Notes on Set Theory

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Contents

1 Chapter 6: Cardinal Numbers and The Axim of Choice

1.1 Material Notes

1.1.1 FINITE SETS

In the Book Page 134 to 135, while proving the case 1, the book mentioned

> **Pigeonhole Principle:** No natural number is equinumerous to a proper subset of itself.
>
> **Proof** Assume that f is a one-to-one function from the set n into the set n. We will show that ran$f$ is all of the set n (and not a proper subset of n). This suffices to prove the theorem. We use induction on n. Define:
>
> $$T = \{n \in \omega \mid \text{any one-to-one function from } n \text{ into } n \text{ has range } n\}.$$
>
> Then $0 \in T$; the only function from the set 0 into the set 0 is $\emptyset$ and its range is the set 0. Suppose that $k \in T$ and that f is a one-to-one function from the set k^+ into the set k^+. We must show that the range of f is all of the set k^+; this will imply that $k^+ \in T$. Note that the restriction $f \upharpoonright k$ of f to the set k maps the set k one-to-one into the set k^+.
>
> **Case 1** Possibly the set k is closed under f. Then $f \upharpoonright k$ maps the set k into the set k. Then because $k \in T$ we may conclude that ran$(f \upharpoonright k)$ is all of the set k. Since f is one-to-one, the only possible value for $f(k)$ is the number k. Hence ran$f$ is $k \cup \{k\}$, which is the set k^+.

**[Ray's Note 1: Here the Case 1 should have more explanation:
We know that k is closed under f and ran$(f \upharpoonright k) = k$. Then why do we have ran$f = k \cup \{k\}$? This is because of the following argument:
f is one-to-one. We also know that $k \notin k$ (otherwise we would form Russell's paradox). The preimage $f^{-1}[\{f(k)\}]$ (the preimage of $f(k)$ under f) can only contain one element since f is one-to-one, and $k \in f^{-1}[\{f(k)\}]$ because the preimage of $f(k)$ must contain k. Thus, ran$f$ = ran$(f \upharpoonright k) \cup$ ran$(f \upharpoonright \{k\}) = k \cup \{k\}$.]**

1.1.2 CARDINAL ARITHMETIC

In Page 139 to 140, while proving Theorem 6H, the book mentioned that

> **Theorem 6H** Assume that $K_1 \approx K_2$ and $L_1 \approx L_2$.
>
> (a) If $K_1 \cap L_1 = K_2 \cap L_2 = \emptyset$, then $K_1 \cup L_1 \approx K_2 \cup L_2$.
>
> (b) $K_1 \times L_1 \approx K_2 \times L_2$.
>
> (c) ${}^{(L_1)}K_1 \approx {}^{(L_2)}K_2$.

[Ray's Note 2: Theorem 6H: More perspectives]

We may also prove that H is a bijection by using the theorem that a function is a bijection if and only if it has an inverse (i.e., its left and right inverses coincide). Thus, the remaining task is to find the inverse. This is straightforward to do. Since $H(j) = f \circ j \circ g^{-1}$, we can express j as $j = f^{-1} \circ H(j) \circ g$, which implies that $H^{-1}(i) = f^{-1} \circ i \circ g$. It is easy to verify that such an H^{-1} is indeed the inverse by composing H with H^{-1} both on the left and the right.

Another approach to prove that H is a bijection between two function spaces is by demonstrating that H is both injective and surjective:

- H **is surjective:** For every $i \in {}^{(L_2)}K_2$, there exists a $j \in {}^{(L_1)}K_1$ such that $j = f^{-1} \circ i \circ g$. It is straightforward to verify that $H(j) = i$.

- H **is injective:** We need to show that if $H(j) = i$, then j must have the form $j = f^{-1} \circ i \circ g$. In other words, $H(j) = i$ if and only if $j = f^{-1} \circ i \circ g$. This is not difficult to verify: $H(f^{-1} \circ i \circ g) = i$. Additionally, since $H(j) = f \circ j \circ g^{-1} = i$, it follows that $j = f^{-1} \circ i \circ g$. After establishing this, if H maps j_1 and j_2 to the same element i, then we must have $j_1 = f^{-1} \circ i \circ g$ and $j_2 = f^{-1} \circ i \circ g$. Furthermore, since function composition is a function, we conclude that $j_1 = j_2$.

In both of the above methods, we do not need to explicitly unfold what i and j are; we treat them as objects themselves rather than focusing on their relations. This, in a certain sense, makes the argument more abstract and simpler. However, in both approaches, we need to use the fact that function composition is itself a function, meaning that $K_g : f \mapsto f \circ g$ is a function of f, and similarly, $K'_g : f \mapsto g \circ f$ is also a function of f.

In Page 141 , the book mentioned that

> **5.** Recall that $\emptyset^K = \{\emptyset\}$ for any set K and that $K^\emptyset = \emptyset$ for nonempty K. In terms of cardinal numbers, these facts become
>
> $$\kappa^0 = 1 \quad \text{for any } \kappa,$$
>
> $$0^\kappa = 0 \quad \text{for any nonzero } \kappa.$$
>
> In particular, $0^0 = 1$.

[Ray's Note 3: More on cardinal arithmetic]

1. Understanding ${}^{\emptyset}K = \{\emptyset\}$

Notation: ${}^{A}B$ represents the set of all functions from set A to set B.
Case: When $A = \emptyset$ (the empty set) and $B = K$ (any set).
Explanation:
Definition of a Function: A function $f : A \to B$ is a set of ordered pairs $(a, f(a))$ where each $a \in A$ is paired with exactly one $f(a) \in B$.

When $A = \emptyset$:
There are **no elements** in A to pair with elements in B.
Therefore, the **only possible function** is the **empty function**, which is the empty set $\emptyset$.
Conclusion: Since there is exactly **one** function from $\emptyset$ to K, we have:

$${}^{\emptyset}K = \{\emptyset\}$$

Cardinal Arithmetic Interpretation:

- The number of such functions is **1**.
- Hence, for any cardinal κ:

$$\kappa^0 = 1$$

2. Understanding ${}^{K}\emptyset = \emptyset$ for Nonempty K
Notation: ${}^{K}\emptyset$ represents the set of all functions from K to the empty set $\emptyset$.
Case: When K is **nonempty** and the codomain is $\emptyset$.
Explanation:
Definition of a Function: A function $f : K \to \emptyset$ must assign to **every** element $k \in K$ an element $f(k) \in \emptyset$.

