Oct. 3rd : Orthogonal functions

Let $f, g : [a,b] \to \mathbb{R}$

SCALAR PRODUCT:

$$\langle f, g \rangle_w := \int_a^b f(x)\, g(x)\, w(x)\, dx$$

WHERE $w : [a,b] \to \mathbb{R}$ such that $w(x) > 0$ if $x \in (a,b)$ } w IS THE "WEIGHT" FUNCTION

The largest class of functions for wich $\langle\, , \rangle_w$ makes sense is

$$L^2_w([a,b]) = \left\{ f : [a,b] \to \mathbb{R} \;\middle|\; \underbrace{\int_a^b |f(x)|^2 w(x)\, dx}_{\text{SHOULD BE A LEBESGUE INTEGRAL}} \in \mathbb{R}^+ \right\}$$

If $f, g \in L^2_w$ then $\langle f, g \rangle_w \in \mathbb{R}$.

PROOF: $(f-g)^2 = f^2 + g^2 - 2fg \Rightarrow$

$|fg| \leq |f|^2 + |g|^2$. Multiply by w, integrate this shows that $\int_a^b |fg|\, w\, dx$ exists. Therefore also $\langle f, g \rangle_w$ exists.

THE THEORY MAY BE EXTENDED TO COMPLEX-VALUED FUNCTIONS:

$f, g : [a,b] \to \mathbb{C}$

$$\langle f, g \rangle_w = \int_a^b f(x)\, \overline{g}(x)\, \underbrace{w(x)}_{\text{STILL A REAL, POSITIVE FUNCTION.}}\, dx$$

THEN:

$\langle f, g \rangle_w = \overline{\langle g, f \rangle_w}$ SESQUILINEAR INNER PRODUCT.

I WILL NOT CONSIDER THIS CASE HERE.

f AND g ARE "ORTHOGONAL" WITH RESPECT TO THE WEIGHT w IF

$$\langle f, g \rangle_w = 0$$

EXAMPLES:

INTERVAL: $[0, \pi]$
WEIGHT: $w(x) = 1$

$\{ \sin((n+1)x) \}$ $n = 0, 1, \ldots$

INTERVAL: $[0, \pi]$
WEIGHT: $w(x) = 1$

$\{ \cos(nx) \}$ $n = 0, 1, \ldots$

INTERVAL: $[0, 2\pi]$
WEIGHT: $w(x) = 1$

$\{ \cos(nx), \sin((n+1)x) \}$ $n = 0, 1, \ldots$

THIS IS:

$1, \sin(x), \cos(x), \sin(2x), \cos(2x), \sin(3x), \ldots$

INTERVAL: $[-1, 1]$
WEIGHT: $w(x) = \dfrac{1}{\sqrt{1-x^2}}$

$\{ T_n(x) \}$ $n = 0, 1, \ldots$

CHEBYSHEV POLYNOMIALS

INTERVAL: $[-1, 1]$
WEIGHT: $w(x) = 1$

$\{ L_n(x) \}$ $n = 0, 1, \ldots$

LEGENDRE POLYNOMIALS

\+ MANY OTHER EXAMPLES (BESSEL FUNCTIONS ARE QUITE NOTABLE)

– WHERE DO THESE SEQUENCES COME FROM?

FROM STURM-LIOUVILLE PROBLEMS

– A S-L PROBLEM IS AN EQUATION OF THIS SORT IN THE UNKNOWN FUNCTION u

$$\frac{d}{dx}\left(p(x)\,\frac{du(x)}{dx}\right) + q(x)\,u(x) = -\lambda\, w(x)\, u(x); \qquad \text{OR} \qquad \mathcal{L}u = -\lambda w u$$

$\mathcal{L}$: STURM LIOUVILLE OPERATOR $\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)$; u: UNKNOWN; λ: EIGENVALUE; w: WEIGHT; u: UNKNOWN

$x \in (a,b)$ ← INTERVAL OVER WICH THE EQUATION IS DEFINED.

