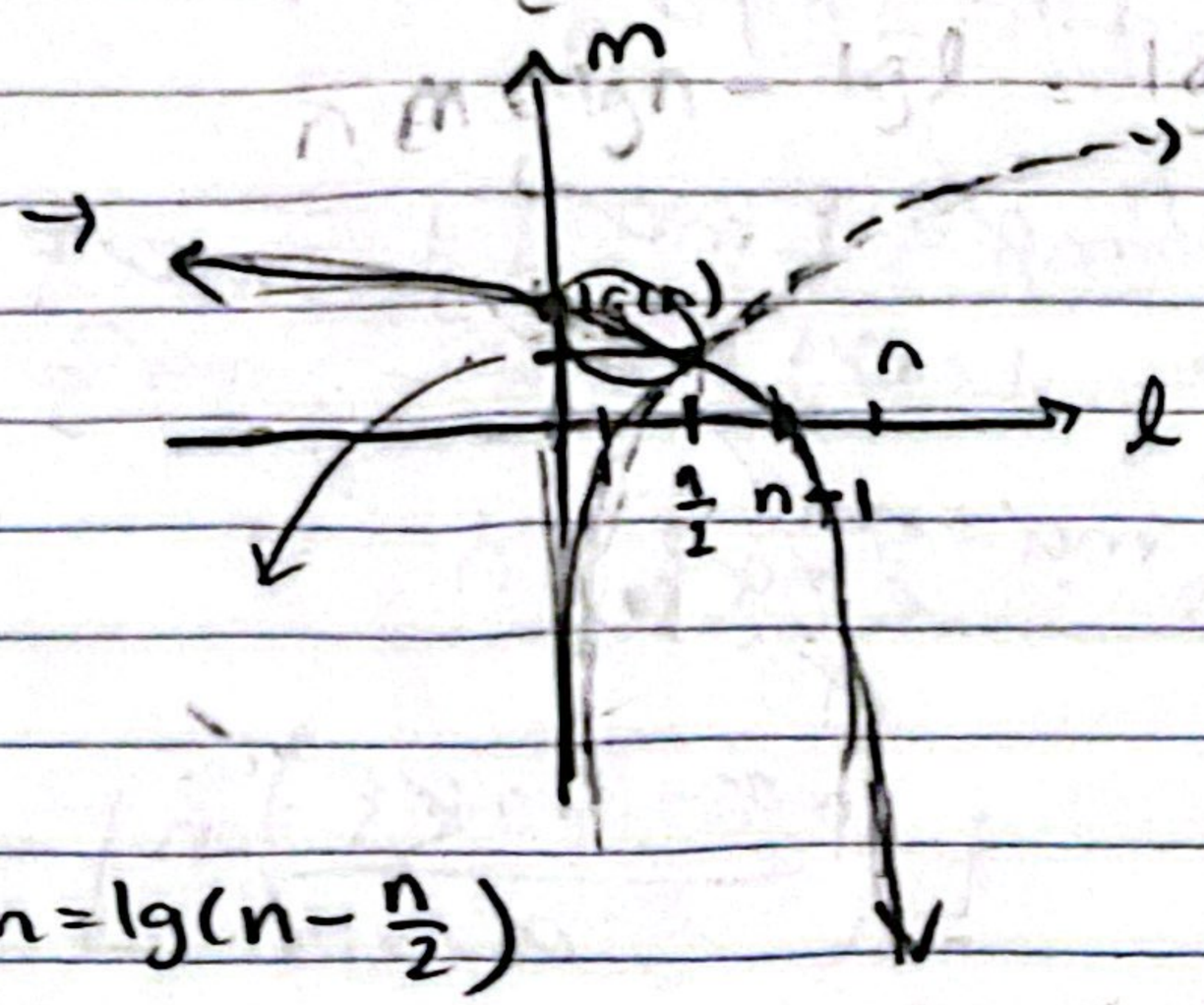
Chapter 3

$n = 2^m + l, \quad 0 \le l < 2^m$

$\rightarrow \Rightarrow$ treat n as an arbitrary integer

$2^m = n - l \rightarrow m = \lg(n-l)$



$0 = \lg(n-l)$

$1 = n - l$

$l = n - 1$

$l = 2^m$

$m = \lg(l)$

$m = \lg(n - \frac{n}{2})$

$\Rightarrow m = \lg(\frac{n}{2})$

$= \lg(n) - 1$

$\lg(n-1) < m \le \lg(n)$

$l \lg(l) = \lg(n-l)$

$l = n - l$

$l = \frac{n}{2}$

$m = \lg(n-l)$

$\Rightarrow \quad 2^m = n - 1$

$\Rightarrow \quad l = n - 2^m$

So we get $n = 2^m + l, \; 0 \le l < 2^m$

$\Rightarrow \quad \lg(n) - 1 < m \le \lg(n), \; l = n - 2^m$

$$\boxed{m = \lfloor \lg n \rfloor \text{ and } l \text{ is then trivial: } l = n - 2^{\lfloor \lg n \rfloor}}$$

2. (a) $\lfloor x + .5 \rfloor$ rounds down always.
(b) $\lceil x - 0.5 \rceil$ rounds up always.

3. $\lfloor \lfloor m\alpha \rfloor n / \alpha \rfloor$

if m, n positive integers, α is irrational > 1.

Then $m, n \geq 1$, and $\alpha > 1$.

Thus: $\lfloor \lfloor m\alpha \rfloor n / \alpha \rfloor$

write $m\alpha = \lfloor m\alpha \rfloor + \{m\alpha\}$

$\Rightarrow \lfloor m\alpha \rfloor = m\alpha - \{m\alpha\}$

$\Rightarrow \dfrac{(m\alpha - \{m\alpha\})\, n}{\alpha}$

and: $\left\lfloor \left(\dfrac{m\alpha - \{m\alpha\}}{\alpha} \right) n \right\rfloor$

we know $1 \leq n < \alpha$

$\Rightarrow = mn + \left\lfloor \dfrac{-\{m\alpha\} n}{\alpha} \right\rfloor$

$= mn - \left\lceil \dfrac{\{m\alpha\} n}{\alpha} \right\rceil$ $\quad n < \alpha$, $\frac{n}{\alpha} < 1$

notice $1 \leq n < \alpha$:

$\frac{1}{\alpha} \leq \frac{n}{\alpha} < 1$

and $\dfrac{\{m\alpha\} n}{\alpha} \leq \dfrac{\{m\alpha\} n}{\alpha} < \{m\alpha\}$

$\Rightarrow$ since $\{m\alpha\}$ is a fraction (< 1)

$\Rightarrow 0 < \left\lceil \dfrac{\{m\alpha\} n}{\alpha} \right\rceil \leq 1$

and: $\boxed{mn - 1}$

5. $\lfloor nx \rfloor = \lfloor n\lfloor x \rfloor + n\{x\} \rfloor$

$= n\lfloor x \rfloor + \lfloor n\{x\} \rfloor$

So if we wish $\lfloor nx \rfloor = n\lfloor x \rfloor$

$\Rightarrow$ $n\lfloor x \rfloor + \lfloor n\{x\} \rfloor = n\lfloor x \rfloor$

and we need $0 \le n\{x\} < 1$

meaning $\boxed{\{x\} < \frac{1}{n} \text{ assuming } n \text{ is a positive integer.}}$

6. here is what we know $f(x) \Rightarrow$ (1) (continuous, monotone decreasing $\hookrightarrow f(x+1) < f(x)$. for all x.

(2) Also: for $x \in \mathbb{Z}$, $f(x) \in \mathbb{Z}$

what can we conclude $\lfloor f(x) \rfloor$?

Let's try to find a function first.

$y = x$ is increasing but fits (2).

Then, what about $y = -x \Rightarrow$ satisfies (1) and (2).

Then $\lfloor -x \rfloor = -\lceil x \rceil$.

notice $g(x) = x$ is monotonically increasing!

$\lfloor f(x) \rfloor$

So we have a start: $\lfloor f(x) \rfloor = -\lceil g(x) \rceil$

$= -\lceil g(\lceil x \rceil) \rceil = \lfloor -g(\lceil x \rceil) \rfloor = \boxed{\lfloor f(\lceil x \rceil) \rfloor}$

Now take $x_n = n$, for $0 \le n < m$, $x_n = x_{n-m} + 1$ for $n \ge m$.

we wish a closed x_n.

Consider: $x_n = x_{n-m} + 1$

when $n \ge m$, $n - m \ge 0$.

Then, assume that $0 \le n-m < m$ or $m \le n < 2m$

$x_n = n, \quad 0 \le n < m$

$x_n = n - m + 1, \quad m \le n < 2m$

$x_n = n - 2m + 2, \quad 2m \le n < 3m$

It follows: $x_{n-m} = n - m, \quad m \le n < 2m$

Now, if $n \ge 2m \Rightarrow$ $n - m \ge m$. Assume $m \le n - m < 2m$

$x_n = x_{n-m} + 1$

$0 + (n - 2m + 1) + 1 = n - 2m + 2$

Then it follows that $x_{n-m} = (n-m) - m + 1 = n - 2m + 1$

So $x_n = n - 2m + 2, \quad 2m \le n < 3m$

we see

It appears $x_n = n, \quad 0 \le n < m$

$x_n = n - Km + K, \quad Km \le n < (K+1)m$

where $K \ge 1$.

But wait, we see $K+1$ and K, we immediately think of $\lfloor\ \rfloor$ and $\lceil\ \rceil$.

