Syllabus Introduction to Analysis
Syllabus Inleiding Analyse

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Part I

Course Manual

Chapter 1

Course Information

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Question hours: Fridays, see timetable

1.1 Required literature (Verplichte literatuur)

- BUNDLE: Stewart - CALCULUS EARLY TRANS 8E + PRECALCULUS 7E + EWA HW PAC (EWA = Electronic WebAssign). ISBN 9781473786424 **The Calculus book is also the required textbook for the course Calculus in block 3**.
- This Syllabus

1.2 Objectives (Doelen)

For a detailed week-by-week description, please consult the document "Course Objectives". The most important objectives of the course can be summarized as follows:

- Bring abstract thinking to a higher level
- Learn the concepts behind high-school mathematics
- Take the first steps in mathematical proofs with correct mathematical logic
- Complementing your mathematical toolkit needed for the Econometrics study

1.3 Topics (Onderwerpen)

- Sets and logic
- Mathematical and complete induction
- Sequences, Functions and their limits
- Defining derivatives
- Functions of multiple variables and their derivatives
- The Lagrange multiplier method
- Integration

1.4 Credits and required presence (Studiepunten en verplichte aanwezigheid)

One can earn 6 ECTS points which is the equivalent of approximately 21 hours of work per week. There are twice two hours of plenary lectures, two hours of exercise tutorials and two hours of exercise lectures weekly, which adds up to 8 hours per week. You have 13 hours to work at home.

As opposed to high school, self-study at the university is of crucial importance! Please do not underestimate the work you need to put in yourself.

There is a written examination at the end of the first block with a re-examination possibility in July. There are also two intermediate tests. The fist midterm is at the end of week 3, on Saturday, September 19, which will determine 10% of the final grade. The second midterm is at the end of week 5, on Saturday, October 3, and it will determine 15% of the final grade (see your timetable for the times). No-show means zero points and there is NO possibility for resit of the midterms. The final exam determines 70% of the final grade. The last 5% of the final grade will be determined by the weekly teamwork exercises. You can collect a maximum of 5 points with deligent, serious work during the tutorials.

Only those will be admitted to take the exam who participated in at least 5 of the 7 tutorials. Special cases are to be judged by the Examination Board.

1.5 Sessions of a week (Sessies in de week)

The first week is differently scheduled than the rest of the block, because of the Take-off on Monday. Weeks 2 to 7 have the following structure:

- Monday: 2 hours plenary lecture
- Tuesday: 2 hours plenary lecture

- Thursday: 2 hours practice tutorial
- Friday: 2 hours exercise lecture and a question hour.

All lectures will be blended: each week a link will be published on the Canvas page of this course, where the lecture will be streamed live through Zoom. With the same link you can rewatch the lecture. Following the lecture live is however strongly advised because then you can ask your questions instantly. This way you can avoid legging behind. I trust that you will ask a LOT of questions whether you are physically present at the lectures or following it live through Zoom; students tend to think that they are the only ones who do not understand a certain issue, whereas experience shows that many of the students will have the same questions, only no one dares to ask. Please be brave and always ask your questions: there are NO stupid questions!

1.6 Question hours (Vragenuur)

The question hour will be organized through Zoom. The Zoom link will also be published on Canvas.

You can "drop in" anytime during the question hours to ask any question about the material. During the office hours you can ask questions about any part of the material, as opposed to the tutorials where you should always restrict your questions to the material of the week. Please DO MAKE USE of the question hours. Is there anything which you do not fully understand? Ask it during the question hours! Also, when you are not sure whether a result or proof you wrote is correct, or whether the notation you used is correct, we can discuss it together.

1.7 Plenary and exercise lectures (Hoorcolleges en sommencolleges)

Each week, during the plenary lectures the most important theory is explained with a few examples. During the lectures it is of crucial importance that you stay focused. Tuning out for a few minutes might have the consequence that you miss important details and you cannot follow the lecture further. That is why it is important not to talk to your neighbors during lectures. If you follow the lecture online, it is important that you turn off all notifications and you put your mobile out of reach, do not let yourself distracted.

Please be prepared that the pace of the lectures is very high compared to high-school lessons. Here we need to tackle large amounts of material each week. That is quite a challenge in the beginning, but you will get used to it, as all students do every year.

You also need to keep in mind that it is perfectly normal if you do not understand something at first sight during the lecture. There is no need to start panicking. The material is sometimes quite abstract, that is why you need to study quite some hours at home for yourself. When you do not understand something, here is what you can do.

- Try to write it down at home, without looking at your notes or the book/syllabus.
- Compare what you have written down afterwards to your notes and/or book/syllabus. Did you miss some parts? Did you write it down differently? What is the importance of the part you missed or wrote down differently? Can you find it out yourself?
- Try to find a "baby-example" for the problem at hand: an example which is very simple, so you can see how the theory works in the case of such a simple example. This might be an eye-opener. Try to generalize the problem afterwards.
- Try to make a drawing of the situation, which helps you visualize things: a powerful method!
- Do you still have problems understanding it? Set it aside and try doing something else for a while, but DO REVISIT it again. You will find that at a certain moment suddenly everything becomes clear if you keep thinking about it. That is how abstract thinking works.
- Is it Friday and you still did not fully get to understand the problem? Then ask further explanation during the question hours.

The exercise lectures on Fridays are meant for explaining the most difficult details, according to the needs. You can always send an email before Friday to request certain problems and/or part of the theory you would like to revisit during the exercise lecture on Friday.

1.8 Tutorials (Practica)

Prepare your homework before the tutorials, so we can see which were experienced as the most difficult problems, and the tutors can discuss them. When you need help with your homework, you can also use WebAssign, where you can get help, feedback and we can also monitor your progress.

1.8.1 Structure of the tutorials

A tutorial consists of two 45-minutes sessions, with a break of at most 15 minutes between the two sessions.

The first 45-minute session is an active discussion of exercises.

It is imperative that you have prepared your homework beforehand and tried your very best to find the solutions. The minimal requirement is that you have *a written answer so that you can point out exactly why and where you got stuck. "I did not know how to do this exercise" does not qualify as pinpointing the problem.*

The second 45-minute session is dedicated to teamwork:

- You will work in groups of 3 students.
- If you follow the tutorial online, you will be assigned to a 'break-out room' within Zoom (a separate session where only you and your teammate can see and talk to each other, see more in "Technical instructions for Zoom" below)
- You can ask for help from the tutor if you have any questions.
- Your tutor will also monitor the teams to see how you progress.
- The teamwork exercises need to be handed in at the end of the tutorial via ANS.
- The teamwork exercises will be graded, and this grade will be part of the final grade for this course.
- For a maximum grade it is sufficient that you and your teammate both try your very best and show that you studied the topics; we do not expect perfect answers yet, as the team exercises are an essential part of your training process.

1.8.2 Rules for the tutorials

Tutorials are mandatory: in order to be admitted to the exam you need to be present at least 5 of the 7 tutorials. Being present means that you join the online session, satisfying the following requirements:

Rules for Online attendance:

- You join the Zoom session with your own FIRST & LAST name visible on the screen (see "Technical instructions for Zoom" below).
- You have your camera ON during the entire session.
- You participate actively during the entire session: you unmute yourself to ask questions or to answer the questions of your tutor.

You will only be registered present at the tutorial when these requirements are all met.

Further requirements for both online and physical participation:

- The 15-minutes rule when being late:

  Should you want to join the tutorial session after the scheduled time, then you will need to wait outside (or in the Zoom waiting room) and you will be admitted by your tutor at exactly 15 minutes after the scheduled time. If you are more than 15 minutes late, then you will only be admitted to join for the team assignment: you will be allowed to make the team assignment for the grade, but you will be registered absent for that tutorial.

- Negotiating about presence with your tutor is strictly forbidden!
- Presence will be actively and randomly checked by your tutor at the beginning, during and at the end of the tutorial: active participation throughout the whole tutorial is mandatory.
- Prepare your homework before the tutorial.
- You are NOT allowed to switch tutorial groups unless you have explicit permission from the lecturers.
- Feedback should always be constructive; everyone is kind to each other.
- In the International classes English is the only language that is used, also in the breaks.
- Free riding is not tolerated: during the team assignments everyone works together on the assignments, you should never split the assignments between the team members.
- To create a safe environment in which all students feel comfortable to participate, recording is not allowed during the sessions, neither by the tutor nor by the students.

1.8.3 Technical instructions for Zoom

How to join a tutorial session?

1. You will receive a link to a Zoom meeting from your TA via a Canvas announcement.
2. When your tutorial starts, click on the link to join the tutorial session. The same link can be used each week.
3. Make sure your first and last name are set correctly in the Zoom session (see instructions below).
4. You enter a virtual waiting room. Wait for your TA to admit you to the session.
5. Click ‘join audio’ to make sure you can hear others during the session.
6. Your microphone will be muted by default. Unmute it if you want to say something.
7. Make sure your video is on.

8. Your TA broadcasts his/her writing. You can 'pin' this screen by hovering over the small video that you want to pin, click the three horizontal dots and select 'pin video'. In the app: double click on a small video to pin.

How to set your name in Zoom?
There are two possibilities.
1. You set up your own free account in Zoom (you do not need a paid account). Go to Settings, choose Profile, then edit your profile: First and Last name, and upload your picture (not an avatar).
2. When you click "Join a meeting", you can adjust your name on the spot (see screenshot).
Please note that because of security issues the TA will only admit you to the tutorial when he/she can check based on your name that you belong to that specific tutorial group.

1.8.4 Technical instructions for ANS - digital tests

- Please use Google Chrome for this purpose.
- Go to www.ans-delft.nl
- Click "Sign in" in the upper right corner.
- Under "Log in with your school account" click "Erasmus University Rotterdam" (if the name of a different school is displayed, then click first "Select another educational institution" and scroll until you find "Erasmus University Rotterdam").
- Log in with your ERNA number and your password.
- The digital test should be visible on your screen now.
- After you solved a certain exercise of the test on paper, start uploading your answers in ANS:
  - Please use only a pen with black or blue ink (preferably a fine liner);
  - Take a picture with your webcam of your hand-written solution and add this picture to the respective answer box;
  - Take a picture of max one $\frac{1}{2}A4$ in portrait orientation at a time;
  - Clearly number the pictures in the correct order, if you upload more than 1 per picture per question;
  - Make sure to begin a new line when you upload your next picture, otherwise you overwrite your previous picture! You should see all the pictures you have uploaded on your screen, otherwise you should be aware that something went wrong;
  - Write as clearly as possible. For some of you this might imply writing with larger letters; Answers that are not readable will be assigned 0 points;

– When you finished your assignment check carefully whether you answered all questions. Then, click ‘Exit Test’ in the lower right corner. After this, a new window opens in which you can see whether you have answered all questions. Click ‘Close’ if you want to continue editing; Click ‘Submit Exam’ if you are finished and want to submit your answers.

Sample webcam pictures:

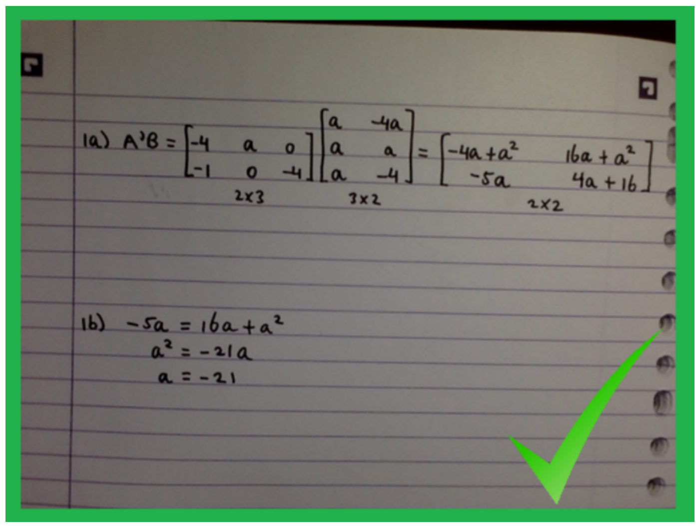


Figure 1.1: Correct

Correct: $\frac{1}{2}A4$ in the picture, paper in portrait orientation, clearly written and with a black fine-liner, see figure 1.1.

