Chapter 5 CM

1. $(11)^4 = (10+1)^4 = \binom{4}{0}(10)^4 + \binom{4}{1}(10)^3 + \binom{4}{2}(10)^2 + \binom{4}{3}(10) + \binom{4}{4}(10)^0$

$= 10^4 + 4\cdot 10^3 + 6\cdot 10^2 + 4\cdot 10 + 1$

$= \boxed{14641}$

$(11)^4_r = (14641)_r$ in any radix $r \geq 7$.

2. $\binom{n}{k}$ is maximized when $n \geq 0$?

$\frac{n!}{n!(n-k)!} = \frac{n^{\underline{k}}}{k!} = S_k$ we wish to maximize k.

notice: $\Delta S_k = \frac{n^{\underline{(k+1)}}}{(k+1)!} - \frac{n^{\underline{k}}}{k!}$

$\frac{n^{\underline{k}}}{k!}\left(\frac{n-k}{k+1} - 1\right)$

$= \frac{n^{\underline{k}}}{k!}\left(\frac{n-k-k-1}{k+1}\right)$

$= \frac{(n-2k-1)}{(k+1)!}\, n^{\underline{k}}$.

we have

$\frac{(n-2k-1)\, n(n-1)\ldots(n-k+1)}{(k+1)!} = 0$

first: $k = \frac{n-1}{2}$,

this requires n to be odd

$\boxed{k = \lfloor \frac{n}{2} \rfloor}$ $\boxed{k = \lceil \frac{n}{2} \rceil}$

3. $\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{(n-1)}{k}\binom{n+1}{k+1}\binom{n}{k-1}$

$$\Rightarrow \frac{(n-1)^{\underline{k-1}}(n)^{\underline{k+1}}(n+1)^{\underline{k}}}{\cancel{(k-1)!(k+1)!\,k!}} = \frac{(n-1)^{\underline{k}}(n+1)^{\underline{k+1}}(n)^{\underline{k-1}}}{\cancel{(k)!(k+1)!(k-1)!}}$$

$$\Rightarrow \frac{\cancel{(n-1)(n-2)\dots(n-1-k+1+1)}(n}{\cancel{n(n-1)\dots(n-k)}} = \frac{\cancel{(n-1)\dots(n-k)}}{\cancel{(n+1)(n)\dots(n-k+1)}}$$

$$\frac{\cancel{(n+1)(n)\dots(n-k+2)}}{\cancel{n(n-1)\dots(n-k+2)}}$$

✓.

4. $\binom{-1}{k}$. $\boxed{y = p}$

$$\Rightarrow (-1)^k\binom{-k+1-1}{k} = (-1)^k\binom{-k}{k}$$

$$= (-1)^k\frac{(1)}{k!} = \boxed{(-1)^k}\frac{(-k)(-k-1)\dots(-2k+1)}{(k)(k-1)\dots 1}$$

5. define p be prime:

Consider $\binom{p}{k} \bmod p = \frac{p^{\underline{k}}}{k!} \bmod p$.

we know $0 < k < p$. Therefore $k > 0$, $k < p$

$\Rightarrow p - k > 0$.

$$\Rightarrow \frac{p(p-1)\dots(p-k+1)}{k!} \bmod p.$$

The numerator is nonzero, $k!$ is always nonzero. Thus, it carries a factor of p

so $\frac{p(p-1)\dots(p-k+1)}{k!} \mid p \Rightarrow \boxed{\frac{p^{\underline{k}}}{k!} \bmod p = 0}$.

Now consider $\binom{p-1}{k} = \frac{(p-1)^{\underline{k}}}{k!} = \frac{(p-1)(p-2)\cdots(p-k)}{k!}$

$$\equiv \frac{(-1)(-2)\cdots(-k+1)(-k)}{k!} \bmod p$$

$$\equiv (-1)^k \frac{(1)(2)(3)\cdots(k)}{k!} \bmod p \equiv \boxed{(-1)^k \bmod p}$$

6. $\sum_{k\geq 0} \binom{n+k}{2k}\binom{2k}{k} \frac{(-1)^k}{k+1}$

$(n+k)^{\underline{k}} = (n)(n-1)\cdots(n+1)$

$(n)^{\underline{k}} = n(n-1)\cdots(n-k+1)$

$$= \frac{(n+k)^{\underline{2k}}}{(2k)!} \cdot \frac{(2k)^{\underline{k}}}{k!} = \frac{(n+k)^{\underline{2k}}}{k!\,k!} = \frac{(n+k)^{\underline{k}}(n)^{\underline{k}}}{k!\,k!}$$

$$\left((2k)^{\underline{2k}} = (2k)^{\underline{k}} \cdot (k)^{\underline{k}} \quad (2k)(2k-1)\cdots(k+1) \right)$$

$$\Rightarrow \binom{n+k}{k}\binom{n}{k}$$

$$\Rightarrow \sum_{k\geq 0} \binom{n+k}{k}\binom{n}{k} \frac{(-1)^k}{k+1}$$

$$\frac{1}{n}\binom{n}{k+1} = \frac{1}{k+1}\binom{n-1}{k}$$

$$\frac{1}{n}\binom{n}{k+1} = \frac{1}{k+1}\binom{n-1}{k}$$

$$\Rightarrow \frac{1}{n+1}\binom{n+1}{k+1} = \frac{1}{k+1}\binom{n}{k}$$

$$= \frac{1}{n+1}\sum_{k\geq 0} \binom{n+k}{k}\binom{n+1}{k+1}(-1)^k$$

$$= \frac{1}{n+1}\sum_{k\geq 0} \binom{n+k}{n}\binom{n+1}{k+1}(-1)^k$$

$$= -\frac{1}{n+1}\binom{n-1}{n}\binom{n+1}{0}(-1)^{-1} = [n=0].$$

OR: $\sum_{k\geq 0} \binom{n+k}{k}\binom{n}{k} \frac{(-1)^k}{k+1}$

$= \sum_k \binom{n+k}{n}\binom{n}{k} \frac{(-1)^k}{k+1}$ $\qquad \frac{(n+k)^{\underline{n}}}{n!}$

$x^{\underline{n}} = \sum_k \binom{n}{k}(-1)^k \left(\frac{\binom{n+k}{n}}{k+1}\right)$ $\qquad \cancel{n+k=n+1+1}$

$\frac{(n+k)(n+k-1)\ldots\cancel{(k+1)}}{\cancel{k+1}}$

$= (n+k)^{\underline{n-1}}$

which has first term

$0k^n + 1k^{n-1} + \ldots$

But we gotta be careful. When $n=0$

$\binom{n+k}{n} = \binom{0+k}{0}$

$= 1$

$\Rightarrow \frac{1}{k+1}$

$= (k)^{\underline{-1}}$

$\Rightarrow (-1)^n \Delta^n (f(x))|_{x=0}$

$= \sum_k \binom{n}{k}(-1)^k f(k)$

Let $f(k) = \frac{\binom{n+k}{n}}{k+1}$

$\Rightarrow f(x) = \frac{\binom{n+x}{n}}{x+1}$

$= \begin{cases} (n+x)^{\underline{n-1}} & \text{if } n\neq 0 \\ \frac{1}{x+1} = (x)^{\underline{-1}} & \text{if } n=0 \end{cases}$

$\Delta f(x) = \begin{cases} (n-1)(n+x)^{\underline{n-2}} & \text{if } n\neq 0 \\ \frac{1}{x+2} - \frac{1}{x+1} = \frac{(-1)}{(x+1)(x+2)} & \text{if } n\neq 0 \end{cases}$

$\Rightarrow \Delta^n = \begin{cases} \frac{(n-1)^{\underline{n}}(n+x)^{\underline{n-1-n}}}{} & \text{if } n\neq 0 \\ \frac{(-1)^n}{(x+n+1)^{\underline{n+1}}} & \text{if } n=0 \end{cases}$

$\Rightarrow \Delta^n f(x) = \begin{cases} 0 & \text{if } n\neq 0 \\ \frac{(-1)^0}{(x+1)^1} & \text{if } n=0 \end{cases}$ $\Rightarrow \Delta^n f(0) = \begin{cases} 0 & \text{if } n\neq 0 \\ \frac{(-1)^0}{(0+1)^1} & \text{if } n=0 \end{cases}$

$\Rightarrow [n=0]$

7. $r^{\underline{k}}\left(r-\frac{1}{2}\right)^{\underline{k}} = (2r)^{\underline{2k}}/2^{2k}$, $k \geq 0$ (5.34

want to know $k<0$. Assume $k \geq 0$

Let $m = -k$

$\Rightarrow r^{\underline{-k}} = \frac{(-1)^k}{(-r-1)^{\underline{k}}}$ holds true

8. $\sum_k \binom{n}{k}(-1)^k \underbrace{\left(1-\frac{k}{n}\right)^n}_{\text{first } \binom{n}{0}\left(\frac{-k}{n}\right)^n (1)^0 = (-1)^n \frac{k^n}{n^n}}$

$= \sum_k \binom{n}{k}(-1)^k \left((-1)^n \frac{k^n}{n^n} + \dots\right)$

$\Rightarrow (-1)^n \cdot \frac{1}{n^n} = a_n$.

Then $\exists\, c_n = \frac{n!}{n^n}(-1)^n$.

$\Rightarrow$ The sum is: $(-1)^n(-1)^n \frac{n!}{n^n} = \boxed{\frac{n!}{n^n}}$

when n gets large

9. $\mathcal{E}_t(z) = \mathcal{E}(tz)^{1/t}$

$\Rightarrow \sum_{k\geq 0}(tk+1)^{k-1}\cdot\frac{z^k}{k!} = \left(\sum_{k\geq 0}(tk+1)^{k-1}\frac{(tz)^k}{k!}\right)^{1/t}$

10. we wish to show $\frac{-2(\ln(1-z)+z)}{z^2}$ is hypergeometric.

$$\Rightarrow \ln(1-z) = \sum_{k\geq 1} \frac{-z^k}{k} = -z - \frac{z^2}{2} - \frac{z^3}{3} - \dots$$

So $\ln(1-z)+z = -\frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} - \dots$

and dividing: $-\frac{1}{2} - \frac{z}{3} - \frac{z^2}{4} - \dots$

Thus: $$\frac{\ln(1-z)+z}{z^2} = \sum_{k\geq 0} \frac{-z^k}{k+2}$$

and: $$-2\left(\frac{\ln(1-z)+z}{z^2}\right) = \sum_{k\geq 0} \frac{2z^k}{k+2}.$$

Let $t_k = \frac{2z^k}{k+2}$.

Then $$\frac{t_{k+1}}{t_k} = \frac{z(k+2)}{(k+3)} = \frac{z(k+2)(k+1)}{(k+3)(k+1)}$$

$$= \boxed{F(2,1;3;z)}$$ yes.

11. $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$

$$= \sum_{k \geq 0} \frac{(z^{2k+1})(-1)^k}{(2k+1)!}$$

$$t_k = \frac{(z^{2k+1})(-1)^k}{(2k+1)!}$$

$$\Rightarrow \frac{t_{k+1}}{t_k} = \frac{z^{2k+3}(-1)^{k+1}}{(2k+3)!} \cdot \frac{(2k+1)!}{z^{2k+1}(-1)^k} = \frac{(-1)z^2}{(2k+3)(2k+2)}$$

$$= \frac{(-1)z^2 \cdot \frac{1}{2 \cdot 2}}{(k+\frac{3}{2})(k+1)} = \frac{-\frac{1}{4}z^2}{(k+\frac{3}{2})(k+1)}$$

$$= \frac{-4z^2(k+1)}{(k+\frac{3}{2})(k+1)(k+1)}$$

$$\Rightarrow t_0 \cdot F\left(1, \frac{3}{2}, 1; -z^2 \cdot \frac{1}{4}\right) = \sum_{k \geq 0} t_k$$

$$\Rightarrow \boxed{z F\left(1; \frac{3}{2}, 1; \frac{-z^2}{4}\right) = \sin(z)}$$

$$\arcsin z = z + \frac{1 \cdot z^3}{2 \cdot 3} + \frac{1 \cdot 3 z^5}{2 \cdot 4 \cdot 5} + \cdots$$

$$= \sum_{k \geq 0} \frac{z^{2k+1}\left(\frac{1}{2}\right)^k}{(2k+1)k!} = z F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2\right).$$

yes.

