Notes on Set Theory

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Contents

1 Chapter 6: Cardinal Numbers and The Axim of Choice

1.1 Material Notes

1.1.1 FINITE SETS

In the Book Page 134 to 135, while proving the case 1, the book mentioned

> **Pigeonhole Principle:** No natural number is equinumerous to a proper subset of itself.
>
> **Proof** Assume that f is a one-to-one function from the set n into the set n. We will show that $\operatorname{ran} f$ is all of the set n (and not a proper subset of n). This suffices to prove the theorem. We use induction on n. Define:
> $$T = \{n \in \omega \mid \text{any one-to-one function from } n \text{ into } n \text{ has range } n\}.$$
> Then $0 \in T$; the only function from the set 0 into the set 0 is $\emptyset$ and its range is the set 0. Suppose that $k \in T$ and that f is a one-to-one function from the set k^+ into the set k^+. We must show that the range of f is all of the set k^+; this will imply that $k^+ \in T$. Note that the restriction $f \upharpoonright k$ of f to the set k maps the set k one-to-one into the set k^+.
>
> **Case 1** Possibly the set k is closed under f. Then $f \upharpoonright k$ maps the set k into the set k. Then because $k \in T$ we may conclude that $\operatorname{ran}(f \upharpoonright k)$ is all of the set k. Since f is one-to-one, the only possible value for $f(k)$ is the number k. Hence $\operatorname{ran} f$ is $k \cup \{k\}$, which is the set k^+.

[Ray's Note 1: Here the Case 1 should have more explanation:
We know that k is closed under f and $\operatorname{ran}(f \upharpoonright k) = k$. Then why do we have $\operatorname{ran} f = k \cup \{k\}$? This is because of the following argument:
f is one-to-one. We also know that $k \notin k$ (otherwise we would form Russell's paradox). The preimage $f^{-1}[\{f(k)\}]$ (the preimage of $f(k)$ under f) can only contain one element since f is one-to-one, and $k \in f^{-1}[\{f(k)\}]$ because the preimage of $f(k)$ must contain k. Thus, $\operatorname{ran} f = \operatorname{ran}(f \upharpoonright k) \cup \operatorname{ran}(f \upharpoonright \{k\}) = k \cup \{k\}$.]

1.1.2 CARDINAL ARITHMETIC

In Page 139 to 140, while proving Theorem 6H, the book mentioned that

> **Theorem 6H** Assume that $K_1 \approx K_2$ and $L_1 \approx L_2$.
>
> (a) If $K_1 \cap L_1 = K_2 \cap L_2 = \emptyset$, then $K_1 \cup L_1 \approx K_2 \cup L_2$.
>
> (b) $K_1 \times L_1 \approx K_2 \times L_2$.
>
> (c) $^{(L_1)}K_1 \approx {}^{(L_2)}K_2$.

[Ray's Note 2: Theorem 6H: More perspectives]
We may also prove that H is a bijection by using the theorem that a function is a bijection if and only if it has an inverse (i.e., its left and right inverses coincide). Thus, the remaining task is to find the inverse. This is straightforward to do. Since $H(j) = f \circ j \circ g^{-1}$, we can express j as $j = f^{-1} \circ H(j) \circ g$, which implies that $H^{-1}(i) = f^{-1} \circ i \circ g$. It is easy to verify that such an H^{-1} is indeed the inverse by composing H with H^{-1} both on the left and the right.

Another approach to prove that H is a bijection between two function spaces is by demonstrating that H is both injective and surjective:

- **H is surjective:** For every $i \in {}^{(L_2)}K_2$, there exists a $j \in {}^{(L_1)}K_1$ such that $j = f^{-1} \circ i \circ g$. It is straightforward to verify that $H(j) = i$.