Problem: The empty set $\emptyset$ has **no elements**. Therefore, there is **no possible way** to assign a value $f(k)$ for any $k \in K$.
Conclusion: Since it is **impossible** to define such a function when K is nonempty, there are **no functions** from K to $\emptyset$:

$$^{K}\emptyset = \emptyset$$

Cardinal Arithmetic Interpretation: - The number of such functions is **0**. - Hence, for any nonzero cardinal κ:

$$0^{\kappa} = 0$$

1.1.3 ORDERING CARDINAL NUMBERS

Page 146, in part

> **Examples 1.**
>
> 1. If $A \subseteq B$, then card $A \leq$ card B. Conversely, whenever $\kappa \leq \lambda$, there exist sets $K \subseteq L$ with card $K = \kappa$ and card $L = \lambda$. To prove this, start with any sets C and L of cardinality κ and λ, respectively. Then $C \subseteq L$, so there is a one-to-one function f from C into L. Let $K = \operatorname{ran} f$; then $C \approx K \subseteq L$.
>
> 2. For any cardinal κ, we have $0 \leq \kappa$.
>
> 3. For any finite cardinal n, we have $n < \aleph_0$. (Why?) For any two finite cardinals m and n, we have:
>
> $$m \in n \implies m \subseteq n \implies m \leq n.$$

About the '(Why?)' part. This is because any natural number n is a proper subset of ω. Thus the embedding map is the injection. Furthermore there is no surjection between n to ω since $n^+ = n \cap \{n\} \notin n$. Thus $n \subsetneq \omega \& n \neq \omega$.

1.1.4 AXIOM OF CHOICE

In Page 156, for the Theorem 6N:

> (a) For any infinite set A, we have $\omega \leq A$.
>
> (b) $\aleph_0 \leq \kappa$ for any infinite cardinal κ.

[Ray's Note 4: Intuitive idea on Theorem 6N]

$$\begin{tikzcd}
B + \omega \arrow[r, leftrightarrow, "bijection"'] \arrow[d, leftrightarrow, "bijection"'] & B + (\omega - C) \arrow[d, leftrightarrow, "bijection"] \\
A \arrow[r, "?"'] & A - C
\end{tikzcd}$$

where $C \neq \emptyset$. Clearly since composition of bijections are bijection, A and $A - C$ has a bijection.

1.1.5 COUNTABLE SETS

Page 160:

Theorem 6Q *A countable union of countable sets is countable. That is, if $\mathcal{A}$ is countable and if every member of $\mathcal{A}$ is a countable set, then $\bigcup \mathcal{A}$ is countable.*

Proof: We may suppose that $\emptyset \notin \mathcal{A}$, for otherwise we could simply remove it without affecting $\bigcup \mathcal{A}$. We may further suppose that $\mathcal{A} \neq \emptyset$, since $\bigcup \emptyset$ is certainly countable. Thus, $\mathcal{A}$ is a countable (but nonempty) set from $\omega \times \omega$ onto $\bigcup \mathcal{A}$. We already know of functions from ω onto $\omega \times \omega$, and the composition will map ω onto $\bigcup \mathcal{A}$, thereby showing that $\bigcup \mathcal{A}$ is countable.

Since $\mathcal{A}$ is countable but nonempty, there is a function G from ω onto $\mathcal{A}$. Informally, we may write

$$\mathcal{A} = \{G(0), G(1), \ldots\}.$$

(Here G might not be one-to-one, so there may be repetitions in this enumeration.) We are given that each set $G(m)$ is countable and nonempty.

Hence for each m there is a function from ω onto $G(m)$. We must use the axiom of choice to select such a function for each m.

Because the axiom of choice is a recent addition to our repertoire, we will describe its use here in some detail. Let $H : \omega \to \omega(\bigcup \mathcal{A})$ be defined by

$$H(m) = \{g \mid g \text{ is a function from } \omega \text{ onto } G(m)\}.$$

We know that $H(m)$ is nonempty for each m. Hence there is a function F with domain ω such that for each m, $F(m)$ is a function from ω onto $G(m)$.

To conclude the proof we have only to let $f(m, n) = F(m)(n)$. Then f is a function from $\omega \times \omega$ onto $\bigcup \mathcal{A}$.

[Ray's Note 5: Intuation on countable set of countable sets is countable]

To gain a more intuitive understanding, observe the illustration: Since $\mathcal{A}$ is countable, it has an enumeration given by $\mathcal{A} = \{G(0), G(1), \ldots\} = \{G_0, G_1, \ldots\}$, where each G_m is another countable set. More explicitly, we can represent each set as follows:

$$G_0 = \{G_0^0, G_0^1, G_0^2, G_0^3, \ldots, G_0^m, \ldots\}$$
$$G_1 = \{G_1^0, G_1^1, G_1^2, G_1^3, \ldots, G_1^m, \ldots\}$$
$$\vdots$$
$$G_n = \{G_n^0, G_n^1, G_n^2, G_n^3, \ldots, G_n^m, \ldots\}$$
$$\vdots$$

Thus, this construction clearly forms an injection into $\omega \times \omega$.

1.2 Excercise Answers

Question 1. E 4 *Construct a one-to-one correspondence between the closed unit interval*

$$[0, 1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$$

and the open unit interval $(0, 1)$.

Proof. To construct a one-to-one correspondence between the closed unit interval $[0, 1]$ and the open unit interval $(0, 1)$, we proceed as follows:

1. We first define a bijection between the sequences $\{0, 1, 1/2, 1/3, 1/4, \ldots\}$ and $\{1/2, 1/3, 1/4, \ldots\}$. Let f be defined as:

$$f(x) = \begin{cases} 1/2 & \text{if } x = 0, \\ 1/3 & \text{if } x = 1, \\ \frac{1}{n+2} & \text{if } x = \frac{1}{n} \text{ for } n \geq 2. \end{cases}$$

The inverse function f^{-1} is given by:

$$f^{-1}\left(\frac{1}{x}\right) = \begin{cases} 0 & \text{if } x = 2, \\ 1 & \text{if } x = 3, \\ \frac{1}{x-2} & \text{if } x > 3. \end{cases}$$

2. For the remaining points in $[0, 1]$ that are not in the sequence $\{0, 1, 1/2, 1/3, 1/4, \ldots\}$, we define a function g by:

$$g(x) = \begin{cases} f(x) & \text{if } x \text{ is in the sequence,} \\ x & \text{otherwise (identity map).} \end{cases}$$

This function g forms a bijection between $[0, 1]$ and $(0, 1)$.