12. a. $\frac{t_{k+1}}{t_k} = \frac{n^{k+1}}{n^k} = n$.

b. k^n: $\frac{(k+1)^n}{k^n} = \left(\frac{k+1}{k}\right)^n = \left(1+\frac{1}{k}\right)^n \Rightarrow$ yes.

c. yes, ratio $= \frac{(k+1)(k+3)}{k+2}$

d. $H_k \Rightarrow \frac{H_{k+1}}{H_k} = \frac{\frac{1}{1}+\frac{1}{2}+\dots+\frac{1}{k}+\frac{1}{k+1}}{\frac{1}{1}+\frac{1}{2}+\dots+\frac{1}{k}}$

$= 1 + \frac{\frac{1}{k+1}}{1+\frac{1}{2}+\dots+\frac{1}{k}} = 1 + \frac{H_k}{k+1}$

since H_k isn't a rational function $\Rightarrow$ no.

e. $\frac{1}{\binom{n}{k}} = \frac{k!}{n^{\underline{k}}} \Rightarrow \frac{(k+1)!}{n^{\underline{k+1}}} \cdot \frac{n^{\underline{k}}}{k!}$

$= \frac{k+1}{n-k+1} \Rightarrow$ yes.

(f) yes.

(g) ~~yes~~ not when $t(k)=2^k$ and $T(k)=1$.

(h) yes.

(i) yes.

(j) No

(k) No

13. $P_n = \prod_{k=1}^{n} k!$, $Q_n = \prod_{k=1}^{n} k^k$

$R_n = \prod_{k=0}^{n} \binom{n}{k}$.

$\Rightarrow R_n = \prod_{0 \le k \le n} \binom{n}{k} = \binom{n}{0}\binom{n}{1}\binom{n}{2} \cdots \binom{n}{n}$.

$= \dfrac{(n^0)(n^1)(n^2) \cdots (n^n)}{0!\,1!\,2!\,3!\,4! \cdots n!}$

$[(1)][(n)]((n)(n-1))(n(n-1)(n-2)) \cdots (n(n-1) \cdots 1)$

n appears n times.

n-1 appears n-1 times

n-2 appear n-2 times

$\Downarrow$

$Q_n = \prod_{1 \le k \le n} k^k$.

$\Rightarrow \boxed{R_n = \dfrac{Q_n}{P_n}}$

$R_n = \prod_{0 \le k \le n} \dfrac{n!}{k!\,(n-k)!} \Longrightarrow \dfrac{(n!)^{n+1}}{P_n\, P_{n-k}} = \boxed{\dfrac{(n!)^{n+1}}{P_n^2}}$

$\Downarrow$

$\dfrac{Q_n^2}{(n!)^{n+1}}$

15. what is $\sum_k \binom{n}{k}^3 (-1)^k$

$$\binom{n}{k}^2 = \frac{n!^2}{(k)!^2 (n-k)!^2}$$

if

n odd, sum = zero.

if $n=2m$,

use 5.29: $(-1)^m (3m)!$

16. $\sum_k \binom{2a}{a+k}\binom{2b}{b+k}\binom{2c}{c+k}(-1)^k$

$$\frac{(2a)!}{(a+k)!(2a-(a+k))!}$$

$a-k$

$$\frac{t_{k+1}}{t_k} = \frac{(a+k)!(a-k)!(b+k)!(b-k)!(c+k)!(c-k)!}{(a+k+1)!(a-k-1)!(b+k+1)!(b-k-1)!(c+k+1)!(c-k-1)!}$$

$$= \frac{(a-k)(b-k)(c-k)}{(a+k+1)(b+k+1)(c+k+1)}(-1)$$

$$\binom{2a}{a}$$

$$= \frac{(k-a)(k-b)(k-c)\cancel{(-1)^k}}{(a+k+1)(b+k+1)(c+k+1)}(k+1)$$

$$\binom{2a}{a}\binom{2b}{b}\binom{2c}{c} F\left(\begin{matrix}-a, -b, -c, 1\\ a+1, b+1, c+1\end{matrix}\middle| 1\right)$$

20. $G\left(\begin{matrix} a_1, \dots, a_m \\ b_1, \dots, b_n \end{matrix} \middle| z\right)$

$$= \sum_{k \geq 0} \frac{a_1^{\underline{k}} \cdots a_m^{\underline{k}}}{b_1^{\underline{k}} \cdots b_n^{\underline{k}}} \frac{z^k}{k!}$$

$$a_1^{\underline{k}} = a(a-1)\dots(a-k+1)$$

$$= (a_1 - k + 1)^{\overline{k}}$$

$$= (-a_1)^{\overline{k}} (-1)^k$$

$$\hookrightarrow (-a_1)(-a_1+1)\dots(-a_1+k-1)(-1)^k$$

$$= a_1(a_1-1)\dots(a_1-k+1)$$

$$\Rightarrow \boxed{F\left(\begin{matrix} -a_1, \dots, -a_m \\ -b_1, \dots, -b_n \end{matrix} ; (-1)^{m+n} z\right)}$$

21. $$\frac{1}{z!} = \lim_{n \to \infty} \binom{n+z}{n} n^{-z}$$

assume $z = m$ is positive int.

$$\Rightarrow \lim_{n \to \infty} \frac{(n+m)^{\underline{n}}}{n!} n^{-m} = \lim_{n \to \infty} \frac{(n+m)(n+m-1)\dots(m+1)}{n!} \cdot n^{-m}$$

$\Rightarrow \lim_{n \to \infty} \binom{n+z}{n} n^{-z}$, if m is positive integer, we can

$$\lim \binom{n+m}{m} n^{-m} = \frac{(n+m)^{\underline{m}}}{m!} \cdot n^{-m}$$

22. Attempt 2:

$$x!\,(x-\tfrac{1}{2})! = (2x)!\,(-\tfrac{1}{2})!\,/\,2^{2x}.$$

Consider: $$2^{2x} = \frac{(-\frac{1}{2})!\,(2x)!}{x!\,(x-\frac{1}{2})!}$$

$$\Rightarrow \frac{(-\frac{1}{2})!}{x!\,(x-\frac{1}{2})!} = \lim_{n\to\infty} \frac{\binom{n+x}{n} n^{-x} \binom{n+n-\frac{1}{2}}{n} n^{-x+\frac{1}{2}}}{\binom{n-\frac{1}{2}}{n} n^{\frac{1}{2}}}$$

$$= \lim_{n\to\infty} \binom{2n+2x}{2n} n^{-2x}.$$

Now $$\frac{1}{(2x)!} = \lim_{n\to\infty} \binom{n+2x}{n} (n)^{-2x}$$

$$\Rightarrow \lim_{2n\to\infty} \binom{2n+2x}{2n} (2n)^{-2x}$$

$$\Rightarrow \text{thus } \frac{(-\frac{1}{2})!\,(2x)!}{x!\,(x-\frac{1}{2})!}$$

$$= \lim_{n\to\infty} \frac{\cancel{\binom{2n+2x}{2n}}\, \cancel{n^{-2x}}}{\cancel{\binom{2n+2x}{2n}} (2n)^{-2x}}$$

$$= \quad = \boxed{2^{2x}}.$$

23. we wish to evaluate $F(-n, 1; ; 1)$

$$= F(-n, 1, 1; 1; 1)$$

$$= \sum_k \frac{(-n)^{\overline{k}} \cancel{(1)^{\overline{k}}} \cancel{(1^{\overline{k}})}}{\cancel{(1)^{\overline{k}}}} \cdot \cancel{\frac{1}{k!}}$$

$$= \sum_k \frac{(-n)^{\overline{k}}}{1} = \sum_k (-n)^{\overline{k}}$$

$$\Rightarrow \cancel{\Delta (k)^{\overline{n}}} =$$

$$= \sum_k (-n)(-n+1)(-n-2)\dots(-n+k-1)$$

$$= \sum_k n(n-1)(n-2)\dots(n-k+1)(-1)^k$$

$$= \sum n^{\underline{k}} (-1)^k$$

$$= \sum_k \binom{n}{k} (-1)^k (k!) = \boxed{(-1)^n n!}$$

$k(k-1)(k-2)\dots 1$

$$\Delta^n (k!)\big|_{k=0} = (k+1)! - k! =$$

24. Find $\sum_k \binom{n}{m+k}\binom{m+k}{2k} 4^k$.

let $t_k = \binom{n}{m+k}\binom{m+k}{2k} 4^k$.

Then: $$\frac{t_{k+1}}{t_k} = \frac{\binom{n}{m+k+1}\binom{m+k+1}{2k+2} 4^{k+1}}{\binom{n}{m+k}\binom{m+k}{2k} 4^k}$$

$$= \frac{\cancel{(m+k+1)!}\,\cancel{(m+k)!}\,\cancel{(n-m-k)!}\,\cancel{(2k)!}\,(m-k)!}{\cancel{(m+k+1)!}\,\cancel{(n-m-k-1)!}\,\cancel{(m+k)!}\,(2k+2)!\,(m-k-1)!}$$

$$= \frac{(n-m-k)\,(m-k)\,4 \cdot \frac{1}{4}}{(2k+2)(2k+1) \cdot \frac{1}{4}}$$

$$= \frac{(k-m)(k-n+m)\,\cancel{4 \cdot \frac{1}{4}}}{(k+1)(k+\frac{1}{2})}$$

$$= \binom{n}{m} F\left(\begin{matrix} -m, -n+m \\ \frac{1}{2} \end{matrix} \middle| 1\right)$$

$$= \binom{n}{m} \frac{\Gamma(\frac{1}{2}+m+n-m)}{\Gamma(\frac{1}{2}+m)} \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+n-m)}$$

$$\downarrow \qquad\qquad \downarrow$$

$$(\tfrac{1}{2}+m)^{\overline{n-m}} \qquad \frac{1}{(\frac{1}{2})^{\overline{n-m}}}$$

$$= \binom{n}{m} \frac{(\frac{1}{2}+m)^{\overline{n-m}}}{(\frac{1}{2})^{\overline{n-m}}} = \binom{n}{m} \frac{(n-\frac{1}{2})^{\underline{n-m}}}{(m-n-\frac{1}{2})^{\underline{n-m}}}$$

$$\binom{n}{m} \frac{(n-\frac{1}{2})^{\underline{n-m}}}{(n-m-\frac{1}{2})^{\underline{n-m}}}$$

recall the identity

$$r^{\underline{k}}(r-\tfrac{1}{2})^{\underline{k}} = \frac{(2r)^{\underline{2k}}}{2^{2k}}$$

$$\Rightarrow \binom{n}{m} \frac{(2n)^{\underline{2n-2m}} \cancel{(2n-2m)^{\underline{2n-2m}}}}{(n)^{\underline{n-m}}\,(2n-2m)^{\underline{2n-2m}}} \quad \frac{(n-m)^{\underline{n-m}}(n-\frac{1}{2})^{\underline{n-m}}}{(n-m-\frac{1}{2})^{\underline{n-m}}} = \frac{(2n)^{\underline{2(n-m)}}}{2^{2(n-m)}(n)^{\underline{n-m}}} = \frac{(2(n-m))^{\underline{2(n-m)}}}{2^{2(n-m)}(n-m)^{\underline{n-m}}}$$

$$\binom{n}{m} \frac{(2n)(2n-1)\dots(2m+1)\cdot(n-m)\dots 1}{(n)(n-1)\dots(m+1)\cdot(2n-2m)\dots(1)}$$

$n^{\underline{m}} = n(n-1)\dots(n-m+1)$, combines with second term yields $n!$

$m!$ lifts with first term on denom.

yields: $n!$

$$\Rightarrow \binom{n}{m} \frac{(2n)(2n-1)\dots(2m+1)\cdot(n-m)\dots 1}{n(n-1)\dots(m+1)\cdot(2n-2m)\dots(1)}$$

$$\Rightarrow \frac{(2n)^{\underline{2n-2m}} \cdot \cancel{n^{\underline{m}}(n-m)\dots 1}}{\cancel{n(n-1)\dots(m+1)}\,m! \cdot (2n-2m)\dots 1}$$

$$= \frac{(2n)^{\underline{2n-2m}}}{(2n-2m)!} = \binom{2n}{2n-2m} = \boxed{\binom{2n}{2m}}$$

