APPENDIX

Proposition 1. Given an arbitrary controller C, the induced closed-loop system $\mathcal{T}_C$, its constructed belief transition system $\mathcal{B}_C$ and LTL formula ϕ, if $\exists\tau \in \mathsf{Path}(\mathcal{B}_C)$ satisfies $\tau \notin (Q_{win})^\omega$, then $\mathsf{Trace}(\mathcal{T}_C) \nsubseteq \mathsf{Word}(\phi)$.

Proof. From Belta et al. (2017), we can draw a conclusion that for any finite product path $\tilde{\tau}^* = \tilde{x}_0\tilde{x}_1\tilde{x}_2\ldots\tilde{x}_m$, $\tilde{X}_{win} = \mathtt{Win}(\tilde{X}_F)$, if $\tilde{x}_m \notin \tilde{X}_{win}$, then $\forall a_n \in Act.n \geq m \geq 0, \exists\tilde{\tau} = \tilde{x}_0\tilde{x}_1\tilde{x}_2\ldots\tilde{x}_m\cdots \in \mathsf{Path}(\tilde{T}), \tilde{x}_n \xrightarrow{a_n} \tilde{x}_{n+1}$ such that $inf(\tilde{\tau}) \cap \tilde{X}_F = \emptyset$ and $\mathsf{Trace}(\tilde{\tau}) \nsubseteq \mathsf{Word}(\phi)$.

This conclusion can be directly extended to belief space, given a finite set Q_F, if an arbitrary finite belief path $\tau_B^* = q_0q_1q_2\ldots q_m$ with $q_m \notin R_0 = \mathtt{Win}(Q_F)$, then $\forall a_n \in Act, n \geq m \geq 0, \exists\tau_B = q_0q_1q_2\ldots q_m\ldots, q_n \xrightarrow{a_n} q_{n+1}$ such that $inf(\tau_B) \cap Q_F = \emptyset$. Also, for an arbitrary infinite path of belief states τ_B, if for $\forall\tilde{\tau} \in \tilde{\tau}(\tau_B), \mathsf{Trace}(\tilde{\tau}) \subseteq \mathsf{Word}(\phi)$, there must be $inf(\tau_B) \cap Q_F \neq \emptyset$, because $\mathsf{Trace}(\tilde{\tau}) \subseteq \mathsf{Word}(\phi)$ leads to $inf(\tilde{\tau}) \cap \tilde{X}_F \neq \emptyset$ and $Q_F = \{q \in Q \mid \exists\tilde{x} \in q, \tilde{x} \in \tilde{X}_F\}$. So, if an arbitrary controller cannot deterministically avoid the system from visiting $Q\backslash R_0$, then the closed-loop system cannot satisfy $\mathsf{Trace}(\mathcal{T}_C) \subseteq \mathsf{Word}(\phi)$.