- **H is injective:** We need to show that if $H(j) = i$, then j must have the form $j = f^{-1} \circ i \circ g$. In other words, $H(j) = i$ if and only if $j = f^{-1} \circ i \circ g$. This is not difficult to verify: $H(f^{-1} \circ i \circ g) = i$. Additionally, since $H(j) = f \circ j \circ g^{-1} = i$, it follows that $j = f^{-1} \circ i \circ g$. After establishing this, if H maps j_1 and j_2 to the same element i, then we must have $j_1 = f^{-1} \circ i \circ g$ and $j_2 = f^{-1} \circ i \circ g$. Furthermore, since function composition is a function, we conclude that $j_1 = j_2$.

In both of the above methods, we do not need to explicitly unfold what i and j are; we treat them as objects themselves rather than focusing on their relations. This, in a certain sense, makes the argument more abstract and simpler. However, in both approaches, we need to use the fact that function composition is itself a function, meaning that $K_g : f \mapsto f \circ g$ is a function of f, and similarly, $K_g' : f \mapsto g \circ f$ is also a function of f.

In Page 141 , the book mentioned that

5. Recall that $\emptyset^K = \{\emptyset\}$ for any set K and that $K^\emptyset = \emptyset$ for nonempty K. In terms of cardinal numbers, these facts become

$$\kappa^0 = 1 \quad \text{for any } \kappa,$$

$$0^\kappa = 0 \quad \text{for any nonzero } \kappa.$$

In particular, $0^0 = 1$.

[Ray's Note 3: More on cardinal arithmetic]

1. Understanding $^\emptyset K = \{\emptyset\}$

Notation: $^A B$ represents the set of all functions from set A to set B.
Case: When $A = \emptyset$ (the empty set) and $B = K$ (any set).
Explanation:
Definition of a Function: A function $f : A \to B$ is a set of ordered pairs $(a, f(a))$ where each $a \in A$ is paired with exactly one $f(a) \in B$.

When $A = \emptyset$:
There are **no elements** in A to pair with elements in B.
Therefore, the **only possible function** is the **empty function**, which is the empty set $\emptyset$.
Conclusion: Since there is exactly **one** function from $\emptyset$ to K, we have:

$$^\emptyset K = \{\emptyset\}$$

Cardinal Arithmetic Interpretation:

- The number of such functions is **1**.

- Hence, for any cardinal κ:
$$\kappa^0 = 1$$

2. Understanding $^K\emptyset = \emptyset$ for Nonempty K
Notation: $^K\emptyset$ represents the set of all functions from K to the empty set $\emptyset$.
Case: When K is **nonempty** and the codomain is $\emptyset$.
Explanation:
Definition of a Function: A function $f : K \to \emptyset$ must assign to **every** element $k \in K$ an element $f(k) \in \emptyset$.

Problem: The empty set $\emptyset$ has **no elements**. Therefore, there is **no possible way** to assign a value $f(k)$ for any $k \in K$.

Conclusion: Since it is **impossible** to define such a function when K is nonempty, there are **no functions** from K to $\emptyset$:

$$^K\emptyset = \emptyset$$

Cardinal Arithmetic Interpretation: - The number of such functions is **0**. - Hence, for any nonzero cardinal κ:

$$0^\kappa = 0$$

1.1.3 ORDERING CARDINAL NUMBERS

Page 146, in part

> **Examples 1.**
>
> 1. If $A \subseteq B$, then card $A \leq$ card B. Conversely, whenever $\kappa \leq \lambda$, there exist sets $K \subseteq L$ with card $K = \kappa$ and card $L = \lambda$. To prove this, start with any sets C and L of cardinality κ and λ, respectively. Then $C \subseteq L$, so there is a one-to-one function f from C into L. Let $K = \mathrm{ran}\, f$; then $C \approx K \subseteq L$.
>
> 2. For any cardinal κ, we have $0 \leq \kappa$.
>
> 3. For any finite cardinal n, we have $n < \aleph_0$. (Why?) For any two finite cardinals m and n, we have:
>
> $$m \in n \implies m \subseteq n \implies m \leq n.$$

About the '(Why?)' part. This is because any natural number n is a proper subset of ω. Thus the embedding map is the injection. Furthermore there is no surjection between n to ω since $n^+ = n \cap \{n\} \notin n$. Thus $n \subsetneq \omega \& n \neq \omega$.