Incorrect: full A4 in the picture, paper in landscape orientation, not clearly written and with a pencil, see figure 1.2.

Please try how to take pictures, already before the first tutorial, in the following practice test: https://secure.ans-delft.nl/universities/5/courses/39714/assignments

There is also an instruction video available on Panopto (just click the link and log in with your ERNA) on taking pictures with your webcam in ANS: https://eur.cloud.panopto.eu/Panopto/Pages/Viewer.aspx?id=c82e0abc-df5f-4a0c-8a28-abdc00f16ec7

PLEASE MAKE SURE YOU HAVE REGISTERED A ZOOM ACCOUNT, YOU KNOW HOW TO LOG IN TO ANS, AND YOU HAVE TRIED TO UPLOAD PICTURES IN THE PRACTICE TEST.

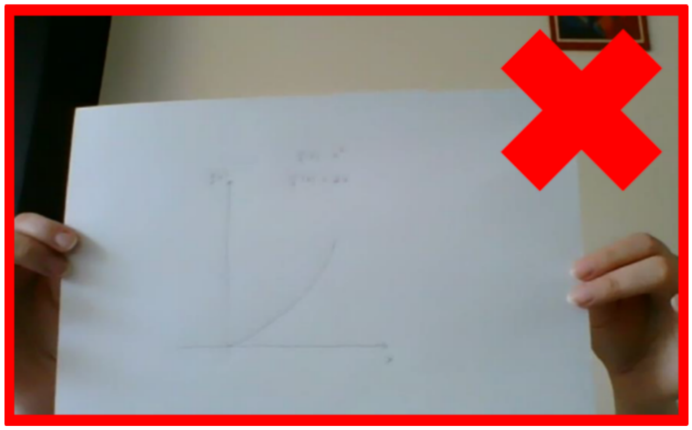


Figure 1.2: Incorrect

1.9 Rules for all lectures and tutorials (Spelregels voor alle colleges en practica)

1. **The 15-minutes rule holds for lectures when you visit them phisically**:

   Should you arrive late to any of the lectures, then please wait and only enter the room all together exactly 15 minutes after starting time, as discretely as possible. If you arrive later than that, then you will only be able to enter during the break. This 15 minute-rule also holds for the second hour.

2. When you do not understand something and/or you have a question or a comment, then ask the lecturer and NOT your neighbour.

3. During NONE of the sessions is allowed to use a calculator, and during the physical lectures it is not allowed to use a smartphone, tablet or computer!

4. Homework is prepared at home.

5. During the sessions everyone works seriously and interactively with an active attitude.

6. Feedback should always be constructive, everyone is kind to each other.

7. In the International classes English is the only language that is used in class, also in the breaks.

8. Free-riding is not tolerated.

1.10 Questions and answers during sessions (Vragen en antwoorden tijdens de sessies)

All sessions are organized to be interactive. The reason behind asking questions is to try to make you think important parts through in order to help your understand the important details. If you only consume the material you would miss important connections and details, your understanding would be much more superficial. Besides making you think deeper about the subjects, your answer to my questions or your questions help your fellow students and me too. For my part, I can see in which direction you think and I can adjust my explanation, I can tailor it to your needs, hence making my explanation much more efficient. **There are NO wrong answers and NO stupid questions**. An answer might be incorrect, which helps all students and me to get the chance to explain why you need to think differently. **These answers are therefore even more valuable than correct answers!**

You might find it stressful to speak up in front of a large lecture room. Do keep in mind that your answer/question can never be wrong and how everyone appreciates that you break through your own limitations in order to help us all. We can only create a good, informative, efficient and pleasant lecture TOGETHER! Be brave and do appreciate all students trying to do the same.

I would also like to get to know as many of you as possible, so I will try to learn your names. Please tell me your name every time you come to me to ask something.

I look forward to meet you and to work with you!

1.11 How can you study efficiently? (Hoe moet je effeciënt studeren?)

- **Rewrite** your notes of the lecture and the syllabus before the first tutorial. Reading mathematics like a novel does not work! Check continuously whether you have used the correct notation. You can also bring your work to the office hour so I can check your notation or your results.
- Complement your knowledge from the book where needed.
- Make the homework exercises.
- Make use of discussion board.
- Try to study efficiently. When you get the feeling your concentration slackens, do not force it further, try instead half an hour active relaxing: take a walk, go jogging or biking – the more intense the activity, the better your concentration will get afterwards!
- Do not read the material like a novel: always use pen and paper and try to make for instance the examples from the text first yourself or try to redo it

without looking in the text; try to write down the important results with the CORRECT NOTATION from your memory. This is the only way to make sure you will be able to write down the correct answers at the exam!

Not so strong math background? Please make the tests at the end of each chapter from the book to find out where the gaps are. Use WebAssign efficiently for your advantage (see the document about WebAssign on Canvas).

	WebAssign	In your notebook
Week 1 **Chapter 1:** Fundamentals **Syllabus:** Part II: Sets	*-Precalculus:* CH 1.1: Pages 10-12: 3, 41-46, 49, 51, 59, 60, 61, 63, 65, 69, 81, 82	**From the syllabus (Sets)** CH 3.8: 1 – 4, 6 CH 4.6: 1, 2, 4, 5, 8, 9 CH 5: 1, 2 CH 7: 1, 2
Week 2 **Syllabus:** Part III: Proposition Logic **Chapter 1:** Fundamentals, **Chapter 12:** Sequences and Series **Syllabus:** Part IV: Induction	*-Precalculus:* CH 1.8: Pages 88-91: 103 CH 12.5: Pages 878-879: 3, 5, 21 CH 12.1: Pages 850-852: 50, 72 CH 12.2: Pages 856-858: 47, 65 CH 12.3: Pages 864-867: 65, 77	*-Precalculus:* CH 12.5: Pages 878-879: 22, 30, 37c CH 1.8: Pages 88-91: 128, 129 CH 12.2: Pages 856-858: 69, 70 CH 12.3: Pages 864-867: 103 CH 12.6: Pages 886-887: 49 **All exercises from Syllabus part IV**
Week 3 **Chapter 2:** Functions **Chapter 3:** Polynomial and Rational Functions **Chapter 4:** Exponential and Logarithmic Functions **Chapter 5:** Trigonometric Functions **Chapter 7:** Analytic Trigonometry **Syllabus:** Part V: Functions	*-Precalculus:* CH 2.8: Pages 226-234: 6, 11, 87, 102 CH 3.1: Pages 251-254: 49 CH 5.1: Pages 407-409: 21, 22, 42, 46, 47 CH 5.2: Pages 416-419: 45, 50 CH 7.4: Pages 568-570: 41	*-Precalculus:* CH 2.8: Pages 226-234: 73 CH 3.6: Pages 308-311: 95c CH 4.2: Pages 341-343: 17 CH 4.3: Pages 351-354: 94 CH 4.4: Pages 358-360: 71 CH 5.1: Pages 407-409: 59 CH 5.2: Pages 416-419: 66, 75 CH 5.5: Pages 444-445: 50 - Syllabus: CH 3.8: 6 (with formal proof)
Week 4 **Chapter 13: 13.1,13.2 &13.4** **Syllabus:** Part VI: Limits	*-Precalculus:* CH 13.2: Pages 912-914: 27, 30, 37-39 CH 13.4: Pages 930-931: 23, 24, 28 CH 13.4: Pages 930- 931: 9, 10, 12, 14, 28, 29	*-Precalculus:* CH 13.4: Pages 930-931: 29, 34 CH 13.2: Pages 912-914: 28, 40-42 CH 13.4: Pages 930- 931: 16, 17, 31, 32 **All exercises from Syllabus Part VI**
Week 5 **Chapter 13:** Tangent lines & derivatives **Chapter 14 (Calculus):** Partial Derivatives **Syllabus:** Part VII	*-Calculus:* CH 14.3: Pages 923-927: 10, 17, 27, 31, 35 CH 14.6: Pages 956-959: 4, 5, 7, 13 CH 14.8: Pages 977-978: 1, 3	*-Calculus:* CH 14.3: Pages 923-927: 39, 43, 45, 51, 53, 82 CH 14.4: Pages 934-936: 31 CH 14.6: Pages 956-959: 9, 11, 23 CH 14.8: Pages 977-978: 9, 19, 29
Week 6 **Chapter 13:** Areas **Chapter 5 (Calculus):** Integrals **Chapter 7 (Calculus):** See Canvas for details	*-Precalculus:* CH 13.5: Pages 938-939: 9, 13 *-Calculus:* CH 5.2: Pages 388-391: 2, 19, 44, 47, 49, 50, 57 CH 5.4: Pages 408-411: 5, 9, 27, 31 CH 5.5: Page 419: 55, 57, 68 CH 7.1: 7.1.11, 7.1.12, 7.1.27 CH 7.2: Page 484: 1, 2 CH 7.4: Page 501: 12, 14 CH 7.8: Page 534: 11, 32	*-Calculus:* CH 5.1: Pages 375-378: 7, 25 CH 5.2: Pages 388-391: 9, 23, 37, 61 CH 5.4: Pages 408-411: 37, 43, 45 CH 5.5: Page 419: 59, 61, 71 CH 7.2: Page 484: 11 CH 7.4: Page 501: 19, 24 CH 7.8: Page 534: 7, 19, 31

Chapter 2

Exam and Midterm (Tentamen en Tussentoets)

There is a midterm test for this course. The aim of this test is not only to make you study from the very beginning but primarily to make you acquainted with the manner exams are taken at university level. In this manner, you will know what to expect from the exam and what we expect of you at the exam.

2.1 During the midterm and the exam

The midterm and final examination for Introduction to Analysis take place at the following times:

- Midterm: Saturday, September 18, 2021 from 09:30-10:30h
- Final examination: Thursday, October 21, 2021 from 13:30-15:30h.

For these examinations the digital environment ANS will be used (when you take the exam online, then in combination with Proctor Exam). The exam will contain of one of the teamwork and/or one of the homework exercises (NOT the midterm, only the final exam). Besides the standard rules, hereby some of the rules for Introduction to Analysis which I would like to highlight. Please also study the rules and instructions for the exam on the first two pages of the examination sheets.

- All answers need to be explained / motivated adequately, otherwise the answer will not be accepted.
- It is not allowed to use books, notes or calculators.
- You are only allowed to use the trigonometric functions sin(x), cos(x), tan(x) and their inverses, no other trigonometric functions (never use cot, sec, csc, and so on, to avoid confusion).

- You should only use the table of **standard derivatives** (see chapter 16) - the reverse is the table of standard antiderivatives. All other antiderivatives you need to deduce yourself clearly in the exam.

2.2 After the midterm and the exam

MIDTERM

The midterm is mandatory and counts for 15% of the final grade. Please note that you cannot deviate from the following dates and times, so mark them in your agenda right away:

- You will be able to view your own midterm-answers through www.ans-delft.nl on Wednesday, September 29, for the Dutch students 12:00 - 15:00, for the Internationals 13:30 - 16:30 h.
- The correct solutions and the grading will be explained on this same day, September 29, for the Dutch students 13:00 - 14:00, for the Internationals from 14:30-15:30 h.
- For all students it is essential to listen carefully to the presentation of the solutions and grading, otherwise you might make the same mistakes later on if you do not realize why and where exactly your solution went wrong.
- You will be able to ask questions through ANS regarding the grading on September 29, the Dutch students between 14:00 - 15:00, the Internationals between 15:30 - 16:30. You can also ask questions through Zoom during the presentation of the solutions.
- Please note that I will only answer questions through ANS which have not specifically been discussed during the presentation of the solutions. If you did not listen to the presentation and your question is not answered through ANS, then you should know that your question has been discussed there and therefore not answered in ANS.
- **Questions can only be asked through Zoom or through ANS, you are not allowed to send emails with questions about your midterm or exam: any such emails will be deleted without reading!**
- Please read and heed the guidelines for perusals (at the end of this chapter), to avoid not getting an answer to your questions.

EXAM

The final course grade is composed as follows: 0.15*Midterm grade + 0.8*Exam grade + 0.05*Tutorial grade.

Please note that you cannot deviate from the following dates and times, so mark them in your agenda right away:

- You will be able to view your own midterm-answers through www.ans-delft.nl on Wednesday, November 3, for the Dutch students 12:00 - 15:00, for the Internationals 13:30 - 16:30 h.