26, $F\left(\begin{matrix} a_1, \dots, a_m \\ b_1, \dots, b_n \end{matrix} \middle| z\right) = 1 + G(z)$

$$= \sum_k \frac{a_1^{\overline{k}} \cdots a_m^{\overline{k}}}{b_1^{\overline{k}} \cdots b_n^{\overline{k}}} \cdot \frac{z^k}{k!}$$

$$= 1 + \frac{a_1 \cdots a_m}{b_1 \cdots b_n} z F\left(\begin{matrix} a_1+1, \dots, a_m+1, 1 \\ b_1+1, \dots, b_n+1, 2 \end{matrix} \middle| z\right)$$

28. Prove $F(a, b \mid z) = (1-z)^{c-a-b} F(c-a, c-b \mid z)$

$$\frac{1}{(1-z)^a} F\left(a, b \mid \frac{-z}{1-z}\right) = F(a, c-b \mid z)$$

let $c-b = b$
$\Rightarrow b$

$$\frac{1}{(1-z)^a} F\left(a, b \mid \frac{-z}{1-z}\right) = F(a, c-b \mid z).$$

$$\Rightarrow \frac{1}{(1-z)^a} F\left(a, c-b \mid \frac{-z}{1-z}\right) = F(a, b \mid z).$$

So: $F(a, b \mid z) = \frac{1}{(1-z)^a} F\left(a, c-b \mid \frac{-z}{1-z}\right)$

$$\Rightarrow \frac{1}{(1-z)^a} F\left(\begin{matrix} c-b, & a \\ & c \end{matrix} \middle| \frac{-z}{1-z}\right)$$

$$= \boxed{(1-z)^{c-a-b} F\left(\begin{matrix} c-a, & c-b \\ & c \end{matrix} \middle| z\right)}$$

29. confluent hypergeometrics : $e^z F\left(\begin{matrix} a \\ b \end{matrix} \middle| -z\right) = F\left(\begin{matrix} b-a \\ b \end{matrix} \middle| z\right)$.

$$\rightarrow e^z \sum_k \frac{a^{\overline{k}}}{b^{\overline{k}}} \frac{(-z)^k}{k!} = \sum_{k=0}^{n} \frac{(b-a)^{\overline{k}}}{b^{\overline{k}}} \frac{z^k}{k!}$$

$$\downarrow$$

$$\sum_k \frac{z^k}{k!}$$

$$\Rightarrow \sum_k \frac{z^k}{k!} \sum_k \frac{a^{\overline{k}}}{b^{\overline{k}}} \frac{(-z)^k}{k!}$$

$$= \sum_{k=0} \left(\sum_{k=0}^{n} \frac{1}{k!} \cdot \frac{(-1)^{n-k}}{(n-k)!} \cdot \frac{a^{\overline{n-k}}}{b^{\overline{n-k}}} \right)$$

coefficients of z^n

$$\Rightarrow \text{LHS}: [z^n] \frac{(b-a)^{\overline{n}}}{b^{\overline{n}}} \cdot \frac{1}{n!}$$

$$\text{RHS}: [z^n] \sum_k \frac{z^k}{k!} \sum_k \frac{a^{\overline{k}}}{b^{\overline{k}}} \frac{(-z)^k}{k!}$$

$$= \sum_{k=0}^{n} \frac{1}{k!} \cdot \frac{a^{\overline{n-k}}}{b^{\overline{n-k}}} \cdot \frac{(-1)^{n-k}}{(n-k)!}$$

29. $e^z F\left(\begin{matrix} a \\ b \end{matrix} \middle| -z\right) = F\left(\begin{matrix} b-a \\ b \end{matrix} \middle| z\right)$

$$\Rightarrow \sum_{k\geq 0} \frac{1}{k!} \cdot z^k \, F\left(\begin{matrix} a \\ b \end{matrix} \middle| -z\right) = F\left(\begin{matrix} b-a \\ b \end{matrix} \middle| z\right)$$

$$\Rightarrow \sum_{k\geq 0} \frac{1}{k!} z^k \sum_{k\geq 0} \frac{a^{\overline{k}}}{b^{\overline{k}}} \frac{(-z)^k}{k!} = \sum_{k\geq 0} \frac{(b-a)^{\overline{k}}}{b^{\overline{k}}} \cdot \frac{z^k}{k!}$$

looking at z^n coefficients:

$$[z^n] \sum_{k\geq 0} \frac{1}{k!} z^k \sum_{k\geq 0} \frac{a^{\overline{k}}}{b^{\overline{k}}} \frac{(-z)^k}{k!}$$

$$= \sum_{k=0}^{n} \frac{1}{(n-k)!} \cdot \frac{(-1)^k}{k!} \cdot \frac{a^{\overline{k}}}{b^{\overline{k}}}$$

$$= \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!\,k!} \frac{a^{\overline{k}}}{b^{\overline{k}}} \Rightarrow \text{needs to equal } [z^n] \sum_{k\geq 0} \frac{(b-a)^{\overline{k}}}{b^{\overline{k}}} \cdot \frac{z^k}{k!}$$

$\Rightarrow$ consider $\displaystyle\sum_{k=0}^{n} \frac{(-1)^k}{k!(n-k)!} \frac{a^{\overline{k}}}{b^{\overline{k}}} = \frac{(b-a)^{\overline{n}}}{b^{\overline{n}}} \cdot \frac{1}{n!}$

$$\Rightarrow \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{a^{\overline{k}}}{b^{\overline{k}}} = \frac{(b-a)^{\overline{n}}}{b^{\overline{n}}}.$$

recall vandermonde's convolution

$$F\left(\begin{matrix} a, -n \\ b \end{matrix} \middle| 1\right) = \sum_{k\geq 0} \frac{(a^{\overline{k}})}{b^{\overline{k}}} \cdot (-n)^{\overline{k}} \cdot \frac{1^k}{k!}$$

$$= \sum_{k\geq 0} \frac{a^{\overline{k}}}{b^{\overline{k}}} \cdot (-n)(-n+1)(-n+2)\cdots(-n+k-1) \cdot \frac{1}{k!}$$

$$= \sum_{k\geq 0} \frac{a^{\overline{k}}}{b^{\overline{k}}} (-1)^k \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$$

Thus: $F\left(\begin{matrix} a, -n \\ b \end{matrix} \middle| 1\right) = \sum \frac{a^{\overline{k}}}{b^{\overline{k}}} (-1)^k \binom{n}{k}$

... which is exactly what we have!

This is $\frac{(b-a)^{\overline{n}}}{b^{\overline{n}}}$.

So $\boxed{e^z F\left(\begin{matrix} a \\ b \end{matrix} \middle| -z\right) = F\left(\begin{matrix} b-a \\ b \end{matrix} \middle| z\right)}$

30. $zF'(z) + F(z) = \frac{1}{1-z}$

$$F(z) = \sum_{k \geq 0} \frac{a_1^{\overline{k}} a_2^{\overline{k}} \cdots a_m^{\overline{k}}}{b_1^{\overline{k}} \cdots b_n^{\overline{k}}} \cdot \frac{z^k}{k!}$$

$$\cancel{zF(z) = \sum_{k \geq 0} (\cdots) \frac{z^{k+1}}{k!}}$$

$$\Rightarrow F'(z) = \sum_{k \geq 0} (\cdots) \frac{k z^{k-1}}{k!}$$

$k = k+1$

$$= \sum_{k \geq 0} (\cdots) \frac{z^{k-1}}{(k-1)!}$$

$$\Rightarrow zF'(z) = \sum_{k \geq 0} (\cdots) \frac{z^k}{(k-1)!}$$

$$\Rightarrow zF'(z) + F(z) = \sum_{k \geq 0} \left[(\cdots) \frac{z^k}{(k-1)!} + (\cdots) \cdot \frac{z^k}{k!} \right]$$

$$zF'(z)+F(z) = \sum_{k\geq 0} (\dots)\left(\frac{z^k}{(k-1)!}+\frac{z^k}{k!}\right)$$

$$\Rightarrow \sum_{k\geq 0} (\dots)\left(\frac{1}{(k-1)!}+\frac{1}{k!}\right) z^k$$

$$= \sum_{k\geq 0} z^k = \frac{1}{1-z}.$$

This implies, by equating coeff.:

$$\frac{a_1^{\bar{k}} \dots a_m^{\bar{k}}}{b_1^{\bar{k}} \dots b_n^{\bar{k}}} \cdot \left(\frac{1}{(k-1)!}+\frac{1}{k!}\right) = 1.$$

$$\Rightarrow \cancel{\frac{a_1^{\bar{n}} \dots a_m^{\bar{n}}}{b_1^{\bar{n}} \dots b_q^{\bar{n}}}} \qquad \frac{k}{k!}+\frac{1}{k!} = \frac{(k+1)}{k!}$$

$$\Rightarrow \frac{a_1^{\bar{k}} \dots a_m^{\bar{k}}}{b_1^{\bar{k}} \dots b_n^{\bar{k}}} = \frac{k!}{(k+1)}.$$

atleast, we need $a_1 = 1 \Rightarrow 1^{\bar{k}} = k!$

The bottom is not possible unless we introduce

$$\Rightarrow \frac{k!\,k!}{(k+1)!} \Rightarrow \left.\begin{matrix} a_1=1,\ a_2=1 \\ b_1=2 \end{matrix}\right\} \boxed{F(z) = F\left(\begin{matrix}1,1\\2\end{matrix}\middle| z\right)}$$

31. Condition $f(k) = \Delta T(k)$

$$\Rightarrow \frac{f(k+1)}{f(k)} = \frac{\frac{T(k+2)}{T(k+1)} - 1}{1 - \frac{T(k)}{T(k+1)}}$$

$T(k)$ is hypergeometric $\Longleftrightarrow \frac{T(k+1)}{T(k)}$ rational.

Thus $f(k)$ is also hypergeometric

$\Rightarrow$ our expression is a rational function (k).

32. $\sum_k k^2 \delta k$ by Gosper's method

$$\frac{t_{k+1}}{t_k} = \frac{(k+1)^2}{k^2} \rightarrow p(k) = (k)^2,\ q(k) = 1,\ r(k+1) = 1.$$

$$t(k) = T(k+1) - T(k)$$

$$T(k) = \frac{r(k)\, s(k)\, t(k)}{p(k)}$$

$Q = 0,\ R = 0.$

$$d = \deg(P) - \deg(Q)$$

$$\rightarrow s(k) = \alpha_3 k^3 + \alpha_2 k^2 + \alpha_1 k + \alpha_0 = 2 - (-1) = 3.$$

$$k^2 = s(k+1) - s(k) = \alpha_3 (k+1)^3 + \alpha_2 (k+1)^2 + \alpha_1 (k+1) + \alpha_0 - \alpha_3 k^3 - \alpha_2 k^2 - \alpha_1 k - \alpha_0$$

$$k^2 = \alpha_3 (\cancel{k^3} - \cancel{k^3} + 3k^2 + 3k + 1) + \alpha_2 (2k+1) + \alpha_1 (1) + \cancel{\alpha_0 - \alpha_0}$$

$$k^2 = k^2 (3\alpha_3) + k (3\alpha_3 + 2\alpha_2) + \alpha_3 + \alpha_2 + \alpha_1$$

$$3\alpha_3 = 1,\quad 3\alpha_3 + 2\alpha_2 = 0,\quad \alpha_3 + \alpha_2 + \alpha_1 = 0$$

$$\Rightarrow \alpha_3 = \frac{1}{3},\quad \alpha_2 = -\frac{1}{2},\quad \alpha_1 = \frac{1}{6},\ \text{let } \alpha_0 = 0.$$

so $s(k) = \frac{1}{3}k^3 - \frac{1}{2}k^2 + \frac{1}{6}k + 0$

$$\Rightarrow \quad T(k) = \frac{r(k)\,s(k)\,t(k)}{p(k)}$$

$$= \frac{(1)\left(\frac{1}{3}k^3 - \frac{1}{2}k^2 + \frac{1}{6}k\right)\cancel{(k^2)}}{\cancel{k^2}}$$

$$= \frac{1}{3}k^3 - \frac{1}{2}k^2 + \frac{1}{6}k.$$

Therefore $t(k) = \Delta T(k)$

$$\Rightarrow \sum_k t(k)\,\delta k = T(k) + C$$

$$\Rightarrow \boxed{\sum_k k^2\,\delta k = \frac{1}{3}k^3 - \frac{1}{2}k^2 + \frac{1}{6}k + C}$$

33 $\sum \frac{5k}{k^2-1}$.

we have $t_k = \frac{1}{k^2-1}$

$$\Rightarrow \frac{t_{k+1}}{t_k} = \frac{k^2-1}{(k+1)^2-1} = \frac{(k+1)(k-1)}{k(k+2)}$$

$$= \frac{p(k+1)}{p(k)} \cdot \frac{q(k)}{r(k+1)}$$ $\Rightarrow$ above where $\exists\, p, q, r$ such that $(k+\alpha) \mid q(k)$ and $(k+\beta) \mid r(k)$ and $\alpha-\beta$ is not positive int.