Also, we can extend this conclusion in another direction, from product path visiting states $\tilde{x}_m \notin \tilde{X}_{win}$ to belief path visiting belief containing $\tilde{x}_m \notin \tilde{X}_{win}$. We denote $\{\tilde{\tau}\}_{H(\tau_B)}$ as the set of paths having the same observation with belief path τ_B. Given an arbitrary finite belief path $\tau_B^* = q_0q_1q_2\ldots q_m$, if $\exists\tilde{x}_m \notin \tilde{X}_{win}, \tilde{x}_m \in q_m$, then $\exists\tilde{\tau}^* \in \{\tilde{\tau}\}_{H(\tau_B^*)}, \tilde{\tau}^* = \tilde{x}_0\tilde{x}_1\tilde{x}_2\ldots\tilde{x}_m$. Then we have $\forall a_n \in Act, n \geq m, \exists\tau_B = q_0q_1q_2\ldots q_m\cdots \in \mathsf{Path}(\mathcal{B}), q_n \xrightarrow{a_n} q_{n+1}, \exists\tilde{\tau} \in \{\tilde{\tau}\}_{H(\tau_B)}$ such that $\mathsf{Trace}(\tilde{\tau}) \nsubseteq \mathsf{Word}(\phi)$, which leads to $\mathsf{Trace}(\tau_B) \nsubseteq \mathsf{Word}(\phi)$. The conclusion can be extended to belief path visiting such belief's adversary proper attractor. We denote the set of states occurs in path τ as $\mathtt{Occ}(\tau)$. According to the definition of adversary proper attractor, given a belief set Q_a, assuming the adversary proper attractor converges at step m', where $\mathtt{Avoid}(Q_a) = \mathtt{Avoid}^{(n)}(Q_a) = \mathtt{Avoid}^{(m')}(Q_a), \forall n \geq m' \geq 0$, then for $\forall q_k \in \mathtt{Avoid}(Q_a), \forall a_m \in Act, 0 \leq m \leq m'-1$, such that $\exists\tau_B^* = q_kq_{k+1}\ldots q_{k+m'}, q_{k+m} \xrightarrow{a_m} q_{k+m+1}, \mathtt{Occ}(\tau_B) \cap Q_a \neq \emptyset$. Given an arbitrary finite belief path $\tau_B^* = q_0q_1q_2\ldots q_m$ and a belief set Q_n, where $\forall q_n \in Q_n, \exists\tilde{x} \in q_n$ such that $\tilde{x} \notin \tilde{X}_{win}$. If $q_m \in \mathtt{Avoid}(Q_n)$, where $\mathtt{Avoid}(Q_n)$ converges at step t, then $\forall a_{m'} \in Act, \exists m \leq m' \leq m+t-1, \exists\tau_B = q_0q_1q_2\ldots q_m\cdots \in \mathsf{Path}(\mathcal{B}), q_{m'} \xrightarrow{a_{m'}} q_{m'+1}$ such that $\mathtt{Occ}(\tau_B) \cap Q_n \neq \emptyset$, which leads to the conclusion of $\mathsf{Trace}(\tau_B) \nsubseteq \mathsf{Word}(\phi)$. So, if an arbitrary controller cannot deterministically avoid the system from visiting $x \notin \tilde{X}_{win}$, then the closed-loop system cannot satisfy $\mathsf{Trace}(\mathcal{T}_C) \subseteq \mathsf{Word}(\phi)$.

Based on the above discussions, we discuss two situations of belief deletion during the ith iteration of Q_{win}'s construction, given R_i, R_{i+1} and $\widehat{\mathtt{Win}}_{R_i}(\hat{Q}_{F,R_i})$.