$\Rightarrow$ let's just look at this bound: $Km \le n < (K+1)m$

Then $K \le \frac{n}{m} < K+1$ and $\lfloor \frac{n}{m} \rfloor = K$

$\Rightarrow$ $x_n = n, \quad 0 \le n < m$

$x_n = n - \lfloor \frac{n}{m} \rfloor m + \lfloor \frac{n}{m} \rfloor, \quad \lfloor \frac{n}{m} \rfloor = K, \quad K \ge 1.$

But this yields something better,

Scanned with CamScanner

Consider definition of mod:

$x_n = n = m\lfloor \frac{n}{m} \rfloor + n \bmod m$

$\Rightarrow \quad n \bmod m = n - m\lfloor \frac{n}{m} \rfloor.$

Then: $\quad x_n = n = m\lfloor \frac{n}{m} \rfloor + n \bmod m, \quad 0 \le n < m$

$x_n = n - \lfloor \frac{n}{m} \rfloor m + \lfloor \frac{n}{m} \rfloor$

$= n \bmod m + \lfloor \frac{n}{m} \rfloor, \quad \lfloor \frac{n}{m} \rfloor = k,$

$k \ge 1$

or $\lfloor \frac{n}{m} \rfloor \ge 1$

or simply $n \ge m$.

And notice something really mostly; $\quad x_n = \underbrace{m\lfloor \frac{n}{m} \rfloor}_{\text{always } 0 \text{ due to } \lfloor \frac{n}{m} \rfloor} + n \bmod m, \quad 0 \le n < m$

Moreover, if $x_n = n \bmod m + \lfloor \frac{n}{m} \rfloor$. even if $0 \le n < m$, $\lfloor \frac{n}{m} \rfloor = 0$ so they are equal!

Thus: $\boxed{x_n = n \bmod m + \lfloor \frac{n}{m} \rfloor,}$ generally.

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9. $\frac{m}{n} = \frac{1}{q} + \left\{ \frac{m}{n} - \frac{1}{q} \right\}, \quad q = \lceil \frac{n}{m} \rceil.$

with $\quad 0 < \frac{m}{n} < 1.$

Recall the identity: $\lfloor x \rfloor + \{x\} = x$

$\Rightarrow \quad \{x\} = x - \lfloor x \rfloor$

$\Rightarrow \quad \frac{m}{n} = \cancel{\frac{1}{q}} + \frac{m}{n} - \cancel{\frac{1}{q}} - \lfloor \frac{m}{n} - \frac{1}{q} \rfloor$

$\Rightarrow \quad \frac{m}{n} = \frac{m}{n} - \lfloor \frac{m}{n} - \frac{1}{q} \rfloor$

Now $\frac{m}{n} = \frac{m}{n} - \lfloor \frac{m}{n} - \frac{1}{q} \rfloor$,

Now we must simply show that

$\lfloor \frac{m}{n} - \frac{1}{q} \rfloor = 0$.

consider $\frac{m}{n} - \frac{1}{q} = \frac{m}{n} - \frac{1}{\lceil \frac{n}{m} \rceil}$

let $x = \lceil \frac{n}{m} \rceil \Rightarrow \frac{n}{m} \leq x < \frac{n}{m} + 1$

if $n, m > 0$ $\therefore$ $x \geq \frac{n}{m}$

$\Rightarrow \frac{m}{n} \geq \frac{1}{x}$

$\Rightarrow -\frac{1}{x} \geq -\frac{m}{n}$

$\Rightarrow \frac{m}{n} - \frac{1}{x} \geq \frac{m}{n} - \frac{m}{n}$

$\Rightarrow \frac{m}{n} - \frac{1}{\lceil \frac{n}{m} \rceil} \geq 0$

$x < \frac{n}{m} + 1$

$x < \frac{n+m}{m}$

$\frac{m}{n+m} < \frac{1}{x}$

$-\frac{1}{x} < \frac{-m}{n+m}$

$\frac{m}{n} - \frac{1}{x} < \frac{m}{n} - \frac{m}{n+m}$

consider $\frac{m}{n} - \frac{m}{n+m}$

$\frac{\cancel{nm} + m^2 - \cancel{nm}}{n(n+m)}$

$\Rightarrow \frac{m^2}{n(n+m)}$.

Now: $\frac{m}{n} - \frac{1}{\lceil \frac{n}{m} \rceil} \geq 0$,

$\frac{m}{n} - \frac{1}{\lceil \frac{n}{m} \rceil} < \frac{m^2}{n(n+m)}$.

$\Rightarrow$ we know $0 < \frac{m}{n} < 1 \Rightarrow n > m$

$\Rightarrow$ since $n, m > 0 \Rightarrow n^2 > m^2$.

$\Rightarrow \frac{m^2}{n(n+m)} < \frac{n^2}{n(n+m)} = \frac{n^2}{n^2+nm} < 1$

and $\frac{m^2}{n(n+m)} < 1$ by transitivity.

Thus: $0 \leq \frac{m}{n} - \frac{1}{\lceil \frac{n}{m} \rceil} < \frac{m^2}{n(n+m)} < 1$

Now, if $0 \le \frac{m}{n} - \frac{1}{\lceil \frac{n}{m} \rceil} < \frac{m^2}{n(n+m)} < 1$

$\Rightarrow$ $0 \le \frac{m}{n} - \frac{1}{\lceil \frac{n}{m} \rceil} < 1$ by trans, and it follows that

$\left\lfloor \frac{m}{n} - \frac{1}{\lceil \frac{n}{m} \rceil} \right\rfloor = 0$ by definition.

Thus: $\frac{m}{n} = \frac{1}{q} + \{\frac{m}{n} - \frac{1}{q}\} = \frac{1}{q} + \frac{m}{n} - \frac{1}{q} - \left\lfloor \frac{m}{n} - \frac{1}{q} \right\rfloor$

$= \frac{m}{n} - \underbrace{\left\lfloor \frac{m}{n} - \frac{1}{\lceil \frac{n}{m} \rceil} \right\rfloor}_{0} = \frac{m}{n}$. QED.

10. Show that $\lceil \frac{2x+1}{2} \rceil - \lceil \frac{2x+1}{4} \rceil + \left\lfloor \frac{2x+1}{4} \right\rfloor$
$=$ either $\lceil x \rceil$ or $\lfloor x \rfloor$

We have a few cases:

① Assume $\frac{2x+1}{4} \in \mathbb{Z}$. Then $\lceil \frac{2x+1}{2} \rceil - \underbrace{\cancel{\lceil \frac{2x+1}{4} \rceil + \left\lfloor \frac{2x+1}{4} \right\rfloor}}_{0}$

$= \lceil x + \frac{1}{2} \rceil \Rightarrow$ if $\frac{2x+1}{4} = n$, then $x = 2n - \frac{1}{2}$

Thus: $\lceil x + \frac{1}{2} \rceil = \lceil 2n - \frac{1}{2} + \frac{1}{2} \rceil = \lceil 2n \rceil = 2n$.

And $\lceil x \rceil = \lceil 2n - \frac{1}{2} \rceil = 2n$ so $\lceil x + \frac{1}{2} \rceil = 2n = \lceil x \rceil$

and it holds true.

② Assume $\frac{2x+1}{4} \notin \mathbb{Z}$.

Recall $\lceil n \rceil - \lfloor n \rfloor = 1$ if n is not an integer.

Thus: $\lceil \frac{2x+1}{2} \rceil - \left(\lceil \frac{2x+1}{4} \rceil - \left\lfloor \frac{2x+1}{4} \right\rfloor \right)$

$= \lceil x + \frac{1}{2} \rceil - 1 = \lceil x + \frac{1}{2} - 1 \rceil = \lceil x - \frac{1}{2} \rceil$

$= \lceil x \rceil$ for $\{x\} > \frac{1}{2}$, $\{x\} < \frac{1}{2}$, $x = \frac{1}{2}$ (by inspection)

And QED.

12. $\lceil \frac{n}{m} \rceil = \lfloor \frac{n+m-1}{m} \rfloor$

consider $\lfloor \frac{n+m-1}{m} \rfloor = \lfloor \frac{n}{m} + 1 - \frac{1}{m} \rfloor$

we have a few cases:

① $\frac{n}{m}$ is an integer: LHS $= \frac{n}{m}$, RHS $= \lfloor \frac{n}{m} + \underbrace{1 - \frac{1}{m}}_{<1} \rfloor$

$= \frac{n}{m} + \lfloor 1 - \frac{1}{m} \rfloor$

$= \frac{n}{m} + 0$ ✓.

② $\frac{n}{m}$ is not an integer:

$\lceil x \rceil - \lfloor x \rfloor = 1$ Then $\lceil \frac{n}{m} \rceil - \lfloor \frac{n}{m} \rfloor = 1$

and $\lfloor \frac{n}{m} \rfloor = \lceil \frac{n}{m} \rceil - 1$

RHS $= \lfloor \lceil \frac{n}{m} \rceil \cancel{-1} + 1 - \frac{1}{m} \rfloor = \lfloor \lceil \frac{n}{m} \rceil - \frac{1}{m} \rfloor$

$= \lceil \frac{n}{m} \rceil - \lfloor \frac{1}{m} \rfloor = \lceil \frac{n}{m} \rceil$ ✓

and so $\lceil \frac{n}{m} \rceil = \lfloor \frac{n+m-1}{m} \rfloor$ Now we prove for $\frac{n}{m} \in \mathbb{Z}$ and for $\frac{n}{m} \notin \mathbb{Z}$, or really, simply $\lceil \frac{n}{m} \rceil = \lfloor \frac{n+m-1}{m} \rfloor$.