+ BOUNDARY CONDITIONS

(• PRESCRIPTIONS ON u AT THE BOUNDARIES
e.g. $u(a) = u(b) = 0$ OR $u'(a) = u'(b) = 0$
OR $[u(a) = u(b)$ AND $u'(a) = u'(b)]$
OR $|u(a)|, |u(b)| < +\infty$)

– $u(x) = 0$ IS AN OBVIOUS SOLUTION. FOR MOST VALUES OF λ AND MOST BOUNDARY CONDITIONS, IT'S ALSO THE ONLY SOLUTION

– UNDER MILD CONDITIONS ON p, q, w ($p \in C^1((a,b))$; $p \geq 0$, q, w CONTINUOUS) WE HAVE THE FOLLOWING:

- THERE EXISTS ONLY A COUNTABLY INFINITE SET OF VALUES OF λ THAT ALLOW FOR NON-ZERO SOLUTION. THEY ARE REAL, AND HAVE A MINIMUM. THUS WE CAN ORDER THEM AS:

$$\lambda_0 < \lambda_1 < \lambda_2 < \dots < \lambda_n < \dots$$

- WITH SUITABLE B.C. (e.g. not the periodic b.c.) A NON-ZERO SOLUTION u_n ASSOCIATED TO λ_n IS UNIQUE UP TO MULTIPLICATION BY A SCALAR MEANING THAT ALL OTHER SOLUTIONS ASSOCIATED TO λ_n ARE OF THE FORM αu_n WITH $\alpha \in \mathbb{R}$

- SOLUTIONS CORRESPONDING TO DISTINCT EIGENVALUES ARE ORTHOGONAL:

$$\langle u_i, u_j \rangle_w \begin{cases} = 0 & \text{if } i \neq j \\ \neq 0 & \text{if } i = j \end{cases}$$

← THIS IS THE ONLY EASY THING TO PROVE. IT'S JUST AN INTEGRATION BY PARTS, AND SHOWS THE ROLE OF THE B.C.

- A SOLUTION u_n HAS n SINGLE ZEROS IN (a,b)

(nothing is said about $u_n(a)$ and $u_n(b)$)

– BACK TO THE EXAMPLES

$p(x) = w(x) = 1$
$q(x) = 0$
$x \in [0, \pi]$

$$\frac{d^2 u}{dx^2} = -\lambda u$$
$$u(0) = u(\pi) = 0$$

EIGENVALUES: $\lambda_n = (n+1)^2$

SOLUTIONS: $u_n(x) = \sin((n+1)x)$

$p(x) = w(x) = 1$
$q(x) = 0$
$x \in [0, \pi]$

$$\frac{d^2 u}{dx^2} = -\lambda u$$
$$u'(0) = u'(\pi) = 0$$

ONLY DIFFERENCES FROM ABOVE

EIGENVALUES: $\lambda_n = n^2$

SOLUTIONS: $u_n(x) = \cos(nx)$

$p(x) = w(x) = 1$
$q(x) = 0$
$\underbrace{x \in [0, 2\pi]}_{\text{LARGER INTERVAL}}$

$$\frac{d^2 u}{dx^2} = -\lambda u$$
$$\begin{cases} u(0) = u(\pi) \\ u'(0) = u'(\pi) \end{cases}$$

PERIODIC B.C.

EIGENVALUES: $\lambda_n = n^2$

SOLUTIONS: $u_0(x) = 1$ → UNIQUE SOLUTION CORRESPONDING TO THE EIGENVALUE $\lambda_0 = 0$

$$\left.\begin{aligned} u_n^{(c)}(x) &= \cos(nx) \\ u_n^{(s)}(x) &= \sin(nx) \end{aligned}\right\}$$