Firstly, we consider beliefs in $R_{i,unq}$ and $R_{i,rem}$. For an arbitrary $q \in R_{i,unq}$, assuming $\tilde{x} \in q, (\tilde{x}, q) \notin \widehat{\mathtt{Win}}_{R_i}(\hat{Q}_{F,R_i})$. From the definition of interconnected transition system's transition function, if $(\tilde{x}', q') \in \hat{\Delta}((\tilde{x}, q), a)$, then $\Delta_B(q, a) \subseteq R_i$. For any $\hat{\tau}^* = \hat{q}_0\hat{q}_1\ldots\hat{q}_m$, if $\hat{q}_m \notin \widehat{\mathtt{Win}}_{R_i}(\hat{Q}_{F,R_i})$, then for $\forall a_n \in Act, n \geq m \geq 0, \exists\hat{\tau} = \hat{q}_0\hat{q}_1\ldots\hat{q}_m\cdots \in \mathsf{Path}(\mathcal{B}(R_i)), \hat{q}_{n+1} \in \hat{\Delta}^i(\hat{q}_n, a_n)$, such that $inf(\hat{\tau}) \cap \hat{Q}_{F,R_i} = \emptyset$. So for any infinite belief path $\tau_B = q_0q_1\ldots q_m\ldots, q_m \in R_{i,unq}$, if $\forall a_n \in Act, n \geq m \geq 0, q_n \xrightarrow{a_n} q_{n+1}$ such that $\Delta_B(q_n, a_n) \subseteq R_i$, then $\exists\tilde{\tau} \in \{\tilde{\tau}\}_{H(\tau_B)}, inf(\tilde{\tau}) \cap \tilde{X}^i_{CF} = \emptyset, \tilde{X}^i_{CF} = \{\tilde{x}_F \in \tilde{X}_F \mid (\tilde{x}_F, q) \in \hat{Q}_{F,R_i}\}$. Also, because $\forall q_n \in \tau_B, \Delta_B(q_n, a_n) \subseteq R_i$, then $\forall\tilde{\tau} \in \{\tilde{\tau}\}_{H(\tau_B)}, \mathtt{Occ}(\tilde{\tau}) \cap (\tilde{X}_F\backslash\tilde{X}^i_{CF}) = \emptyset$. So, for any infinite belief path $\tau_B = q_0q_1\ldots q_m\ldots, q_m \in R_{i,unq}$, if $\forall a_n \in Act, n \geq m \geq 0, q_n \xrightarrow{a_n} q_{n+1}$ such that $\Delta_B(q_n, a_n) \subseteq R_i$, then $\exists\tilde{\tau} \in \{\tilde{\tau}\}_{H(\tau_B)}, inf(\tilde{\tau}) \cap \tilde{X}_F = \emptyset$, $\mathsf{Trace}(\tau_B) \nsubseteq \mathsf{Word}(\phi)$. This also means that if $\exists a_{n'} \in Act, n' \geq m \geq 0, q_{n'} \xrightarrow{a_{n'}} q_{n'+1}$ such that $\forall\tilde{\tau} \in \{\tilde{\tau}\}_{H(\tau_B)}, inf(\tilde{\tau}) \cap \tilde{X}_F \neq \emptyset$, then $\exists\tau_B$, where $q_{n'+1} \notin R_i$, that is $(\exists i' < i, q_{n'+1} \in (R_{i'}\backslash R_i)) \vee (q_{n'+1} \in (Q\backslash R_0))$. Set $R_{i,unq}$ is extended to $R_{i,rem}$ because for any belief path $\tau_B^* = q_0q_1\ldots q_m, q_m \in R_{i,rem}$, then $\forall a_n \in Act, n \geq m, \exists\tau_B = q_0q_1q_2\ldots q_m\cdots \in \mathsf{Path}(\mathcal{B}), q_n \xrightarrow{a_n} q_{n+1}$ such that $\mathtt{Occ}(\tau_B) \cap (R_{i,unq} \cup (Q\backslash R_i)) \neq \emptyset$.

Secondly, we consider beliefs in $(R_i\backslash R_{i,rem})\backslash R_{i+1}$. For an arbitrary finite belief path $\tau_B^* = q_0q_1\ldots q_m, q_m \in R_i\backslash R_{i,rem}$, if $q_m \notin R_{i+1}$, then $\forall a_n \in Act, n \geq m$ such that $\exists\tau_B = q_0q_1\ldots q_m\cdots \in \mathsf{Path}(\mathcal{B}), q_n \xrightarrow{a_n} q_{n+1}, \Delta_B(q_n, a_n) \subseteq R_i\backslash R_{i,rem}, inf(\tau_B) \cap ((R_i\backslash R_{i,rem}) \cap Q_F) = \emptyset$. Also, $inf(\tau_B) \cap ((Q\backslash(R_i\backslash R_{i,rem})) \cap Q_F) = \emptyset$ because $inf(\tau_B) \cap (Q\backslash(R_i\backslash R_{i,rem})) = \emptyset$. So $\forall a_n \in Act, n \geq m$ such that $\exists\tau_B = q_0q_1\ldots q_m\cdots \in \mathsf{Path}(\mathcal{B}), q_n \xrightarrow{a_n} q_{n+1}, \Delta_B(q_n, a_n) \subseteq R_i\backslash R_{i,rem}$, $inf(\tau_B) \cap Q_F = \emptyset$. This means that if $\exists a_{n'} \in Act, n' \geq m, q_{n'} \xrightarrow{a_{n'}} q_{n'+1}$, such that $\forall\tau_B, inf(\tau_B) \cap Q_F \neq \emptyset$, then $\exists\tau_B$, where $q_{n'+1} \notin R_i\backslash R_{i,rem}$, that is $(\exists i' < i, q_{n'+1} \in (R_{i'}\backslash R_i)) \vee (q_{n'+1} \in Q\backslash R_0) \vee (q_{n'+1} \in R_{i,rem})$.