$\Rightarrow$ $p(k) = k$, $q(k) = k-1$, $r(k) = k+1$.

we see! $q(k) = k-1$, $r(k) = k+1$ which means $\exists$ only unique $\alpha = -1$, $\beta = 1$ so $-1-1=-2$ ✓

$Q(k) = (k-1) - (k+1) = -2$

$R(k) = 2k \cancel{+1}$

since $\deg(Q) < \deg(R) = 1$

$$\Rightarrow d = \frac{-2\lambda'}{\lambda} = \frac{-2(-2)}{2} = 2.$$

or $d = \deg(P) - \deg(R) + 1 = 1-1+1 = 1$.

let $s(k) = \alpha_1 k + \alpha_0$.

Then $\beta(k) = q(k)\, s(k+1) - r(k)\, s(k)$

$$\Rightarrow k = (k-1)(\alpha_1 k + \alpha_1 + \alpha_0) - (k+1)(\alpha_1 k + \alpha_0)$$

$$= \cancel{\alpha_1 k^2} + (\alpha_0 + \alpha_1)k - \alpha_1 k - \alpha_1 - \alpha_0$$

$$- \cancel{\alpha_1 k^2} - \alpha_0 k - \alpha_1 k - \alpha_0$$

$$= (\cancel{\alpha_0} + \cancel{\alpha_1} - \cancel{\alpha_1} - \cancel{\alpha_0} - \alpha_1)k - \alpha_1 - 2\alpha_0 = k$$

$$\Rightarrow (-\alpha_1)k - \alpha_1 - 2\alpha_0 = k$$

$$\Rightarrow -\alpha_1 = 1, \quad -2\alpha_0 = -1 \Rightarrow \alpha_0 = \frac{1}{2}$$

$\Rightarrow \quad -k+\frac{1}{2} = s(k).$

$$\Rightarrow \quad T(k) = \frac{r(k)\, s(k)\, t(k)}{p(k)}$$

$$= \frac{(k+1)\left(-k+\frac{1}{2}\right)\left(\frac{1}{k^2-1}\right)}{k} = \frac{\left(-k+\frac{1}{2}\right)\left(\frac{1}{k-1}\right)}{k}$$

$$= \boxed{\frac{(-2k+1)}{2k^2-k}.}$$

$$\Rightarrow \quad \sum_k \frac{\delta k}{k^2-1} = \sum_k t_k\, \delta k = T(k)$$

$$= \boxed{\frac{(-2k+1)}{2k(k-1)} + C}$$

34. yes... all $k > c$ vanish

$$\Rightarrow \quad \lim_{\varepsilon \to 0} F(-m, -n; \varepsilon - m; 1)$$

$$= \lim_{\varepsilon \to 0} \frac{(\varepsilon - n - m)^{\overline{m}}}{(\varepsilon - m)^{\overline{m}}} = \boxed{(-1)^m \binom{n-1}{m}}$$

35. We are tasked:

$$\sum_{k \le n} \binom{n}{k} 2^{k-n}.$$

a. Sum on k: $\sum_{k \le n} \binom{n}{k} 2^{k-n}$

$$= \sum_{0 \le k \le n} \binom{n}{k} 2^{k-n}$$

$$\Rightarrow \cancel{\text{consider } \sum_{k} \binom{n}{k} 2^{k-n} \, \delta k.}$$

consider $\sum_{k} \binom{r}{k} x^k y^{r-k} = (x+y)^r$

$$\Rightarrow \cancel{\sum_{k} \binom{n}{k} (1)^k (2)^{n-k} = (1+2)^n}$$

$$\cancel{= \sum_{k} \binom{n}{k} 2^{n-k}}$$

$$\Rightarrow \sum_{k \le n} \binom{n}{k} 2^{k-n} = 2^{-n} \sum_{k \le n} \binom{n}{k} 2^k$$

$$\Rightarrow \sum_{k} \binom{n}{k} 2^k (1)^{n-k} = (2+1)^n = 3^n$$

$$\Rightarrow 2^{-n} 3^n \text{ (assuming } n \ge 0).$$

$$\Rightarrow \boxed{2^{-n} \cdot 3^n \; [n \ge 0]}$$

b. $\sum_{k \le n} \binom{n}{k} 2^{k-n}$ on n:

$$\Rightarrow \sum_{n \ge k} \binom{n}{k} 2^{k-n} = \sum_{n+k \ge k} \binom{n+k}{k} 2^{-n}$$

$$= \sum_{n \ge 0} \binom{n+k}{k} 2^{-n} = \sum_{n \ge 0} \binom{n+k}{k} \left(\tfrac{1}{2}\right)^n$$

$\Rightarrow$ recall $\frac{1}{(1-z)^{n+1}} = \sum_{k \ge 0} \binom{n+k}{n} z^k$

$$\Rightarrow \sum_{n \ge 0} \binom{n+k}{k} \left(\tfrac{1}{2}\right)^n$$

$$= \frac{1}{\left(1-\frac{1}{2}\right)^{k+1}} = \frac{1}{\left(\frac{1}{2}\right)^{k+1}} = \boxed{2^{k+1}}$$

36. p^k is largest power of a prime p divdg $\binom{m+n}{m}$, prove k is # carries occurs when m added to n in radix p.

$$\frac{(m+n)^{\underline{m}}}{m!} = \frac{(m+n)(m+n-1)\cdots(m+n-m+1)}{m!}$$

$$= \frac{(m+n)(m+n-1)\cdots(n+1)}{m!}$$

$\exists\, p^k$ for some k such that

we wish

$$\frac{(m+n)(m+n-1)\cdots(n+1)}{m!} \mod (p^k) \equiv 0.$$

~~observe that this p^k must exist somewhere in generation~~
~~↳ one of the terms~~

→ the sum of digits of $m+n$ = sum of digits of m + sum digits of n

→ minus $p-1$ (# carries).

37. $\sum_k \binom{n}{k} x^{\underline{k}} y^{\underline{n-k}} = A(k)$

$$\Rightarrow \frac{A(k)}{n!} = \sum_k \binom{n}{k} \left(\frac{x^{\underline{k}} y^{\underline{n-k}}}{n!} \right)$$

$$= \sum_k \binom{n}{k} \left(\frac{x(x-1)\dots(x-k+1)(y)(y-1)\dots(y-n+k+1)}{n!} \right)$$

$$= \sum_k \frac{\cancel{n!}}{k!(n-k)!} \frac{x^{\underline{k}} y^{\underline{n-k}}}{\cancel{n!}}$$

$$= \sum_k \binom{x}{k} \binom{y}{n-k}$$

Vandermonde convolution.

let's do by hand:

let $A(z) = \sum_k \binom{x}{k} z^k$, $B(z) = \sum_k \binom{y}{k} z^k$.

$$\Rightarrow A(z)(B(z)) = \left(\sum_k \binom{x}{k} z^k\right)\left(\sum_k \binom{y}{k} z^k\right)$$

$\Rightarrow$ Binomial theorem tells us

$$(x+y)^r = \sum_k \binom{r}{k} y^k x^{r-k}.$$

convolution

$$\left\{ \begin{array}{l} \text{setting: } (1+z)^r = \sum_k \binom{r}{k} z^k \\ \Rightarrow (1+z)^x = \sum_k \binom{x}{k} z^k. \\ (1+z)^y = \sum_k \binom{y}{k} z^k \end{array} \right.$$

$$\sum_k \binom{x}{k} z^k = (1+z)^x, \quad \sum_k \binom{y}{k} z^k = (1+z)^y$$

$$A(z)B(z) = (1+z)^x (1+z)^y = (1+z)^{x+y} = \sum_k \binom{x+y}{k} z^k$$

but, looking at t^n term:

$$[z^n] A(z)(B(z)) = [z^n] \sum_k \binom{x+y}{k} z^k$$

$$= \binom{x+y}{n} = \sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k}$$

$\Rightarrow$ but $$\frac{A(k)}{n!} = \sum_k \binom{x}{k}\binom{y}{n-k}$$

$$= \sum_{0 \le k \le n} \binom{x}{k}\binom{y}{n-k}$$

$$= \binom{x+y}{n}.$$

So $$A(k) = \binom{x+y}{n} n! = \frac{(x+y)^{\underline{n}}}{n!} \cdot n!$$

$$= \boxed{(x+y)^{\underline{n}}}$$

38. prove $n = \binom{a}{1} + \binom{b}{2} + \binom{c}{3}$, with $0 \le a < b < c$.

Choose a c As large as possible : $\binom{c}{3} \le n$.

$\Rightarrow \quad n - \binom{c}{3} = \binom{a}{1} + \binom{b}{2}$

$\Rightarrow \quad n - \binom{c}{3} = a + \binom{b}{2}$.

choose now any greatest b such That

$\binom{b}{2} \le n - \binom{c}{3}$.

$\Rightarrow \quad n - \binom{c}{3} - \binom{b}{2} = a$.

Now, because a can be zero, this can always be possible.

if

39. $xy = ax + by$, show

$$x^n y^n = \sum_{k=1}^{n} \binom{2n-1-k}{n-1} (a^n b^{n-k} x^k + a^{n-k} b^n y^k)$$

$(xy)^n = (ax+by)^n$

$\Rightarrow$ we see we can use The binomial Theorem

$$\sum_{k=0}^{n} \binom{n}{k} (ax)^{n-k} (by)^k$$

$1 \le j \le m$

$1 \le j+1 \le m$

40. $(-1)^m \sum_{j=1}^{m} \binom{-D-1}{j} \binom{m-E-1}{m-j}$

$\rightarrow (-1)^m \sum_{j=0}^{m-1} \binom{-D-1}{j+1} \binom{m-E-1}{m-j-1}$

$\cancel{\binom{n}{k} k = n \binom{n-1}{k-1}}$

$\cancel{\Rightarrow \binom{-D-1}{j+1} = \frac{-D-1}{j+1} \binom{-D-2}{j}}$

$\Rightarrow (-1)^m \sum_{j=1}^{m} \binom{r}{j} \binom{m-rk-s-1}{m-j}$

$\rightarrow (-1)^m \left(\sum_{j=0}^{m} \binom{r}{j} \binom{m-rk-s-1}{m-j} - \binom{r}{0} \binom{m-rk-s-1}{m} \right)$

$\rightarrow (-1)^m \left(\underbrace{\sum_{j=0}^{m} \binom{r}{j} \binom{m-rk-s-1}{m-j}}_{\sum_k \binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}} - \binom{m-rk-s-1}{m} \right)$

$\Rightarrow (-1)^m \left(\binom{r+m-rk-s-1}{m} - \binom{m-rk-s-1}{m} \right)$

$\Rightarrow (-1)^m \left(\binom{m+r(1-k)-s-1}{m} - \binom{m-rk-s-1}{m} \right)$

So: $(-1)^{m+1} \sum_{k=1}^{n} \left(\binom{m+r(1-k)-s-1}{m} - \binom{m-rk-s-1}{m} \right)$