1.2 Excercise Answers

> **Question 1. E 6** *Let κ be a nonzero cardinal number. Show that there does not exist a set to which every set of cardinality κ belongs.*

Proof. Proof by contradiction (BWOC): Assume that such a set exists. Let

$$A = \{K \mid |K| = \kappa\}$$

be the set containing all sets with cardinality κ.

Consider

$$\bigcup A,$$

the union of all sets in A. By construction, $\bigcup A$ would be a set that contains every possible set of cardinality κ.

However, this leads to a contradiction since such a set does not exist in the framework of standard set theory (e.g., due to limitations implied by Russell's paradox or cardinality constraints).

Therefore, there does not exist a set A such that it contains every set of cardinality κ. $\qquad\square$

> **Question 2. E 7** *Assume that A is finite and $f : A \to A$. Show that f is one-to-one if and only if ran $f = A$.*

Given that ran $f = A$, this implies that f is surjective. Thus, the statement is equivalent to proving that: *"Assume that A is finite and $f : A \to A$. Show that f is one-to-one if and only if f is surjective."*

Proof. Assume A is finite.

- ($\Rightarrow$) **(By way of contradiction):** Assume that f is injective but not surjective. Then, the image of f under A, denoted by $f[A]$, is a proper subset of A, meaning $f[A] \subsetneq A$. Thus, we have an injective function $f : A \to f[A]$. By construction, $f[A] = f[A]$, and therefore, f is surjective onto $f[A]$. Hence, f is a bijection from A to $f[A]$, where $f[A] \subsetneq A$. According to Corollary 6D, this implies that A is equinumerous to a proper subset of itself, which contradicts the assumption that A is finite.

- ($\Leftarrow$) **(By way of contradiction):** Assume that f is surjective but not injective. By the axiom of choice, there exists an injection, call it g, from A to the preimage of A under f, which is a proper subset of A. Such a function g would then be a bijection from the preimage of A to A. This again implies that A is equinumerous to a proper subset of itself, leading to the conclusion that A is infinite. This contradicts our original assumption that A is finite.

$\square$

> **Question 3. E 8** *Prove that the union of two finite sets is finite (Corollary 6K), without any use of arithmetic.*

Proof. Let $|A| \approx n$ and $|B| \approx m$, where $n, m \in \omega$. Assume that A and B are disjoint. By the definition of equinumerosity, there exist functions $h : A \to n$ and $g : B \to m$. We define a function $f : A \cup B \to n + m$ (as defined earlier) such that

$$f(x) = \begin{cases} h(x) & \text{if } x \in A \\ n + g(x) & \text{if } x \in B \end{cases}$$

This function is well-defined since A and B are disjoint. It is straightforward to verify that such an f is a bijection with an inverse $f^{-1} : n + m \to A \cup B$ defined as follows:

$$f^{-1}(k) = \begin{cases} h^{-1}(k) & \text{if } k \leq n \\ g^{-1}(k - n) & \text{if } k > n \end{cases}$$

If A and B are not disjoint, then let $A \cap B = C \neq \emptyset$. Replace each element in C by elements not present in $A \cup B$ to form a new set C'. Let $A' = (A - C) \cup C'$. We have $|A'| = |A|$ and $A \cup B \subseteq A' \cup B$, which forms a disjoint union. By Lemma 6F, there exists a $k < n + m$ such that $|A \cup B| = k$.

Therefore, in both cases, $A \cup B$ is finite. $\square$

> **Question 4. E 9** *Prove that the Cartesian product of two finite sets is finite (Corollary 6K), without any use of arithmetic.*

Proof. We will use induction to prove this statement:
Let
$$S = \{m \mid \forall n \in \omega, \forall A, B \text{ such that } |A| = m, |B| = n, |A \times B| = m \times n\},$$

where $\times$ is defined as the Cartesian product for natural numbers.