Based on the two situations discussed above, we can make a deduction that, given any finite belief path $\tau_B^* = q_0q_1\ldots q_m, q_m \in R_i\backslash R_{i+1}$, if $\exists a_n \in Act, n \geq m$ such that $\forall\tau_B = q_0q_1\ldots q_m\ldots, q_n \xrightarrow{a_n} q_{n+1}, \mathsf{Trace}(\tau_B) \subseteq \mathsf{Word}(\tau_B)$, then must $\exists q_{n'} \in \tau_B$ such that $(\exists i' < i, q_{n'} \in (R_{i'}\backslash R_i)) \vee (q_{n'} \in (Q\backslash R_0))$. And for ${\tau_B^*}' = q_0q_1\ldots q_{n'}, q_{n'} \in R_{i'}\backslash R_i$, assuming $q_{n'} \in R_{i'}\backslash R_{i'+1}$, then $\exists q_{n''} \in \tau_B'$ such that $(\exists i'' < i', q_{n''} \in (R_{i''}\backslash R_{i'})) \vee (q_{n''} \in (Q\backslash R_0))$. This leads to a convergence of given any finite belief path $\tau_B^* = q_0q_1\ldots q_m, q_m \in R_i\backslash R_{i+1}$, if $\exists a_n \in Act, n \geq m$ such that $\forall\tau_B = q_0q_1\ldots q_m\ldots, q_n \xrightarrow{a_n} q_{n+1}, \mathsf{Trace}(\tau_B) \subseteq \mathsf{Word}(\tau_B)$, then must $\mathtt{Occ}(\tau_B) \cap Q\backslash R_0 \neq \emptyset$, which is contradict to $\mathsf{Trace}(\tau_B) \subseteq \mathsf{Word}(\phi)$. So we prove that if $\exists\tau \in \mathsf{Path}(\mathcal{B}_C), \tau \notin (Q_{win})^\omega$, then $\mathsf{Trace}(\mathcal{T}_C) \nsubseteq \mathsf{Word}(\phi)$.

Proposition 2. Given a belief transition system $\mathcal{B}$, an arbitrary initial belief state $q \in Q_{win}$ and a non-blocking controller C, if the system visits a state q' with $\forall(x_{\mathtt{int}}, x_{\mathtt{cur}}, b) \in q_{\mathtt{aug}}, b = 1$ then all finite paths from q to q' have reached accepting states.