□

Question 2. E 6 *Let κ be a nonzero cardinal number. Show that there does not exist a set to which every set of cardinality κ belongs.*

Proof. Proof by contradiction (BWOC): Assume that such a set exists. Let

$$A = \{K \mid |K| = \kappa\}$$

be the set containing all sets with cardinality κ.

Consider

$$\bigcup A,$$

the union of all sets in A. By construction, $\bigcup A$ would be a set that contains every possible set of cardinality κ.

However, this leads to a contradiction since such a set does not exist in the framework of standard set theory (e.g., due to limitations implied by Russell's paradox or cardinality constraints).

Therefore, there does not exist a set A such that it contains every set of cardinality κ. □

Question 3. E 7 *Assume that A is finite and $f : A \to A$. Show that f is one-to-one if and only if $\operatorname{ran} f = A$.*

Given that $\operatorname{ran} f = A$, this implies that f is surjective. Thus, the statement is equivalent to proving that: *"Assume that A is finite and $f : A \to A$. Show that f is one-to-one if and only if f is surjective."*

Proof. Assume A is finite.

- ($\Rightarrow$) **(By way of contradiction):** Assume that f is injective but not surjective. Then, the image of f under A, denoted by $f[A]$, is a proper subset of A, meaning $f[A] \subsetneq A$. Thus, we have an injective function $f : A \to f[A]$. By construction, $f[A] = f[A]$, and therefore, f is surjective onto $f[A]$. Hence, f is a bijection from A to $f[A]$, where $f[A] \subsetneq A$. According to Corollary 6D, this implies that A is equinumerous to a proper subset of itself, which contradicts the assumption that A is finite.

- ($\Leftarrow$) **(By way of contradiction):** Assume that f is surjective but not injective. By the axiom of choice, there exists an injection, call it g, from A to the preimage of A under f, which is a proper subset of A. Such a function g would then be a bijection from the preimage of A to A. This again implies that A is equinumerous to a proper subset of itself, leading to the conclusion that A is infinite. This contradicts our original assumption that A is finite.

$\square$

Question 4. E 8 *Prove that the union of two finite sets is finite (Corollary 6K), without any use of arithmetic.*

Proof. Let $|A| \approx n$ and $|B| \approx m$, where $n, m \in \omega$. Assume that A and B are disjoint. By the definition of equinumerosity, there exist functions $h : A \to n$ and $g : B \to m$. We define a function $f : A \cup B \to n + m$ (as defined earlier) such that

$$f(x) = \begin{cases} h(x) & \text{if } x \in A \\ n + g(x) & \text{if } x \in B \end{cases}$$

This function is well-defined since A and B are disjoint. It is straightforward to verify that such an f is a bijection with an inverse $f^{-1} : n + m \to A \cup B$ defined as follows:

$$f^{-1}(k) = \begin{cases} h^{-1}(k) & \text{if } k \leq n \\ g^{-1}(k - n) & \text{if } k > n \end{cases}$$

If A and B are not disjoint, then let $A \cap B = C \neq \emptyset$. Replace each element in C by elements not present in $A \cup B$ to form a new set C'. Let $A' = (A - C) \cup C'$. We have $|A'| = |A|$ and $A \cup B \subseteq A' \cup B$, which forms a disjoint union. By Lemma 6F, there exists a $k < n + m$ such that $|A \cup B| = k$.

Therefore, in both cases, $A \cup B$ is finite. $\square$

Question 5. E 9 *Prove that the Cartesian product of two finite sets is finite (Corollary 6K), without any use of arithmetic.*

Proof. We will use induction to prove this statement:

Let

$$S = \{m \mid \forall n \in \omega, \forall A, B \text{ such that } |A| = m, |B| = n, |A \times B| = m \times n\},$$

where $\times$ is defined as the Cartesian product for natural numbers.

1. **Base Case:** $0 \in S$. For any set B, $\emptyset \times B = \emptyset$. Thus, $|\emptyset \times B| = 0 \times |B| = 0$ since B is finite and therefore $|B| \in \omega$. Furthermore, $1 \in S$ since for any singleton $\{a\}$ and any set B, there exists a bijection between $\{a\} \times B$ and B given by the function $f : \{a\} \times B \to B$ defined as $f(a, x) = x$. The inverse function is $g(x) = (a, x)$. Therefore, $|\{a\} \times B| = 1 \times |B| = |B|$.

2. **Inductive Step:** Assume $k \in S$. We want to show that $k^+ \in S$, where k^+ denotes the successor of k.

   To show $k^+ \in S$, we need to prove that for all $m \in \omega$ and for all sets A and B such that $|A| = k^+$ and $|B| = m$, we have $|A \times B| = k^+ \times m$.

   Since we have already proven that $0 \in S$, the induction starts with $k \geq 1$, meaning $k^+ \geq 2$.

Let A be a set with $|A| = k^+$. Since $k^+ \geq 2$, there exists an element $a \in A$. Let $A' = A \setminus \{a\}$. Therefore, $|A'| = k$, and by the induction hypothesis, $|A' \times B| = k \times m$. Since $\{a\}$ is a singleton, we have $|\{a\} \times B| = 1 \times m = m$ by the induction hypothesis.

Therefore,

$$|A \times B| = |(A' \times B) \cup (\{a\} \times B)| = k \times m + m = k^+ \times m$$

by the definition of multiplication for natural numbers.