- The correct solutions and the grading will be explained on this same day, November 3, for the Dutch students 13:00 - 14:00, for the Internationals from 14:30-15:30 h.

- For all students it is essential to listen carefully to the presentation of the solutions and grading, otherwise you might make the same mistakes later on if you do not realize why and where exactly your solution went wrong.

- You will be able to ask questions through ANS regarding the grading on November 3, the Dutch students between 14:00 - 15:00, the Internationals between 15:30 - 16:30. You can also ask questions through Zoom during the presentation of the solutions.

- Please note that I will only answer questions through ANS which have not specifically been discussed during the presentation of the solutions. If you did not listen to the presentation and your question is not answered through ANS, then you should know that your question has been discussed there and therefore not answered in ANS.

- **Questions can only be asked through Zoom or through ANS, you are not allowed to send emails with questions about your midterm or exam: any such emails will be deleted without reading!**

- Please read and heed the guidelines for perusals (at the end of this chapter), to avoid not getting an answer to your questions.

2.3 Material for the midterm (Stof voor de tussentoets)

- everything we discussed during lectures

- Syllabus parts II, III, IV and V & all documents on Canvas under modules week 1, week 2 & week 3

- Book Precalculus:
  - Chapters 1, 2 (except sections 1.6 and 1.7)
  - Chapter 3: sections 3.1, 3.2, 3.6 & 3.7
  - Chapter 4, except 4.6 & 4.7
  - Chapter 5, except 5.6

- Chapters 6 & 7: trigonometric formulae need not to be memorized, you only need to know $\sin^2(x) + \cos^2(x) = 1$ and the expressions for $\sin(x \pm y)$ and $\cos(x \pm y)$.
- Chapter 12, except section 12.4
- All parts "Focus on modeling" are optional

2.4 Material for the exam (Stof voor het tentamen)

- All documents on Canvas and everything we discussed during lectures; please refer to the Course Manual and the week modules for all the info and documents.

- Referring to Syllabus: you can refer to all results (including the exercises) which are in the syllabus.

- You need to know all result from this Syllabus; for instance, you need to know all the standard derivatives of the elementary functions listed in this syllabus. Please do note that if you reverse this table then you get the table of standard ANTIDERIVATIVES which you also need to know: these are also the only antiderivatives you do not need to derive yourselves during the exam. Example: the derivative $(\arctan x)' = \frac{1}{1+x^2}$, so the antiderivative of $\frac{1}{1+x^2}$ is $\arctan x$.

- BOOK PRECALCULUS:
  - Chapters 1 & 2 (all sections)
  - Chapter 3: all sections, except chapter 3.5.
  - Chapers 4, except 4.6 & 4.7
  - Chapter 5, except 5.6
  - Chapters 6 & 7: the trigonometric formulae need not to be memorized, only the formula $\sin^2(x) + \cos^2(x) = 1$ and the expressions for $\sin(x \pm y)$ and $\cos(x \pm y)$.
  - Chapter 9, except 9.5 & 9.6
  - Chapter 10: only 10.1, 10.2 & 10.7
  - Chapter 12, except section 12.4
  - Chapter 13 (all sections)
  - All parts "Focus on modeling" are optional

- BOOK CALCULUS:
  - Chapter 5, except the Net change theorem
  - Chapter 7: only 7.1, 7.2, 7.4 & 7.8

- Chapter 14 except:
  * Limits and Continuity, section 14.2
  * Higher derivatives & Partial differential equations pages 918 until 923; also not: Example 5 from page 917.
  * The chain rule & Implicit differentiation, section 14.5
  * Tangent planes to level surfaces pages 954 until 956
  * Second Derivative test pages 961 until 967

You will be able to inspect your corrected exam online, by logging in on www.ans-delft.nl with your erna account, by first selecting your school (Erasmus University Rotterdam). Please heed the following rules concerning the perusal.

2.5 Perusal guidelines (Richtlijnen voor de Inzage)

Below you can find some rules and guidelines for the perusal in Ans. We provide you with these guidelines to make the perusal process smooth, efficient, fair and enjoyable for both the students and the reviewers. Please look carefully at the criteria on the right when reviewing your exam. Be aware that they sometimes only serve as orientation for the grading team. That is, in case your solution method deviates from the answer model, you might still get points for (partly) correct answers. Please note that those partly correct answers do need to be of sufficient importance for the solution at hand. Just listing some definitions, for instance, however correct, will not guarantee partial points.

Important rules:

- There is a strict time window for the perusal. The time window of the perusal is communicated via CANVAS, make sure that you check Canvas for the exact date and time. Deviating from this date and time is not possible and you will get no second chance for inspecting your exam.
- There is no room for negotiations, the perusal is not a marketplace.
- Enquiries can only be made via Ans, and thus not via email. Emails concerning the perusal in ANS will be automatically deleted without reading.
- The reviewers will solely answer enquiries which are well founded, clear, concise and polite. Enquiries that are not well founded, clear, concise and/or polite, will not be considered and will also not be answered.

Further guidelines to these rules:

1. Be polite.

   However frustrated, when you write a message in ANS do realize that there are real people on the receiving end, so please adjust your tone and re-read your message before clicking the send button. Please also refrain from using all capital letters or multiple exclamation marks (like shouting), or similar manifestations.
   Don’ts:

   - “The notation is not super neat, but it is there!!!! See at the very top right of the box. I understand that this is not immediately noticeable (my mistake). Please check it again, because I understand!”
   - “I hear several people talking about this and I think that the last word has not yet been said about this!”

   Do’s:

- “According to the scoring I did not write that [...] but in the top right of my answer I did. Can it be true that I still earn these points? Thank you very much for your time and best regards,”

2. Be clear in your message: refer to the criteria that your comment is about.

   The recipient of the message must be able to see at a glance which part of the question has been assessed incorrectly. Hence, clearly refer to the relevant criterion.
   Don’ts:

   - “I have answered all criteria points”
   - “Why is this wrong?”
   - “I don’t understand why you need to look at the values [. . .]. Can you explain this to me?”
   - “Recorrect my answer to this exercise”

   Do’s:

   - “I think you might have missed the point for criterion 1.”
   - “I do not understand why I did not get the points for criterion 2. I understand that I don’t get all points because [. . .], but I got the steps for finding the answer right. Thank you in advance for looking over this again.”
   - “As I see now, I have not received any points for criteria 4 and 5. However, line 7 of my answer does state the steps that I carry out that are mentioned in criteria 4 and 5. Could you reconsider this? Thanks in advance.”

3. Explain why, in your opinion, the criteria were not applied fairly.

   You may disagree with the criteria, but we cannot deviate from that, as that is the standard we have decided on for every single student. Explain clearly how and why you believe that we have made incorrect use of the given criteria. Or explain why you have used an alternative correct method that does not match the criteria.
   Don’ts:

   - “Why are no points awarded for steps 2 and 3?”
   - “I personally believe I deserve more points than I got, because pretty much the only thing I did wrong is [. . .] I just don’t think awarding me 3.5 points is fair. Please review this for me.”

   Do’s:

   - “For the first point [. . .], I wrote something very similar on the 8th and 9th line of this section, with the only variation that I wrote the last part with absolute values, but this should still be the same right?”

- “I think I did find the answers to criteria 4 and 5. I’m sorry for not pointing that out clearly.”
- “I see that my approach is different from the one in the criteria [...]. And even though I understand that according to the criteria I should only be rewarded three points, I do not agree with this. [... explanation of alternative approach ...] Furthermore I do not believe that I have made any mistakes in my calculations. Is it possible to revisit my answer and if it would still be wrong, explain to me why my answer would be incorrect?”

4. Be concise in your question.

   Do not write a ”diary report” of what you did.
   Don’ts:

   - “The final answer is unfortunately not right !!! But look at why [...] Unfortunately I forgot to put this 1 / in my calculator (oops!) And therefore came to .... instead of ... Just do the math. [...] Please calculate yourself in the calculator. If remove the 1 / you end up with [...]. I hope I have made it clear so that my error is only a calculation error. I even think a calculation error is a big word, because it was only a mistake when typing the calculator.”

5. Be critical of your own answer. If it is not there, it is not there.

   Re-read your answer critically and compare carefully with the criteria. Do not assume in advance that you are right and the reviewer is wrong. Crossed out answers cannot and will not be considered.
   Don’ts:

   - “In line 4 I have: [...] which is in my opinion practically the same as [answer in criterion]. I also have in line 7 that [...] what is therefore both essentially correct, I just have not entered any value.”
   - “If you look closely to my crossed-out answer, you can see I wrote down [...]. But I crossed this because [...]. So no points should be subtracted here...”
   - “I had the right answer but I got confused [...] You should understand that I understand that I have to compare the final answer with [...], it is unfair that I don’t get a point for that.”

Part II

Sets (Verzamelingen)

Chapter 3

Descriptions and Definitions (Beschrijving en Definities)

3.1 Definition and notation (Definitie en notatie)

The idea of a set is familiar in everyday life. Do you have a set of books or a set of plates? Each of these are physical units, however sets need NOT consist of physical objects. For example, the laws of gravity or integer numbers.

Many mathematical theories are based on the idea of sets. But what is a set more precisely?

> A set is a **well-defined** collection of objects, called elements of the set. Sets are denoted by capital letters, as A, B, C, ..., while their elements are denoted by lowercase letters, e.g. $a, b, c, \ldots$

The most important property in this definition is that a set is WELL-DEFINED: this means that it must always be clear objectively to anyone whether a unit is an element of a set or not. For example, the set of all integer numbers is well defined, but the set of good books is not: one person might find a book entertaining while another person would find it boring.

Example of well-defined sets: the set of even numbers, the set of first year Econometrics students in Rotterdam, numbers which solve the identity $x^2 = 4$.

3.2 Describing sets (Beschrijving van verzamelingen)

Sets can be described by a verbal or written description, which we will denote mathematically in either of the following two ways:

1. *LIST (Roster method):*

The set of positive integers less than 5	$\{1, 2, 3, 4\}$
The set of countries of the BeNeLux	{Belgium, The Netherlands, Luxemburg}
The set of positive integers	$\{1, 2, 3, 4, 5, \ldots\}$

2. *SET-BUILDER NOTATION:*

The set of positive integers less than 5	$\{x \in \mathbb{N} \mid 0 < x < 5\}$
The set of countries of the BeNeLux	$\{x$ is a country$\mid x$ is part of the BeNeLux$\}$
The set of positive integers	$\{x \in \mathbb{Z} \mid x > 0\ \}$

A set with **NO ELEMENTS** is called the **EMPTY SET**, and is denoted by the symbol $\emptyset$ or $\{\}$.

E.g. The following set is empty: $\{x \in \mathbb{N} | x < 0\} = \emptyset$.

Remark 3.1. It does NOT matter in which order the elements of a set are listed. Furthermore, the elements of a set are only listed once!

Example. Ann lives in Amsterdam. Let A be the set containing the capitals of the BeNeLux, plus the city where Ann lives. Then $A = \{\text{Amsterdam, Brussels, Luxembourg}\}$, and Amsterdam is only listed once! Besides, one could also write just as well $A = \{\text{Brussels, Luxembourg, Amsterdam}\}$.

3.3 Equality of sets (Gelijke verzamelingen)

Definition 3.2. Two sets A and B are equal, denoted by $\mathbf{A} = \mathbf{B}$, if they have the same elements, not necessarily listed in the same order.
Example. $\{\text{Amsterdam, Brussels, Luxembourg}\} = \{\text{Brussels, Luxembourg, Amsterdam}\}$.
Or, $\{x | 0 \leq x < 3\} = [0, 3)$, where $[0, 3)$ is notation for the interval closed from the left at 0 and open from the right at 3 (that, is, containing 0, but excluding 3).

3.4 Subsets (Deelverzamelingen)

The following symbols will be used extensively throughout the course:

$\forall$ stands for "for all", "any", "arbitrary"

$\exists$ stands for "exists", "there is";

$\iff$ stands for "equivalent", "if and only if"

$\Rightarrow$ stands for "implies that", "then", "follows that".