13.

numelements in spec$(\alpha) \leq n$: $N(\alpha, n) = \lceil \frac{n+1}{\alpha} \rceil - 1$

$N(\alpha, n) + N(\beta, n) = n$ for spec(α) and spec(β) to partition $\mathbb{Z}^+$:

$\Rightarrow \lceil \frac{n+1}{\alpha} \rceil + \lceil \frac{n+1}{\beta} \rceil - 2 = n$

$\Rightarrow -\lfloor -\frac{n+1}{\alpha} \rfloor - \lfloor -\frac{n+1}{\beta} \rfloor - 2 = n$

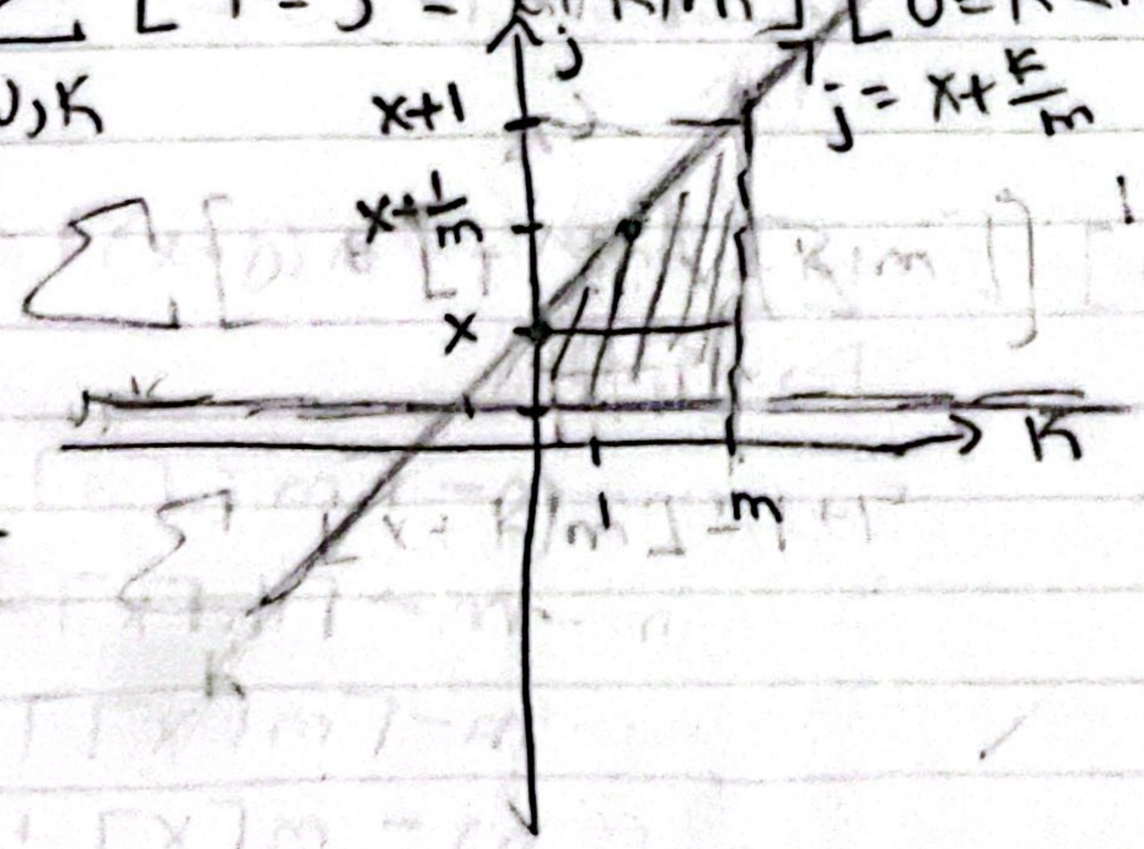
$\Rightarrow -\left(-\left(\frac{n+1}{\alpha}\right) - \left\{-\frac{n+1}{\alpha}\right\}\right) - \left(-\frac{n+1}{\beta} - \left\{-\frac{n+1}{\beta}\right\}\right) - 2 = n$

17. $\sum_{0\le k<m} \lfloor x+k/m \rfloor$ with $x \ge 0$ by sub.

$$\sum_{j} [1 \le j \le x + k/m] \text{ for } \lfloor x+k/m \rfloor$$

we will first eval., Then explain why this sub. works.

$$\Rightarrow \sum_{j,k} [1 \le j \le x + k/m][0 \le k < m]$$



$$\rightarrow [0 \le k < m][1 \le j \le x] \qquad j = x + \frac{k}{m} \quad m(j-x) = k$$

$$+ [x \le j \le x+1][m(j-x) \le k < m]$$

$$\Rightarrow \underbrace{\sum_{j,k} [0 \le k < m][1 \le j < x]}_{①} + \underbrace{\sum_{j,k} [x \le j < x+1][m(j-x) \le k < m]}_{②}$$

$$① \sum_{j,k} [0 \le k < m][1 \le j < x]$$

$$= \sum_{j} [1 \le j < x] \sum_{k} [0 \le k < m] = \sum_{1 \le j < x} m$$

$\sum_{1 \le j < x} m \Rightarrow$ since x is not an integer

$$= \sum_{1 \le j \le \lceil x \rceil - 1} m = m(\lceil x \rceil - 1 - 1 + 1) = m(\lceil x \rceil - 1)$$

Now ②: $\sum_{j,k} [x \le j < x+1][m(j-x) \le k < m]$

$$= \sum_{j,k} [\lceil x \rceil = j][m(j-x) \le k < m]$$

$$= \sum_{[j = \lceil x \rceil]} \cancel{m - x - \lceil m(j-x) \rceil + 1}$$

$$= m - m - \lceil m(\lceil x \rceil - x) \rceil$$

$$= m - m - \lceil m\lceil x \rceil - mx \rceil$$

$$= m - m - \lceil m\lceil x \rceil \rceil - \lceil -mx \rceil$$

$$= m - m - m\lceil x \rceil + \lfloor mx \rfloor$$

adding ① and ②:

$$\cancel{m(\lceil x \rceil - 1)} + \cancel{m} - \cancel{m\lceil x \rceil} - \lfloor mx \rfloor$$

$$= m\lceil x \rceil - \cancel{m\lceil x \rceil} \cancel{- m + m} + \boxed{\lfloor mx \rfloor}$$

There is another way:

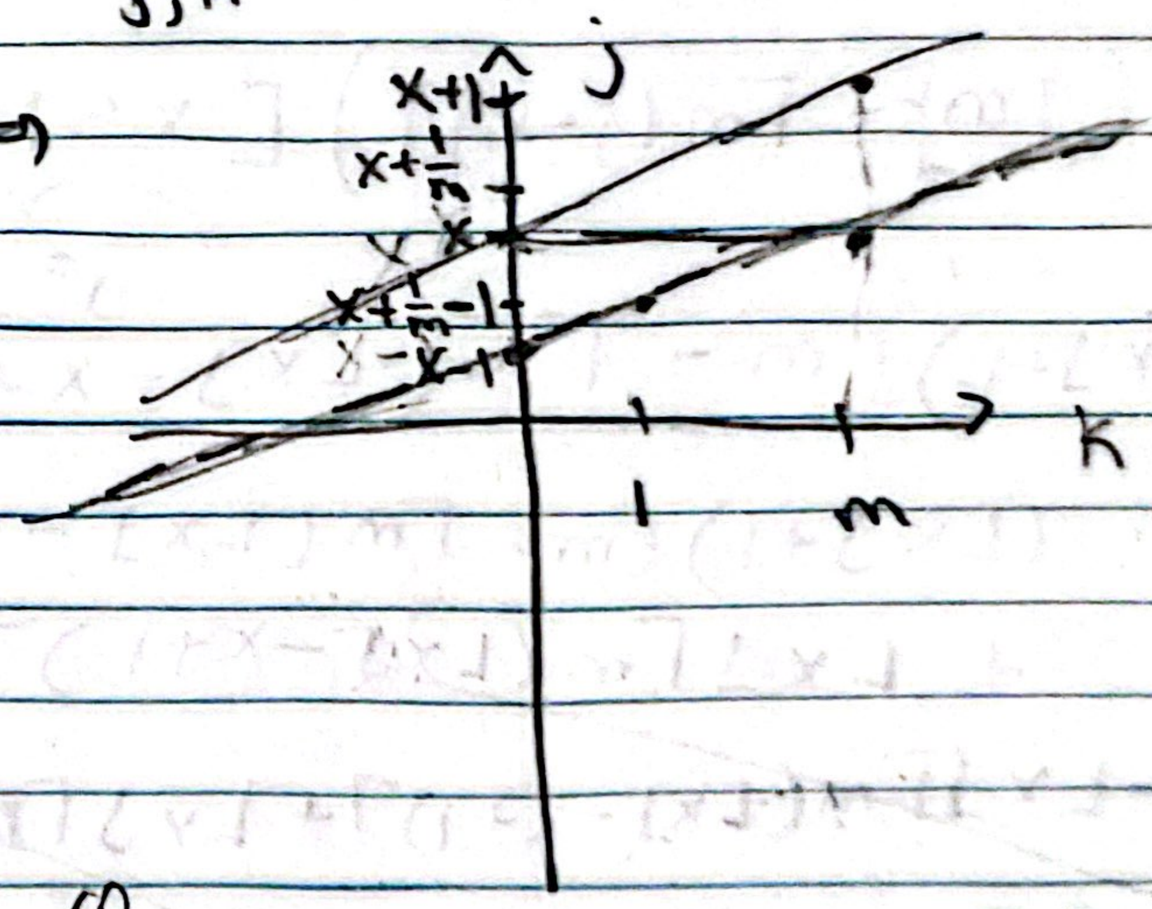
$\sum_{0 \le k < m} \lfloor x + \frac{k}{m} \rfloor$ let $j = \lfloor x + \frac{k}{m} \rfloor$

$$\Rightarrow \sum_{0 \le k < m} j[j = \lfloor x + \tfrac{k}{m} \rfloor] = \sum_{0 \le k < m} j[x + \tfrac{k}{m} - 1 < j \le x + \tfrac{k}{m}]$$

$$= \sum_{j,k} j[x + \tfrac{k}{m} - 1 < j \le x + \tfrac{k}{m}][0 \le k < m]$$

$$\Rightarrow \sum_{j,k} j\left[x+\frac{k}{m}-1 < j \le x+\frac{k}{m}\right]\left[0 \le k < m\right]$$