Hence, by induction, the Cartesian product of two finite sets is finite (since natural number is closed under multiplication). □

Question 6. E 10 to 12 *For Excercises 10 to 12, proving all the Theorem 6I is suffices.*

In Page 142

> **Theorem 6I** For any cardinal numbers κ, λ, and μ:
>
> 1. $\kappa + \lambda = \lambda + \kappa$ and $\kappa \cdot \lambda = \lambda \cdot \kappa$.
> 2. $\kappa + (\lambda + \mu) = (\kappa + \lambda) + \mu$ and $\kappa \cdot (\lambda \cdot \mu) = (\kappa \cdot \lambda) \cdot \mu$.
> 3. $\kappa \cdot (\lambda + \mu) = \kappa \cdot \lambda + \kappa \cdot \mu$.
> 4. $\kappa^{\lambda+\mu} = \kappa^\lambda \cdot \kappa^\mu$.
> 5. $(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$.
> 6. $(\kappa^\lambda)^\mu = \kappa^{\lambda \cdot \mu}$.
>
> **Proof** Take sets K, L, and M with card $K = \kappa$, card $L = \lambda$, and card $M = \mu$; for convenience, choose them in such a way that any two are disjoint. Then each of the equations reduces to a corresponding statement about equinumerous sets. For example, $\kappa \cdot \lambda =$ card $(K \times L)$ and $\lambda \cdot \kappa =$ card $(L \times K)$; consequently, showing that $\kappa \cdot \lambda = \lambda \cdot \kappa$ reduces to showing that $K \times L \approx L \times K$. Listed in full, the statements to be verified are:
>
> 1. $K \cup L \approx L \cup K$ and $K \times L \approx L \times K$.
> 2. $K \cup (L \cup M) \approx (K \cup L) \cup M$ and $K \times (L \times M) \approx (K \times L) \times M$.
> 3. $K \times (L \cup M) \approx (K \times L) \cup (K \times M)$.
> 4. ${}^{(L \cup M)}K \approx {}^K L \times {}^k M$.
> 5. ${}^M(K \times L) \approx {}^M K \times {}^M L$.
> 6. ${}^M({}^L K) \approx {}^{(L \times M)}K$.

I will use the second part of 1 to 6 (on the set) to prove the theorem.

Proof. For the simple ones I will simple list the fucntion without justifying that it is bijection.

1. $K \cup L = L \cup K$ by algebra of set (Page 27). f: $K \times L \to L \times K$ s.t. $< k, l > \mapsto < l, k >$
2. $K \cup (L \cup M) = (K \cup L) \cup M$ by algebra of set (Page 27). f: $K \times (L \times M) \to (K \times L) \times M$ s.t. $< k, < l, m >> \mapsto << k, l >, m >$.

3. $K \times (L \cup M) = (K \times L) \cup (K \times M)$ by Excercise 2 Chapter3.

4. $H : {}^{(L \cup M)}K \to {}^{K}L \times {}^{k}M \; s.t. f \mapsto < f \upharpoonright_L, f \upharpoonright_{M-L} >$.

5. $H : {}^{M}(K \times L) \to {}^{M}K \times {}^{M}L$. Let $f \in {}^{M}(K \times L)$, then $\forall m \in Mf : m \mapsto f(m) =< k, l >=< f_1(m), f_2(m) >$. Thus let $H : f \mapsto < f_1, f_2 >$.

6. Proven in book.

□

Question 7. E 13 *Show that a finite union of finite sets is finite. That is, show that if B is a finite set whose members are themselves finite sets, then $\bigcup B$ is finite.*

Proof. In Chapter 6 E 3, we have proved that two finite set's union is finite. Do induction on natural number, we can prove that any finite union of finite sets is finite. □

Question 8. 14 *Define a permutation of K to be any one-to-one function from K onto K. We can then define the factorial operation on cardinal numbers by the equation*

$$\kappa! = card\{f \mid f \text{ is a permutation of } K\},$$

where K is any set of cardinality κ. Show that $\kappa!$ is well defined, i.e., the value of $\kappa!$ is independent of just which set K is chosen.

Proof. To show that $\kappa!$ is well-defined, let κ be a cardinal number, and consider any two sets A and B such that $|A| = |B| = \kappa$.

Since $A \approx A$, let f_A be a bijection $f_A : A \to A$. Similarly, let f_B be a bijection $f_B : B \to B$. Since $A \approx B$, there exists a bijection $f : A \to B$.

We can establish a bijection between the set of permutations of A (denoted by perm(A) or sym(A)) and the set of permutations of B (denoted by perm(B) or sym(B)). This bijection can be constructed as follows: for any $f_A \in$ perm(A), we define a corresponding permutation of B by $f'_A = f \circ f_B \circ f^{-1}$ (this is a conjugation by f).

This construction is clearly a bijection between perm(A) and perm(B). Thus, the value of $\kappa!$ is independent of the particular choice of the set K with cardinality κ, proving that the factorial operation on cardinal numbers is well-defined. □

Question 9. E 15 *Show that there is no set $\mathcal{A}$ with the property that for every set there is some member of $\mathcal{A}$ that dominates it.*

Proof. To prove that there is no such set $\mathcal{A}$ with the stated property, we proceed by contradiction.

By way of contradiction (BWOC): Assume that such a set $\mathcal{A}$ exists. Let $\mathcal{A}' = \bigcup \mathcal{A}$. By the definition of $\mathcal{A}$, there exists a set $K \in \mathcal{A}$ such that K dominates $\mathcal{A}'$. Therefore, there is an injection from $\mathcal{A}'$ to K.

By the definition of $\mathcal{A}$, the power set 2^K (the set of all subsets of K) is dominated by some element $K' \in \mathcal{A}$. This means that there exists an injection from 2^K to K'. Since $K' \in \mathcal{A}$ and $\mathcal{A}' = \bigcup \mathcal{A}$, we have $K' \subseteq \mathcal{A}'$. Therefore, the embedding of K' into $\mathcal{A}'$ is a natural injection.

Furthermore, we know that $\mathcal{A}'$ injects into K, and K injects into 2^K, since the power set of any set strictly dominates the set itself. Thus, we have an injection from 2^K into $\mathcal{A}'$. By the transitivity of injections (i.e., the property that injections are preserved under composition), this implies that 2^K injects into K, which contradicts the fact that the power set of a set always strictly dominates the set itself.

Therefore, our initial assumption that such a set $\mathcal{A}$ exists must be false. $\square$

Question 10. E 16 *Show that for any set S we have $S \preceq 2^S$, but $S \not\approx 2^S$.*

Proof. To prove that for any set S we have $S \preceq 2^S$ but $S \not\approx 2^S$, we proceed as follows:

1. Injection from S into 2^S: Define a function $H : S \to {}^S 2$ such that $H(x) \mapsto f_x$, where $f_x : S \to \{0, 1\}$ is defined by:

$$f_x(y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}$$

This function H is injective since distinct elements $x, x' \in S$ will be mapped to distinct functions f_x and $f_{x'}$.