Definition 3.3 (**Subset**). The set A is a subset of B, denoted by $\mathbf{A} \subseteq \mathbf{B}$ if and only if every element of A is also an element of B. Formally,

$$A \subseteq B \iff (\forall x \in A \Rightarrow x \in B)$$

Example. If $A = \{1, 2\}, B = \{1, 2, 3\}, C = \{2\}$, then $A \subseteq B$, $C \subseteq A$ and $C \subseteq B$. ALSO, trivially $A \subseteq A, B \subseteq B, C \subseteq C$, since every element of A is also an element of A, and so on.

> A set A is said to be a **proper subset** of set B, denoted by $\mathbf{A} \subset \mathbf{B}$, if A is a subset of B but $A \neq B$.

From the previous example, A and C are proper subsets of B, and C is a proper subset of A.

Definition 3.4 (**Alternative definition of Subsets**). The set A is a subset of B, denoted by $\mathbf{A} \subseteq \mathbf{B}$ if and only if there is no element of A that is not an element of B. Formally,

$$A \subseteq B \iff (\forall x \notin B \Rightarrow x \notin A)$$

As a consequence, for any set A, $\emptyset \subseteq A$, because there is no element of $\emptyset$ that is not in A. This is an important result!
Example. If $A = \{1, 2\}$, then the proper subsets of A are $\emptyset, \{1\}$ and $\{2\}$, and $\{1, 2\}$ is a subset.

Definition 3.5 (**Power set**). The set of all subsets of a set A is called the **power set** and is denoted by $\mathcal{P}(\mathbf{A})$.
Example. If $A = \{1, 2\}$, then the power set of A, $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.
Attention! $\mathcal{P}(A)$ is a set containing sets, be very careful about the notation! Begin with an accolade, end with an accolade, write the elements also in accolades, and put a comma between the elements!
Incorrect notation:

$$\mathcal{P}(A) = \emptyset, \{1\}, \{2\}, \{1, 2\}$$

$$\mathcal{P}(A) = \{\emptyset \ \{1\} \ \{2\} \ \{1, 2\}\}$$

or any combination or version of these. Avoid incorrect notation, be precise!

Definition 3.6 (**Universal set**). The **universal set** $\mathcal{U}$ is the set of all elements under discussion.
For the previous example we could define the universal set $\mathcal{U} = \{1, 2, 3, 4, 5\}$ but it could also be $\mathcal{U} = \mathbb{N}$; the universal set is always GIVEN at the beginning of the discussion.

3.5 Cardinality: The number of elements in a set (Kardinaliteit)

Definition 3.7 (**Cardinal number**). For any set A one can or cannot count its elements: A can have a *finite* number of elements, A can have a *countably infinite* number of elements or A can have *uncountably infinite* number of elements. The number of elements of A, denoted as $|\mathbf{A}|$, is called the **cardinal number** of A.

Finite: If $A = \{0, @, \$, \#\}$, then $|A| = 4$.

Countably infinite: If $A = \{0, 1, 2, 3, \ldots\} = \mathbb{N}$ then $|\mathbb{N}| = |A| = \aleph_0$ (read "aleph null", $\aleph$ is the first letter of the Hebrew alphabet), the so-called **tansfinite cardinal** defined by George Cantor.

Uncountably infinite: If $A = (0, 1)$ then the cardinal number assigned to it is c, the so-called "*continuum*" (George Cantor, 1874).

Theorem 3.8 (**Cardinality of the power set**). *The cardinal number of the power set of a set A is given by $|\mathcal{P}(\mathbf{A})| = 2^{|A|}$.*

Example. If $A = \{1, 2\}$, then $|A| = 2$ and the power set of A, $|\mathcal{P}(A)| = 2^{|A|} = 2^2 = 4$. Indeed, $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$, has 4 elements.

3.6 Infinite sets (Oneindige verzamelingen)

In the previous section we have spoken about infinite sets: the set of natural numbers, $\mathbb{N}$ is countably infinite, and the set of real numbers, $\mathbb{R}$ is uncountably infinite. Let us define an infinite set more properly.

> A **One-to-one Correspondence between two sets** A **and** B means a function which matches each element of a set A to EXACTLY ONE (NOT MORE AND NOT LESS THAN ONE) element of B.

Example. Let $A = \{a, b, c, d\}$ and $B = \{@, \#, \$, \&\}$, then there are many ways to define a one-to-one correspondence between the two sets. One of these possibilities is the following:

$$\begin{array}{lccccl} A = \{ & a, & b, & c, & d & \} \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ B = \{ & @, & \#, & \$, & \& & \} \end{array}$$

The function matching this correspondence, if we denote it by f, is defined by $f(a) = @$, $f(b) = \#$, $f(c) = \$$, and $f(d) = \&$. Another possibility would be:

$$\begin{array}{lccccl} A = \{ & a, & b, & c, & d & \} \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ B = \{ & \&, & \#, & \$, & @ & \} \end{array}$$

The function matching this second correspondence, if we denote it by g, is defined by $g(a) = \&$, $g(b) = \#$, $g(c) = \$$, and $g(d) = @$. Do realize that this function g is different from f. Between the sets $C = \{a, b, c\}$ and $B = \{@, \#, \$, \&\}$ there is NO one-to-one correspondence possible because the number of elements are finite and the cardinality of the two sets are different. Anyhow you try it, you would get something like this:

$$\begin{array}{lccccl} C = \{ & a, & b, & c & & \} \\ & \updownarrow & \updownarrow & \updownarrow & ? & \\ B = \{ & @, & \#, & \$, & \& & \} \end{array}$$

so we CANNOT define a function as above which would realize a one-to-one correspondence between the two sets.

But what about infinite sets? Consider $N = \{0, 1, 2, 3, \ldots\}$, the set of natural numbers, and $E = \{0, 2, 4, 6, \ldots\}$ the set of even natural numbers. Let us try to define a one-to-one correspondence between the two sets:

$$\begin{array}{lcccccl} N = \{ & 0, & 1, & 2, & 3 & \ldots & \} \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ E = \{ & 0, & 2, & 4, & 6 & \ldots & \} \end{array}$$

The function matching this correspondence, if we denote it by f, is defined by $f(n) = 2n$, $\forall n \in N$, and this is a well-defined one-to-one correspondence.

ATTENTION: Although at first this might look completely impossible to you, the fact that both sets are INFINITE makes it possible to construct this one-to-one correspondence. If the two sets N and E were finite, then we would finally arrive to a conflict, because the number of elements in the two sets would not match. This property is exactly that makes finite and infinite sets so very different!

Definition 3.9 (**Equivalent sets or equinumerous sets**). If between two sets A and B there exists a one-to-one correspondence, then we say that the two sets are equivalent, and we denote it by $\mathbf{A} \sim \mathbf{B}$. This also means that two equivalent sets have the same cardinal number.

Careful! If two sets are equivalent, $A \sim B$, this absolutely does not mean that $A = B$. Also, equivalence between sets has nothing to do with the equivalence between statements, $\Longleftrightarrow$: it is incorrect to write $A \Longleftrightarrow B$, since A and B are objects and not statements!

Example. Let $A = \{a, b, c, d\}$ and $B = \{@, \#, \$, \&\}$, then $A \sim B$ (see the one-to-one correspondence we established earlier), and also $|A| = |B|$ but $A \neq B$.
Example. If $N = \{0, 1, 2, 3, \ldots\}$, and $E = \{0, 2, 4, 6, \ldots\}$, then $N \sim E$ (see the one-to-one correspondence we established earlier), and also $|N| = |E|$ but $N \neq E$.

Definition 3.10 (**Infinite set**). A set is infinite if it is equivalent to one of its proper subsets.

Example. Clearly if $N = \{0, 1, 2, 3, \ldots\}$, and $E = \{0, 2, 4, 6, \ldots\}$, then $N \neq E$ (for instance, $1 \notin E$ but $1 \in N$), hence $E \subset N$, a proper subset. However, $N \sim E$, so N is an infinite set.
Example. Consider the set of real numbers, $\mathbb{R}$, and the open interval $P = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, clearly $P \subset \mathbb{R}$. The function $f(x) = \arctan(x)$ is a one-to-one correspondence between $\mathbb{R}$ and P (we will see this in more detail in Week 3 of the course when we learn about functions). Hence, $\mathbb{R} \sim P$, and $\mathbb{R}$ is infinite. In fact, $\mathbb{R}$ is equivalent to all intervals.
Note: When we want to show that a set A is equivalent to one of its proper subsets, the chosen proper subset needs to have the same cardinal number as A, otherwise no equivalence can be established! The set of natural numbers, $\mathbb{N}$ is a proper subset of $\mathbb{R}$, but $|\mathbb{N}| = \aleph_0$, whereas $|\mathbb{R}| = c$ and $c > \aleph_0$: a one-to-one correspondence is impossible to establish!

3.7 Venn Diagrams

In order to visualize sets Venn diagrams can be used (see page 588 of your "Introduction to Probability and Mathematical Statistics" book). Please note that Venn diagrams are only meant for illustration purposes. A proof in this course, Introduction to Analysis, is only accepted through formal mathematical methods. For the Introduction to Statistics course an illustration by Venn Diagrams is accepted instead of a formal proof because the objective of that course differs from the objective of the Introduction to Analysis course. For Introduction to Statistics sets are only a tool for anlyzing statistical and probabilistic problems, whereas for Introduction to Analysis, Sets are one of the central subjects. Here we also aim to develop/improve abstract and sound mathematical thinking.

3.8 How to construct a proof? (Hoe moet een bewijs opgezet worden?)

From the next chapter on we will conduct formal proofs. The question imminently arises: how to conduct a formal proof? First of all, Venn Diagrams are meant to help you with intuition, that is NOT a formal proof. Here you find a **preliminary set of tools** which can help you with proofs, the formal proofs are presented in the next chapter.

Let A, B, C arbitrary subsets of the universal set $\mathcal{U}$.

1. How to prove that $A = B$? A technique you could use here is to prove that $A \subseteq B$ and $B \subseteq A$, then the conclusion is that $A = B$.

2. How to prove that $A \subseteq B$? One possibility is to use the definition: if it is true that $(\forall x \in A \Rightarrow x \in B)$, then $A \subseteq B$.

3. How to prove that a set A is empty, that is, $A = \emptyset$? One of the easiest ways is to use "**Proof by Contradiction**": assume that there exists an element

in A (meaning that you assume that A is NOT empty), mathematically: $\exists x \in A$, then use all the facts and definitions that are given and try to arrive to a contradiction with facts that are known true. If you reached a contradiction then you can conclude that your assumption is invalid; that is, you know that there is NO element in A, meaning $A = \emptyset$.

4. How to prove set equivalence (two sets being equinumerous)? Try to construct a one-to-one correspondence or try to prove that the cardinal numbers are equal.

5. How to prove that a set is infinite? Show that it is equivalent to one of its own proper subsets.

In general: try to use the definitions, try to use all facts that are given in the assumptions, try to divide the problem into subproblems and prove those first. If you have no idea how to start, construct then a simple example and see how the statement works for that simple example. This might already give you an idea for the proof. Do remember though that a specific example can NOT be considered as a proof!

When trying to prove a statement or solving a problem, the most important is that you are NOT only staring at the paper but you start WRITING and/or drawing. Write down everything you can, that is the way to learn it!

Exercises

1. List all proper subsets of the set $A = \{a, 1, b, 2\}$.

2. Use a list notation in order to determine the set $\mathcal{P}(A)$ of the set $A = \{a, 1, b, 2\}$. Verify afterwards whether you obtained all possible elements of $\mathcal{P}(A)$.

3. Show that the sets $O = \{1, 3, \ldots, 2n+1, \ldots\}$ and $E = \{0, 2, 4, \ldots, 2n, \ldots\}$ have the same cardinal number.

4. Find the cardinality of the set $B = \{0, 1, 4, \ldots, n^2, \ldots, 144\}$.

5. Find the cardinality of the set $C = \{0, 1, 4, \ldots, n^2, \ldots\}$.

6. Prove that the set $C = \{0, 1, 4, \ldots, n^2, \ldots\}$ is infinite by using the definition of infinite sets. Note: after week 3, you need to prove bijectivity formally for such an exercise!

Chapter 4

Set Operations

4.1 Union (Vereniging)

Definition 4.1 (**Union**). If A and B are two arbitrary sets from the universal set $\mathcal{U}$, the union of A and B, denoted by $\mathbf{A} \cup \mathbf{B}$, is the set of all elements that are either in A or in B or in both A and B, formally

$$A \cup B = \{x \in \mathcal{U} | x \in A \text{ OR } x \in B\}$$

NOTE that we use the inclusive "OR", meaning "either in A or in B or in both A and B".