$\Rightarrow$



$j = x + \frac{k}{m} - 1$

$m(j-x+1) = k$

$j = x + \frac{k}{m}$

$m(j-x) = k$

$$\Rightarrow \overset{①}{\left[0 \le k < m(j-x+1)\right]\left[x-1 < j \le x\right]}$$

$$+ \overset{②}{\left[m(j-x) \le k \le m\right]\left[x < j \le x+1\right]}$$

$$\Rightarrow ①: \sum_{j,k} j\left[0 \le k < m(j-x+1)\right]\left[x-1 < j \le x\right]$$

$$= \sum_{j,k} j\left[x-1 < j \le x\right]\left[k \in \left[0 \ldots m(j-x+1)\right)\right]$$

$$= \sum_{j} j\left[x-1 < j \le x\right]\left(\lceil m(j-x+1)\rceil - 1 - 0 + 1\right)$$

$$= \sum_{x-1 < j \le x} j\lceil m(j-x+1)\rceil$$

$$= \sum_{j=\lfloor x \rfloor} j\lceil m(j-x+1)\rceil$$

$$= \lfloor x \rfloor \lceil m(\lfloor x \rfloor - x + 1)\rceil$$

$$② \sum j\left[m(j-x) < k \le m\right]\left[x < j \le x+1\right]$$

$x-1 < j-1 \le x$
$j-1 = \lfloor x \rfloor$

$$= \sum_j j\,[m(j-x) \le k \le m]\,[x < j \le x+1]$$

$$= \sum_j j\left(\lfloor m \rfloor - \lceil m(j-x) \rceil\right)\underbrace{[x < j \le x+1]}_{j = \lfloor x \rfloor + 1}$$

$$= (\lfloor x \rfloor + 1)\left(m - \lceil m(\lfloor x \rfloor - x + 1) \rceil\right)$$

① + ② : $(\lfloor x \rfloor + 1)(m - \lceil m(\lfloor x \rfloor - x + 1) \rceil)$
$+ \lfloor x \rfloor \lceil m(\lfloor x \rfloor - x + 1) \rceil$

$\Rightarrow \cancel{-\lfloor x \rfloor \lceil m(\lfloor x \rfloor - x + 1) \rceil + \lfloor x \rfloor \lceil m(\lfloor x \rfloor - x + 1) \rceil}$
$+ m\lfloor x \rfloor + m - \lceil m(\lfloor x \rfloor - x + 1) \rceil$

$\Rightarrow m\lfloor x \rfloor + m - \lceil \underset{\text{int}}{m\lfloor x \rfloor} - mx + \underset{\text{int}}{m} \rceil$

$\Rightarrow m\lfloor x \rfloor + m - \lceil m\lfloor x \rfloor \rceil - \lceil -mx \rceil - \lceil m \rceil$

$\Rightarrow \cancel{m\lfloor x \rfloor} - \cancel{m\lfloor x \rfloor} + \cancel{m} - \cancel{m} + \boxed{\lfloor mx \rfloor}$

See Next Page.

20. $\sum_{k}$ [all multiples of x] [between α and β]

$= \sum_{k} [\alpha \le kx \le \beta]\, kx$, k is an integer.

$= x \sum_{k} k[\alpha \le kx \le \beta] = x \sum_{k} k\left[\frac{\alpha}{x} \le k \le \frac{\beta}{x}\right]$

$= x \left(\frac{\lceil \frac{\alpha}{x} \rceil + \lfloor \frac{\beta}{x} \rfloor}{2}\right)\left(\lfloor \frac{\beta}{x} \rfloor - \lceil \frac{\alpha}{x} \rceil + 1\right)$

$= \boxed{\frac{1}{2} x \left(\lfloor \frac{\beta}{x} \rfloor \left(\lfloor \frac{\beta}{x} \rfloor + 1\right) - \lceil \frac{\alpha}{x} \rceil \left(\lceil \frac{\alpha}{x} \rceil - 1\right)\right)}$

21. how many 2^m for $0 \le m \le M$ have leading 1 in decimal notation.

How do we find the leading digit in mathematical terms: 2^m; first digit...

If we take $\lfloor \log_{10} 2^m \rfloor + 1$, we obtain the # of digits 2^m has: example:

$\lfloor \log_{10}(2^4) \rfloor + 1 = 2$.

Now, $10^{\lfloor \log_{10} 2^m \rfloor}$ will always have a leading digit of 1 with $\lfloor \log_{10} 2^m \rfloor + 1$ digits

Then, we want $10^{\lfloor \log_{10} 2^m \rfloor} \le 2^m <$?

what ? has a preserved leading digit of 1.

Our task is to find this ? → solve a counting sum.

consider $\underbrace{1\,0\,0\,0\,0 \ldots 0}_{\lfloor \log_{10} 2^m \rfloor}$. Then, all other values can range from 0 to

Then $10^{\lfloor \log_{10} 2^m \rfloor} \leq 2^m < ?$

where $? = 10^{\lfloor \log_{10} 2^m \rfloor} + K$.

But $0 \leq K < 10^{\lfloor \log_{10} 2^m \rfloor}$

Thus: $\sum_{\substack{0 \leq m \leq M, \\ K}} \left[10^{\lfloor m \log_{10} 2 \rfloor} \leq 2^m < 10^{\lfloor m \log_{10} 2 \rfloor} + K \right]$, $\left[0 \leq K < 10^{\lfloor \log_{10} 2^m \rfloor} \right]$

$$= \sum_{0 \leq m \leq M} \left[10^{\lfloor m \log_{10} 2 \rfloor} \leq 2^m < 2 \cdot 10^{\lfloor \log_{10} 2^m \rfloor} \right]$$

$$\rightarrow \sum_{0 \leq m \leq M} \left[\lfloor m \log_{10} 2 \rfloor \log_2 10 \leq m < 1 + \frac{\lfloor \log_{10} 2^m \rfloor}{\log_2 10} \right]$$

$$\Rightarrow \sum_{0 \leq m \leq M} \left[0 \leq m - \lfloor m \log_{10} 2 \rfloor \log_2 10 < 1 \right]$$

$\Rightarrow$ let $n = \lfloor m \log_{10} 2 \rfloor$

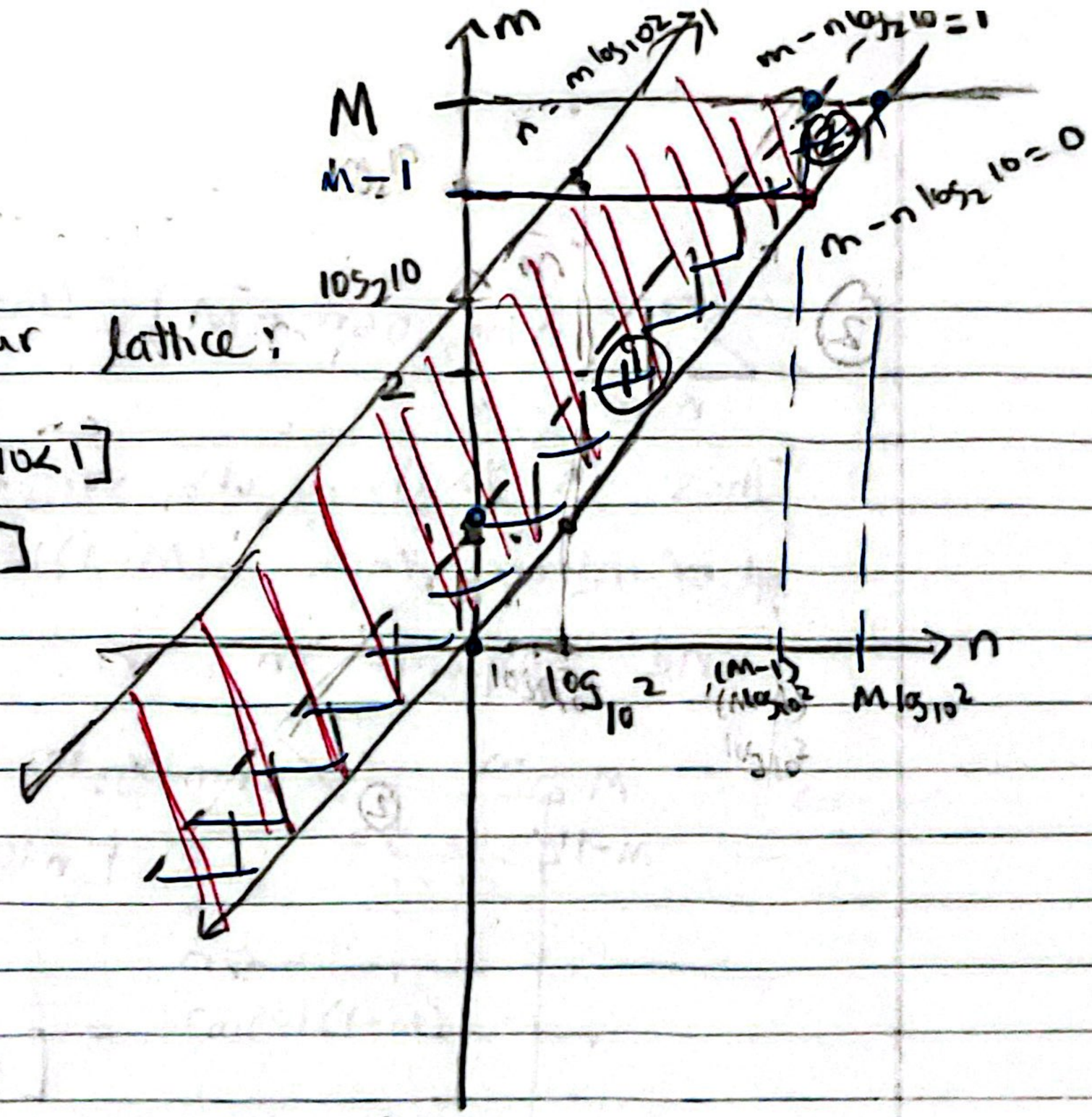
$$\Rightarrow \sum_{\substack{0 \leq m \leq M, \\ n}} \left[0 \leq m - \lfloor n \log_2 10 < 1 \right] \left[n = \lfloor m \log_{10} 2 \rfloor \right]$$

$$= \sum_{\substack{0 \leq m \leq M, \\ n}} \left[0 \leq m - n \log_2 10 < 1 \right] \left[m \log_{10} 2 - 1 < n \leq m \log_{10} 2 \right]$$