$\Rightarrow (-1)^{m+1} \sum_{k=1}^{n} \left(\binom{m-r(k-1)-s-1}{m} - \binom{m-rk-s-1}{m} \right)$

$$\rightarrow (-1)^{m+1} \sum_{k=1}^{n} \left(\binom{m-r(k-1)-s-1}{m} - \binom{m-rk-s-1}{m} \right)$$

$$= (-1)^{m+1} \left(\binom{m-s-1}{m} - \cancel{\binom{m-r-s-1}{m}} + \cancel{\binom{m-r-s}{m}} - \cancel{\binom{m-2r-s-1}{m}} + \cancel{\binom{m-2r-s-1}{m}} - \cancel{\binom{m-3r-s-1}{m}} + \dots + \cancel{\binom{m-r(n-2)-s-1}{m}} - \cancel{\binom{m-r(n-1)-s-1}{m}} + \cancel{\binom{m-r(n-1)-s-1}{m}} - \binom{m-rn-s-1}{m} \right)$$

$$m-(m-s-1)-1$$
$$s+1-1$$

$$= (-1)^{m+1} \left(\binom{m-s-1}{m} - \binom{m-rn-s-1}{m} \right)$$

$$= (-1)^{m+1} \left((-1)^{m} \binom{s}{m} - (-1)^{m} \binom{m-s}{m} \right)$$

$$= (-1)^{m+1+m} \left(\binom{s}{m} - \binom{m-s}{m} \right)$$

$$= \boxed{\binom{m-s}{m} - \binom{s}{m}}$$

41. $$\sum_k \frac{\binom{n}{k} k!}{(n+1+k)!} = \sum_k \frac{n^{\underline{k}}}{(n+1+k)!}$$

$$= \sum_k \frac{n^{\underline{k}}}{\cancel{(n+1+k)(n+k)} \dots (n+1)\, n \dots (n-k+1)(n-k)!}$$

consider $$\sum_k \frac{n^{\underline{k}}}{(n+1+k)!} = \sum_k t_k$$

$$\frac{t_{k+1}}{t_k} = \frac{n^{\underline{k+1}}}{(n+1+k+1)!} \cdot \frac{(n+1+k)!}{n^{\underline{k}}} = \frac{(n-k)}{(n+k+2)}.$$

$\Rightarrow p(k) = 1, \quad q(k) = n-k, \quad r(k) = n+k+1$

$= (-1)(k-n).$

$\Rightarrow -n-(n+1) = -1 < 0.$

$Q = n-k-(n+k+1) = -2k-1$

$R = 2n+1.$

$\deg(Q) \geq \deg(R)$

$d =$

Try Gosper Zeilberger

$$\sum_k \frac{n^{\underline{k}}}{(n+1+k)!} = \sum_k t(n,k).$$

Let $\hat{t}(n,k) = \beta_0 t(n,k) + \beta_1 t(n+1,k).$

$$\frac{t(n+1,k)}{t(n,k)} = \frac{(n+1)^{\underline{k}}}{(n+2+k)!} \cdot \frac{(n+1+k)!}{n^{\underline{k}}} = \frac{(n+1)}{(n+2+k)(n-k+1)}$$

$$t(n+1,k) = t(n,k)\,\frac{(n+1)}{(n+2+k)(n-k+1)}$$

$$\Rightarrow \hat{t}(n,k) = t(n,k)\left(\beta_0 + \beta_1\left(\frac{n+1}{(n+2+k)(n-k+1)}\right)\right)$$

$$\rightarrow \hat{t}(n,k) = \frac{t(n,k)}{(n+k+2)(n-k+1)}\underbrace{\left(\beta_0(n+k+2)(n-k+1) + \beta_1(n+1)\right)}_{\hat{p}(n,k)}.$$

$$\frac{\hat{t}(n,k+1)}{\hat{t}(n,k)} = \frac{\hat{p}(n,k+1)}{\hat{p}(n,k)} \cdot \frac{q(n,k)}{r(n,k+1)} \cdots$$

$$\frac{\hat{t}(n,k+1)}{\hat{t}(n,k)} = \frac{t(n,k+1)}{t(n,k)} \cdot \frac{(n+k+2)(n-k+1)}{(n+k+3)(n-k)}.$$

$$= \frac{n^{\underline{k+1}}}{(n+1+k+1)!} \cdot \frac{(n+1+k)!}{n^{\underline{k}}} \cdot \frac{(n+k+2)(n-k+1)}{(n+k+3)(n-k)}$$

$$= \frac{\cancel{(n-k)}}{\cancel{(n+k+2)}} \cdot \frac{\cancel{(n+k+2)}(n-k+1)}{(n+k+3)\cancel{(n-k)}}$$

$q(n,k) = n-k+1$

$r(n,k+1) = n+k+3 \rightarrow r(n,k) = n+k+2.$

$\Rightarrow Q(n,k) = (n-k+1) - (n+k+2) = -2k-1$

$R(n,k) = 2n+3$

$\rightarrow \deg(Q) > \deg(R) = 1$

Thus $d = \deg(\hat{p}) - \deg(Q) = 2 - 1 = 1$

So $s(n,k) = \alpha_1 k + \alpha_0$

and we wish to solve

$$\hat{p}(n,k) = q(n,k)\,s(n,k+1) - r(n,k)\,s(n,k)$$
$$= (n-k+1)(\alpha_1(k+1)+\alpha_0)$$
$$- (n+k+2)(\alpha_1 k + \alpha_0).$$
$$= \beta_0(n+k+2)(n-k+1) + \beta_1(n+1)$$

$\Rightarrow$ ~~[illegible]~~ doesn't work.

$$\sum_{k\geq 0} \frac{\binom{n}{k} k!}{(n+1+k)!} = \sum_{k\geq 0} \frac{n!}{(n-k)!\,(n+1+k)!}$$

$$= \frac{n!}{(2n+1)!} \sum_{k\geq 0} \binom{2n+1}{k} = \boxed{\frac{2^{2n}\, n!}{(2n+1)!}}$$

42. $$\sum_x \left((-1)^x \Big/ \binom{n}{x} \right) \delta x$$

$$= \sum \frac{(-1)^x}{\binom{n}{x}} \delta x$$

$$= \sum \binom{n}{x}(-1)^x \cdot \left(\frac{1}{\binom{n}{x}}\right)^2 \delta x$$

$$= \sum \binom{n}{x}(-1)^x \left(\frac{(x!)^2}{(n^{\underline{x}})^2}\right) \delta x$$

$$\sum \frac{(-1)^x}{\binom{n}{x}} \delta_x \qquad t_x = \frac{(-1)^x}{\binom{n}{x}}$$

$$\frac{t_{x+1}}{t_x} = -\frac{\binom{n}{x}}{\binom{n}{x+1}} = -\frac{\left(\frac{n^{\underline{x}}}{x!}\right)}{\frac{n^{\underline{x+1}}}{(x+1)!}}$$

$$\sum_{k=0}^{n} \frac{(-1)^k}{\binom{n}{k}} = (-1)^{x-1} \frac{n+1}{n+2} \left|\binom{n+1}{x}\right|_0^{n+1} = -\frac{(x+1)}{n-x} = \frac{x+1}{x-n}$$

$$= \boxed{2\,\frac{n+1}{n+2}\ [n \text{ even}]}$$

$Q = 1+n+1 = n+2$
$R = 2x-n$

$\left\{ \begin{array}{l} q = x+1 \\ r = x-n-1 \end{array} \right.$

$\deg(Q) < \deg(R) = 1$

$d = 0 - 1 + 1 = 0.$

$s_x = \alpha_0.$

$$p(x) = q(x)\, s(x+1) - r(x) s(x)$$

$$= (x+1)\alpha_0 - (x-n-1)\alpha_0 = 1$$

$$\rightarrow \cancel{\alpha_0 x} + \alpha_0 - \cancel{\alpha_0 x} + n\alpha_0 + \alpha_0 = 1$$

$$n\alpha_0 + 2\alpha_0 = 1$$

$$\alpha_0 = \frac{1}{n+2}$$

so $$\sum \frac{(-1)^x}{\binom{n}{x}} \delta_x = (x-n-1)\left(\frac{1}{n+2}\right)\left(\frac{(-1)^x}{\binom{n}{x}}\right)$$

$$= \frac{(n-x+1)}{n+2} \frac{(-1)^{x-1}}{\binom{n}{x}} = \frac{1}{n+2} \frac{(-1)^{x-1}}{\frac{n^{\underline{x-1}}}{x!}}$$

$$= \boxed{\frac{n+1}{n+2} \cdot \frac{(-1)^{x-1}}{\binom{n+1}{x}}}$$

43. $\sum_k \binom{m-r+s}{k}\binom{n+r-s}{n-k}\binom{r+k}{m+n}$

$= \binom{r}{m}\binom{s}{n}$

consider $\binom{r+k}{m+n} = \sum_j \binom{r}{m+n-j}\binom{k}{j}$

$\rightarrow \sum_k \binom{m-r+s}{k}\binom{n+r-s}{n-k}\sum_j \binom{r}{m+n-j}\binom{k}{j}$

$\rightarrow \sum_j\sum_k \binom{m-r+s}{k}\binom{k}{j}\binom{n+r-s}{n-k}\binom{r}{m+n-j}$

$= \sum_j \binom{r}{m+n-j}\sum_k \binom{m-r+s}{j}\binom{m-r+s-j}{k-j}\binom{n+r-s}{n-k}$

$= \sum_j \binom{r}{m+n-j}\binom{m-r+s}{j}\sum_k \binom{m-r+s-j}{k-j}\binom{n+r-s}{n-k}$

$= \sum_j \binom{r}{m+n-j}\binom{m-r+s}{j}\sum_k \binom{m-r+s-j}{k-j}\binom{n+r-s}{n-k}$

$= \sum_j \binom{r}{m+n-j}\binom{m-r+s}{j}\binom{m+n-j}{n-j}$

$= \sum_j \binom{r}{m+n-j}\binom{m+n-j}{n-j}\binom{m-r+s}{j}$

$= \sum_j \binom{r}{n-j}\binom{r-n+j}{m+n-j-(n-j)}\binom{m-r+s}{j}$

$$= \sum_j \binom{r}{n-j}\binom{r-n+j}{m}\binom{m-r+s}{j}$$

$$= \sum_j \binom{r}{r-n+j}\binom{r-n+j}{m}\binom{m-r+s}{j}$$

$$= \sum_j \binom{r}{m}\binom{r-m}{r-n+j-m}\binom{m-r+s}{j}$$

$$= \binom{r}{m}\left(\sum_j \binom{r-m}{r-n+j-m}\binom{m-r+s}{j}\right)$$

$$= \binom{r}{m}\cancel{\binom{r-m+m-r+s}{r-n+j+m+j}}$$

$$= \binom{r}{m}\sum_j \binom{r-m}{n-j}\binom{m-r+s}{j}$$

$$= \boxed{\binom{r}{m}\binom{s}{n}}$$

例45. $$\sum_{k\le n}\binom{2k}{k}4^{-k} = \sum_{k\le n}\frac{(2k)^{\underline{k}}}{k!}\cdot 4^{-k}$$

$$= \sum_{k\le n}\frac{(2k)(2k-1)(2k-3)\cdots(k+1)}{4^k\, k!}$$

$$= \sum_{k\le n}\binom{2k}{k}\Big/2^{2k} = \sum_{k\le n}\binom{k-\frac{1}{2}}{k} = \binom{n+\frac{1}{2}}{n}$$

46. $\sum_k \binom{2k-1}{k}\binom{4n-2k-1}{2n-k}\frac{(-1)^{k-1}}{(2k-1)(4n-2k-1)}$

$$B_{-1}(z) = \sum_k \binom{2k-1}{k}\frac{(-z)^k}{1-2k}$$

$$= \sum_k \binom{2k-1}{k}\frac{(-1)^{k-1}(z)^k}{2k-1}.$$

let $b_k = \binom{2k-1}{k}\frac{(-1)^{k-1}}{2k-1}$.

Then: $\sum_k b_k \binom{4n-2k-1}{2n-k}\cdot\frac{1}{4n-2k-1}$.

$$= \sum_k b_k \binom{2(2n-k)-1}{2n-k}\frac{1}{2(2n-k)-1}$$

$$= -\sum_k b_k \binom{2(2n-k)-1}{2n-k}\frac{1}{1-2(2n-k)}$$

$$B_{-1}(-z) = \sum_k \binom{2k-1}{k}\frac{z^k}{1-2k}$$

$\Rightarrow$ let $a_k = \binom{2k-1}{k}\cdot\frac{1}{1-2k}$.