2. Proving $S \not\approx 2^S$: To show that S is not equinumerous to 2^S, we use a diagonal argument. Consider the function $g : S \to \{0, 1\}$ defined by:

$$g(x) = 1 - H(x)(x)$$

We claim that the set $B = \{g : g(x) = 1 - H(x)(x)\}$ is a subset of ${}^S 2$ but is not in the range of H.

Proof. We prove by two parts:
$B \subseteq {}^S 2$: This is straightforward to show. Since $H(x)(x) \in \{0, 1\}$, we have $g(x) \in \{0, 1\}$ for all $x \in S$. Therefore, g is a valid function from S to $\{0, 1\}$, implying $B \subseteq {}^S 2$.
B is not in the range of H: Suppose, for the sake of contradiction (BWOC), that there exists $x \in S$ such that $H(x) = g$. Then we have:

$$H(x)(x) = 1 - H(x)(x)$$

This implies:

$$H(x)(x) = \frac{1}{2}$$

which is a contradiction, since $H(x)(x) \in \{0, 1\}$.

Thus, we have shown that $S \preceq 2^S$ but $S \not\approx 2^S$. $\square$

$\square$

Question 11. E 17 *Give counterexamples to show that we cannot strengthen Theorem 6L by replacing "$\le$" by "$<$" throughout.*

Theorem 6L in Page 149:

> **Theorem 6L** Let κ, λ, and μ be cardinal numbers.
>
> (a) $\kappa \le \lambda \implies \kappa + \mu \le \lambda + \mu$.
>
> (b) $\kappa \le \lambda \implies \kappa \cdot \mu \le \lambda \cdot \mu$.
>
> (c) $\kappa \le \lambda \implies \kappa^\mu \le \lambda^\mu$.
>
> (d) $\kappa \le \lambda \implies \mu^\kappa \le \mu^\lambda$; if not both κ and μ equal zero.

Proof. **A Concrete Example:**

Consider the sets $A = \{0, 1\}$ and $B = \{0, 1, 2\}$. We have $|A| < |B|$, yet $A \cup \omega \approx B \cup \omega$, which implies $|A| + \aleph_0 = |B| + \aleph_0$. The same holds for the other operations as well.

A More General and Abstract Example:

For any $n, m \in \omega$ with $n < m$, we have:

$$n + \aleph_0 = m + \aleph_0, \quad n \times \aleph_0 = m \times \aleph_0,$$

and similarly for the other cases.

□

Question 12. E 18 *Prove that the following statement is equivalent to the axiom of choice: For any set $\mathcal{A}$ whose members are nonempty sets, there is a function f with domain $\mathcal{A}$ such that $f(X) \in X$ for all $X \in \mathcal{A}$.*

Proof. To prove that the axiom of choice is equivalent to the given statement, we proceed as follows:

Axiom of Choice: The axiom of choice states that for any relation R, there exists a function $f \subset R$ such that $\operatorname{dom} f = \operatorname{dom} R$.

To Show:

$$\text{Axiom of Choice} \iff \text{Given Statement}$$

(Forward Implication): Assume the axiom of choice holds. Let

$$R := \{\langle X, x\rangle \mid x \in X \in \mathcal{A}\}.$$

By the axiom of choice, there exists a function $f \subset R$ such that $\operatorname{dom} f = \operatorname{dom} R = \mathcal{A}$. This function f satisfies the required condition: for all $X \in \mathcal{A}$, $f(X) \in X$.

(Reverse Implication): Assume the given statement holds. Let R be a relation and $K = \operatorname{dom} R$. Define

$$\mathcal{A} = \{X_{x_r} \subseteq K \mid x_r \in \operatorname{ran} R, \forall x \in X, xRr_x \text{ and } X \neq \emptyset\}.$$

By the assumption, there exists a function f such that for all $X \in \mathcal{A}$, $f(X) \in X$. This function f has $\operatorname{dom} f = \mathcal{A} = \operatorname{dom} R$.

□

Question 13. E 19 *Assume that H is a function with finite domain I and that $H(i)$ is nonempty for each $i \in I$. Without using the axiom of choice, show that there is a function f with domain I such that $f(i) \in H(i)$ for each $i \in I$. [Suggestion: Use induction on card I.]*

Question 14. E 20 *Assume that A is a nonempty set and R is a relation such that*

$$\forall x \in A, \exists y \in A \text{ such that } yRx.$$

Show that there exists a function $f : \omega \to A$ with

$$f(n^+) R f(n) \quad \text{for all } n \in \omega.$$

Proof. Given that $R \subset A \times A$ is a "surjective" relation on A, we can define $R' = R^{-1} := \{\langle x, y\rangle \mid \langle y, x\rangle \in R\}$, which is also a relation on A. By the axiom of choice, there exists a function $g \subset R'$ such that $\operatorname{dom} g = \operatorname{dom} R' = A$.

Since A is nonempty, it contains at least one element. Let $x \in A$ be such an element. We can define a function $f : \omega \to A$ recursively as follows:

1. $f(0) = x$.
2. $f(n^+) = g(f(n))$, where g is the function induced by R'.

Since g is a function derived from the inverse relation R', it follows that $f(n^+)\, R\, f(n)$ for all $n \in \omega$.

By a simple induction, we can also show that $\operatorname{dom} f = \omega$, ensuring that f is defined on all natural numbers. □

Question 15. E 21 Teichmüller–Tukey Lemma: *Assume that $\mathcal{A}$ is a nonempty set such that for every set B,*

$$B \in \mathcal{A} \iff \text{every finite subset of } B \text{ is a member of } \mathcal{A}.$$

Show that $\mathcal{A}$ has a maximal element, i.e., an element that is not a subset of any other element of $\mathcal{A}$.

Zorn's Lemma: Let $\mathcal{A}$ be a set such that for every chain $\mathcal{B} \subseteq \mathcal{A}$, we have $\bigcup \mathcal{B} \in \mathcal{A}$. ($\mathcal{B}$ is called a **chain** if for any C and D in $\mathcal{B}$, either $C \subseteq D$ or $D \subseteq C$.) Then $\mathcal{A}$ contains a maximal element M such that M is not a subset of any other set in $\mathcal{A}$.

Our goal is to verify the conditions of Zorn's Lemma for the set $\mathcal{A}$, and then conclude that $\mathcal{A}$ contains a maximal element.

To apply Zorn's Lemma, we need to show that for every chain $\mathcal{B} \subseteq \mathcal{A}$, the union $\bigcup \mathcal{B}$ is in $\mathcal{A}$. This means we need to prove:

1. $\mathcal{B}$ is a chain in $\mathcal{A}$.
2. $\bigcup \mathcal{B} \in \mathcal{A}$.