Example. Let $\mathcal{U} = \{a, b, c, \ldots, z, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be the universal set and the sets $A = \{a, b, c, d\} \subset \mathcal{U}$, $B = \{0, a, 1, b, 2\} \subset \mathcal{U}$. Then $A \cup B = \{a, b, c, d, 0, 1, 2\}$, we list the common elements only once!

4.2 Intersection (Doorsnede)

In the previous example the *common* elements of the two sets A and B were mentioned.

Definition 4.2 (**Intersection**). If A and B are two arbitrary sets from the universal set $\mathcal{U}$, the intersection of A and B, denoted by $\mathbf{A} \cap \mathbf{B}$, is the set of all elements that are both in A and in B simultaneously, formally

$$A \cap B = \{x \in \mathcal{U} | x \in A \text{ AND } x \in B\}$$

Example. Let $\mathcal{U} = \{a, b, c, \ldots, z, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be the universal set and the sets $A = \{a, b, c, d\} \subset \mathcal{U}$, $B = \{0, a, 1, b, 2\} \subset \mathcal{U}$. Then $A \cap B = \{a, b\}$. Consider now also $C = \{0, 1, 2, 3, 4\}$, then $A \cap C = \emptyset$.

Definition 4.3 (**Disjoint sets (Disjuncte verzamelingen)**). Two sets with no elements in common are disjoint, that is, if the sets A and B are disjoint then $A \cap B = \emptyset$.

4.3 The complement of a set and De Morgan's laws (Het complement en de wetten van De Morgan)

We often need to determine elements from a universal set $\mathcal{U}$, which do not posses a certain property A.

Definition 4.4 (**Complement**). Let A be an arbitrary set in the universal set $\mathcal{U}$. The complement of A, denoted by $\mathbf{A^c}$, is the set of elements in $\mathcal{U}$ that are not in A, formally,

$$A^c = \{x \in \mathcal{U} | x \notin A\}$$

Example. Let $\mathcal{U} = \{a, b, c, d, e, 0, 1, 2, 3\}$ be the universal set and the sets $A = \{a, b, c, d\} \subset \mathcal{U}$, $B = \{0, a, 1, b, 2\} \subset \mathcal{U}$. Then $A^c = \{e, 0, 1, 2, 3\}$ and $B^c = \{c, d, e, 3\}$.

Let us now determine the following sets: 1, $(A \cup B)^c$, 2, $A^c \cup B^c$, 3, $(A \cap B)^c$ and 4, $A^c \cap B^c$.

1. $(A \cup B)^c = \{a, b, c, d, 0, 1, 2\}^c = \{e, 3\}$
2. $A^c \cup B^c = \{e, 0, 1, 2, 3\} \cup \{c, d, e, 3\} = \{c, d, e, 0, 1, 2, 3\}$
3. $(A \cap B)^c = \{a, b\}^c = \{c, d, e, 0, 1, 2, 3\}$
4. $A^c \cap B^c = \{e, 0, 1, 2, 3\} \cap \{c, d, e, 3\} = \{e, 3\}$.

We can observe here, that the sets $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$. The question is whether this is a coincidence or this holds for all subsets A, B of an arbitrary universal set $\mathcal{U}$? It is very important to realize that an example is NOT a proof, even a thousand examples cannot prove a statement! We need to conduct a formal proof.

Theorem 4.5 (**De Morgan's laws**). *Let A, B be arbitrary subsets of a universal set $\mathcal{U}$. Then the following statements are true:*

a. $(A \cup B)^c = A^c \cap B^c$

b. $(A \cap B)^c = A^c \cup B^c$

Formal Proof. a. Please look in section 3.8, there you find that one of the common techniques to prove a set identity is to prove that the two sets are each other's subsets, that is, if $(A \cup B)^c \subseteq A^c \cap B^c$ and also $A^c \cap B^c \subseteq (A \cup B)^c$ then $(A \cup B)^c = A^c \cap B^c$.

1. Prove that $(A \cup B)^c \subseteq A^c \cap B^c$. Look again in section 3.8: $\forall x \in (A \cup B)^c$ we need to show that $x \in A^c \cap B^c$. Let us start with selecting an arbitrary element, then:

$\forall x \in (A \cup B)^c \overset{\text{def.c.}}{\iff} x \notin A \cup B$, ($\overset{\text{def.c.}}{\iff}$ means: by the definition of the complement this is equivalent to the statement...)
Now we need to tread very carefully here: we cannot use the definition of union directly for "not element" only for "element"! If you think logically, the union of two sets means that the elements are either in one or the other, or in both simultanously. That is, if x is not in the union, then it cannot be in either of the sets, otherwise it will be in the union. We can conclude thus that, $x \notin A \cup B$ means that $x \notin A$ and simultaneously, $x \notin B$. By the definition of the complement, this means that, $x \in A^c$ and simultaneously, $x \in B^c$. By the definition of the intersection this means that $x \in A^c \cap B^c$. Writing this out only with mathematical symbols, this becomes:

$$\begin{aligned}
\forall x \in (A \cup B)^c \quad & \overset{\text{def.c.}}{\iff} \quad x \notin A \cup B \\
& \overset{?}{\iff} \quad (x \notin A \text{ and } x \notin B) \\
& \overset{\text{def.c.}}{\iff} \quad (x \in A^c \text{ and } x \in B^c) \\
& \overset{\text{def.}\cap}{\iff} \quad x \in A^c \cap B^c.
\end{aligned}$$

We have just shown that $\forall x \in (A \cup B)^c \iff x \in A^c \cap B^c$, which means not only that $(A \cup B)^c \subseteq A^c \cap B^c$, but also the inverse, $A^c \cap B^c \subseteq (A \cup B)^c$, because of the equivalence sign $\iff$. We can conclude that $(A \cup B)^c = A^c \cap B^c$.

One of the equivalence relations above is marked by a question mark. The reason for this is that although we can reason with common sense that this equivalence is true, there is also a rigorous way to prove it, by the rules of proposition logic. We will see this in the next part of this Syllabus. After you have learned proposition logic, you must prove this step ($\overset{?}{\iff}$) rigorously, as it is done in Chapter 7.

b. For statement $(A \cap B)^c = A^c \cup B^c$ we proceed in the same way. $\forall x \in (A \cap B)^c$ we need to show that $x \in A^c \cup B^c$. Let us start with selecting an arbitrary element: $\forall x \in (A \cap B)^c \overset{\text{def.c.}}{\iff} x \notin A \cap B$. In words, this means that x cannot be simultanously in A and in B, which means that x is either not in A or not in B, or in either of them. We write again the proof formally:

$$\begin{aligned}
\forall x \in (A \cap B)^c \quad & \overset{\text{def.c.}}{\iff} \quad x \notin A \cap B \\
& \overset{?}{\iff} \quad (x \notin A \text{ or } x \notin B) \\
& \overset{\text{def.c.}}{\iff} \quad (x \in A^c \text{ or } x \in B^c) \\
& \overset{\text{def.}\cup}{\iff} \quad x \in A^c \cup B^c.
\end{aligned}$$

We showed that $\forall x \in (A \cap B)^c \iff x \in A^c \cup B^c$, which means not only that $(A \cap B)^c \subseteq A^c \cup B^c$, but also the inverse, $A^c \cup B^c \subseteq (A \cap B)^c$, because of the equivalence sign $\iff$. We can conclude that $(A \cap B)^c = A^c \cup B^c$.

Once again, one of the equivalence relations above is marked by a question mark. The reason for this is that although we can reason with common sense that this equivalence is true, there is also a rigorous way to prove it, by the rules of proposition logic. We will see this in the next part of this Syllabus. Do note that after you have learned proposition logic, you must be able to prove this step ($\overset{?}{\Longleftrightarrow}$) rigorously, as it is done in Chapter 7.

4.4 Distributive laws (Distributive wetten)

In the previous chapter we have seen how to determine the complement of a union and an intersection, but what about combining intersection with union? More precisely, for any A, B and C subsets in a universal set $\mathcal{U}$, how to determine $A \cup (B \cap C)$ or $A \cap (B \cup C)$? Just as the simple operations with real numbers, it turns out that set operations also possess the distributive property.

Theorem 4.6 (**Distributive Laws**). *For arbitrary subsets A, B and C of a universal set $\mathcal{U}$ the following holds properties hold.*

a. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

b. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Example. Let $\mathcal{U} = \{a, b, c, d, e, 0, 1, 2, 3\}$ be the universal set and the subsets $A = \{a, b, c, d\}$, $B = \{0, a, 1, b, 2\}$, $C = \{2, 3\}$. Then $A \cup B = \{a, b, c, d, 0, 1, 2\}$, $A \cup C = \{a, b, c, d, 2, 3\}$, $B \cup C = \{a, b, 0, 1, 2, 3\}$, $A \cap B = \{a, b\}$, $A \cap C = \emptyset$, $B \cap C = \{2\}$, and the distributive laws:

a. $A \cup (B \cap C) = \{a, b, c, d\} \cup \{2\} = \{a, b, c, d, 2\}$, and the right side $(A \cup B) \cap (A \cup C) = \{a, b, c, d, 0, 1, 2\} \cap \{a, b, c, d, 2, 3\} = \{a, b, c, d, 2\}$, so indeed $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

b. $A \cap (B \cup C) = \{a, b, c, d\} \cap \{a, b, 0, 1, 2, 3\} = \{a, b\}$ and also $(A \cap B) \cup (A \cap C) = \{a, b\} \cup \emptyset = \{a, b\}$.

This is an example, not a proof. The formal proof is conducted in the same manner as for De Morgan's Laws.

4.5 Set Difference (Verschil van verzamelingen)

Definition 4.7 (**Set difference**). If A and B are two arbitrary subsets of the universal set $\mathcal{U}$, the difference of A and B, denoted by $\mathbf{A} \setminus \mathbf{B}$, is the set of all elements that are in A and not in B, formally

$$A \setminus B = \{x \in \mathcal{U} | x \in A \text{ and } x \notin B\}.$$

Careful! Set difference is completely different from difference between numbers!

Example. Let $\mathcal{U} = \{a, b, c, d, e, 0, 1, 2, 3\}$ be the universal set and the subsets $A = \{a, b, c, d\}$, $B = \{0, a, 1, b, 2\}$, $C = \{2, 3\}$, then

$A \setminus B = \{c, d\}$

$B \setminus A = \{0, 1, 2\}$

$A \setminus C = \{a, b, c, d\} = A$, because $A \cap C = \emptyset$.

$B \setminus C = \{0, a, 1, b\}$, and so on

Theorem 4.8 (**Identities with differences**). *Let A and B are two arbitrary subsets of the universal set $\mathcal{U}$. The following statements hold.*

a. $A^c = \mathcal{U} \setminus A$

b. $A \setminus B = A \cap B^c$

Formal proof. a. The definition of the complement is $A^c = \{x \in \mathcal{U} | x \notin A\}$, that is

$$\begin{aligned} \forall x \in A^c &\overset{\text{def.c.}}{\iff} x \in \mathcal{U} \text{ and } x \notin A \\ &\overset{\text{def.diff.}}{\iff} x \in \mathcal{U} \setminus A, \end{aligned}$$

which proves that $A^c \subseteq \mathcal{U} \setminus A$, and because of the equivalence, also that $\mathcal{U} \setminus A \subseteq A^c$, that is, $A^c = \mathcal{U} \setminus A$.

b.

$$\begin{aligned} A \setminus B &= \{x \in \mathcal{U} | x \in A \text{ and } x \notin B\} \\ &\overset{\text{def.c.}}{=} \{x \in \mathcal{U} | x \in A \text{ and } x \in B^c\} \\ &\overset{\text{def.}\cap}{=} \{x \in \mathcal{U} | x \in A \cap B^c\} = A \cap B^c. \end{aligned}$$

Consequences:

1. $\mathcal{U}$ and $\emptyset$ are each other complements, $\mathcal{U}^c = \emptyset$ and $\emptyset^c = \mathcal{U}$.