1. **Base Case:** $0 \in S$. For any set B, $\emptyset \times B = \emptyset$. Thus, $|\emptyset \times B| = 0 \times |B| = 0$ since B is finite and therefore $|B| \in \omega$. Furthermore, $1 \in S$ since for any singleton $\{a\}$ and any set B, there exists a bijection between $\{a\} \times B$ and B given by the function $f : \{a\} \times B \to B$ defined as $f(a, x) = x$. The inverse function is $g(x) = (a, x)$. Therefore, $|\{a\} \times B| = 1 \times |B| = |B|$.

2. **Inductive Step:** Assume $k \in S$. We want to show that $k^+ \in S$, where k^+ denotes the successor of k.

   To show $k^+ \in S$, we need to prove that for all $m \in \omega$ and for all sets A and B such that $|A| = k^+$ and $|B| = m$, we have $|A \times B| = k^+ \times m$.

   Since we have already proven that $0 \in S$, the induction starts with $k \geq 1$, meaning $k^+ \geq 2$.

   Let A be a set with $|A| = k^+$. Since $k^+ \geq 2$, there exists an element $a \in A$. Let $A' = A \setminus \{a\}$. Therefore, $|A'| = k$, and by the induction hypothesis, $|A' \times B| = k \times m$. Since $\{a\}$ is a singleton, we have $|\{a\} \times B| = 1 \times m = m$ by the induction hypothesis.

   Therefore,
   $$|A \times B| = |(A' \times B) \cup (\{a\} \times B)| = k \times m + m = k^+ \times m$$

   by the definition of multiplication for natural numbers.

Hence, by induction, the Cartesian product of two finite sets is finite (since natural number is closed under multiplication). $\qquad\square$

Question 5. E 10 to 12 *For Excercises 10 to 12, proving all the Theorem 6I is suffices.*

In Page 142

> **Theorem 6I** For any cardinal numbers κ, λ, and μ:
>
> 1. $\kappa + \lambda = \lambda + \kappa$ and $\kappa \cdot \lambda = \lambda \cdot \kappa$.
>
> 2. $\kappa + (\lambda + \mu) = (\kappa + \lambda) + \mu$ and $\kappa \cdot (\lambda \cdot \mu) = (\kappa \cdot \lambda) \cdot \mu$.
>
> 3. $\kappa \cdot (\lambda + \mu) = \kappa \cdot \lambda + \kappa \cdot \mu$.
>
> 4. $\kappa^{\lambda + \mu} = \kappa^\lambda \cdot \kappa^\mu$.
>
> 5. $(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$.
>
> 6. $(\kappa^\lambda)^\mu = \kappa^{\lambda \cdot \mu}$.
>
> **Proof** Take sets K, L, and M with card $K = \kappa$, card $L = \lambda$, and card $M = \mu$; for convenience, choose them in such a way that any two are disjoint. Then each of the equations reduces to a corresponding statement about equinumerous sets. For example, $\kappa \cdot \lambda =$ card $(K \times L)$ and $\lambda \cdot \kappa =$ card $(L \times K)$; consequently, showing that $\kappa \cdot \lambda = \lambda \cdot \kappa$ reduces to showing that $K \times L \approx L \times K$. Listed in full, the statements to be verified are:
>
> 1. $K \cup L \approx L \cup K$ and $K \times L \approx L \times K$.

2. $K \cup (L \cup M) \approx (K \cup L) \cup M$ and $K \times (L \times M) \approx (K \times L) \times M$.

3. $K \times (L \cup M) \approx (K \times L) \cup (K \times M)$.

4. $^{(L \cup M)}K \approx {}^{K}L \times {}^{k}M$.

5. $^{M}(K \times L) \approx {}^{M}K \times {}^{M}L$.

6. $^{M}(^{L}K) \approx {}^{(L \times M)}K$.