So we have a lattice over:

$$\left[0 \leq m \leq M \right] \left[0 \leq m - n \log_2 10 < 1 \right] \left[m \log_{10} 2 - 1 < n \leq m \log_{10} 2 \right]$$

Graphing our lattice:



$[0 \le m \le M][0 \le m - n\log_2 10 < 1]$

$[m\log_{10}2 - 1 < n \le m\log_{10}2]$

simplifying our lattice field:

① $[0 \le n \le (M-1)\log_{10}2][n\log_2 10 \le m < 1 + n\log_2 10]$

$+$ ② $[n\log_2 10 \le m \le M][(\log_{10}2)(M-1) < n \le M\log_{10}2]$

① : $\sum_{n,m} [0 \le n \le (\log_{10}2)(M-1)][n\log_2 10 \le m < 1 + n\log_2 10]$

$\rightarrow \sum_{\substack{n \\ 0 \le n \le (M-1)\log_{10}2}} \sum_{m} [n\log_2 10 \le m < 1 + n\log_2 10]$

$= \sum_{0 \le n \le (M-1)\log_{10}2} (\lceil 1 + n\log_2 10 \rceil \cancel{-1} - \lceil n\log_2 10 \rceil \cancel{+1})$

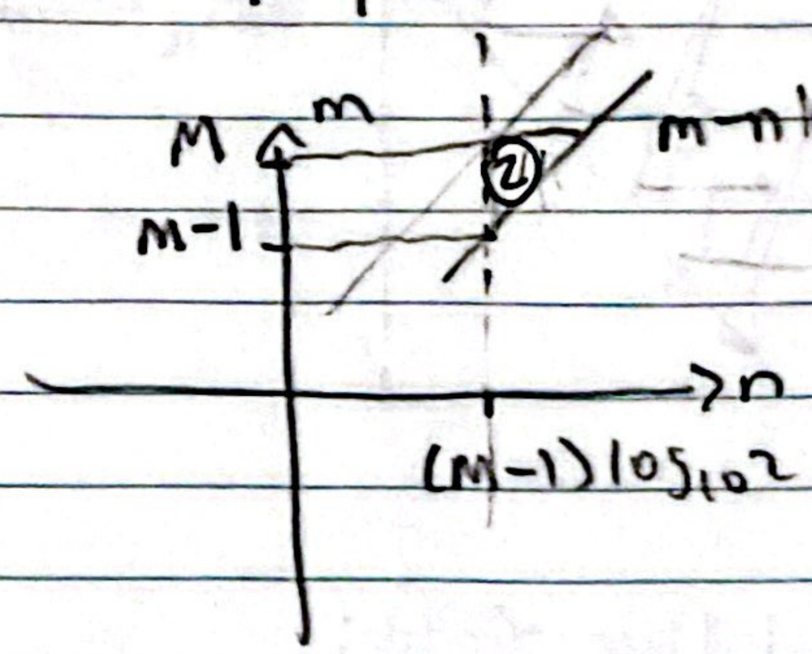
$= \sum_{0 \le n \le (M-1)\log_{10}2} (1 + \cancel{\lceil n\log_2 10 \rceil} - \cancel{\lceil n\log_2 10 \rceil})$

$= \lceil (M-1)\log_{10}2 \rceil \cancel{-1} - 0 \cancel{+1}$

② $\sum_{n,m} [n\log_2 10 \le m \le M][(M-1)\log_{10}2 < n \le M\log_{10}2]$.

This is tough... its very hard to count the # of integers from $(M-1)\log_{10}2 < n \le M\log_{10}2$. and pinpoint an n.

$\Rightarrow$



$m - n\log_2 10 = 0 \Rightarrow n = m\log_{10}2$

$[n\log_2 10 \le m \le M][(M-1)\log_{10}2 < n \le M\log_{10}2]$

$= [(M-1)\log_{10}2 < n \le m\log_{10}2]$
$[M-1 \le m \le M]$

$\Rightarrow$ ② $= \sum_{n,m} [(M-1)\log_{10}2 < n \le m\log_{10}2][M-1 \le m \le M]$

$= \sum_{M-1 \le m \le M} \lceil m\log_{10}2 \rceil \cancel{-1} - \lceil (M-1)\log_{10}2 \rceil \cancel{+1}$

$= \cancel{\lceil (M-1)\log_{10}2 \rceil} - \cancel{\lceil (M-1)\log_{10}2 \rceil}$
$+ \lceil M\log_{10}2 \rceil - \lceil (M-1)\log_{10}2 \rceil$

$= \underbrace{\lceil M\log_{10}2 \rceil - \lceil (M-1)\log_{10}2 \rceil}$

So ① + ② : $\cancel{\lceil (M-1)\log_{10}2 \rceil} + \lceil M\log_{10}2 \rceil - \cancel{\lceil (M-1)\log_{10}2 \rceil}$

$= \lceil M\log_{10}2 \rceil \rightarrow$ since $M \in \mathbb{Z}$, then $M\log_{10}2 \notin \mathbb{Z}$

and $= \boxed{\lfloor M\log_{10}2 \rfloor + 1}$

and we're done.

23. show nth element of:

$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, \ldots$

is $\lfloor \sqrt{2n} + \frac{1}{2} \rfloor$

notice: $\underbrace{1}_{1}, \underbrace{2, 2}_{2}, \underbrace{3, 3, 3}_{3}, \underbrace{4, 4, 4, 4}_{4}$

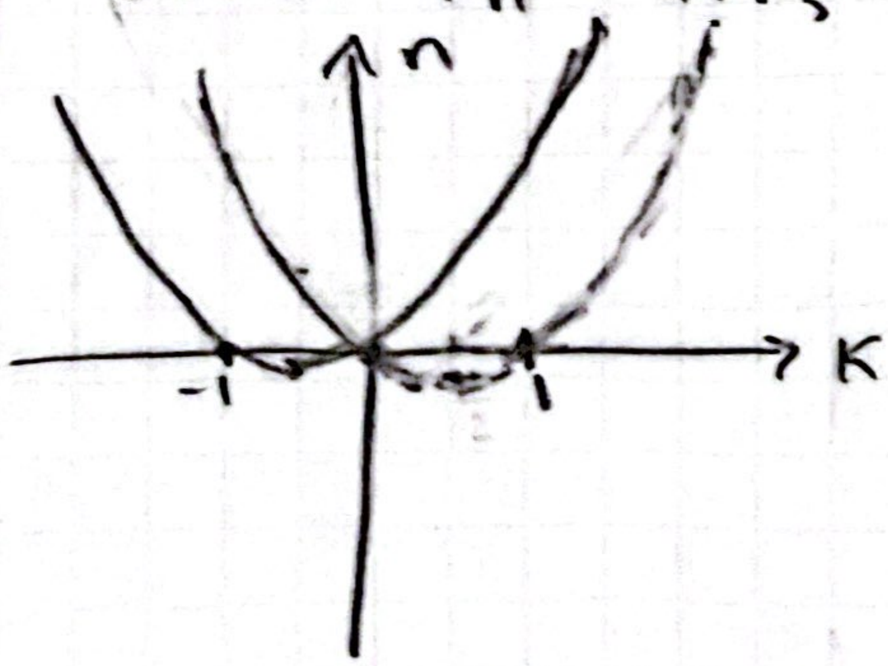
if I wanted, term number x_3: $x_{1+2} \Rightarrow 2$.

consider 4th term: x_4: $x_4 = x_{1+2+3} = x_6 \Rightarrow 3$

$x_n \Rightarrow$ say $1+2+3+4+\ldots+k-1 < n \leq 1+2+3+4+\ldots+k$

$x_1 = 1$

define: $x_n = k$, $1+2+3+4+\ldots+k-1 < n \leq 1+2+3+4+\ldots+k$.



$$\left(\frac{k}{2}\right)(k-1) < n \leq \frac{(k+1)k}{2}$$

$$\frac{k^2}{2} - \frac{k}{2} < n \leq \frac{k^2}{2} + \frac{k}{2}$$

$$\Rightarrow k^2 - k < 2n \leq k^2 + k$$

Now, we rewrite: $-\frac{1}{2} + \frac{\sqrt{1+8y}}{2} \leq k < \frac{1}{2} + \frac{\sqrt{1+8y}}{2}$

and $\left(0 < k < 1 \quad \frac{k^2-k}{1} < n \leq 0\right)$ which we don't consider as $k \in \mathbb{Z}$

Now: $x_1 = 1$

$x_n = k$, $-\frac{1}{2} + \frac{\sqrt{1+8n}}{2} \leq k < \frac{1}{2} + \frac{\sqrt{1+8n}}{2}$

$$\Rightarrow \left\lceil -\frac{1}{2} + \frac{\sqrt{1+8n}}{2} \right\rceil = k$$

So $\boxed{x_n = \left\lceil -\frac{1}{2} + \frac{\sqrt{1+8n}}{2} \right\rceil}$ (you can verify).

Now we must show $\left\lceil -\frac{1}{2} + \frac{\sqrt{1+8n}}{2} \right\rceil = \lfloor \sqrt{2n} + \frac{1}{2} \rfloor$

consider: $\left\lceil \frac{-1+\sqrt{1+8n}}{2} \right\rceil \overset{?}{=} \left\lfloor \sqrt{2n}+\frac{1}{2} \right\rfloor$

both appear to yield correct solutions.

