Notes on Set Theory

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Contents

1 Chapter 6: Cardinal Numbers and The Axim of Choice

1.1 Material Notes

In the Book Page 134 to 135, while proving the case 1, the book mentioned

> **Pigeonhole Principle:** No natural number is equinumerous to a proper subset of itself.
>
> **Proof** Assume that f is a one-to-one function from the set n into the set n. We will show that ran$f$ is all of the set n (and not a proper subset of n). This suffices to prove the theorem. We use induction on n. Define:
>
> $$T = \{n \in \omega \mid \text{any one-to-one function from } n \text{ into } n \text{ has range } n\}.$$
>
> Then $0 \in T$; the only function from the set 0 into the set 0 is $\emptyset$ and its range is the set 0. Suppose that $k \in T$ and that f is a one-to-one function from the set k^+ into the set k^+. We must show that the range of f is all of the set k^+; this will imply that $k^+ \in T$. Note that the restriction $f \upharpoonright k$ of f to the set k maps the set k one-to-one into the set k^+.
>
> **Case 1** Possibly the set k is closed under f. Then $f \upharpoonright k$ maps the set k into the set k. Then because $k \in T$ we may conclude that ran$(f \upharpoonright k)$ is all of the set k. Since f is one-to-one, the only possible value for $f(k)$ is the number k. Hence ran$f$ is $k \cup \{k\}$, which is the set k^+.

[Ray's Note 1: Here the Case 1 should have more explanation:
We know that k is closed under f and ran$(f \upharpoonright k) = k$. Then why do we have ran$f = k \cup \{k\}$? This is because of the following argument:
f is one-to-one. We also know that $k \notin k$ (otherwise we would form Russell's paradox). The preimage $f^{-1}[\{f(k)\}]$ (the preimage of $f(k)$ under f) can only contain one element since f is one-to-one, and $k \in f^{-1}[\{f(k)\}]$ because the preimage of $f(k)$ must contain k. Thus, ran$f$ = ran$(f \upharpoonright k) \cup$ ran$(f \upharpoonright \{k\}) = k \cup \{k\}$.]

1.2 Excercise Answers

2 Ray's Notes Summary

[Ray's Note 1 (Page 2): Here the Case 1 should have more explanation:
We know that k is closed under f and ran$(f \upharpoonright k) = k$. Then why do we have ran$f = k \cup \{k\}$? This is because of the following argument:
f is one-to-one. We also know that $k \notin k$ (otherwise we would form Russell's paradox). The preimage $f^{-1}[\{f(k)\}]$ (the preimage of $f(k)$ under f) can only contain one element since f is one-to-one, and $k \in f^{-1}[\{f(k)\}]$ because the preimage of $f(k)$ must contain k. Thus, ran$f$ = ran$(f \upharpoonright k) \cup$ ran$(f \upharpoonright \{k\}) = k \cup \{k\}$.]