Proof. Assume that $x_{\mathtt{int}} \in q, (x_{\mathtt{int}}, x_{\mathtt{cur}}, b) \in q_{\mathtt{aug}}$ and $x_{\mathtt{act}} = x_{\mathtt{cur}}$ is randomly chosen for action generation according to the algorithm. And after m steps of transition, $\forall(x_a, x_b, b) \in q_{\mathtt{aug}}$ if $x_a = x_{\mathtt{int}}$ then $b = 1$. According to the algorithm, $x_{\texttt{act}} = x'_{\texttt{cur}}$ is picked from $(x'_{\texttt{int}}, x'_{\texttt{cur}}, b) \in q_{\texttt{aug}}$ where $b = 0$. We will firstly prove that after m steps, all possible path starting from initial inner state $x_{\texttt{int}}$ must contain at least one accepting state. We denote $\{\tilde{\tau}\}^{m+1}_{x_{\texttt{int}}}$ and $\{\tau_B\}^{m+1}_{q}$ as set of all paths starting from $x_{\texttt{int}}$ and q with length equals to $m+1$. $\tilde{\tau} = \tilde{x}_0\tilde{x}_1 \dots \tilde{x}_m$ is valid, so $\forall \tilde{x}_n \in \tilde{\tau}, \tilde{x}_n \in q_n, 0 \leq n \leq m$. We denote $\{x_{\texttt{cur}}\}^{n}_{x_{\texttt{int}}} = \{x_{\texttt{cur}} \in q_n \mid \exists (x_{\texttt{cur}}, x_{\texttt{int}}, 0) \in q_{\texttt{aug}}\}$ as the current inner state set originate from $x_{\texttt{int}}$ in q_n that have not visited accepting states And we have $\{x_{\texttt{cur}}\}^{m+1}_{x_{\texttt{int}}} = \emptyset$ according to the assumption. If there exists a path of product states $\tilde{\tau} \in \{\tilde{\tau}\}^{m+1}_{x_{\texttt{int}}}$ that $\tilde{\tau}$ visits none of the accepting states in $\tilde{X}_{acc}$. Then for $\forall \tilde{x}_n \in \tilde{\tau}$, if $\tilde{x}_n \in \{x_{\texttt{cur}}\}^{n}_{x_{\texttt{int}}}$, we have $\tilde{x}_{n+1} \in \{x_{\texttt{cur}}\}^{n+1}_{x_{\texttt{int}}}$ under controller C because of the augment state update function. So we can derive the fact that $\tilde{x}_m \in \{x_{\texttt{cur}}\}^{m+1}_{x_{\texttt{int}}}$, which is contradict to the condition that after m steps of transition, $\{x_{\texttt{cur}}\}^{m+1}_{x_{\texttt{int}}} = \emptyset$. So if $\forall (x_a, x_b, b) \in q_{\texttt{aug}}, x_a = x_{\texttt{int}}, b = 1$, then paths of product states originated from $x_{\texttt{int}}$ must all have reached accepting states. This leads to a further conclusion that if the system reaches a belief state with $\forall x_{\texttt{int}} \in q, (x_{\texttt{int}}, x_{\texttt{cur}}, b) \in q_{\texttt{aug}}, b = 1$ then all paths originated from any potential initial inner states $x_{\texttt{int}} \in q$ must all have reached accepting states.

Proposition 3. (non-blockingness) Given a belief transition system $\mathcal{B}$ with an arbitrary belief state $q_{init} \in Q_{win}$ and a LTL specification ϕ, under C the closed-loop system $\mathcal{T}_C$ satisfies $\mathsf{Trace}^*(\mathcal{T}_C) \subseteq \overline{\mathsf{Word}(\phi)}$.

Proof. From Proposition 2, we can conclude that, given a finite belief path $\tau^*_B = q_0 \dots q_m \in \mathsf{Path}(\mathcal{B}_C)^*$ with q_m satisfying $\forall (x_{\texttt{int}}, x_{\texttt{cur}}, b) \in q_{\texttt{aug}}, b = 1$ then for $\forall \tau^* \in \mathsf{Path}(\mathcal{T}_C)^*, H(\tau^*) = H(\tau^*_B)$, this finite path τ^* must have reached accepting states. So, any finite belief path $\tau^*_B \in \mathsf{Path}(\mathcal{B}_C)^*$ is in the prefix set of all finite paths that have reached accepting states at least once, which is a subset of the prefix set of all infinite paths infinitely visiting accepting states. Also, from Belta et al. (2017) we know that if an infinite path $\tau \in \mathsf{Path}(\mathcal{T}_C)$ infinitely visit accepting states then $\mathsf{Trace}(\tau) \in \mathsf{Word}(\phi)$. So, we prove that $\mathsf{Trace}^*(\mathcal{T}_C) \subseteq \overline{\mathsf{Word}(\phi)}$.

Proposition 4. Given a belief transition system $\mathcal{B}$ under a non-blocking controller C, if $\forall q \in Q_{win}$, the system will always visits states satisfying $\forall (x_{\texttt{int}}, x_{\texttt{cur}}, b) \in q_{\texttt{aug}}, b = 1$ in finite steps then C is a sure winning controller.

Proof. Combining Proposition 3, we know that if from $q \in Q_{win}$ under controller C, the system can visits a state with $\forall (x_{\texttt{int}}, x_{\texttt{cur}}, b) \in q_{\texttt{aug}}, b = 1$ in finite steps, then all product paths originated from states in q have reached accepting states in finite steps. According to the algorithm, after reaching a state with $\forall (x_{\texttt{int}}, x_{\texttt{cur}}, b) \in q_{\texttt{aug}}, b = 1$, the computation will be repeated starting from current belief state towards accepting states. So, as the algorithm conducting infinitely, all product paths of the system will surely visit accepting states infinitely such that $\mathsf{Trace}(\mathcal{T}_C) \subseteq \mathsf{Word}(\phi)$.