→ TWO LINEARLY INDEPENDENT SOLUTIONS CORRESPONDING TO THE SAME EIGENVALUE $\lambda_n = n^2$

$p(x) = \sqrt{1-x^2}$

$q(x) = 0$

$w(x) = \frac{1}{\sqrt{1-x^2}}$

$x \in [-1, 1]$

$$\frac{d}{dx}\left(\sqrt{1-x^2}\,\frac{du}{dx}\right) = -\frac{\lambda}{\sqrt{1-x^2}}\,u$$

$$|u(-1)|, |u(1)| < +\infty$$

EIGENVALUES: $\lambda_n = n^2$

SOLUTIONS: $u_n(x) = T_n(x)$ (CHEBYSHEV POLYNOMIALS)

$p(x) = 1-x^2$

$q(x) = 0$

$w(x) = 1$

$x \in [-1, 1]$

$$\frac{d}{dx}\left((1-x^2)\,\frac{du}{dx}\right) = -\lambda u$$

$$|u(-1)|, |u(1)| < +\infty$$

EIGENVALUES: $\lambda_n = n(n+1)$

SOLUTIONS: $u_n(x) = L_n(x)$ (LEGENDRE POLYNOMIALS)

— WHAT CAN WE DO WITH A FINITE NUMBER OF ORTHOGONAL FUNCTIONS ?

ANSWER: APPROXIMATE SQUARE-INTEGRABLE FUNCTIONS!

- BECAUSE OF THE DOT PRODUCT, WE HAVE A NORM:

$$\|f\| = \sqrt{\langle f, f\rangle}$$

- BECAUSE OF THE NORM, WE HAVE A DISTANCE:

$$d(f,g) = \|f-g\| = \sqrt{\langle f-g, f-g\rangle}$$

THIS STANDS FOR "DISTANCE"

— LET $\{e_1, \ldots, e_m\}$ BE A FINITE COLLECTION OF ORTHOGONAL FUNCTIONS

FOR SIMPLICITY OF LATER CALCULATIONS, ASSUME $|e_i| = 1$

DEFINE $\quad g(x) = c_1 e_1(x) + \cdots + c_m e_m(x)$

GIVEN f, WHAT IS THE CHOICE OF COEFFICIENTS $c_1 \cdots c_m$ THAT MINIMIZES

$$d(f,g) \qquad ?$$

— IT SHOULD BE CLEAR THAT WHATHEVER g MINIMIZES d, IT ALSO MINIMIZES d^2. SO NOW WE'LL WORK WITH d^2 TO AVOID THATH PESKY $\sqrt{\ }$.

$$d^2(f,g) = \langle f-g, f-g\rangle = \langle f,f\rangle + \langle g,g\rangle - 2\langle f,g\rangle =$$

$$= \|f\|^2 + \left\langle \sum_{i=1}^m c_i e_i, \sum_{i=1}^m c_i e_i \right\rangle - 2\left\langle f, \sum_{i=1}^m c_i e_i\right\rangle =$$

$$= \|f\|^2 + \sum_{i=1}^m c_i^2 - 2\sum_{i=1}^m c_i \langle f, e_i\rangle$$

So d^2 is a function of $c_1, \ldots, c_m$. Let's minimize it:

$$0 = \frac{d}{dc_j} d^2(c_1, \ldots, c_m) = 2c_j - 2\langle f, e_j\rangle .$$

THUS: $\quad c_j = \langle f, e_j\rangle$

– We are led to conclude that the minimizer is the function

$$\tilde{f} = \sum_{i=1}^{m} \langle f, e_i \rangle e_i$$

$\tilde{f}$ IS CALLED PROJECTION OF f ONTO THE SUBSPACE SPANNED BY $\{e_1, \ldots, e_m\}$

BUT.. IS IT REALLY A MINIMUM? COULD IT BE A MAX? IF IT'S A MIN, IS IT THE ONLY ONE?

