1. Let $Q(t)$denote the number of customers in the system at time t. Then $\{Q(t), t \geqslant 0\}$ is a Markov Chain with state space {0,1,...,N}.

2. Let $Q_i(t)$denote the number of customers in the system at time t. Then $\{(Q_1(t), Q_2(t)), t \geqslant 0\}$ is a Markov Chain with state space $\{(i, j)$, i, j = 0,1,2,..$\}$. $\pi_{i,j} = Pr(Q_1(\infty) = i, Q_2(\infty) = i)$.

M/M/1 $\rho \leqslant 1$	$\pi_0 = 1 - \rho$ $\pi_i = (1-\rho)\rho^i$	$L = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu-\lambda}$ $L_q = \frac{\rho^2}{1-\rho} = \frac{\lambda^2}{\mu(\mu-\lambda)} = \lambda W_q$ $W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{L_q}{\lambda}$ $W = \frac{L}{\lambda} = \frac{1}{\mu-\lambda}$
M/M/1/N $\rho \leqslant 1$	$\pi_0 = \frac{1-\rho}{1-\rho^{N+1}}$ $\pi_i = \frac{1-\rho}{1-\rho^{N+1}}\rho^i$ $if \rho = 1, \pi = \frac{1}{N+1}$	$L = \frac{\rho(1+N\rho^{N+1}-(N+1)\rho^N)}{(1-\rho)(1-\rho^{N+1})}$ $L = \frac{N}{2}\ if\ \rho = 1$ $\lambda_\alpha = \lambda(1-\pi_N)$ $W = \frac{L}{\lambda(1-\pi_N)}$ $L_q = \lambda(1-\pi_N)W_q$ $W_q = W - \frac{1}{\mu}$
M/M/K $\frac{\lambda}{k\mu} < 1$	$\pi_0 = \frac{1}{\sum_{n=0}^{s-1}\frac{(\frac{\lambda}{\mu})^n}{n!} + \frac{(\frac{\lambda}{\mu})^s}{s!} * \frac{1}{1-(\frac{\lambda}{s\mu})}}$ $\pi_i = \frac{(\frac{\lambda}{\mu})^i}{i!} * \pi_0\ if\ i = 1, ..s-1$ $\pi_i = \frac{(\frac{\lambda}{\mu})^i}{s!s^{i-s}} * \pi_0\ if\ i = s, s+1, ...$	$L_q = \frac{(\frac{\lambda}{\mu})^s\lambda\mu}{(s-1)!(s\mu-\lambda)^2} * \pi_0$ $W_q = \frac{L_q}{\lambda} = \frac{(\frac{\lambda}{\mu})^s\mu}{(s-1)!(s\mu-\lambda)^2} * \pi_0$ $W = W_q + \frac{1}{\mu}$ $L = \lambda W$ $= \frac{\lambda}{\mu} + \frac{(\frac{\lambda}{\mu})^s\lambda\mu}{(s-1)!(s\mu-\lambda)^2}\pi_0$
Preemptive $\rho_1 + \rho_2 < 1$	$Let\ \lambda = \lambda_1 + \lambda_2$ $\alpha(x) = \begin{cases} \lambda\pi_{0,0} = \mu_1\pi_{1,0} + \mu_2\pi_{0,1} \\ (\lambda+\mu_1)\pi_{i,0} = \lambda_1\pi_{i-1,0} + \mu_1\pi_{i+1,0}, i \geqslant 1 \\ (\lambda+\mu_2)\pi_{0,j} = \lambda_2\pi_{0,j-1} + \mu_2\pi_{0,j+1} \\ +\mu_1\pi_{1,j}, j \geqslant 1 \\ (\lambda+\mu_1)\pi_{i,j} = \lambda_1\pi_{i-1,j} + \lambda_2\pi_{i,j-1} \\ +\mu_1\pi_{i+1,j}, i, j \geqslant 1 \\ \sum_{i,j}^{\infty}\pi_{i,j} = 1 \end{cases}$	$L_1 = \frac{\rho_1}{1-\rho_1}$ $L_2 = \frac{\rho_2}{1-\rho_1-\rho_2}(1 + \frac{\mu_2}{\mu_1}\frac{\rho_1}{1-\rho_1})$ $L_1^q = \frac{(\rho_1)^2}{1-\rho_1}$ $L_2^q = L_2 - \frac{\rho_2}{1-\rho_1}$ $L_q = L_1^q + L_2^q$
Non-preemptive, priority $\rho_1 + \rho_2 < 1$		$\rho = \rho_1 + \rho_2$ $L_1^q = \lambda_1\frac{\frac{\rho_1}{\mu_1}+\frac{\rho_2}{\mu_2}}{1-\rho_1}$ $L_2^q = \frac{\rho_2}{1-\rho_1} * \frac{\frac{\rho_1}{\mu_1}+\frac{\rho_2}{\mu_2}}{1-\rho_1-\rho_2}$ $L_q = L_1^q + L_2^q$
non-priority $\rho_1 + \rho_2 < 1$		$\rho = \rho_1 + \rho_2$ $L_1^q = \rho_1\frac{\lambda}{\mu_1}\frac{1-(1-\frac{\mu_1}{\mu_2})\rho_2}{1-\rho_1-\rho_2}$ $L_2^q = \frac{\lambda_2\lambda}{(\mu_1)^2}\frac{\frac{\mu_1^2}{\mu_2^2}+(1-\frac{\mu_1}{\mu_2})(\frac{\lambda_1}{\mu_2})}{1-\rho_1-\rho_2}$ $L_q = L_1^q + L_2^q$
Network $\frac{\lambda}{k\mu} < 1$ for each node		$L = \frac{\rho_1}{1-\rho_1} + \frac{\rho_2}{1-\rho_2} + ...$ $= \frac{\lambda_1}{\mu_1-\lambda_1} + \frac{\lambda_2}{\mu_2-\lambda_2} + ...$ $W = \frac{L}{\sum\alpha}$