Since $\mathcal{B}$ is any chain in $\mathcal{A}$, we focus on proving that $\bigcup \mathcal{B} \in \mathcal{A}$.

According to the defining property of $\mathcal{A}$:

$$B \in \mathcal{A} \iff \text{every finite subset of } B \text{ is in } \mathcal{A}.$$

So, to show that $\bigcup \mathcal{B} \in \mathcal{A}$, we need to demonstrate that **every finite subset of $\bigcup \mathcal{B}$ is in $\mathcal{A}$**.

Proof. Let $\mathcal{B}$ be a chain s.t. $\mathcal{B} \subseteq \mathcal{A}$. Let F be any finite subset of $\bigcup \mathcal{B}$. For each element $x \in F$, there exists a set $B_x \in \mathcal{B}$ such that $x \in B_x$. This is because x is in $\bigcup \mathcal{B}$, so it must belong to at least one member of $\mathcal{B}$.

Since $\mathcal{B}$ is a chain, the sets B_x are comparable by inclusion. For any two elements $x, y \in F$, either $B_x \subseteq B_y$ or $B_y \subseteq B_x$. Because F is finite, the collection $\{B_x : x \in F\}$ is finite. We can find a maximal set B_0 among $\{B_x : x \in F\}$ with respect to inclusion.

Since $B_0 \in \mathcal{A}$, and $F \subseteq B_0$, we can conclude that $F \in \mathcal{A}$. Thus $\bigcup \mathcal{B} \in \mathcal{A}$, which satisfies conditions for zorn's lemma. □

Question 16. E 22 *Show that the following statement is another equivalent version of the axiom of choice: For any set A there is a function F with $\operatorname{dom} F = \bigcup A$ and such that $x \in F(x) \in A$ for all*

$x \in \bigcup A$.

Proof. To show: **Axiom of Choice (AC)** $\iff$ statement.

($\Rightarrow$) Assume A is a set. Consider the relation $R = \{\langle x, X\rangle \mid X \in A, x \in X\}$. By the Axiom of Choice, there exists a function $f \subseteq R$ such that $\operatorname{dom} f = \operatorname{dom} R$. Clearly, $\operatorname{dom} R = A$. Additionally, for all $x \in \bigcup A$, we have $x \in f(x) \in A$. Thus, we have found such a function f.

($\Leftarrow$) Let R be a relation. Let $K = \operatorname{dom} R$. Define

$$A = \{X_r \subseteq K \mid x_r \in \operatorname{ran} R, \forall x \in X_r, xRx_r\}.$$

By assumption, there exists a function f with $\operatorname{dom} f = \bigcup A = \operatorname{dom} R$.

□

Question 17. E 26 *Prove the following generalization of Theorem 6Q: If every member of a set $\mathcal{A}$ has cardinality κ or less, then*

$$\operatorname{card} \bigcup \mathcal{A} \leq (\operatorname{card} \mathcal{A}) \cdot \kappa.$$

Question 18. E 27

1. *Let A be a collection of circular disks in the plane, no two of which intersect. Show that A is countable.*
2. *Let B be a collection of circles in the plane, no two of which intersect. Need B be countable?*
3. *Let C be a collection of figure eights in the plane, no two of which intersect. Need C be countable?*

Proof. Use Rational numbers.

1. Every disck include at least one rational number. If it is countable, then there must be two circle containing same rational number, contardiction to pair wise disjoint.
2. No. Consider $B = \{B_r(0) | r \in (0, 1]\}$
3. Yes. Pick one eight shape in the connlections. Three conditions for second eight, either in one of the circle of first eight or outside of the first eight: if all the eight is pair wise outside, then it is the same as eight shaped disk, appying the first arguement it is countable. If on the other hand in one of the circles, then the second disk's diameter (greatest length) is decreased by 2 wetice at least. This will form a sequence for eight shapes that is countable (since there is at least one rational number that is between two shapes.). Or in another way we can argue that there must be a rational number included in the difference of two eight, thus must be countable since difference shape is more than eight shape.

□

2 Chapter 7: ORDERINGS AND ORDINALS

2.1 Excercise Answers

Question 19. E 4 *Let $<$ be the usual ordering on the set P of positive integers. For $n \in P$, let $f(n)$ be the number of distinct prime factors of n. Define the binary relation R on P by*

$$mRn \iff \text{either } f(m) < f(n) \text{ or } [f(m) = f(n) \wedge m < n].$$

Show that R is a well-ordering on P. Does $\langle P, R\rangle$ resemble any of the pictures in Fig. 45 (p. 185)?

Proof. To demonstrate that the relation R is linear:

1. **Transitivity**: Assume $x\,R\,y$ and $y\,R\,z$. Clearly, by the definition of R, $x\,R\,z$ holds, satisfying the transitivity condition.

2. **Totality**: For all $x, y \in P$, if $x \neq y$, then exactly one of the following must hold:

- If $f(x) \neq f(y)$, then either $x\,R\,y$ or $y\,R\,x$, depending on whether $f(x) < f(y)$ or $f(x) > f(y)$.
- If $f(x) = f(y)$, then $x \neq y$ implies either $x < y$ or $y < x$ (by the natural ordering on P).

Thus, R is a linear relation.

For well-ordering:

Let $A \subseteq P$. Define subsets $S_k = \{x \in A \mid f(x) = k\}$, which clearly form a partition of A based on the values of $f(x)$.

- Observe that there is a natural ordering on these subsets S_k induced by the ordering of their indices k: $S_k < S_m$ if $k < m$. This ordering is linear since $k \in \mathbb{N}$.
- The collection $\{S_k\}$ can be embedded into $\mathbb{N}$ via a natural projection $S_k \mapsto k$. Since $\mathbb{N}$ is well-ordered, there exists a least k such that $S_k \neq \emptyset$. Let this least subset be S_k.
- Within S_k, $S_k \subseteq \mathbb{N}$ implies that S_k inherits the natural ordering of $\mathbb{N}$. Hence, S_k contains a least element.

This least element of S_k is the least element of A under R, proving that R is a well-ordering on P. □

Question 20. E 10 *For any set S, we can define the relation $\in_S$ by the equation:*

$$\in_S = \{\langle x, y\rangle \in S \times S \mid x \in y\}$$

(a) Show that for any natural number n, the $\in$-image of $\langle n, \in_n\rangle$ is n.