BY CONTRADICTION SAY THAT g ACHIEVES A SMALLER DISTANCE. DEFINE $u = g - \tilde{f}$

THEN:

$$d^2(f,g) = d^2(f, \tilde{f}+u) = \langle f-\tilde{f}-u, f-\tilde{f}-u \rangle = \langle f-\tilde{f}, f-\tilde{f} \rangle + \langle u,u \rangle - 2\langle f-\tilde{f}, u\rangle =$$

$$= d^2(f,\tilde{f}) + \|u\|^2 - 2\left(\underbrace{\langle f,u\rangle}_{} - \underbrace{\langle \tilde{f},u\rangle}_{}\right) \rightarrow 0$$

$$\langle f,u\rangle = \langle f, \sum_{i=1}^{m} u_i e_i \rangle = \sum_{i=1}^{m} u_i \langle f, e_i \rangle = \sum_{i=1}^{m} u_i c_i$$

$$\langle \tilde{f},u\rangle = \langle \tilde{f}, \sum_{i=1}^{m} u_i e_i \rangle = \sum_{i=1}^{m} u_i \langle \tilde{f}, e_i \rangle = \sum_{i=1}^{m} u_i c_i$$

THUS: $d^2(f,g) = d^2(f,\tilde{f}) + \|u\|^2$

$\tilde{f}$ IS REALLY A MINIMUM, BECAUSE $\|u\|^2$ IS NON-NEGATIVE AND IT IS UNIQUE, BECAUSE $\|u\|^2 = 0 \Rightarrow \tilde{f} = g$

– ANOTHER NICE PROPERTY IS THE FOLLOWING:

Let $\tilde{f}_m$ be the projection of f onto $\{e_1, \ldots, e_m\}$

Let $\tilde{f}_{m+1}$ be the projection of f onto $\{e_1, \ldots, e_m, e_{m+1}\}$

we have added an extra orthogonal function here.

Then: $d(f, \hat{f}_m) \geq d(f, \hat{f}_{m+1})$. PROOF: SIMPLE EXERCISE.

– FINALLY, WE HAVE BESSEL'S INEQUALITY:

$$\|\tilde{f}\|^2 \leq \|f\|^2$$

PROOF: NOTICE THAT: $\langle f, \tilde{f} \rangle = \langle f, \sum_{i=1}^{m} c_i e_i \rangle = \sum_{i=1}^{m} c_i \langle f, e_i \rangle = \sum_{i=1}^{m} c_i^2 = \langle \tilde{f}, \tilde{f} \rangle = \|\tilde{f}\|^2$

THEN:

$$0 \leq \|f-\tilde{f}\|^2 = \langle f-\tilde{f}, f-\tilde{f} \rangle = \langle f,f \rangle + \langle \tilde{f},\tilde{f} \rangle - 2\langle f,\tilde{f} \rangle = \|f\|^2 - \|\tilde{f}\|^2 \Rightarrow \|f\|^2 \geq \|\tilde{f}\|^2$$

THESE RESULTS MAY BE SUMMARIZED BY SAYING THAT AN APPROXIMATION IN TERMS OF ORTHOGONAL FUNCTIONS IS ALWAYS "FROM BELOW" (IT UNDERESTIMATES THE NORM OF THE FUNCTION) AND, IF YOU ADD MORE ORTHOGONAL FUNCTIONS IT MAY IMPROVE (OR STAY THE SAME), BUT NOT DECREASE IN QUALITY.

— FINALLY LET ME STATE A FINAL (VERY TOUGH) THEOREM ABOUT ORTHOGONAL FUNCTIONS GENERATED BY STURM-LIOUVILLE PROBLEMS.

IF THE SEQUENCE OF FUNCTIONS $e_0, e_1, \dots e_n, \dots$ IS THE SET OF ALL LINEARLY INDEPENDENT SOLUTIONS OF A S-L PROBLEM, THEN IT IS A BASE OF $L^2(a,b)$

THIS MEANS THAT, FOR ANY SQUARE INTEGRABLE f in (a,b):

Define $c_i = \langle f, e_i \rangle$

— THE SERIES $\sum_{i=0}^{\infty} c_i e_i$ CONVERGES (NEARLY EVERYWHERE)

$$- \quad d^2\left(f, \sum_{i=0}^{\infty} c_i e_i\right) = \int_a^b \left(f(x) - \sum_{i=0}^{\infty} c_i e_i(x)\right)^2 dx = 0$$

$$- \quad \|f\|^2 = \left\|\sum_{i=0}^{\infty} c_i e_i\right\|^2 = \sum_{i=0}^{\infty} c_i^2$$

← THIS IS CALLED PARSEVAL IDENTITY.