Then: $-\sum_k b_k a_{2n-k}$.

$$= -[z^{2n}]\, B_{-1}(z)\, B_{-1}(-z)$$

we know $B_{-1}(z) = \sum_k \binom{2k-1}{k} \left(\frac{(-z)^k}{1-2k}\right)$

$$= \frac{1+\sqrt{1+4z}}{2}$$

Thus: $-[z^{2n}]\, B_{-1}(z)\, B_{-1}(-z) = [z^{2n}] \left(\frac{-1-\sqrt{1+4z}}{2}\right)\left(\frac{1+\sqrt{1+4z}}{2}\right)$

$$= [z^{2n}] \left(\frac{-1-\sqrt{1-4z} - \sqrt{1+4z} - \sqrt{1-16z^2}}{4}\right)$$

$$(1-4z)^{1/2} = \sum_k \binom{1/2}{k} (-4z)^k$$

$$(1+4z)^{\frac{1}{2}} = \sum_k \binom{1/2}{k} (4z)^k$$

$$(1-16z^2)^{1/2} = \sum_k \binom{1/2}{k} (-16z^2)^k$$

$$\Rightarrow [z^{2n}] \left(\frac{-1 - \sum_k \binom{1/2}{k} (-4z)^k - \sum_k \binom{1/2}{k} (4z)^k - \sum_k \binom{1/2}{k} (-16z^2)^k}{4}\right)$$

$$= \frac{-\binom{1/2}{2n}(4)^{2n} - \binom{1/2}{2n}(-4)^{2n} - \binom{1/2}{n}(-16)^n}{4}$$

(we omit the 1 because it has no z^{2n} term.

$$= \frac{16^n\left(-\binom{1/2}{2n} - \binom{1/2}{2n} - \binom{1/2}{n}(-1)^n\right)}{4}$$

$\frac{1}{2}(1-4n+2)$

$$\Rightarrow \binom{1/2}{2n} = \frac{(1/2)(-\frac{1}{2})\cdots(\frac{1}{2}-2n+1)}{(2n)!}$$

$$= \frac{(1)(-1)(-3)(-5)\cdots(3-4n)}{2^{(2n)}(2n)!}$$

$$= \frac{(-1)^{2n-1}2^{2n-1}(1)(3)(5)\cdots(4n-3)(2n-1)!}{2^{2n-1}\cdot 2^{2n}(2n)!(2n-1)!}$$

$$= \frac{(-1)^{2n-1}(1)(3)(5)\cdots(4n-3)(2)(4)(6)\cdots(4n-2)}{2^{4n-1}(2n)!(2n-1)!}$$

$$= \frac{(-1)^{2n-1}(4n-2)!}{2^{4n-1}(2n)(2n-1)!^2}$$

$$= \frac{(-1)^{2n-1}}{2^{4n}n}\binom{4n-2}{2n-1}$$

$$= \frac{(-1)^{2n-1}}{2^{4n-1}}\cdot\binom{4n-1}{2n}\frac{\cancel{2n}}{4n-1}$$

$$\underbrace{(4n)\underbrace{(4n-1)(4n-2)\cdots(2n)}_{(2n+1)}}$$

$$= \frac{(-1)^{2n-1}}{2^{4n-1}}\binom{4n}{2n}\cdot\frac{1}{\cancel{4n}}\cdot\cancel{2n}\cdot\frac{1}{4n-1}$$

$$= \frac{(-1)^{2n-1}}{4^{2n}}\binom{4n}{2n}\cdot\frac{1}{4n-1}$$

So: $$\frac{-\frac{\binom{4n}{2n}(-1)^{2n-1}}{\cancel{4^{2n}}(4n-1)}\cancel{(4^{2n})} - \frac{\binom{4n}{2n}(-1)^{2n-1}}{\cancel{4^{2n}}(4n-1)}\cancel{(-4)^{2n}}}{4}$$

$$= \frac{-\frac{\binom{2n}{n}(-1)^{n-1}}{4^n(2n-1)}(-16)^n}{4}$$

$$= \frac{\frac{2\binom{4n}{2n}(-1)^{2n-1}}{4n-1} - \frac{\binom{2n}{n}(4)^n(-1)^{2n-1}}{2n-1}}{4}$$

$$= \frac{\frac{4\binom{4n-1}{2n}}{4n-1} + \frac{4\binom{2n}{n}4^{n-1}}{2n-1}}{4}$$

$$= \boxed{\frac{\binom{4n-1}{2n}}{4n-1} + \frac{\binom{2n}{n}4^{n-1}}{2n-1}}$$

47. $\displaystyle\sum_k \binom{rk+s}{k}\binom{rn-rk-s}{n-k}$ $\quad r(n-k)-s$

Vandermonde : $\displaystyle\sum_k \binom{r}{k}\binom{s}{n-k}$

$$= \binom{r+s}{n}$$

this is a convolution between

$$\sum_k \binom{rk+s}{k} z^k = \frac{B_r(z)^s}{1-r+rB_r(z)^{-1}} \qquad \frac{(rk+s)(rk+s-1)\cdots((r-1)k+s+1)}{k!}$$

and

$$\sum_k \binom{rk-s}{k} z^k = \frac{B_r(z)^{-s}}{1-r+rB_r(z)^{-1}}$$

$$\cancel{(x+y)^n = \sum_k \binom{n}{k}(x)^{n-k}(y)^k}$$

$$\therefore \quad [z^n]\ \frac{\cancel{B_r(z)^s}}{1-r+rB_r(z)^{-1}} \cdot \frac{\cancel{B_r(z)^{-s}}}{1-r+rB_r(z)^{-1}} = \sum_k \binom{rk+s}{k}\binom{rn-rk-s}{n-k}$$

Thus, the sum is $[z^n]\dfrac{1}{(1-r\frac{z}{z}+rB_r(z)^{-1})(1-r+rB_r(z))}$

which is independent of s.

49. $\displaystyle\sum_k (-1)^k \binom{x}{k}\binom{x+n-k}{n-k}\frac{y}{y+n-k}$

$$\frac{t_{k+1}}{t_k} = \frac{(-1)^{k+1}\dfrac{x^{\underline{k+1}}}{(k+1)!}\cdot\dfrac{(x+n-k-1)^{\underline{n-k}}}{(n-k-1)!}\cdot\dfrac{y}{y+n-k-1}}{(-1)^k\dfrac{x^{\underline{k}}}{k!}\cdot\dfrac{(x+n-k)^{\underline{n-k}}}{(n-k)!}\cdot\dfrac{y}{y+n-k}}$$

$$= \frac{(-1)(x-k)(n-k)(y+n-k)}{(k+1)(x+n-k)(y+n-k-1)}$$

$$= \frac{\cancel{(-1)^6}(k-x)(k-n)(k-n-y)}{(k+1)(k-x-n)(k-n-y+1)}$$

$$= \frac{y}{y+n}\binom{x+n}{n}F\left(\begin{matrix}-x,-n,-n-y\\-x-n,-n-y+1\end{matrix}\middle|1\right)$$

$$\Rightarrow F\left(\begin{matrix}a,b,-n\\c,a+b-n+1-c\end{matrix}\middle|1\right) = \frac{(c-a)^{\overline{n}}(c-b)^{\overline{n}}}{c^{\overline{n}}(c-a-b)^{\overline{n}}}.$$

$$F\left(\begin{matrix}-x,-n,-(n+y)\\-x-n,\cancel{-x-n}-n-y+1\cancel{+x+n}\end{matrix}\right)$$

$$= \left|\frac{(-x-n+x)^{\underline{n+y}}(-x-n+n)^{\underline{n+y}}}{(-x-n)^{\underline{n+y}}\cancel{(-x-n+x+n)^{\underline{n+y}}}}\right|$$

$$\frac{n^{k+1}}{(k+1)!} \cdot \frac{k!}{n^k}$$

56. $\sum_k \binom{-3}{k} \delta k$

$$\frac{t_{k+1}}{t_k} = \frac{\binom{-3}{k+1}}{\binom{-3}{k}} = \frac{\frac{(-3)^{k+1}}{(k+1)!}}{\frac{(-3)^k}{k!}} = \frac{\frac{(-3)(-4)\dots(-3-k)}{(k+1)!}}{\frac{(-3)(-4)\dots(-3-k+1)}{k!}}$$

$$= \frac{-3-k}{k+1} = \frac{-(k+3)}{k+1} = \frac{p(k+1)}{p(k)} = \frac{q(k)}{r(k+1)}.$$

let $p(k) = (k+1)^{3-1}$

$q(k) = -1$

$r(k) = 1.$

$\Rightarrow s(k) \rightarrow$ degree

$d = 2$

$$\frac{(k+2)^2}{(k+1)^2} = \frac{\cancel{(k+2)}(k+2)}{(k+1)\cancel{(k+1)}}$$

$$(k+2)(k+1) = -\left((k+1)^2\alpha_2 + (k+1)\alpha_1 + \alpha_0\right) = \left(k^2\alpha_2 + k\alpha_1 + \alpha_0\right)$$

$$= \cancel{-(2k+1)\alpha_2 - \alpha_2}$$

$$(k+2)(k+1) = -(k^2+2k+1)\alpha_2 - (k+1)\alpha_1 - \alpha_0 - k^2\alpha_2 - k\alpha_1 - \alpha_0$$

$$k^2+3k+2 = k^2(-\alpha_2-\alpha_2) + k(-2\alpha_2-\alpha_1-\alpha_1) - \alpha_2 - \alpha_1 - \alpha_0 - \alpha_0$$

$$-2\alpha_2 = 1, \quad -2\alpha_2 - 2\alpha_1 = 3, \quad -\alpha_2 - \alpha_1 - 2\alpha_0 = 2$$

$$\Rightarrow \alpha_2 = -\frac{1}{2}, \quad \alpha_1 = -1, \quad \alpha_0 = -\frac{1}{4}$$

$$\Rightarrow \underline{s(k) = -\frac{1}{2}k^2 - k - \frac{1}{4}}$$

$$= \frac{(1)\left(-\frac{1}{2}k^2-k-\frac{1}{4}\right)\binom{-3}{k}}{(k+1)^{\overline{2}}}$$

$$\binom{-3}{k} = \frac{(-3)(-4)\cdots(-2-k)}{k!} = \frac{(-1)^k(k+2)(k+1)\cdots(4)(3)}{k(k-1)\cdots(3)(2)(1)} = \frac{(-1)^k(k+2)(k+1)}{(2)(1)}$$

$$(-3)(-4)\cdots(-3-k+1)$$

$$= \left(-\frac{1}{2}k^2-k-\frac{1}{4}\right)\frac{(-3)}{k!}$$

$$= \frac{(-1)^k\left(-\frac{1}{2}k^2-k-\frac{1}{4}\right)}{(k+1)(k+2)}$$

$$= \boxed{\frac{1}{8}(-1)^{k-1}(2k^2+4k+1)+C = \sum_k \binom{-3}{k}\delta k}$$

57. $\sum_k \binom{n}{k} z^k (k+\theta)\,\delta k.$

$$\frac{t_{k+1}}{t_k} = \frac{\binom{n}{k+1} z^{k+1}(k+1+\theta)}{\binom{n}{k} z^k (k+\theta)}$$

$$= \frac{(n-k)(k+1+\theta)z}{(k+1)(k+\theta)}$$

$q(k) = (n-k)z$ $\quad p(k+1) = k+1+\theta$

$r(k) = k$, $\quad p(k) = k+\theta$

$Q = z(n-k)-k$ $\quad \deg Q \geq \deg R$

$R = (n-k)z+k$

$d = \deg(P) - \deg Q = 1-1 = 0$

$s = \alpha_0 \Rightarrow \quad k+\theta = (n-k)z\alpha_0 - k\alpha_0$

$k+\theta = k(-z\alpha_0-\alpha_0)+nz\alpha_0$

$\Rightarrow -z\alpha_0-\alpha_0 = 1,\ nz\alpha_0 = \theta.$

$$\alpha_0(-z-1)=1 \rightarrow \alpha_0=\frac{-1}{z+1}$$

$$nz\cdot\frac{(-1)}{z+1}=\theta$$

$$=\frac{-nz}{z+1}=\theta$$

$$\Downarrow$$

$$\sum_k \binom{n}{k} z^k \left(k-\frac{nz}{1+z}\right)\delta k$$

$$=\underline{(k)\left(\frac{-1}{z+1}\right)\binom{n}{k}z^k\cancel{\left(k-\frac{nz}{1+z}\right)}}$$

$$\cancel{k-\frac{nz}{z+1}}$$

$$=\boxed{\begin{aligned}&\frac{-k}{z+1}\binom{n}{k}z^k+C\\ &=\frac{-n}{z+1}\binom{n-1}{k-1}z^k+C\end{aligned}}$$

59. $\sum_{k\geq 1} \binom{n}{\lfloor \log_m k \rfloor}$

$$= \sum_{j,k} \binom{n}{j} [j = \lfloor \log_m k \rfloor][k \geq 1].$$

$$= \sum_{j,k} \binom{n}{j} [\log_m k - 1 < j \leq \log_m k][k \geq 1].$$

$$= \sum_{j,k} \binom{n}{j} [\log_m k < j+1, \log_m k \geq j][k \geq 1]$$

$$= \sum_{j,k} \binom{n}{j} [m^j \leq k < m^{j+1}][k \geq 1]$$

$$= \sum_{j,k} \binom{n}{j} [m^j \leq k < m^{j+1}][j \geq 0]$$

$$= \sum_{j,k} \binom{n}{j} [k \in [m^j, m^{j+1})][j \geq 0]$$

$$= \sum_{j} \binom{n}{j} (m^{j+1} - m^j + 1)[j \geq 0]$$

$$= \sum_{j \geq 0} \binom{n}{j} m^j (m-1)$$

$$= (m-1) \sum_{j \geq 0} \binom{n}{j} m^j$$

$$= \boxed{(m-1)(1+m)^n.}$$

61. Prove when $\binom{n}{m} = \binom{\lfloor n/p \rfloor}{\lfloor m/p \rfloor}\binom{n \bmod p}{m \bmod p} \pmod p$

$p\lfloor \frac{n}{p} \rfloor + n \bmod p = n$

consider $(x+1)^n = \sum_{k \geq 0} \binom{n}{k} x^k$

if $m > n$, then $\binom{n}{m} = 0$.

so $0 = \binom{\lfloor n/p \rfloor}{\lfloor m/p \rfloor}\binom{n \bmod p}{m \bmod p} \bmod p$.