I will use the second part of 1 to 6 (on the set) to prove the theorem.

Proof. For the simple ones I will simple list the fucntion without justifying that it is bijection.

1. $K \cup L = L \cup K$ by algebra of set (Page 27). f: $K \times L \to L \times K$ s.t. $< k, l > \mapsto < l, k >$

2. $K \cup (L \cup M) = (K \cup L) \cup M$ by algebra of set (Page 27). f: $K \times (L \times M) \to (K \times L) \times M$ s.t. $< k, < l, m >> \mapsto << k, l >, m >$.

3. $K \times (L \cup M) = (K \times L) \cup (K \times M)$ by Excercise 2 Chapter3.

4. $H : {}^{(L \cup M)}K \to {}^{K}L \times {}^{k}M$ s.t. $f \mapsto < f \upharpoonright_L, f \upharpoonright_{M-L} >$.

5. $H : {}^{M}(K \times L) \to {}^{M}K \times {}^{M}L$. Let $f \in {}^{M}(K \times L)$, then $\forall m \in M f : m \mapsto f(m) = < k, l > = < f_1(m), f_2(m) >$. Thus let $H : f \mapsto < f_1, f_2 >$.

6. Proven in book.

$\square$

Question 6. E 13 *Show that a finite union of finite sets is finite. That is, show that if B is a finite set whose members are themselves finite sets, then $\bigcup B$ is finite.*

Proof. In Chapter 6 E 3, we have proved that two finite set's union is finite. Do induction on natural number, we can prove that any finite union of finite sets is finite. $\square$

Question 7. 14 *Define a permutation of K to be any one-to-one function from K onto K. We can then define the factorial operation on cardinal numbers by the equation*

$$\kappa! = card\{f \mid f \text{ is a permutation of } K\},$$

where K is any set of cardinality κ. Show that $\kappa!$ is well defined, i.e., the value of $\kappa!$ is independent of just which set K is chosen.

Proof. To show that $\kappa!$ is well-defined, let κ be a cardinal number, and consider any two sets A and B such that $|A| = |B| = \kappa$.

Since $A \approx A$, let f_A be a bijection $f_A : A \to A$. Similarly, let f_B be a bijection $f_B : B \to B$. Since $A \approx B$, there exists a bijection $f : A \to B$.

We can establish a bijection between the set of permutations of A (denoted by perm(A) or sym(A)) and the set of permutations of B (denoted by perm(B) or sym(B)). This bijection can be constructed as

follows: for any $f_A \in \text{perm}(A)$, we define a corresponding permutation of B by $f'_A = f \circ f_B \circ f^{-1}$ (this is a conjugation by f).

This construction is clearly a bijection between $\text{perm}(A)$ and $\text{perm}(B)$. Thus, the value of $\kappa!$ is independent of the particular choice of the set K with cardinality κ, proving that the factorial operation on cardinal numbers is well-defined. $\qquad\square$

2 Ray's Notes Summary

[Ray's Note 1 (Page 2):   Here the Case 1 should have more explanation:
We know that k **is closed under** f **and** $\text{ran}(f \upharpoonright k) = k$**. Then why do we have** $\text{ran} f = k \cup \{k\}$**? This is because of the following argument:**
f **is one-to-one. We also know that** $k \notin k$ **(otherwise we would form Russell's paradox). The preimage** $f^{-1}[\{f(k)\}]$ **(the preimage of** $f(k)$ **under** f**) can only contain one element since** f **is one-to-one, and** $k \in f^{-1}[\{f(k)\}]$ **because the preimage of** $f(k)$ **must contain** k**. Thus,** $\text{ran} f = \text{ran}(f \upharpoonright k) \cup \text{ran}(f \upharpoonright \{k\}) = k \cup \{k\}$**.]**

 [Ray's Note 2 (Page 2): Theorem 6H: More perspectives]
 [Ray's Note 3 (Page 3): More on cardinal arithmetic]