In M/M/2, λ, μ $\pi_0 = \frac{2\mu-\lambda}{2\mu+\lambda}$; $L_q = \frac{\lambda^3}{\mu(4\mu^2-\lambda^2)}$; $W_q = \frac{\lambda^2}{\mu((2\mu)^2-\lambda^2)}$; $L = \frac{4\mu\lambda}{4\mu^2-\lambda^2}$

Def. Continues Markov Chain: $Pr(X_{t+s} = j|X_s = i, X_r = i_r, 0 <= r < s) = Pr(X_{t+s} = j|X_s = i)$
Example. Argue for Markov Property: $Pr(X_{t+s} = 0|X_s = 1, X_r = i_r, 0 <= r < s) = Pr(X_{t+s} = 0|X_s = 1)$
The left-hand side of the equation indicates the probability that the salesman is in town A at time (r+s), given that he is in town B at town s, and in town i_r at time r before time s. The event that the salesman is in town A at time (s+t) depends only on how long time the salesman will remain at town B after time s. However, due to the memoryless property of the exponential random variable, the even how long time the salesman will remain at town B after time s is independent of

which town the salesman will stay before time s. Hence, we have that the conditional probability that the salesman in town A at time (t+s), given that he is in town B at time s, and in town i_r at time r before time s is the same as the conditional probability that the salesman is in town A at time (t+s), given that he is in town B at time s, This is equivalent to the equation above.

Possion Distribution. if X is a Possion random variable, the prob mass function is $f(x) = \frac{e^{-\lambda}\lambda^x}{x!}$, where λ is both the mean and variance of X.
Possion Process with rate λ.