(b) Find the $\in$-image of $\langle \omega, \in_\omega\rangle$.

Proof. **Proof for (a):** Let $A = \{x \in \omega \mid \text{the } \epsilon\text{-image of } \epsilon_x = x\}$. We want to show that $A = \omega$.

Proof by induction:

- **Base case:** Consider $0 = \emptyset$. Clearly, ϵ-image of $\epsilon_0 = F(0) = F(\emptyset) = \emptyset = 0$. Thus, $0 \in \epsilon$-image, and $0 \in A$.
- **Inductive step:** Assume $n \in A$, i.e., the ϵ-image of $\epsilon_n = n$. We want to show that $n^+ \in A$, i.e., the ϵ-image of $\epsilon_{n^+} = n^+$.

  From the definition of F, we know that $F(t) = F[\![\text{seg } t]\!]$. Applying this to n, we have $F(n) = F[\![\text{seg } n]\!] = \epsilon$-image of n.

Therefore,

$$\epsilon\text{-image of } n^+ = \epsilon\text{-image of } n \cup \{F(n)\} = (\epsilon\text{-image of } n) \cup \{\epsilon\text{-image of } n\} = n \cup \{n\} = n^+$$

Thus, $n^+ \in A$.

By induction, $A = \omega$.

Proof for (b): We claim that the ϵ-image is ω.

Proof by induction:

- **Base case:** Clearly, $0 = \emptyset \in \epsilon$-image, since $F(0) = F(\emptyset) = \emptyset = 0$.
- **Inductive step:** Assume $n \in \epsilon$-image, i.e., there exists $t \in \omega$ such that $F(t) = F[\![\text{seg } t]\!] = n$. We want to show that $n^+ \in \epsilon$-image.

  Since $n \in \epsilon$-image, there exists $t \in \omega$ such that $F(t) = n$. For t^+, we compute:

  $$F(t^+) = F[\![\text{seg } t^+]\!] = F[\![\text{seg } t]\!] \cup F[\![t]\!].$$

  Substituting $F[\![\text{seg } t]\!] = F(t) = n$, we get:

  $$F(n^+) = n \cup F[\![t]\!].$$

  Since $F[\![t]\!] = \{f(t)\} = \{n\}$, we have:

  $$F(n^+) = n \cup \{n\} = n^+.$$

  Thus, $n^+ \in \epsilon$-image.

By induction, the ϵ-image $= \omega$.

□

Question 21. E 13. *Assume that two well-ordered structures are isomorphic. Show that there can be only one isomorphism from the first onto the second.*

Proof. Let $(A, <_A)$ and $(B, <_B)$ be two isomorphic well-ordered structures, and let f and g be two isomorphisms from A to B. Define the set

$$M = \{x \in A \mid f(x) \neq g(x)\}.$$

We aim to show that $M = \emptyset$, i.e., $f = g$.

Since $M \subseteq A$ and A is well-ordered, the set M, if non-empty, must have a least element. Let $a \in M$ be this least element. By definition of M, we have $f(a) \neq g(a)$. By the linearity of the order on B, either $f(a) <_B g(a)$ or $g(a) <_B f(a)$. Without loss of generality, assume $f(a) <_B g(a)$.

Since f and g are bijections, there exists some $a' \in A$ such that $g(a') = f(a)$. Note that $a' \in M$ because $f(a') \neq g(a')$. Furthermore, since a is the least element of M, we must have $a' <_A a$.

Thus, we have the following relations: 1. $f(a) <_B g(a)$, 2. $g(a') = f(a)$, 3. $a' <_A a$.

Now, apply f and g to the relation $a' <_A a$. Since both f and g preserve the order (as they are isomorphisms), we have:

$$f(a') <_B f(a) \quad \text{and} \quad g(a') <_B g(a).$$

From these, we can form the following chain:

$$g(a) <_B g(a') = f(a) <_B f(a').$$

This implies $g(a) <_B f(a)$, contradicting our earlier assumption that $f(a) <_B g(a)$.

Therefore, $M = \emptyset$, and we conclude that $f = g$. Hence, the isomorphism between the two well-ordered structures is unique.

□

Question 22. E 20. *Show that if R and R^{-1} are both well-orderings on the same set S, then S is finite.*

Proof. By way of contradiction, assume S is not finite. Let $S_0 = S$. Since S is well-ordered with respect to R, let x_0 be the least element of S_0. Define $S_1 = S_0 \setminus \{x_0\}$, and let x_1 be the least element of S_1 with respect to R. Repeating this process, we form a sequence $\{x_i\}_{i=0}^{\infty}$. This is possible because S is infinite, and removing any finite subset from S still leaves an infinite set.

Define $K = \{x_i\}_{i=0}^{\infty}$. Clearly, $K \subseteq S$. Furthermore, for every $x_i \in K$, there exists $x_j \in K$ such that $x_i \, R \, x_j$. This implies $x_j \, R^{-1} \, x_i$.

However, since K has no least element with respect to R^{-1}, it follows that R^{-1} is not a well-ordering on S. This contradicts the assumption that both R and R^{-1} are well-orderings on S.

Thus, S must be finite. □

2.2 Material Notes

2.2.1 ISOMORPHISMS

Around the Book Page 184, we also need to notice that two structures are isomorphic if and only if their ϵ-image equals. Notice that I used **equals**, this is just by extension, the element wise equals.

Proof. Let $(A, <_A)$ and $(B, <_B)$ be two isomorphic well-ordered structures with isomorphism f. Let α be the ε-image of the first, and β be the ε-image of the second. To show that $\alpha = \beta$, we are essentially showing that for all $x \in A$, $E_A(x) = E_B(f(x))$. We can use transfinite induction.

Let $P = \{x \mid E_A(x) = E_B(f(x))\}$. By transfinite induction, assume $\operatorname{seg}(t) \subseteq P$. We want to show that $t \in P$, meaning that $E_A(t) = E_B(f(t))$.

We have:

$$E_A(t) = E_A[[\operatorname{seg}(t)]] = \{E_A(x) \mid x < t\} = \{E_B(f(x)) \mid f(x) < f(t)\} = E_B[[\operatorname{seg}(f(t))]] = E_B(t).$$

Thus, $t \in P$.