$n > m$. since modulo p doesn't change the magnitudes of n and m w/respect to each other, This is 0

Now, if $m \leq n$, Then at The x^m term we have $\binom{n}{m}$ as The coefficient.

But: $(x+1)^n = (x+1)^{p\lfloor \frac{n}{p} \rfloor + n \bmod p} \pmod p$

we mustn't forget The mod p

$\equiv (x+1)^{\lfloor \frac{n}{p} \rfloor}(x+1)^{n \bmod p}$

$\equiv \sum_{k \geq 0} \binom{\lfloor \frac{n}{p} \rfloor}{k} x^k \sum_{k \geq 0} \binom{n \bmod p}{k} x^k \pmod p$

$$(x+1)^n = (x+1)^{p\lfloor \frac{n}{p} \rfloor + n \bmod p} \pmod p$$

$$= (x+1)^{p\lfloor \frac{n}{p} \rfloor} (x+1)^{n \bmod p} \pmod p$$

$$= \left((x+1)^p\right)^{\lfloor \frac{n}{p} \rfloor} (x+1)^{n \bmod p} \pmod p$$

$$\Rightarrow (x+1)^p \bmod p \equiv \binom{p}{0}x^p + \binom{p}{1}x^{p-1} + \dots + \binom{p}{p}x^0 \bmod p$$

only first and last terms have no "p".

so: $= (x^p + 0 + 1)^{\lfloor \frac{n}{p} \rfloor} (x+1)^{n \bmod p} \pmod p$

$$= (x^p+1)^{\lfloor \frac{n}{p} \rfloor} (x+1)^{n \bmod p}$$

$$= \sum_{k \geq 0} \binom{\lfloor \frac{n}{p} \rfloor}{k} x^{pk} \sum_{k \geq 0} \binom{n \bmod p}{k} x^k \pmod p$$

but $a^p \bmod p \equiv x^p \bmod p \equiv x \bmod p$.

$$= \sum_{k \geq 0} \binom{\lfloor \frac{n}{p} \rfloor}{k} x^k \sum_{k \geq 0} \binom{n \bmod p}{k} x^k \pmod p$$

$m - pk \leq n \bmod p$

$\Rightarrow$ the x^m term: $\sum_{0 \leq k \leq m} \binom{\lfloor \frac{n}{p} \rfloor}{k} \binom{n \bmod p}{m-k}$

$$\sum_{k \geq 0} \binom{\lfloor \frac{n}{p} \rfloor}{k} \binom{n \bmod p}{m - pk} \bmod p$$

$pk \leq m$

$k \leq \lfloor \frac{m}{p} \rfloor$

but, this is 0 for all k except when $k = \lfloor \frac{m}{p} \rfloor$.

$0 \leq k \leq \lfloor \frac{n}{p} \rfloor$

$0 \leq m - pk \leq n \bmod p$

$$\sum_{k\geq 0} \binom{n \bmod p}{m-pk}\binom{\lfloor \frac{n}{p}\rfloor}{k} \qquad \frac{m+1}{p}$$

β^{-1}

know $\quad 0 \leq k \leq \lfloor \frac{n}{p} \rfloor, \quad 0 \leq \underbrace{m-pk \leq n \bmod p}$

$0 \leq m-pk \leq n \bmod p < p-1$

$\Rightarrow m-pk < p$

$m < p+pk$

$\frac{m}{p}-1 < k.$

$m-pk \leq n-p\lfloor \frac{n}{p}\rfloor$

$m-n+p\lfloor \frac{n}{p}\rfloor \leq pk$

$\frac{m-n}{p}+\lfloor \frac{n}{p}\rfloor \leq k.$

$m-pk \geq 0$

$m \geq pk$

$k \leq \frac{m}{p}.$

So: $\frac{m}{p}-1 < k \leq \frac{m}{p}$. is when k is nonzero.

Hence, $\quad k=\lfloor \frac{m}{p}\rfloor \Leftrightarrow \frac{m}{p}-1 < k \leq \frac{m}{p}$

Thus: $\displaystyle\sum_{k\geq 0} \binom{n \bmod p}{m-pk}\binom{\lfloor \frac{n}{p}\rfloor}{k}$

$$= \binom{n \bmod p}{m-p\lfloor \frac{m}{p}\rfloor}\binom{\lfloor \frac{n}{p}\rfloor}{\lfloor \frac{m}{p}\rfloor}$$

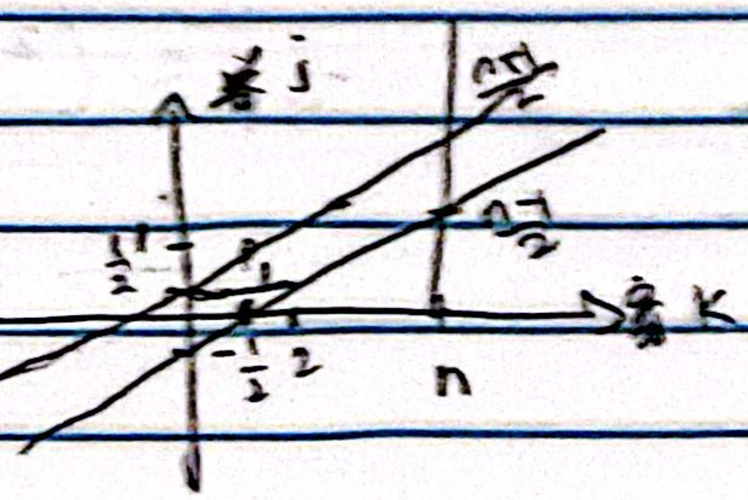
$$= \boxed{\binom{n \bmod p}{m \bmod p}\binom{\lfloor \frac{n}{p}\rfloor}{\lfloor \frac{m}{p}\rfloor}}$$

64. $\displaystyle\sum_{k=0}^{n} \binom{n}{k} \Big/ \left\lceil \frac{k+1}{2} \right\rceil$

$$= \sum_{k,j} \frac{\binom{n}{k}}{j} \left[j = \left\lceil \frac{k+1}{2} \right\rceil \right] [0 \le k \le n]$$

$$= \sum_{k,j} \frac{\binom{n}{k}}{j} \left[\frac{k-1}{2} < j \le \frac{k+1}{2} \right] [0 \le k \le n]$$

$$= \sum_{k,j} \frac{\binom{n}{k}}{j} \left[2j-1 \le k < 2j+1 \right]$$



$$= \sum_{j \ge 1} \frac{\binom{n}{2j-1} + \binom{n}{2j}}{j}$$

$$= \sum_{j \ge 0} \frac{\binom{n}{2j} + \binom{n}{2j+1}}{j+1} = \sum_{j \ge 0} \frac{\binom{n+1}{2j+1}}{j+1}$$

$$= \sum_{j \ge 0} \frac{(n+1)^{\underline{2j+1}}}{(2j+1)!\,(j+1)}$$

Recall $\binom{r}{k} = \frac{r}{k}\binom{r-1}{k-1}$

$$= \sum_{j \ge 0} \frac{2}{n+2} \binom{n+2}{2j+2} \quad \Rightarrow \quad \binom{n+2}{2j+2} = \frac{n+2}{2j+2}\binom{n+1}{2j+1}$$

$$\frac{2j+2}{n+2}\binom{n+2}{2j+2} = \binom{n+1}{2j+1}$$

$$= \frac{2}{n+2} \sum_{j \ge 0} \binom{n+2}{2j+2}$$

$$= \frac{2^{n+2} - 2}{n+2}$$

$= \frac{2}{n+2} \sum_{j \geq 0} \binom{n+2}{2j+2}$. Looks like binom Theorem...

consider: $\sum_{j \geq 0} \binom{n}{j} (1)^{n-j} (1)^j = 2^n$

Hence $\sum_{j \geq 0} \binom{n+2}{j} = 2^{n+2}$.

Now, consider even and odd terms.

$E(x) = \sum_{j \geq 0} \binom{n+2}{2j} x^{2j}$, $O(x) = \sum_{j \geq 0} \binom{n+2}{2j+1} x^{2j+1}$.

Notice $E(1) + O(1) = 2^{n+2}$.

Now, consider $\sum_{j \geq 0} \binom{n+2}{j} (1)^{n+2-j} (1)^j = 2^{n+2}$

$= \sum_{j \geq 0} \binom{n+2}{j} \cancel{(1)^{n+2-j}} (-1)^j = 0$

$= \binom{n+2}{0} + \binom{n+2}{1}(-1) + \binom{n+2}{2} + \dots$

So, $E(-1) + O(-1) = 0$

But $E(-1) = E(1)$ $\Rightarrow E(1) = -O(-1)$

And: $E(1) + O(-1) = 0$

$E(1) + O(1) = 2^{n+2}$

$O(1) - O(-1) = 2^{n+2}$.

Thus, ~~contradiction~~ Therefore $O(1) - (-O(1)) = 2^{n+2}$

$O(1) = 2^{n+1}$.

And $E(1) = 2^{n+1}$.

$$\sum_{j\geq 0} \binom{n+2}{2j} = 2^{n+1} = 1 + \binom{n+2}{2} + \binom{n+2}{4} + \cdots$$

So, $\displaystyle\sum_{j\geq 0} \binom{n+2}{2j+1} = \binom{n+2}{2} + \binom{n+2}{4} + \cdots$

So, $\displaystyle\sum_{j\geq 0} \binom{n+2}{2j+1} = \sum_{j\geq 0} \binom{n+2}{2j} - 1 = \boxed{2^{n+1} - 1.}$

Thus: Ans: $\displaystyle\frac{2}{n+2} \sum_{j\geq 0} \binom{n+2}{2j} = \boxed{\frac{2^{n+2} - 2}{n+2}}$

65. $\sum_k \binom{n-1}{k} n^{-k} (k+1)!$

$$= \sum_{0 \le k \le n-1} \frac{(n-1)^{\underline{k}}}{k!} n^{-k} \cdot (k+1)!$$

$$= \left(\sum_k \frac{(n-1)^{\underline{k}}}{n^k} \cdot (k+1) \right) \delta k$$

$$\frac{t_{k+1}}{t_k} = \frac{(n-1)^{\underline{k+1}}}{n^{k+1}} (k+2) \cdot \frac{n^k}{(k+1)(n-1)^{\underline{k}}}$$

$$= \frac{(n-k-1)(k+2)}{n(k+1)} = \frac{p(k+1)}{p(k)} \cdot \frac{q(k)}{r(k+1)}$$

aha! This is easily evaluated using Gosper!

we have $p(k) = k+1$, $q(k) = n-k-1$
$r(k) = n$.

$Q = -k-1$, $R = 2n-k-1$

$\deg Q = \deg R$

so $d = \deg P - \deg Q = 0$

Thus $p(k) = s(k+1)q(k) - s(k)r(k)$

$\Rightarrow k+1 = \alpha_0(n-k-1) - \alpha_0 n$

$k+1 = (-k-1)\alpha_0$

so $\alpha_0 = -1$.