$\Rightarrow$ assume equality holds:

$\frac{-1+\sqrt{1+8n}}{2}$ and $\sqrt{2n}+\frac{1}{2}$ can only differ at most by <1.

$\left\lceil \frac{-1+\sqrt{1+8n}}{2} \right\rceil \le \sqrt{2n}+\frac{1}{2} < \left\lceil \frac{-1+\sqrt{1+8n}}{2} \right\rceil + 1$

$\Rightarrow \quad 0 \le \sqrt{2n}+\frac{1}{2} - \left\lceil \frac{-1+\sqrt{1+8n}}{2} \right\rceil < 1$

$\Rightarrow \quad 0 \le \sqrt{2n}+\frac{1}{2} - \left\lceil -\frac{1}{2}+\sqrt{\frac{1}{4}+2n} \right\rceil < 1$

consider $\left\lceil -\frac{1}{2}+\sqrt{\frac{1}{4}+2n} \right\rceil$

$\Rightarrow$

$-\frac{1}{2}+\sqrt{\frac{1}{4}+2n} \le \left\lceil -\frac{1}{2}+\sqrt{\frac{1}{4}+2n} \right\rceil < \frac{1}{2}+\sqrt{\frac{1}{4}+2n}$

Now we verify extremes:

at minimum: $\left\lceil -\frac{1}{2}+\sqrt{\frac{1}{4}+2n} \right\rceil = -\frac{1}{2}+\sqrt{\frac{1}{4}+2n}$

and: $0 \le \sqrt{2n}+\frac{1}{2} - \left(-\frac{1}{2}+\sqrt{\frac{1}{4}+2n}\right) < 1$

$\rightarrow \quad 0 \le 1 + \underbrace{\sqrt{2n} - \sqrt{\frac{1}{4}+2n}}_{<0} < 1$ so This is true.

at max: $\left\lceil -\frac{1}{2}+\sqrt{\frac{1}{4}+2n} \right\rceil < \frac{1}{2}+\sqrt{\frac{1}{4}+2n} < n$

28. $a_0 = 1, \quad a_n = a_{n-1} + \lfloor \sqrt{a_{n-1}} \rfloor$

1 $a_0 = 1$

$a_1 = 1 + \lfloor \sqrt{1} \rfloor = 2$ 1

$a_2 = 2 + \lfloor \sqrt{2} \rfloor = 3$ 1

$a_3 = a_2 + \lfloor \sqrt{a_2} \rfloor = 3 + \lfloor \sqrt{3} \rfloor = 4$ 2^2

$a_4 = 4 + \lfloor \sqrt{4} \rfloor = 6$ 2

$a_5 = 6 + \lfloor \sqrt{6} \rfloor = 8$ 2

$a_6 = 8 + \lfloor \sqrt{8} \rfloor = 10$ 3

4 $a_7 = 10 + \lfloor \sqrt{10} \rfloor = 13$ 3

$a_8 = 13 + \lfloor \sqrt{13} \rfloor = 16$ 4

$a_9 = 16 + \lfloor \sqrt{16} \rfloor = 20$ 4

$a_{10} = 20 + \lfloor \sqrt{20} \rfloor = 24$ 4

$a_{11} = 24 + \lfloor \sqrt{24} \rfloor = 28$ 4

$a_{12} = 28 + \lfloor \sqrt{28} \rfloor = 33$ 5

$a_{13} = 33 + \lfloor \sqrt{33} \rfloor = 38$ 6

8 $a_{14} = 38 + \lfloor \sqrt{38} \rfloor = 44$ 6

$a_{15} = 44 + \lfloor \sqrt{44} \rfloor = 50$ 6

$a_{16} = 50 + \lfloor \sqrt{50} \rfloor = 57$ 7

$a_{17} = 57 + \lfloor \sqrt{57} \rfloor = 64$ 8

$a_{18} = 64 + \lfloor \sqrt{64} \rfloor = 72$

$a_{19} = 72 + \lfloor \sqrt{72} \rfloor = 80$

$a_{20} = 80 + \lfloor \sqrt{80} \rfloor = 88$

$a_{21} = 88 + \lfloor \sqrt{88} \rfloor = 97$

16 $a_{22} = 97 + \lfloor \sqrt{97} \rfloor = 106$

$a_{23} = 106 + \lfloor \sqrt{106} \rfloor = 116$

$a_{24} = 116 + \lfloor \sqrt{116} \rfloor = 126$

$a_{25} = 126 + \lfloor \sqrt{126} \rfloor = 137$

$a_{26} = 137 + \lfloor \sqrt{137} \rfloor = 148$

$a_{27} = 148 + \lfloor \sqrt{148} \rfloor = 160$

$a_{28} = 160 + \lfloor \sqrt{160} \rfloor = 172$

$a_{29} = 172 + \lfloor \sqrt{172} \rfloor = 185$

$a_{30} = 185 + \lfloor \sqrt{185} \rfloor = 198$

$a_{31} = 198 + \lfloor \sqrt{198} \rfloor = 212$

$a_{32} = 212 + \lfloor \sqrt{212} \rfloor = 226$

$a_{33} = 226 + 15 = 241$

$a_{34} = 241 + \lfloor \sqrt{241} \rfloor = 256$

16 $a_{35} = 256 + \lfloor \sqrt{256} \rfloor$

conclusions:

$a_1 = 1 + \lfloor \sqrt{1} \rfloor = 2$

3($a_4 = 6$

5($a_9 = 20$

9($a_{18} = 72$

17($a_{35} = 272$

Notice our special terms.

for an $a_n = m^2$, $a_{n+1} = m^2 + \lfloor \sqrt{m^2} \rfloor = m^2 + m$

we see this for $n=0$, $n=3$, $n=8$, $n=17$, $n=34$

assume $a_n = m^2$, $a_{n+1} = m^2 + m$

The question is then... for what k is $a_k = (2m)^2$ which is always to occur.

$\Rightarrow$ we see between a_0 and a_3 ($2+1$), a_3 and a_8 ($4+1$), a_8 and a_{17} ($8+1$), a_{17} and a_{34} ($16+1$)

$\Rightarrow$

$\Rightarrow$ Thus: a_{n+2k+1} describes it.

also

$$\left.\begin{aligned} a_0 &= 1 = 1^2 \\ a_3 &= 4 = 2^2 \\ a_8 &= 16 = 4^2 \\ a_{17} &= 64 = 8^2 \\ a_{34} &= 256 = 16^2 \end{aligned}\right\} \quad a_{n+2k+1} = (2k)^2$$

well: $a_n = m^2$, $a_{n+2m+1} = (2m)^2$

we see from k&

$$\boxed{a_{n-1} = 2^l + \left\lfloor \left(\frac{n-1}{2}\right)^2 \right\rfloor, \quad 2^l + 1 \le n \le 2^{l+1} + l + 1}$$

32 $\sum_k 2^k \| x/2^k \|^2$, $\|x\| = \min(x - \lfloor x \rfloor, \lceil x \rceil - x)$

let $f(x) = \sum_k 2^k \| x/2^k \|^2$

since $f(x) = f(-x) \Rightarrow$ assume $x \geq 0$.

$$f(2x) = \sum_k 2^k \|(2x)/2^k\|^2 = 2\sum_k 2^{k-1} \|x/2^{k-1}\|^2 = 2f(x)$$

$$f(x) = l(x) + r(x) \rightarrow l(x) = \sum_{k \leq 0} 2^k \|x/2^k\|^2,$$

$$r(x) = \sum_{k > 0} 2^k \|x/2^k\|^2$$

$l(x+1) = l(x) \Rightarrow \leq \frac{1}{2}$. when $0 \leq x < 1 \Rightarrow r(x) = \frac{x^2}{2} + \frac{x^2}{4} + \dots = x^2$

and $r(x+1) = \frac{(x-1)^2}{2} + \frac{(x+1)^2}{4} + \dots$

$= x^2 + 1$

$f(x+1) = f(x) + 1, \quad 0 \leq x < 1$

$\rightarrow$ $f(n) = n$ is easily proved.

$$f(x) = 2^{-m} f(2^m x) = 2^{-m} \lfloor 2^m x \rfloor + 2^{-m} f(\{2^m x\})$$

$$\Rightarrow f(\{2^m x\}) = l(\{2^m x\}) + r(\{2^m x\}) \leq \frac{1}{2} + 1$$

so $|f(x) - x| \leq |2^{-m} \lfloor 2^m x \rfloor - x| + 2^{-m} \frac{3}{2} \leq 2^{-m} \frac{5}{2}$

and $f(x) = |x|$.