2. $A \cup A^c = \mathcal{U}$ and $A \cap A^c = \emptyset$, for all $A \subseteq \mathcal{U}$.

4.6 Further set-proof examples

1. For arbitrary A and B subsets in a universal set $\mathcal{U}$, such that $A \subseteq B$, prove the following statements:

   $$a. A \cup B = B \text{ and } b. A \cap B = A.$$

   Proof. a. The part $B \subseteq A \cup B$ is obvious, so we only need to show that $A \cup B \subseteq B$.

   $$\begin{aligned} \forall x \in A \cup B &\overset{\text{def.}\cup}{\iff} x \in A \text{ or } x \in B \\ &\overset{\text{def.}\subseteq}{\implies} x \in B \text{ or } x \in B \\ &\iff x \in B. \end{aligned}$$

   Since $\forall x \in A \cup B \implies x \in B$, by the definition of subsets it follows that $A \cup B \subseteq B$. This means that $A \cup B = B$.

b. It is again obvious that $A \cap B \subseteq A$, so we only need to prove that $A \subseteq A \cap B$.

$$\forall x \in A \overset{\text{def.}\subseteq}{\Longrightarrow} x \in B.$$

But $x \in A$ and $x \in B$ also implies by the definition of the intersection that $x \in A \cap B$, that is, $A \subseteq A \cap B$. QED.

2. For arbitrary A and B subsets in a universal set $\mathcal{U}$, such that $A \subseteq B$, prove that $A \cap B^c = \emptyset$.

   The easiest method to prove this for me is Proof by contradiction. Assume that $A \cap B^c \neq \emptyset$, that is $\exists x \in A \cap B^c$, then

$$\begin{aligned} \exists x \in A \cap B^c \quad & \overset{\text{def.}\cap}{\Longleftrightarrow} \quad x \in A \text{ and } x \in B^c \\ & \overset{\text{def.c}}{\Longleftrightarrow} \quad x \in A \text{ and } x \notin B \\ & \overset{\text{def.}\subseteq}{\Longleftrightarrow} \quad A \nsubseteq B, \end{aligned}$$

   which is a contradiction with the given fact that $A \subseteq B$, so our assumption that $\exists x \in A \cap B^c$ is incorrect: $A \cap B^c = \emptyset$. QED.

3. For arbitrary A and B subsets in a universal set $\mathcal{U}$ prove that $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

   Proof. We will need to show that a. $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$ and b. $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$.

   a.

$$\begin{aligned} \forall X \in \mathcal{P}(A \cap B) \quad & \overset{\text{def.}\mathcal{P}}{\Longleftrightarrow} \quad X \subseteq A \cap B \\ & \overset{\text{def.}\cap}{\Longleftrightarrow} \quad X \subseteq A \text{ and } X \subseteq B \\ & \overset{\text{def.}\mathcal{P}}{\Longleftrightarrow} \quad X \in \mathcal{P}(A) \text{ and } X \in \mathcal{P}(B) \\ & \overset{\text{def.}\cap}{\Longleftrightarrow} \quad X \in \mathcal{P}(A) \cap \mathcal{P}(B). \end{aligned}$$

   Since we have managed to use $\Longleftrightarrow$ between all statements this proves not only $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$ but also the inverse, $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$, so we can conclude that $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$. QED.

Exercises

Let $\mathcal{U}$ be a universal set and A, B and C arbitrary subsets in $\mathcal{U}$.

1. Prove that $A \cup B$, $A \cap B \subseteq \mathcal{U}$.
2. Prove that $A^c \cap A = \emptyset$.
3. Prove that $A^c \cup A = \mathcal{U}$

4. Prove that $A \cap \emptyset = \emptyset$.
5. Prove that $A \cup \emptyset = A$.
6. Transitivity: Prove that if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.
7. Prove that $A = (A \cap B) \cup (A \cap B^c)$.
8. Prove that $(A \cap B) \cap (A \cap B^c) = \emptyset$.
9. Prove that if $B \subseteq A$ then $A = B \cup (A \cap B^c)$.

Chapter 5

Indicator function (Karakteristieke functie)

Let $\mathcal{U}$ be a universal set and $A \subseteq \mathcal{U}$.

Definition 5.1 (**Indicator function**). The function $1_A : \mathcal{U} \to \{0,1\}$, defined by

$$1_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

is called the indicator function of A.

The indicator function of A indicates, or signals when an element falls within A or outside A - it is like a metal detector: $\mathcal{U}$ is all the sand of the beach, and A is the set of metal pieces under the sand. The indicator function beeps (= takes on value 1) when it finds metal (finds an element of A) and remains silent (=takes on value 0) when there is no metal (= elements of $\mathcal{U}$ which are outside A).

Example. Let $\mathcal{U} = \{1,2,3,4,5\}$, and $A = \{1,3,5\}$. Then $1_A(1) = 1$, because $1 \in A$, $1_A(2) = 0$, because $2 \notin A$, $1_A(3) = 1$, because $3 \in A$, $1_A(4) = 0$, because $4 \notin A$, and $1_A(5) = 1$, because $5 \in A$.

Exercises

Let $\mathcal{U}$ be a universal set and A, B and C arbitrary subsets in $\mathcal{U}$.

1. Determine the value of $1_\emptyset(x)$ for all $x \in \mathcal{U}$.

2. Determine the value of $1_\mathcal{U}(x)$ for all $x \in \mathcal{U}$.

3. Determine a formula for $1_{A\cap B}(x)$ for all $x \in \mathcal{U}$ in terms of $1_A(x)$ and $1_B(x)$.

4. Determine a formula for $1_{A\cup B}(x)$ for all $x \in \mathcal{U}$ in terms of $1_A(x)$ and $1_B(x)$.

5. Determine a formula for $1_{A^c}(x)$ for all $x \in \mathcal{U}$ in terms of $1_A(x)$.

List of symbols

∞	infinity / oneindig (can be $-\infty$ or $+\infty$)
$\forall$	every, arbitrary / alle, willekeurig
$\exists$	there exists / er bestaat
$\nexists$	there **does not** exist / er bestaat **geen**
$\emptyset$	empty set / lege verzameling $\{ \}$
$\in$	element of / element van
$\notin$	not an element of / geen element van
$\subseteq$	subset (may be equal) / deelverzameling (kan gelijk zijn)
$\subset$	a proper subset of / een echte verzameling van
$\bigcup$	sets union / vereniging van verzamelingen
$\bigcap$	sets intersection / doorsnede van verzamelingen
A^c	complement of the set A / complement van de verzameling A
$\vert A\vert$	cardinality (=number of elements) of the set A / kardinaliteit (= hoeveelheid elementen) in de verzameling A
1_A	indicator function for the set A / karakteristieke functie voor de verzameling A
$\Leftrightarrow$	'if and only if' / 'dan en slechts dan als'
$\sum$	sum / som

Part III

Proposition logic (Propositielogica)

Chapter 6

Propositions (Proposities)

Everyone knows what propositions are from languages. Here are a couple of examples:

Today is Monday.
Tomorrow will be Friday.
White is not black.
Black is a bright color.
You obtain snow if water freezes.

The above sentences are all propositions. As you can observe, a proposition does not need to be true, as long as they are declarative sentences. We also need to be able to decide whether a proposition is true or false. If today is indeed Monday, then the first sentence is true. However, the second sentence is then false, unless you stopped reading after the first example and reassumed reading on Thursday. The third sentence is true and the the last two are false again.

Sentences such as "What time is it?" or "Leave me alone!" are not propositions. They are not declarative and we cannot decide whether they are true or false.

Furthermore, the first four of the previous examples are *atomic propositions*, they do not have sentences as parts. The last sentence is not atomic, it contains the two atomic sentences "you obtain snow" and "water freezes", connected by "if". We will denote atomic sentences by letters such as p, q, r, and so on.

6.1 Connectives (Connectieven)

The most important connectives of our interest, which link the atomic sentences, and form together one composed sentence, are the following:

1. negation: NOT, "$\neg$"
2. conjuction: AND, "$\wedge$"
3. disjunction: OR, "$\vee$"

4. implication: IF... THEN..., "$\Rightarrow$"
5. equivalence: IF AND ONLY IF, "$\Leftrightarrow$"

6.2 Truth tables (Waarheidstabellen)

In order to find out whether a composed sentence is true or false, we then only need to know whether the atomic sentences are true or false and deduct from there in which cases the composition is true or false. If we have an atomic sentence, p="the train is coming", then the **negation** of this sentence is $\neg p$ ="the train is not coming". $\neg p$ is true exactly when p is false. In order to represent all possible situations when p and $\neg p$ are true or false, we use truth tables.

p	$\neg p$
T	F
F	T

Consider now two atomic sentences, p="the train is coming" and q="the signals are red". Given the two atomic sentences p and q, we want to know when p *and* q are true or false. p *and* q will be true when p true is, and q true is, simultanuously. We can also build a truth table to this end.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Similary, for a disjunction we can set up a similar table.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

6.3 Implication and equivalence (Implicatie en equivalentie)

An implication can be somewhat counterintuitive. If p then q translates for our example as "If the train is coming then the signals are red". When is this proposition true? When p is true, that is, the train is coming, and also the signals are red, that is, q is also true, then our proposition is indeed true. When p is true, that is, the train is coming, but the signals are not red, that is, q is false, then our proposition is also false. However, if the train is not coming, the signals can be red or not, our proposition will still be true, because we did not mention in our proposition which color the signals should have when the train is not coming. Take a look now at the truth table which belongs to the implication.

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

The $p \Leftrightarrow q$ statement is in fact the composition of $p \Rightarrow q$ and $q \Rightarrow p$. Note that the conjuction is false when both statements that are combined with it, are false, thus the 'if and only if' will only be true when BOTH $p \Rightarrow q$ and $q \Rightarrow p$ are simultaneously true.

p	q	$(p \Rightarrow q)$	$\wedge$	$(q \Rightarrow p)$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	F
F	F	T	T	T

Chapter 7

Equivalent propositions (Gelijkwaardige proposities)

The truth value of a proposition p will be designated by $|p|$. That is, for any particular p it holds that $|p| = 0$ or $|p| = 1$. Two propositions, p and q are equal (or equivalent) when they have the same truth values, that is, when $|p| = |q|$. There are several especially important equivalent propositions we will investigate here.

1. **Double negation (Dubbele ontkenning)**: $(\neg\neg p) \Leftrightarrow p$.

   Although this is a quite obvious equivalence, we can check this by drawing up the corresponding truth table and checking whether indeed $|\neg\neg p| = |p|$

p	$\neg p$	$\neg\neg p$
T	F	T
F	T	F

2. **The law of contraposition (De wet van contrapositie)**:

   $(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$.

   On this law is the proof technique "Proof by contradiction" (bewijs uit het ongerijmde) is based. In order to prove that p implies q one can also prove that not-q implies not-p. You have seen that we have also used this thechnique to set up the **alternative definition for subsets** (check the section Subsets now!).

3. **Commutativity (commutativiteit) of $\wedge$ and $\vee$**

   $(p \wedge q) \Leftrightarrow (q \wedge p)$, and
   $(p \vee q) \Leftrightarrow (q \vee p)$

p	q	$\neg q$	$\neg p$	$p \Rightarrow q$	$\neg q \Rightarrow \neg p$
T	T	F	F	T	T
T	F	T	F	F	F
F	T	F	T	T	T
F	F	T	T	T	T

4. **Associativity (associativiteit) of $\wedge$ and $\vee$**
   $[(p \wedge q) \wedge r] \Leftrightarrow [p \wedge (q \wedge r)]$, and
   $[(p \vee q) \vee r] \Leftrightarrow [p \vee (q \vee r)]$

5. **Distributive laws (Distributive wetten)**
   $[(p \wedge q) \vee r] \Leftrightarrow [(p \vee r) \wedge (q \vee r)]$, and
   $[(p \vee q) \wedge r] \Leftrightarrow [(p \wedge r) \vee (q \wedge r)]$

   Please compare these to the Distributive laws for sets!

6. **De Morgan's laws (De wetten van De Morgan)**
   $\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$, and
   $\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$

   Please compare these to the De Morgan's laws for sets, and also set up the truth table for the second case. We will show here how this works for the first of De Morgan's laws.