- $Pr(N_t = k) = \frac{e^{-\lambda_t}(\lambda_t)^k}{k!}$; $\Pr\{N_{s-u} - N_S = k\} = \frac{e^{-\lambda\mu}(\lambda\mu)^k}{k!}$; $F(t) = 1 - e^{-\lambda t}$
- the event $N_{s+u} - N_s = i$ is independent of the event $N_t = j$ if $t < s$; The prob $\Pr\{N_{s-u} - N_S = i\}$ only depends on the value of μ

Problem 1: Consider a single server queue with Poisson arrivals and exponential service times having the following variation: Whenever a service is completed a departure occurs only with probability p. With probability 1-p the customer, instead of leaving, joins the end of the queue. Note that a customer may be serviced more than once. Let $\{Qt, t \geqslant 0\}$be the number of customers in system at time t. (i) Set up the generator and find the steady-state distribution, stating conditions for it to exist. (ii) Find the expected waiting time of a customer from the time he arrives until he enters service for the first time.

solution. Suppose that the inter-arrival time is $\exp(\lambda)$ and the customer service time follows $\exp(\mu)$. Let Qt be the number of customers in the system at time t. Then we know that$\{Qt, t \geqslant 0\}$ is a time-continuous Markov chain with state space {0,1,2,...}. The changing rate at state 0 is equal to the rate of the customer arrival. Thus $q_0 = \lambda$. When the process $\{Qt, t \geqslant 0\}$ changes its state 0, it will go to state 1 with probability one. Hence $q_{01} = \lambda$.

We summarise these in the matrix version,

$$\begin{bmatrix} -\lambda & \lambda & 0 & 0 & \cdots \\ p\mu & -(\lambda + p\mu) & \lambda & 0 & \cdots \\ 0 & p\mu & -(\lambda + p\mu) & \lambda & \cdots \\ & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

Hence, when $\lambda/p\mu < 1$, the steady-state distribution is given by

$$\pi_0 = 1 - \frac{\lambda}{p\mu},$$
$$\pi_i = \left(\frac{\lambda}{p\mu}\right)^i \times \left(1 - \frac{\lambda}{p\mu}\right), i = 1, 2, \cdots.$$

Note that

$$P\Big(\text{the number of feedbacks an external arrival customer will make} = k\Big)$$
$$= (1-p)^k p, \ k = 0, 1, 2, \cdots.$$

Thus

$$\text{the expected number of feedbacks an external arrival customer will make}$$
$$= \sum_{k=0}^{\infty} k \times (1-p)^k p = \frac{1-p}{p}.$$

Hence, the arrival rate is

$$\lambda + \frac{1-p}{p}\lambda = \frac{\lambda}{p}.$$

The Little formula gives that the expected waiting time for an external arrival customer is give by

$$\frac{1}{\lambda} \times \frac{\lambda^2}{p\mu(p\mu - \lambda)} = \frac{\lambda}{p\mu(p\mu - \lambda)}.$$

Problem 2: Single server queue with reneging: Suppose that every customer has "patience time", i.e., if he has to wait for service initiation longer than the patience time, he will leave system without service. Assume that patience times of all arriving customers are independent and identically distributed exponential random variables with parameter θ. Let the arrival process be a Poisson with parameter λ and service time be independent and identical exponential random variables with parameter μ. Let $Q(t)$ be the number of customers in the system at time t. Show that $\{Q(t), t \geq 0\}$ is a death-birth process. Find the condition such that the steady-state distribution for $\{Q(t), t \geq 0\}$ exists. Under this condition, give the steady-state distribution for $\{Q(t), t \geq 0\}$.

Solution: Let $Q(t)$ represent the queue length at time t. Then similar to the discussion in our class, we know that $\{Q(t), t \geq 0\}$ is a Markov chain.

The changing rate at state 0 is equal to the rate of the customer arrival. Thus

$$q_0 = \lambda.$$

When the process $\{Q(t): t \geq 0\}$ changes its state 0, it will go to state 1 with probability one. Hence

$$q_{01} = \lambda.$$