34. Let $f(n) = \sum_{k=1}^{n} \lceil \lg k \rceil$

$$= \sum_{\substack{k=1 \\ m}}^{k=n} [m = \lceil \lg k \rceil] m = \sum_{k,m} [1 \le k \le n][m = \lceil \lg k \rceil] m$$

$$= \sum_{k,m} [1 \le k \le n][\lg k \le m < \lg(k) + 1] m$$

$m = \lg k$ $\quad$ $m = \lg(k) + 1$

$2^m = k$, $\quad$ $2^{m-1} = k$

$$\Rightarrow ① [1 \le k \le 2^m][0 \le m < 1]$$

$\cancel{\lg(0)}$

$$② + [1 \le m \le \lg(n)][2^{m-1} < k \le 2^m]$$

$$③ + [\lg(n) \le m \le \lg(n) + 1][2^{m-1} < k \le n]$$

$$\Rightarrow ① : \sum_{k,m} [1 \le k \le 2^m][0 \le m < 1] m = 0$$

$$③ \sum_{k,m} [1 \le m < \lg(n)][2^{m-1} < k \le 2^m] m$$

$$= \sum_{1 \le m < \lg(n)} (2^m - (2^{m-1} + 1) + 1) m = \sum_{1 \le m < \lg(n)} m 2^{m-1} (2-1)$$

$$= \sum_{1 \le m < \lg(n)} m 2^{m-1}. \quad \text{we know} \sum_{1 \le m \le N} m 2^{m-1}$$

as: $S_N = \sum_{1 \le m \le N} m 2^{m-1}$. $\quad S_{N+1} = S_N + (N+1) 2^N$

$$= \sum_{1 \le m \le N+1} m 2^{m-1} = 1 + \sum_{1 \le m \le N} (m+1) 2^m = 1 + 2 \sum_{1 \le m \le N} m 2^{m-1} + \sum_{1 \le m \le N} 2^m$$

So $S_N = \sum_{1 \le m \le N} m 2^{m-1} \Rightarrow S_N + (N+1) \cdot 2^N = 1 + 2S_N + \sum_{1 \le m \le N} 2^m$

$$\Rightarrow S_N = 2^N(N+1) - 1 - 2(2^N - 1)$$
$$= N2^N + 2^N - 1 - 2^{N+1} + 2 = N2^N + 2^N(1-2) + 1$$
$$= N2^N - 2^N + 1 = (N-1)2^N + 1$$

Now, ②: $\sum_{1 \le m \le \lg(n)} m 2^{m-1}$.

$$\circ \sum_{1 \le m \le \lceil \lg(n) \rceil - 1} m 2^{m-1} = \underline{(\lceil \lg(n) \rceil - 2) 2^{\lceil \lg(n) \rceil - 1} + 1}.$$

③ $\sum_{k,m} [\lg(n) \le m < \lg(n) + 1][2^{m-1} < k \le n] m$

$$= \sum [m = \lceil \lg(n) \rceil] (n - (2^{m-1} + 1) + 1) m$$
$$= \sum_{k,m} [m = \lceil \lg(n) \rceil] m (n - 2^{m-1})$$
$$= \lceil \lg(n) \rceil (n - 2^{\lceil \lg(n) \rceil - 1})$$

and ① + ② + ③ $= 0 + \lceil \lg(n) \rceil 2^{\lceil \lg(n) \rceil - 1} - 2^{\lceil \lg(n) \rceil} + 1$
$+ \lceil \lg(n) \rceil n - 2^{\lceil \lg(n) \rceil - 1} \lceil \lg(n) \rceil$

$$\Rightarrow \boxed{\lceil \lg(n) \rceil n - 2^{\lceil \lg(n) \rceil} + 1 = f(n)}$$

$$f(\lceil \tfrac{(n+1)}{2} \rceil) + f(\lfloor \tfrac{(n+1)}{2} \rfloor)$$
$$- f($$

b. prove $f(n) = n-1+f(\lceil n/2 \rceil) + f(\lfloor n/2 \rfloor)$ for all $n \geq 1$.

Note $\lceil n/2 \rceil = \lfloor (n+1)/2 \rfloor$.

Then: $$f(n) = n-1+\underbrace{f(\lfloor (n+1)/2 \rfloor) + f(\lfloor n/2 \rfloor)}_{\Delta f(\lfloor n/2 \rfloor)}$$

but $$\Delta f(n) = (n+1)-1-(n-1)+\Delta f(\lfloor n/2 \rfloor)$$
$$= 1+\Delta f(\lfloor n/2 \rfloor).$$

Why? $$\Delta f(n) = (n+1)-1+f(\lceil \tfrac{(n+1)}{2} \rceil) + f(\lfloor \tfrac{(n+1)}{2} \rfloor)$$
$$-\left((n-1)+f(\lceil n/2 \rceil)+f(\lfloor n/2 \rfloor)\right)$$
$$= 1+\underbrace{f(\lceil \tfrac{(n+1)}{2} \rceil)}_{\lfloor \frac{n+1+2-1}{2} \rfloor = \lfloor \frac{n}{2}+1 \rfloor} - f(\lfloor \tfrac{n}{2} \rfloor)$$

35. Simplify: $\lfloor (n+1)^2 n! e \rfloor \bmod n$

$$\cancel{\lfloor (n+1)^2 n! e \rfloor \bmod n = \lfloor (n+1)^2 n! e \rfloor - n \lfloor \frac{(n+1)^2 n! e}{n} \rfloor}$$

Recall $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$\Rightarrow \lfloor (n+1)^2 n! \left(\sum_{n=0}^{n-1} \frac{1}{k!}\right) + \frac{(n+1)^2 n!}{n!} + \frac{(n+1)^2 n!}{(n+1)!} + (n+1)^2 n! \left(\sum_{k=n+2}^{\infty} \frac{1}{k!}\right) \rfloor \bmod n$$

$$\Rightarrow \underbrace{\lfloor (n+1)^2 n! \left(\sum_{n=0}^{n-1} \frac{1}{k!}\right)}_{\text{divisible by } n} + \underbrace{\frac{(n+1)^2 n!}{n!} + \frac{(n+1)^2 n!}{(n+1)!}}_{(n+1)^2+(n+1) = n^2+3n+2 \Rightarrow 2 \bmod n.} + \underbrace{(n+1)^2 n! \sum_{k=n+2}^{\infty} \frac{1}{k!} \rfloor}_{S_5} \bmod n$$

S_1

$$S_3 = (n+1)^2 n! \sum_{k=n+2}^{\infty} \frac{1}{k!}$$

$$= (n+1)^2 n! \left(\frac{1}{(n+4)!} + \frac{1}{(n+5)!} + \dots \right)$$

$$= (n+1)(n+1)! \left(\frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \dots \right)$$

$$= (n+1) \left(\frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \frac{1}{(n+2)(n+3)(n+4)} + \dots \right)$$

$$< (n+1) \left(\frac{1}{(n+2)^2} + \frac{1}{n+2} + \frac{1}{(n+2)^3} + \dots \right)$$

$$= \frac{n+1}{n+2} \cdot \frac{1}{1 - \frac{1}{n+2}} = 1$$

Therefore $0 < S_3 < 1$.

Thus: $\lfloor S_1 + S_2 + S_3 \rfloor \bmod n$

$$\equiv \lfloor 0 \rfloor + \lfloor 2 + S_3 \rfloor \bmod n$$

$$\equiv \boxed{2 \bmod n}$$

36. $$\sum_{1 \le k \le 2^{2^n}} \frac{1}{2^{\lfloor \lg k \rfloor} 4^{\lfloor \lg \lg k \rfloor}}$$

$$\Rightarrow \sum_{1 \le k < 2^{2^n}} \frac{1}{2^l 4^m} [l = \lfloor \lg k \rfloor] [m = \lfloor \lg(l) \rfloor]$$

$\rightarrow$ bounds: $[l = \lfloor \lg k \rfloor][m = \lfloor \lg(l) \rfloor][1 < k < 2^{2^n}]$

$\rightarrow$ $[l \le \lg k \le l+1][m \le \lg l < m+1][1 \le k < 2^{2^n}]$

Think of bounds like almost a triple integral ... k, m, l dimensions.

what do we have?

$$\sum_{k,m,l} \frac{1}{2^l 4^m} [1 \le k < 2^{2^n}] [l = \lfloor \lg k \rfloor] [m = \lfloor \lg(l) \rfloor]$$

$$c_1 < k < c_2 \qquad f(k) < l < f(k) \qquad g(l) < m < g(l)$$

This is the analogous to:

$$\int_{c_1}^{c_2} \int_{l(k)}^{l(k)} \int_{m(l,k)}^{m(l,k)} f(l,m) \, dm \, dl \, dk$$

$\downarrow k \quad dl$

where do we even start... this is horrible.

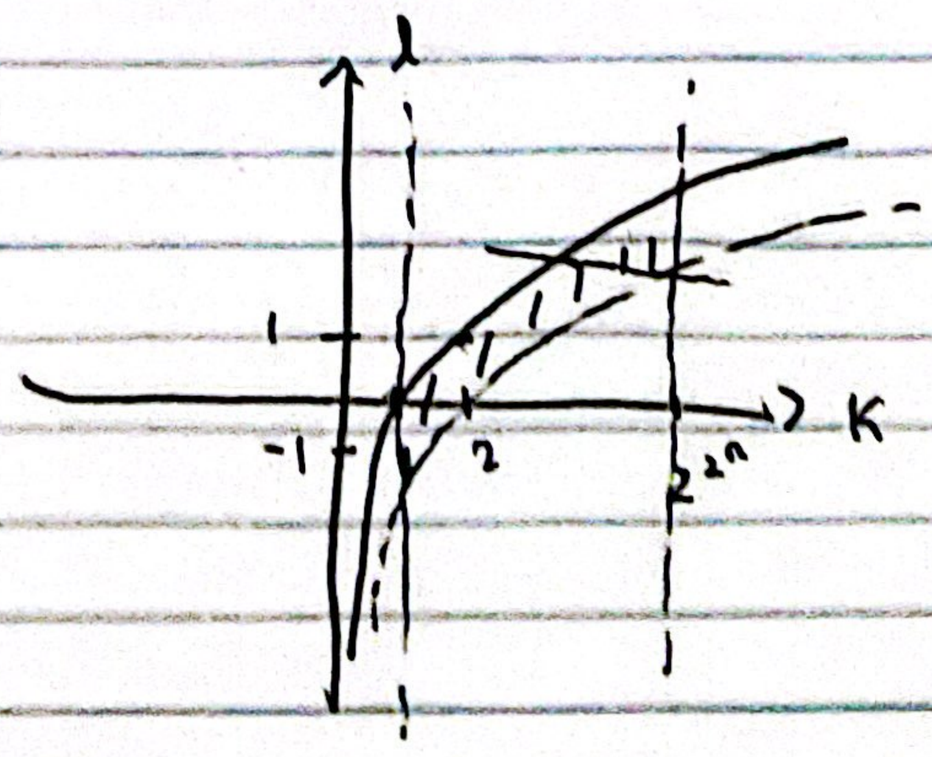
well... what's stopping us from evaluating this sum?