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Example: Let us prove the first of the De Morgan's laws for sets, using equivalent propositions. That is, for two arbitrary subsets A, B of a universal set $\mathcal{U}$,

$$(A \cup B)^c = A^c \cap B^c.$$

Please look back now and check Theorem 4.5 and its proof. We proceed now as follows.

$$\begin{aligned}
\forall x \in (A \cup B)^c \quad & \overset{\text{def.c.}}{\Longleftrightarrow} \quad x \notin A \cup B \\
& \Longleftrightarrow \quad \neg[x \in A \cup B] \\
& \overset{\text{def.} \cup}{\Longleftrightarrow} \quad \neg[(x \in A) \vee (x \in B)] \\
& \overset{\text{law 6}}{\Longleftrightarrow} \quad \neg(x \in A) \wedge \neg(x \in B) \\
& \Longleftrightarrow \quad (x \notin A) \wedge (x \notin B) \\
& \overset{\cap \,\&\, \text{c.}}{\Longleftrightarrow} \quad x \in A^c \cap B^c.
\end{aligned}$$

7. **Necessary and sufficient condition (Voldoende en noodzakelijke conditie)**

   $(p \Leftrightarrow q) \Leftrightarrow [(p \Rightarrow q) \wedge (\neg p \Rightarrow \neg q)]$

   When we need to prove an equaivalence relation, that is, $p \Leftrightarrow q$, then it is said that in this case, p is a necessary and sufficient condition for q. But what does that exactly mean? This is a composition of two statements we have already discussed: we have seen that $p \Leftrightarrow q$ means that $(p \Rightarrow q) \wedge (q \Rightarrow p)$.

   (a) Sufficient (Voldoende): The implication $p \Rightarrow q$ means that p is a sufficient condition for q. However, we have also seen that the implication $p \Rightarrow q$ is also true when p is false. In this case p is sufficient, meaning: if p is true, and we know that $p \Rightarrow q$ is true, then q must be true too. However p is not necessary, meaning that when p is false, and we know that $p \Rightarrow q$ is true, q could still be true, it is not necessarily false.

   (b) Necessary (Noodzakelijk): A necessary condition means that p is necessary for q to be true, that is, when p is not true then q cannot be true either. But does $q \Rightarrow p$ imply that? Yes it does! To see that, check the law of contraposition. That means that $q \Rightarrow p$ is equivalent with the statement $\neg p \Rightarrow \neg q$, that is, knowing that $q \Rightarrow p$ holds, if p is not true, then q cannot be true either. In other words p is a necessary condition for q.

   Note that if p is a necessary and sufficient condition for q, then vica versa, q is also a necessary and sufficient condition for p.

   Example: Let p be the statement "The natural number n is divisible by 5", and q the statement "The last digit of the natural number n is either 0 or 5". Then we say that p and q are necessary and sufficient conditions of each other. When it is true that a natural number n has 0 or 5 as last digit then it is also true that n is divisible by 5 (sufficient condition), but if n has as last digit a number different from 0 or 5, then we know that n is not divisible by 5 (necessary condition).

Although there is implication instead of equivalence between the following two statements, this implication is essential in proofs: the transitivity of implication.

Transitivity of implication (Transitiviteit van implicatie)

$[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (p \Rightarrow r)$.

Draw up the truth table yourself and observe that whenever the left hand side is true, the right hand side is also true, hence the implication will always be true.

Excercises

1. Prove the second part of the De Morgan's laws for sets with equivalent propositions.
2. Prove the Distributive laws for sets with equivalent propositions.
3. Set up the truth tables for the necessary and sufficient conditions, that is, show with a truth table that $(p \Leftrightarrow q) \Leftrightarrow [(p \Rightarrow q) \wedge (\neg p \Rightarrow \neg q)]$.

Part IV

Induction (Inductie)

Chapter 8

Mathematical induction (Wiskundige inductie)

Please read now first Chapter 12.5 (pages 873 - 879), pay particular attention to the cartoon on page 875, to the sums of powers on page 877, and do try to make all worked examples in this chapter by yourself (close the book) and look only afterwards how exactly it is solved in the book. Should you not manage it by yourself, then read it first in the book, CLOSE the book afterwards and redo it yourself! Do compare your solution with that in the book row - by - row: only then you will know whether you fully understood all details. Here, I will only elaborate on some of the mathematical notation you are REQUIRED to use during the exam.

8.1 Divisibility (Deelbaarheid)

Please study the solution to exercise 20 (page 878): *Use mathematical induction to show that* $3^{2n} - 1$ *is divisible by* 8 *for all* $n \in \mathbb{N}$, $n \geq 0$.

Just a short intermezzo before we start with the induction: An integer n is said to be **divisible** by another integer m if there exists an integer d such that $n = m \cdot d$. Do note that here it is crucial that all three numbers n, m and d are integers.
Example: $-24 \in \mathbb{Z}$ is divisible by 8 because $\exists -3 \in \mathbb{Z}$ such that $-24 = 8 \cdot (-3)$. Here we have thus $n = -24$, $m = 8$ and $d = -3$.
Obviously, if we take a different n, let's denote it as $n_2 = 32$, then considering the same $m = 8$, we also obtain a different $d_2 = 4$, because $32 = 8 \cdot 4$.

It is thus important to realize that this is a reason one needs to use different notation for each d whenever the number n changes: if a natural number n_k is divisible by 8 then $\exists\, d_k \in \mathbb{N}$ such that $n_k = 8 \cdot d_k$. We want to emphasize that n_k and d_k belong together.

Solution:

BS: (Base step) for $n = 0$

$3^{2n} - 1 = 3^0 - 1 = 1 - 1 = 0$. Since $0 = 8 \cdot 0$, 0 is indeed divisible by 8.

IH: (Induction Step)

Select an *arbitrary* $k \in \mathbb{N}$, $k \geq 0$ and *assume* that $\exists\, d_k \in \mathbb{N}$ such that $3^{2k} - 1 = 8 \cdot d_k$.

Observe that this is an assumption, we do not want to prove this. Considering that the statement *is true* for k, we want to prove that it is true for $k + 1$.

Proof:

We need to prove that $\exists\, d_{k+1} \in \mathbb{N}$ such that $3^{2(k+1)} - 1 = 8 \cdot d_{k+1}$.
Also remember that in the proof you need to make use of your induction hypothesis otherwise your proof is not done by induction.
Let us examine $3^{2(k+1)} - 1$.

$$3^{2(k+1)} - 1 = 3^{2k+2} - 1 = 3^{2k} \cdot 3^2 - 1.$$

Here we can use our IH for $3^{2k} - 1$, the only problem is that we do not have this exact form. We have $3^{2k} \cdot 9 - 1$ instead. We will use the fact that $9 \cdot 3^{2k} = 8 \cdot 3^{2k} + 3^{2k}$.

$$3^{2(k+1)} - 1 = 8 \cdot 3^{2k} + 3^{2k} - 1 \overset{\text{IH}}{=} 8 \cdot 3^{2k} + 8 \cdot d_k = 8(3^{2k} + d_k).$$

As both 3^{2k} and d_k are natural numbers, $3^{2k} + d_k \in \mathbb{N}$. Hence we found the natural number we were looking for: since there is one single number satisfying $3^{2(k+1)} = 8 \cdot d_{k+1}$, we have found that $d_{k+1} = 3^{2k} + d_k \in \mathbb{N}$.

Conclusion: $3^{2n} - 1$ is divisible by 8 for all $n \in \mathbb{N}$, $n \geq 0$.

8.2 Induction for recursively defined sequences (Inductie voor recursieve rijen)

Attention: Before you start reading the following chapters, please look up *recursively defined sequences (recusrsive rijen)* on pages 845 & 846, and make exercises 15 - 20 on page 850.

See problem 30 on page 878: The Fibonacci sequence is defined as $F_1 = F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$, $n \in \mathbb{N}$.
Use induction to prove that

$$F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2} - 1,$$

for all $n \in \mathbb{N}$*,* $n \geq 1$*.*

Attention: It is really important to realize here that the recursive relation only starts at $n = 3$, so we can only start our proof from $n = 3$, since that is how this sequence is defined. If in the base step we would only do the check for the first n, $n = 1$ then there would be a gap between $n = 1$ and $n = 3$, so we cannot roll on the proof. Think of it like this: a proof by induction is like a domino effect, you check the first domino, make sure that if you push over any arbitrary kth domino, the $k+1$th will also fall over, then by pushing the first domino you know that all the dominoes will fall over. But now, if you have a gap (here $n = 2$ is missing) then the domino-effect will already stop after the first domino you pushed over. How can you repair this problem? By checking $n = 1, n = 2$ and $n = 3$ instead of only $n = 1$. Let us proceed then as follows.

BS:
For $n = 1$, the LHS (left hand side) is $F_1 + F_2 + F_3 + \ldots + F_n = F_1 = 1$.
the RHS (right hand side) is $F_{n+2} - 1 = F_3 - 1 = 2 - 1 = 1$,
since the LHS = RHS, the statement holds for $n = 1$.
For $n = 2$, the LHS is $F_1 + F_2 + F_3 + \ldots + F_n = F_1 + F_2 = 1 + 1 = 2$.
the RHS is $F_{n+2} - 1 = F_4 - 1 = 3 - 1 = 2$,
since the LHS = RHS, the statement also holds for $n = 2$.
For $n = 3$, the LHS is $F_1 + F_2 + F_3 + \ldots + F_n = F_1 + F_2 + F_3 = 1 + 1 + 2 = 4$.
the RHS is $F_{n+2} - 1 = F_5 - 1 = 5 - 1 = 4$,
since the LHS = RHS, the statement also holds for $n = 3$.

IH: Select an arbitrary $k \geq 3$, $k \in \mathbb{N}$, and assume that $F_1 + F_2 + F_3 + \ldots + F_k = F_{k+2} - 1$.

Proof: We need to prove that $F_1 + F_2 + F_3 + \ldots + F_k + F_{k+1} = F_{k+3} - 1$.
We can immediatelly use the IH, since we made an assumption about the first k terms on the LHS:

$$\mathbf{F_1} + \mathbf{F_2} + \mathbf{F_3} + \ldots + \mathbf{F_k} + F_{k+1} \overset{\text{IH}}{=} F_{k+2} - 1 + F_{k+1}.$$

Since F is the Fibonacci sequence, we know that $F_{k+1} + F_{k+2} = F_{k+3}$, which leads to:

$$\mathbf{F_1} + \mathbf{F_2} + \mathbf{F_3} + \ldots + \mathbf{F_k} + F_{k+1} \overset{\text{IH}}{=} F_{k+1} + F_{k+2} - 1 = F_{k+3} - 1,$$

and this is exactly what we needed to prove.
Conclusion: $F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2} - 1$, for all $n \in \mathbb{N}$, $n \geq 1$.

Remark 8.1. **Frequently made mistakes:**

1. The recursion relation is GIVEN, that is how the sequence is defined. Still, I see very often students trying to prove the recurrence relation instead of the statement. Be always careful that you try to prove the statement and you use the recurrence relation as a known fact.

2. If a recursive sequence is defined, for instance as $a_1 = 1, a_2 = 2, a_3 = 4$ and $a_4 = 6$, such that $a_n = 2a_{n-1} + 3a_{n-2} + 4a_{n-3} - 5a_{n-4}$, then it means that you need to do the ckeck a_1, a_2, a_3, a_4 and a_5 in the base step!

3. If a recursive sequence is defined, for instance as $a_1 = 1, a_2 = 2, a_3 = 4$ and $a_4 = 6$, such that $a_n = 2a_{n-1} + 3a_{n-2} + 4a_{n-3} - 5a_{n-4}$, then it means that you will write the IH for the statement for an arbitrary k, $\mathbf{k \geq 5}$.

Chapter 9

Complete or Strong induction (Volledige inductie)

Complete induction is in fact a stronger form of mathematical induction. To see why we need a stronger form of induction, that is, why is mathematical induction not always sufficient, let us take a look at the following problem.

See problem 20 on page 850: Let the "Tribonacci sequence" be defined as $a_1 = a_2 = a_3 = 1$, and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for $n \geq 4$, $n \in \mathbb{N}$.
Use induction to prove that $a_n < 2^n$ for all $n \in \mathbb{N}$, $n \geq 1$.
We will take the steps exactly as we have learnt it in the previous chapter.
BS:
We need to do the check for $n = 1, n = 2, n = 3$ and $n = 4$.