□

2.3 ORDINAL NUMBERS

In Page 191, we have

> **Theorem 7L.** Let α be any transitive set that is well-ordered by $\in$. Then α is an ordinal number; in fact, α is the $\in$-image of $(\alpha, \in_\alpha)$.
>
> **Proof.** Let E be the usual function from α onto its $\in$-image. We can use transfinite induction to show that E is just the identity function on α. Note that for $t \in \alpha$,
>
> $$x \in t \iff x \in_\alpha t$$
>
> because α is a transitive set. As a consequence, we have $\operatorname{seg} t = t$.
>
> If the equation $E(x) = x$ holds for all $x \in \operatorname{seg} t$, then
>
> $$\begin{aligned} E(t) &= \{E(x) \mid x \in_\alpha t\} \\ &= \{x \mid x \in_\alpha t\} \\ &= \operatorname{seg} t \\ &= t. \end{aligned}$$

Yet while I was deducting it myself, I get the following:

$$E(t) = E[\![\text{seg } t]\!] = E[\![t]\!] = \{E(t)\}$$

Which gives that

$$E(t) \in E(t)$$

Can you tell where the problem is?

The answer is that $E[\![t]\!] \neq \{E(t)\}$, $E[\![\{t\}]\!] = \{E(t)\}$, yet $t \neq \{t\}$

2.4 Rank

In Page 202, we have the following

> **Lemma 7R** Let δ and ε be ordinal numbers; let F_δ and F_ε be functions from Lemma 7Q. Then
>
> $$F_\delta(\alpha) = F_\varepsilon(\alpha)$$
>
> for all $\alpha \in \delta \cap \varepsilon$.
>
> *Proof.* By the symmetry, we suppose that $\delta \subseteq \varepsilon$. Hence $\delta \subseteq \varepsilon$ and $\delta \cap \varepsilon = \delta$. We will establish the equation $F_\delta(\alpha) = F_\varepsilon(\alpha)$ by using transfinite induction in $\langle \delta, \in_\delta \rangle$. Define
>
> $$B = \{\alpha \in \delta \mid F_\delta(\alpha) = F_\varepsilon(\alpha)\}.$$
>
> In order to show that $B = \delta$, it suffices to show that B is "$\in_\delta$-inductive," i.e., that
>
> $$\text{seg } \alpha \subseteq B \implies \alpha \in B$$
>
> for each $\alpha \in \delta$.
> We calculate:
>
> $$\begin{aligned} \text{seg } \alpha \subseteq B &\implies F_\delta(\beta) = F_\varepsilon(\beta) \quad \text{for } \beta \in \alpha \\ &\implies \bigcup \mathscr{P} F_\delta(\beta) \mid \beta \in \alpha\} = \bigcup \mathscr{P} F_\varepsilon(\beta) \mid \beta \in \alpha\} \\ &\implies F_\delta(\alpha) = F_\varepsilon(\alpha) \\ &\implies \alpha \in B. \end{aligned}$$
>
> for $\alpha \in \delta$. And so we are done.
>
> In particular (by taking $\delta = \varepsilon$) we see that the function F_δ from Lemma 7Q is unique. We can now unambiguously define V_α.
>
> **Definition** Let α be an ordinal number. Define V_α to be the set $F_\delta(\alpha)$, where δ is any ordinal greater than α (e.g., $\delta = \alpha^+$).
>
> **Theorem 7S** For any ordinal number α,
>
> $$V_\alpha = \bigcup \{\mathscr{P} V_\beta \mid \beta \in \alpha\}.$$
>
> *Proof.* Let $\delta = \alpha^+$. Then $V_\alpha = F_\delta(\alpha)$ and $V_\beta = F_\delta(\beta)$ for $\beta \in \alpha$. Hence the desired equation reduces to Lemma 7Q. □
>
> **Lemma 7T** For any ordinal number α, V_α is a transitive set.
>
> *Proof.* We would like to prove this by transfinite induction over the class of all ordinals. This can be done by utilizing Exercise 25. But we can also avoid that exercise by proving for each ordinal δ that V_α is a transitive set whenever $\alpha \in \delta$. This requires only transfinite induction over $\langle \delta, \in_\delta \rangle$.

Here I provide an alternative proof:

Proof. Let $x \in y \in V_\alpha$, where

$$V_\alpha = \bigcup\{\mathscr{P}(V_\beta) \mid \beta \in \alpha\}.$$

Then, by definition, there exists $\beta \in \alpha$ such that $y \in \mathscr{P}(V_\beta)$.

Since $y \in \mathscr{P}(V_\beta)$, it follows that $y \subseteq V_\beta$. Hence, because $x \in y$, we conclude that $x \in V_\beta$.

Next, note that $\{x\} \subseteq V_\beta$. To justify this step, recall that if $a \in b$, then $\{a\} \subseteq b$.

Since $\{x\} \subseteq V_\beta$, it follows that $\{x\} \in \mathscr{P}(V_\beta)$, because any subset of V_β is an element of its power set $\mathscr{P}(V_\beta)$.

Finally, since $\mathscr{P}(V_\beta) \in \bigcup\{\mathscr{P}(V_\beta) \mid \beta \in \alpha\}$, we conclude that $x \in \bigcup\{\mathscr{P}(V_\beta) \mid \beta \in \alpha\}$. This holds because if $q \in p$ and $p \in n$, then $q \in \bigcup n$.

□

3 Ray's Notes Summary

[Ray's Note 1 (Page 2): Here the Case 1 should have more explanation:
We know that k is closed under f and $\text{ran}(f \upharpoonright k) = k$. Then why do we have $\text{ran}f = k \cup \{k\}$? This is because of the following argument:
f is one-to-one. We also know that $k \notin k$ (otherwise we would form Russell's paradox). The preimage $f^{-1}[\{f(k)\}]$ (the preimage of $f(k)$ under f) can only contain one element since f is one-to-one, and $k \in f^{-1}[\{f(k)\}]$ because the preimage of $f(k)$ must contain k. Thus, $\text{ran}f = \text{ran}(f \upharpoonright k) \cup \text{ran}(f \upharpoonright \{k\}) = k \cup \{k\}$.]

[Ray's Note 2 (Page 2): Theorem 6H: More perspectives]

[Ray's Note 3 (Page 3): More on cardinal arithmetic]

[Ray's Note 4 (Page 4): Intuitive idea on Theorem 6N]

[Ray's Note 5 (Page 5): Intuation on countable set of countable sets is countable]