THE FOURIER BASIS

- DEFINE:

$$A_0 = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} 1 \cdot f(x)\, dx \, ; \qquad A_k = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} \cos(kx)\, f(kx)\, dx, \quad k = 1, 2, \dots$$

$$B_k = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} \sin(kx)\, f(kx)\, dx, \quad k = 1, 2, \dots$$

FOR ANY SQUARE-INTEGRABLE FUNCTION f.

THEN THE FOURIER SERIES CONVERGING (ALMOST EVERYWHERE) TO f IS:

$$f(x) = A_0 \cdot \frac{1}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} A_k \frac{\cos(kx)}{\sqrt{\pi}} + \sum_{k=1}^{\infty} B_k \frac{\sin(kx)}{\sqrt{\pi}}$$

- FOR DISCONTINUOUS FUNCTIONS THERE MAY BE INDIVIDUAL VALUES x WHERE THE FOURIER SERIES DOES NOT CONVERGE TO $f(x)$. E.G. $f(x) = \begin{cases} -1 & x < \pi \\ 1 & x \geq \pi \end{cases}$

THE SERIES CONVERGES TO: $f(x) = \begin{cases} -1 & x < \pi \\ 0 & x = 0 \\ 1 & x > \pi \end{cases}$

- FOR CONTINOUS FUNCTIONS WITH CONTINUOUS FIRST DERIVATIVE AND $f(-\pi) = f(\pi)$ (THAT IS, PERIODIC) THEN THE CONVERGENCE IS UNIFORM.

- EVEN BETTER, FOR PERIODIC C^1 FUNCTIONS YOU CAN COMPUTE THE FOURIER SERIES OF THE DERIVATIVE JUST BY TAKING $\frac{d}{dx}$ INSIDE THE SERIES: (THE PROOF IS NOT DIFFICULT...)

$$\frac{d}{dx} f(x) = \frac{d}{dx}\left[\frac{A_0}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} A_k \frac{\cos(kx)}{\sqrt{\pi}} + \sum_{k=1}^{\infty} B_k \frac{\sin(kx)}{\sqrt{\pi}}\right] =$$

$$= \sum_{k=1}^{\infty} \frac{A_k}{\sqrt{\pi}} \frac{d}{dx}\cos(kx) + \sum_{k=1}^{\infty} \frac{B_k}{\sqrt{\pi}} \frac{d}{dx}\sin(kx) =$$

$$= \sum_{k=1}^{\infty} \underbrace{-kA_k}_{\text{THIS IS } B'_k} \frac{\sin(kx)}{\sqrt{\pi}} + \sum_{k=1}^{\infty} \underbrace{kB_k}_{\text{THIS IS } A'_k} \frac{\cos(kx)}{\sqrt{\pi}}$$

- FINALLY, CONSIDER A PERIODIC C^∞ FUNCTION.
THE COEFFICIENTS OF THE m-th DERIVATIVE WILL BE $\propto k^m$.
THUS A_k, B_k MUST GO TO ZERO FASTER THAN $\frac{1}{k^m}$, OR THE SERIES WON'T BE CONVERGENT.
BECAUSE THIS HAS TO BE TRUE FOR ANY m, THEN

$A_k\ B_k \xrightarrow[k \to \infty]{} 0$ FASTER THAN ANY NEGATIVE POWER OF k.

E.G. EXPONENTIALLY FASTER. THIS MEANS THAT IF I TRUNCATE THE SERIES AFTER N TERMS, THE ERROR DECREASES EXTREMELY RAPIDLY WITH N.

TRIGONOMETRIC APPROXIMATION/INTERPOLATION OF <u>SMOOTH FUNCTIONS</u> IS

EXTREMELY ACCURATE!