$\Rightarrow$ we HAVE to sum on m first, then l, then k.

$\Rightarrow$ objective: disbound and redefine/simplify the bound so that we can sum whichever way.

Now how? l is bounded by functions of k,
k is bounded by constant functions.
m is bounded by function of l.

we would love to sum on k first. Problem is, we can't. lets fix that: consider $[1 < k < 2^{2^n}][l = \lfloor \lg k \rfloor]$

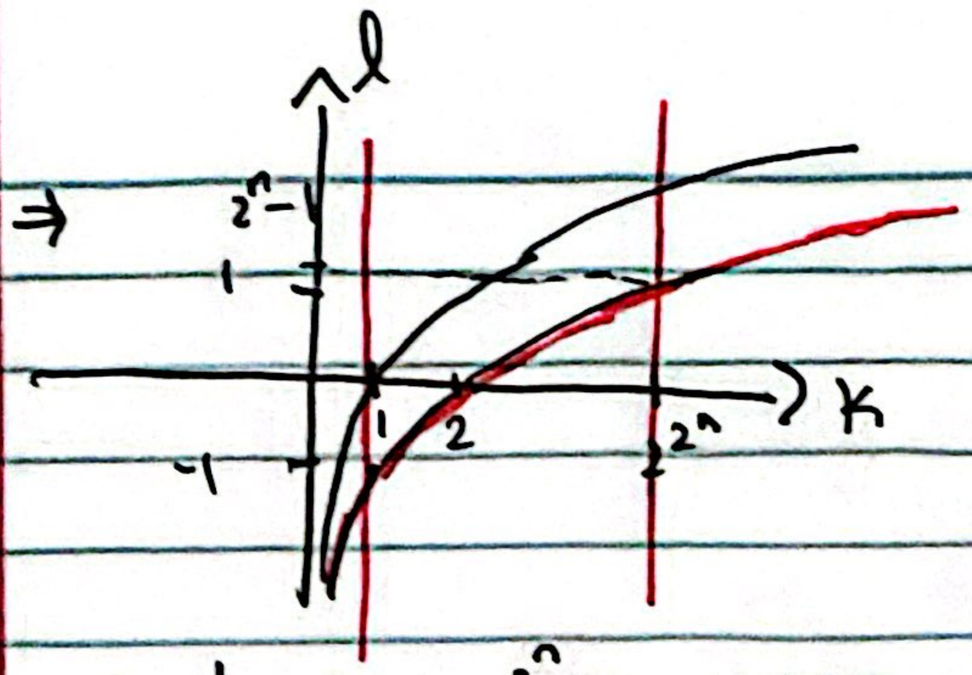


$\rightarrow [l \le \lg k < l+1][1 < k < 2^{2^n}]$

$\rightarrow [\lg(k) - 1 < l \le \lg(k)][1 < k < 2^{2^n}]$

There's a very interesting property with $\lfloor\rfloor$ and $\lceil\rceil$.

$\Rightarrow$



$$[1<k<2^{2^n}][l=\lfloor \lg k\rfloor]$$
$$=[\lg(k)-1<l\le \lg(k)][1<k<2^{2^n}]$$
$$=[2^l\le k<2^{l+1}][0<l<2^n-1] + \underbrace{[2^l\le k<2^{2^n}][2^n-1\le l\le 2^n]}$$

$$\sum_{k,m,l}\frac{1}{2^l 4^m}[1<k<2^{2^n}][l=\lfloor \lg k\rfloor][m=\lfloor \lg l\rfloor]$$

$$=\sum_{k,m,l}\frac{1}{2^l 4^m}[2^l\le k<2^{l+1}][0<l\le 2^n-1][m=\lfloor \lg l\rfloor]$$

notice $[2^n-1\le k<2^{2^n}][l=2^n-1] + \cancel{[2^n\le k<2^{2^n}][l=2^n]}$ 0

$$=[2^l\le k<2^{l+1}][l=2^n-1]$$

$$\Rightarrow \underbrace{[2^l\le k<2^{l+1}][0<l\le 2^{n-1}]}$$

So our transformation: $\displaystyle\sum_{k,m,l}\frac{1}{2^l 4^m}[2^l\le k<2^{l+1}][0<l\le 2^n-1][m=\lfloor \lg l\rfloor]$

$\Rightarrow$ we can sum on k now!

$$\Rightarrow \sum_{m,l}\frac{1}{2^l 4^m}\underbrace{(2^{l+1}-1-2^l+1)}_{[k\in[2^l,\dots,2^{l+1})]}[0<l\le 2^n-1][m=\lfloor \lg l\rfloor]$$

$$=\sum_{m,l}\frac{1}{2^l 4^m}\underbrace{(2^{l+1}-2^l)}_{2^l(2-1)}[0<l\le 2^n-1][m=\lfloor \lg l\rfloor]$$

$$(*)=\sum_{m,l}\frac{1}{4^m}[0<l\le 2^n-1][m=\lfloor \lg l\rfloor] \Rightarrow \text{solvable too!}$$

$$=\sum_{l}\frac{1}{4^{\lfloor \lg l\rfloor}}[0<l\le 2^n-1] = \dots$$ wait. how do we solve this? we can't.

go back to $(*)$

Need to manipulate.

So now: $\sum_{m,l} \frac{1}{4}m\,[0<l\le 2^n-1][m=\lfloor \lg l\rfloor]$

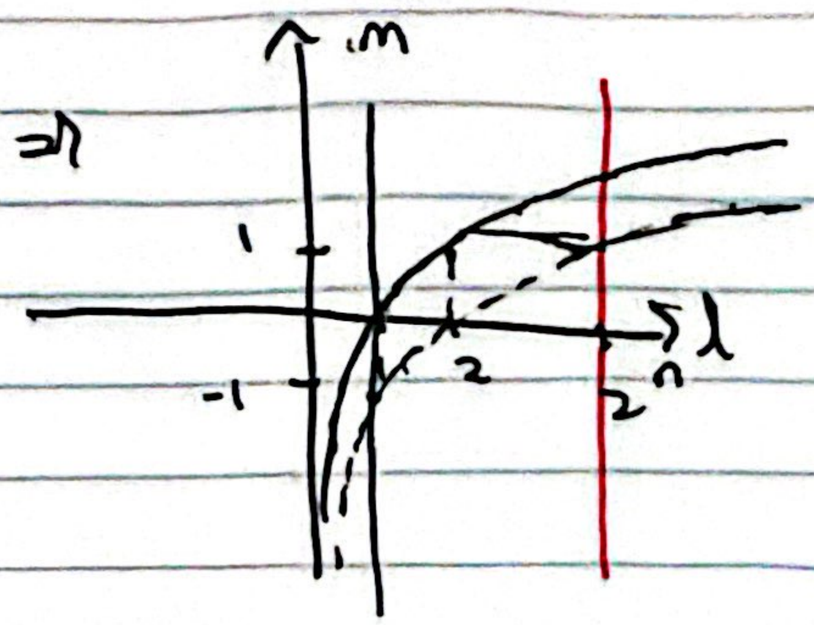
$\Downarrow$

$[\lg(l)-1<m\le \lg(l)]$

$\Downarrow$

$[2^m\le l\le 2^{m+1}]$

$\Rightarrow$



$\Rightarrow [0<l\le 2^n-1][\lg(l)-1<m\le \lg(l)]$

$\Rightarrow [1\le l<2^n][\lg(l)-1<m\le \lg(l)]$

$= [2^m\le l<2^{m+1}][0\le m<n-1]$

$+ \underbrace{[n-1\le m\le n][2^m\le l<2^n]}$

$[2^{n-1}\le l<2^n][m=n-1] =$

$+ \cancel{[2^n\le l<2^n][m=n]} \to 0$

$\Rightarrow [2^m\le l<2^{m+1}][0\le m<n-1]$

$+ [2^{n-1}\le l<2^n][m=n-1]$

$= \underline{[2^m\le l<2^{m+1}][0\le m\le n-1]}$

$$\sum_{m,l} \frac{1}{4}m\,[2^m\le l<2^{m+1}][0\le m\le n-1]$$

$$= \sum_m \frac{1}{4}m\,(2^{m+1}-1-2^m+1)[0\le m\le n-1]$$

$$= \sum_{0\le m\le n-1} \frac{2^m}{4m}$$

$$= \sum_{0\le m<n} \left(\frac{1}{2}\right)^m$$

Let $S_{n-1} = \sum_{0\le m\le n-1} \left(\frac{1}{2}\right)^m$, $S_n = S_{n-1} + \left(\frac{1}{2}\right)^n$

$$= \sum_{0\le m\le n} \left(\frac{1}{2}\right)^m = 1 + \sum_{1\le m\le n} \left(\frac{1}{2}\right)^m$$

$$= 1 + \sum_{0\le m\le n-1} \left(\frac{1}{2}\right)^{m+1} = 1 + \frac{1}{2}\sum_{0\le m\le n-1} \left(\frac{1}{2}\right)^m$$

$$\Rightarrow S_{n-1} + \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2}S_{n-1}$$

$$\frac{1}{2}S_{n-1} = 1 - \left(\frac{1}{2}\right)^n$$

$$\Rightarrow S_{n-1} = \underline{2\left(1-2^{-n}\right)}$$

So $\sum_{1\le K<2^{2n}} \frac{1}{2^{\lfloor \lg K\rfloor}\,4^{\lfloor \lg \lg K\rfloor}}$

$$= \sum_{0\le m<n} \left(\frac{1}{2}\right)^m$$

$$= S_{n-1} = \boxed{2\left(1-2^{-n}\right)}$$