Proposition 5. We define $Actions(x_{\texttt{cur}}, q, \hat{\mathcal{B}}) \subseteq Act$ as the set of all actions possibly generated by function $\texttt{Action}(x_{\texttt{cur}}, q, \hat{\mathcal{B}})$. Given a belief transition system $\mathcal{B}$ under a non-blocking controller C, if $\forall q \in Q_{win}, \forall x_{\texttt{cur}}, x'_{\texttt{cur}} \in q$ satisfy $Actions(x_{\texttt{cur}}, q, \hat{\mathcal{B}}) = Actions(x'_{\texttt{cur}}, q, \hat{\mathcal{B}})$ then C is a sure winning controller.

Proof. We will first prove that if $\forall q \in Q_{win}, \forall x_{\texttt{cur}}, x'_{\texttt{cur}} \in q$ satisfy $Actions(x_{\texttt{cur}}, q, \hat{\mathcal{B}}) = Actions(x'_{\texttt{cur}}, q, \hat{\mathcal{B}})$ then starting from $\forall q \in Q_{win}$ under controller C, the system can always reach a state with $\forall (x_{\texttt{int}}, x_{\texttt{cur}}, b) \in q_{\texttt{aug}}, b = 1$ in finite steps. Assume given an arbitrary $q_0 = \{x_{\texttt{int},1}, \dots, x_{\texttt{int},n}\} \in Q_{win}$, and in interconnected transition system $(x_{\texttt{int},m}, q_0) \in \texttt{Attr}^{(l_m)}(\hat{Q}_{acc}) \backslash \texttt{Attr}^{(l_m - 1)}(\hat{Q}_{acc})$, we denote $L = \max_{0 \leq m \leq n} l_m$. After L steps of transition, the system generates belief path $\tau_B = q_0 q_1 \dots q_L$ corresponding to a set of finite product paths. It can be inferred that $\nexists H(\tilde{\tau}) = H(\tau_B)$ such that $\tilde{\tau}$ visits none of the accepting states in $\tilde{X}_{acc}$. This is because consider an arbitrary $\tilde{\tau} = \tilde{x}_0 \dots \tilde{x}_L$ such that $H(\tilde{\tau}) = H(\tau_B)$, and at step l the algorithm will take an action $a_l \in Actions(x_{\texttt{act}}, q_l, \hat{\mathcal{B}}) = Actions(x_l, q_l, \hat{\mathcal{B}})$ according to the assumption. The initial state $\tilde{x}_0 \in q_0$, assuming $\tilde{x}_0 = x_{\texttt{int},m} \in \texttt{Attr}^{(l_m)}(\hat{Q}_{acc}) \backslash \texttt{Attr}^{(l_m - 1)}(\hat{Q}_{acc})$ and we have $l_m \leq L$. So after l steps and reaching $\tilde{x}_{l-1}$, according to function $\texttt{Action}$, the system will transfer to $(\tilde{x}_l, q_l) \in \texttt{Attr}^{(l_m - l)}(\hat{Q}_{acc})$, and after l_m steps we will have $(\tilde{x}_{l_m}, q_{l_m}) \in \texttt{Attr}^{(0)}(\hat{Q}_{acc}) = \hat{Q}_{acc}$, which means that $\tilde{x}_{l_m} \in \tilde{X}_{acc}$. So the system will visit a state with $\forall (x_{\texttt{int}}, x_{\texttt{cur}}, b) \in q_{\texttt{aug}}, b = 1$ in L steps, and we can make a deduction that the algorithm converges. Combining Proposition 4, we can prove that if $\forall q \in Q_{win}, \forall x_{\texttt{cur}}, x'_{\texttt{cur}} \in q$ satisfy $Actions(x_{\texttt{cur}}, q, \hat{\mathcal{B}}) = Actions(x'_{\texttt{cur}}, q, \hat{\mathcal{B}})$, our proposed controller becomes sure winning controller.

REFERENCES

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