Thus $T(k) = (-1)(n) \frac{(n-1)^{\underline{k}}}{n^k} \cancel{(k+1)}$

$\cancel{k+1}$

So $\displaystyle\sum_k \frac{(n-1)^{\underline{k}}}{n^k}(k+1)\,\delta k$

$$= -\frac{(n-1)^{\underline{k}}}{n^{k-1}} + C.$$

Now, $\displaystyle\sum_{0\le k\le n-1} \frac{(n-1)^{\underline{k}}}{k!}\, n^{-k}\cdot(k+1)!$

$$= \sum_{0\le k\le n} \frac{(n-1)^{\underline{k}}}{k!}\, n^{-k}\,(k+1)!\,\delta k$$

$$= \left[\frac{-(n-1)^{\underline{k}}}{n^{k-1}}\right]_0^n$$

$$= -\frac{(n-1)^{\underline{n}}}{n^{n-1}} + \frac{(n-1)^{\underline{0}}}{n^{0-1}}$$

$$= -\frac{(n-1)(n-2)\dots\cancel{(n-n+1)}^{\,0}}{n^{n-1}} + \frac{1}{n^{-1}}$$

$$= \frac{0}{n^{n-1}} + \frac{1}{n^{-1}} = \boxed{n}$$

$$k[k \le \tfrac{n}{2}] - k(1-[k \le \tfrac{n}{2}])$$
$$-k + [k \le \tfrac{n}{2}](n+1)$$

68. $$\sum_k \binom{n}{k} \underbrace{\min(k, n-k)}_{k[k \le (n-k)] + (n-k)[n-k < k]}$$

$$= \sum_k \binom{n}{k}\left(k[k \le \tfrac{n}{2}] - k[k > \tfrac{n}{2}]\right) \quad ①$$

$$+ \sum_k \binom{n}{k} n[k > \tfrac{n}{2}]. \quad ②$$

① $$\sum_k \binom{n}{k} k\left([k \le \tfrac{n}{2}] - [k > \tfrac{n}{2}]\right)$$

$$= \sum_k \binom{n}{k} k\left([1 \le k \le \tfrac{n}{2}] - [\tfrac{n}{2} < k \le n]\right)$$

$$= \binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}, \ k \ne 0. \qquad \binom{n-1}{n-k-1}$$

$$= \sum_{1 \le k \le \lfloor \frac{n}{2} \rfloor} n\binom{n-1}{k-1} - \sum_{\lfloor \frac{n}{2} \rfloor + 1 \le k \le n} n\binom{n-1}{k-1}$$

$$= \sum_{1 \le k \le \lfloor \frac{n}{2} \rfloor} n\binom{n-1}{k-1} - \sum_{\lfloor \frac{n}{2} \rfloor + 1 \le k \le n} n\binom{n-1}{n-k}$$

$$\binom{n-1}{k-1} = \binom{n-1}{n-1-(k-1)} = \binom{n-1}{n-k}$$

replace k with $1+n-k$.

$$= \sum_{1 \le k \le \lfloor \frac{n}{2} \rfloor} n\binom{n-1}{k-1} - \sum_{\lfloor \frac{n}{2} \rfloor + 1 \le 1+n-k \le n} n\binom{n-1}{k-1}$$

$$\cancel{1+n-k} \ge \lfloor \tfrac{n}{2} \rfloor + 1$$
$$k \le n - \lfloor \tfrac{n}{2} \rfloor$$

$$= \sum_{1 \le k \le \lfloor \frac{n}{2} \rfloor} n\binom{n-1}{k-1} - \sum_{1 \le k \le n - \lfloor \frac{n}{2} \rfloor} n\binom{n-1}{k-1}$$

Thus: $\sum_{1\le k\le \lfloor \frac{n}{2}\rfloor} n\binom{n-1}{k-1} - \sum_{1\le k\le n-\lfloor\frac{n}{2}\rfloor} n\binom{n-1}{k-1}$

if n is even: $n=2j$

$1\le k\le j$, and $1\le k\le 2j-j$

$1\le k\le j$ $\quad j+\frac{1}{2}$

if n is odd: $n=2j+1$ $\quad \frac{n-1}{2}$

$1\le k\le j$ and $1\le k\le 2j+1-\lfloor\frac{2j+1}{2}\rfloor$

$1\le$ $\quad \to 1\le k\le 2j+1-j$

$j=\lfloor\frac{n}{2}\rfloor$. $\quad 1\le k\le j+1$

Thus, the sum ① evaluates to

$$0[n \text{ even}] - n\binom{n-1}{\lfloor\frac{n}{2}\rfloor}[n \text{ odd}]$$

② $\sum_k \binom{n}{k} n[k>\frac{n}{2}]$

n is odd: $\frac{2^n}{2} = 2^{n-1}\cdot n$

n is even: $\left(\frac{2^n - \binom{n}{\frac{n}{2}}}{2}\right)n$

$= n2^{n-1} - \frac{n\binom{n}{n/2}}{2}$

Thus, this evaluates to $n2^{n-1} - \frac{n}{2}\binom{n}{n/2}[n \text{ is even}]$

putting the two together: $n2^{n-1} - \frac{n}{2}\binom{n}{n/2}[n \text{ is even}] - n\binom{n-1}{\lfloor\frac{n}{2}\rfloor}[n \text{ is odd}]$

$= n2^{n-1} - \frac{n}{2}\binom{n}{n/2}[n \text{ is even}] - n\binom{n-1}{\lfloor\frac{n}{2}\rfloor} + n\binom{n-1}{\lfloor\frac{n}{2}\rfloor}[n \text{ is even}]$

$$n\binom{n-1}{\frac{n}{2}-1}$$

$$= n2^{n-1} - n\binom{n-1}{\lfloor \frac{n}{2} \rfloor} + \left(-\frac{n}{2}\binom{n}{n/2} + n\binom{n-1}{\lfloor \frac{n}{2} \rfloor}\right) \text{ (n is even)}$$

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1} \qquad \frac{n^{n/2}}{(n/2)!}$$

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

$$\frac{n(n-1)\dots(n-\frac{n}{2}+1)}{n(n-1)\dots(\frac{n}{2}+1)}$$

$$(n-1)(n-2)\dots(\frac{n}{2})$$

$$(n-1)\dots(\frac{n}{2}+1)(-n+\frac{n}{2})$$

$$= n2^{n-1} - n\binom{n-1}{\lfloor \frac{n}{2} \rfloor} + \left(-\frac{n}{2}\left(\frac{n(n-1)\dots(\frac{n}{2}+1)}{(n/2)!}\right) + n\left(\frac{(n-1)\dots(\frac{n}{2})}{(n/2)!}\right)\right)$$

[n is even]

$$= n2^{n-1} - n\binom{n-1}{\lfloor \frac{n}{2} \rfloor} + \left(-\frac{n}{2}\binom{n}{n/2} + \frac{n}{2}\binom{n}{n/2}\right) \text{ [n is even]}$$

$$= \boxed{n\left(2^{n-1} - \binom{n-1}{\lfloor \frac{n}{2} \rfloor}\right)}$$

69.

$$\min_{k_1,\dots,k_m \geq 0} \sum_{j=1}^{m} \binom{k_j}{2}$$

$$k_1 + \dots + k_m = n$$

$$= \min_{k_1,\dots,k_m \geq 0} \sum_{j=1}^{m} \binom{k_j}{2} [k_1 + \dots + k_m = n]$$

$$= \min_{k_1,\dots,k_m} \left(\sum_{j=1}^{m} \frac{(k_j)(k_j-1)}{2!} [k_1 + \dots + k_m = n] \right)$$

$$= \min_{k_1,\dots,k_m} \left(\frac{1}{2!} \sum_{j=1}^{m} (k_j^2 - k_j) \left[\sum_{j=1}^{m} k_j = n \right] \right)$$

$$= \min_{k_1,\dots,k_m \geq 0} \left(\frac{1}{2!} \sum_{j=1}^{m} k_j^2 \left[\sum_{j=1}^{m} k_j = n \right] \right) - \frac{1}{2!} n$$

$$\sum_{j=1}^{m} k_j \sum_{j=1}^{m} k_j = n^2$$

consider $\sum_{j} k_j x^j = A(x)$

$$A(1) = \sum_{j} k_j = n$$

$$\sum_{j=1}^{m} k_j k_{m-j} = [x^m] A(x) \cdot A(x) \Big|_{x=1}$$

$$= \sum_{j=1}^{m} k_j^2 + \dots \text{ terms.}$$

since terms ≥ 0,

we can't just let them be 0.

70. $\sum_k \binom{n}{k}\binom{2k}{k}\left(-\frac{1}{2}\right)^k$

$\binom{-1/2}{k} = \left(-\frac{1}{4}\right)^k \binom{2k}{k}$

$(-4)^k \binom{-1/2}{k} = \binom{2k}{k}$

$= \sum_k \binom{n}{k}(-1)^k \cdot \binom{2k}{k}\left(\frac{1}{2^k}\right)$

$= \sum_k \binom{n}{k}(2^k)\binom{-1/2}{k}$

$= \sum_k \binom{n}{k} 2^k \binom{-1/2}{k}$

$= F\left(-n, -n; \frac{1}{2}-n; \frac{1}{2}\right)$

$= F\left(-\frac{n}{2}, -\frac{n}{2}; \frac{1}{2}-n; 1\right)$

71. Let $S_n = \sum_{k\geq 0} \binom{n+k}{m+2k} a_k$

where $A(z) = \sum_{k\geq 0} a_k z^k$

a. express $S(z) = \sum_{n\geq 0} S_n z^n$ in terms of $A(z)$.

$n+k \leq m+2k$

$$= \sum_{n\geq 0} \left(\sum_{k\geq 0} \binom{n+k}{m+2k} a_k \right) z^n$$

$0 \leq n \leq m+k$

$$= \sum_{k\geq 0} a_k \underbrace{\sum_{n\geq 0} \binom{n+k}{m+2k} z^n}$$

$\binom{n+k}{n-m-k}$

recall $\sum_{k\geq 0} \binom{n+k}{n} z^k = \frac{1}{(1-z)^{n+1}}$

or $\sum_{n\geq 0} \binom{k+n}{k} z^n = \frac{1}{(1-z)^{k+1}}$

$$\sum_{k\geq n\geq 0} \binom{n}{k} z^n = \frac{z^k}{(1-z)^{k+1}}$$

and $\sum_{0\leq n\leq m+k} \binom{n+k}{m+2k} z^n = \sum_{0\leq n-k\leq m+k} \binom{n}{m+2k} z^{n-k}$

$$= \left(\sum_{k\leq n\leq m+2k} \binom{n}{m+2k} z^n \right) / z^k.$$

$$\sum_{n\geq 0}\binom{n+k}{m+2k}z^n \quad , \quad \sum_{n\geq 0}\binom{n}{k}z^n = \frac{z^k}{(1-z)^{k+1}}$$

$$\cancel{\sum_{n\geq 0}\binom{k+n}{k}z^n}$$

$m+2k=l$

$$= \frac{\sum_{n\geq k}\binom{n}{m+2k}z^n}{z^k} = \frac{\sum_{k\leq n\leq m+2k}\binom{n}{m+2k}z^n}{z^k} = \frac{\left(\frac{z^{m+2k}}{(1-z)^{m+2k+1}}\right)}{z^k}$$

and $\sum_{0\leq n\leq k}\binom{n}{k}z^n$

$$\text{so } S(z) = \sum_{k\geq 0} a_k \frac{z^{m+k}}{(1-z)^{m+2k+1}} = \frac{z^m}{(1-z)^{m+1}}\sum_{k\geq 0} a_k \cdot \frac{z^k}{(1-z)^{2k}}$$

$$= \boxed{\frac{z^m}{(1-z)^{m+1}} A(z/(1-z)^2)}$$

7. $\sum_{k\geq 0}\binom{n+k}{m+2k}\binom{2k}{k}\frac{(-1)^k}{k+1}$

$$a_k = \binom{2k}{k}\frac{(-1)^k}{k+1}$$

$$A(z) = \sum_{k\geq 0}\binom{2k}{k}\frac{(-z)^k}{k+1} = \frac{\sqrt{1+4z}-1}{2z}$$

Hence $A(z/(1-z)^2) = 1-z$

but $\frac{z^m}{(1-z)^m}[z^n] = \binom{n-1}{n-m}$.