1. For $n = 1$, the LHS is $a_n = a_1 = 1$, and the RHS is $2^n = 2^1 = 2$. Obviously, $1 < 2$, i.e. the statement is true for $n = 1$.

2. For $n = 2$, the LHS is $a_n = a_2 = 1$, and the RHS is $2^n = 2^2 = 4$. Obviously, $1 < 4$, i.e. the statement is true for $n = 2$.

3. For $n = 3$, the LHS is $a_n = a_3 = 1$, and the RHS is $2^n = 2^3 = 8$. Obviously, $1 < 8$, i.e. the statement is true for $n = 3$.

4. For $n = 4$, the LHS is $a_n = a_4 = 3$, and the RHS is $2^n = 2^4 = 16$. Obviously, $3 < 16$, i.e. the statement is true for $n = 4$.

IH:
Select an *arbitrary* $k \in \mathbb{N}$, $k \geq 4$ and assume that $a_k < 2^k$.
Proof:
We need to prove the statement for $k + 1$, that is, we need to prove that $a_{k+1} < 2^{k+1}$.

Let us take a look at the LHS; we know that the sequence is defined by the recursion relation, and that is what we can use now:

$$a_{k+1} = a_k + a_{k-1} + a_{k-2} \overset{\text{IH}}{<} 2^k + a_{k-1} + a_{k-2} \dots$$

and now we are stuck, because our induction hypothesis SOLELY holds for k and not for $k-1$ or $k-2$. The only step we could take from here, is to apply again the recurrence relation to the terms a_{k-1} and a_{k-2} until we get back to a_4, which we already checked. However, since k is selected arbitrarily (imagine that $k = 999999999999999999$) this is an impossible suggestion, and we are truly stuck with this proof.

This is exactly the reason why we need complete induction. The difference between mathematical and complete induction lies in the Induction Hypothesis. For mathematical induction the assumption is for one single k, whereas for complete induction the assumption is for all numbers between 4 and k, whatever k may be.

IH for COMPLETE INDUCTION:
Select an *arbitrary* $k \in \mathbb{N}$, $k \geq 4$ and assume that $a_m < 2^m$, for all $m \in \mathbb{N}$, such that $4 \leq m \leq k$.
In other words, this means, that for the arbitrarily selected k we assume that $a_4 < 2^4$, $a_5 < 2^5$, $a_6 < 2^6$, …, $a_{k-1} < 2^{k-1}$, $a_k < 2^k$. The notation with m is just much shorter.

Proof by COMPLETE INDUCTION:

$$a_{k+1} = a_k + a_{k-1} + a_{k-2} \overset{\text{IH}}{<} 2^k + 2^{k-1} + 2^{k-2}$$

Now this looks much more promising, since we only need to show now that $2^k + 2^{k-1} + 2^{k-2} \leq 2^{k+1}$.

$$a_{k+1} \overset{\text{IH}}{<} 2^k + 2^{k-1} + 2^{k-2} = 2^{k+1}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right) = 2^{k+1}\frac{7}{8} < 2^{k+1}.$$

Here we used the induction hypothesis with $m = k, m = k-1$ and $m = k-2$.
Conclusion: $a_n < 2^n$ for all $n \in \mathbb{N}$, $n \geq 1$.

Remark 9.1. Important notes on complete induction

1. In case of strong induction, the Induction Hypothesis is never used for absolutely all $m \leq k$, but we only need some of these numbers. However since we often do not know in advance how exactly we want to prove a statement, and we also persue one uniform model, we assume the statement to be true for all $m \leq k$. In the previous problem we only needed to use the IH for $m = k, m = k-1$ and $m = k-2$, but the induction hypothesis contained the assumption for all $m \in \mathbb{N}$, such that $4 \leq m \leq k$.

2. Often with a recursion relation it is handy and/or necessary to use complete induction, but complete induction is not only used for proving statements involving recursive sequences. However, the lack of recursion does

not imply at all that you can automatically solve the problem with mathematical induction. See Exercise 3 at the end of this chapter, and also the second induction exercise in the Exam of 2018.

3. Do note that in the previous exercise the statement is true for all natural numbers $n \geq 1$, although the recursion only starts with $n \geq 4$.

Exercises

1. Let the sequence $(a_n)_{n\in\mathbb{N}}$ be defined as $a_1 = 1, a_2 = 2, a_3 = 3$, and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for $n \geq 4$, $n \in \mathbb{N}$.

   Use mathematical or complete induction in order to prove that $a_n < 2^n$ for all $n \in \mathbb{N}$, $n \geq 1$. Did you need mathematical or complete induction? Explain why.

2. Let the sequence $(b_n)_{n\in\mathbb{N}}$ be defined through the recursive relation $bn = b_{n-1} + b_{n-2}$, $n \geq 3$, such that $b_1 = 3$ and $b_2 = 6$. Prove that for all $n \in \mathbb{N}$, $n \geq 1$, b_n is divisible by 3. Did you need mathematical or complete induction? Explain why.

3. Let $P(n)$ be the following statement:

$$\forall n \in \mathbb{N}, n \geq 8, \exists\, a, b \in \mathbb{N} \text{ such that } n = 3a + 5b.$$

   Prove that $P(n)$ is true for all $n \geq 8$ natural numbers. Did you need mathematical or complete induction? Explain why.

4. Let the function $f : \mathbb{N} \to \mathbb{N}$ defined by the recursion relation

$$f(1) = 3,\ f(2) = 1,\ f(n) = (f(n-1))^2 + f(n-2) + 1, n \geq 3.$$

   Prove that $f(n)$ is odd for all $n \geq 1$ natural numbers. Did you need mathematical or complete induction? Explain why.

Part V

Functions (Functies)

Chapter 10

Domain, range and codomain (Domein, bereik en codomein)

It is important that you first work your way through Chapter 2 of the book Precalculus. All details from sections 2.1 through 2.8 are absolutely essential, do not leave out any details! No details of the book are repeated here.

Any function is defined by its domain, range and the rule (e.g. $f(x) = \sin(x)$). Always refer to a function as function f or function g, and note that $f(x)$ or $g(x)$ are function **values**, or the rule, but not the functions themselves. The function f means the domain, the range, and the rule, it always comes in a package of three items.

The **domain** of a real-valued real function is a subset of $\mathbb{R}$ for which the function is defined. For instance, the function with rule $g(x) = \frac{1}{x}$ has domain

$$A = (-\infty, 0) \cup (0, +\infty) \text{ or } \mathbb{R} \setminus \{0\}.$$

This means that the function g is not defined for the value $x = 0$, hence the value 0 cannot be part of the domain.

However, the domain can be **restricted**; one could also consider the function

$$h : (0, +\infty) \to (0, +\infty),\ h(x) = \frac{1}{x},$$

which thus becomes a different function even though the function rule stays unchanged.

The **range** of a function is also a subset of $\mathbb{R}$, it is a set that contains ALL the function values that a function takes on. Formally, for a function $f : A \to B$, where A denotes the domain and B denotes the range,

$$B = \{f(x) | x \in A\}.$$

For the function g defined above, the range is the set $(-\infty, 0) \cup (0, +\infty)$, since the function value will equal all real numbers except 0. The formal notation is thus

$$g : (-\infty, 0) \cup (0, +\infty) \to (-\infty, 0) \cup (0, +\infty).$$

For the restriction of g onto $(0, +\infty)$ we obtain $h : (0, +\infty) \to (0, +\infty)$. Do note that restricting the domain will probably effect the range too, only the rule stays unchanged.

Although in our example the particular restriction of g effected the range, this might not always be the case. Consider now the function $k : \mathbb{R} \to [0, +\infty)$, $k(x) = x^2$. We restrict now the function k to the new domain $[0, +\infty)$, then the function becomes $l : [0, +\infty) \to [0, +\infty)$, $l(x) = x^2$, thus the range stayed unchanged.

Why bother about restrictions? That can have many reasons. To name one of them: to make a function bijective because then it becomes invertible. We will see later on that the function k defined above is not bijective, whereas the restriction l is a bijective function.

Although the precise description of a function is through its domain, range, and the rule, we often define functions instead by the domain, **codomain (NL: codomein)** and rule. The **codomain** of a function is a larger set than the range, and the range must be a subset of the codomain. Formally, the codomain is a set C for which it must hold that

$$C \subseteq \mathbb{R}, \text{ and } B \subseteq C,$$

where B denotes the range of a function; or even shorter: $B \subseteq C \subseteq \mathbb{R}$. Any set that fulfills this requirement can be a codomain. For instance, for the function $k : \mathbb{R} \to [0, +\infty)$ $C = \mathbb{R}$ is a possible codomain, since $B = [0, +\infty) \subseteq C = \mathbb{R} \subseteq \mathbb{R}$, then the function can be written as $k : \mathbb{R} \to \mathbb{R}$. This is however not the only possibility, in this case we might have infinitely many possibilities for a codomain, we list only a few: $(-\frac{1}{3}, +\infty)$, $[-2, +\infty)$, $(-\pi, +\infty)$, $(-145, +\infty)$.

The sets $[0, 10000)$ or $(1, +\infty)$ are NOT possible codomains for the function k, because the range of k, $[0, +\infty) \nsubseteq [0, 1000)$ nor $[0, +\infty) \nsubseteq [1, +\infty)$.

Do note the following important remarks:

- The function does not change if we define it with a codomain instead of the range. The function $k : \mathbb{R} \to [0, +\infty)$, $k(x) = x^2$, is the same function as $k : \mathbb{R} \to \mathbb{R}$, $k(x) = x^2$, it is solely **preciser** to define the function with its range.
- The range is the smallest codomain since $B \subseteq B \subseteq \mathbb{R}$ always holds.
- The codomain is thus any set that is "larger or equal" to the range, as long as it contains the range.

So why do we need codomains and why don't we just define all functions with their range? It is because sometimes it is cumbersome to define the range of a function, whereas we do not always need the range anyways. For example, consider the function $\mathcal{N}$ defined by the rule $\mathcal{N}(x) = \frac{1}{n^2+n+1}$, $n \in \mathbb{N}$. What is the precise range of $\mathcal{N}$? Whereas the range is difficult to determine, we can say that $\mathcal{N} : \mathbb{N} \to (0,1]$, but $(0,1]$ is merely a codomain, not the range since for instance $\frac{1}{2}$ is NOT a function value:

$$\forall n \in \mathbb{N} \text{ it holds that } \frac{1}{2} \neq \frac{1}{n^2+n+1}.$$

Formally we could still define the range with setbuilder notation as

$$R = \{a \in \mathbb{R} \,|\, a = \frac{1}{n^2+n+1}, \text{ for all } n \in \mathbb{N}\},$$

but $\mathcal{N} : \mathbb{N} \to (0,1]$, or $\mathcal{N} : \mathbb{N} \to \mathbb{R}$ looks neater as long as we do not explicitly need the range.

Chapter 11

Injective, surjective and bijective functions

11.1 Injective functions (Injectieve functies)

Definition 11.1 (**Injectivity**). A function $f : A \to B$ is called an injective function if for all $x_1, x_2 \in A$ such that $x_1 \neq x_2$ follows that $f(x_1) \neq f(x_2)$.

This definition can be written as a mathematical formula for those who can visulize it better that way:

$$f : A \to B \text{ injective } \overset{\text{DEF}}{\Longleftrightarrow} [\forall x_1, x_2 \in A : x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)]$$

Example Prove that the function f, defined by $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x + 3$, is injective.
Using Definition 11.1 we need to show that $\forall x_1, x_2 \in \mathbb{R}$, sucht that $x_1 \neq x_2$ we must obtain $f(x_1) \neq f(x_2)$, that is, $2x_1 + 3 \neq 2x_2 + 3$.

Although this definition is perfectly fine and mathematically correct, it is rather unnatural (and sometimes difficult) to use in practice.

Think now back to the proposition logic, and call the statement "$x_1 \neq x_2$" statement p, and the statement "$f(x_1) \neq f(x_2)$" statement q. Apply now the Law of Contraposition to $p \Rightarrow q$ to obtain $\neg q \Rightarrow \neg p$, that is,

$$f : A \to B \text{ injective } \overset{\text{ALT. DEF.}}{\Longleftrightarrow} [\forall x_1, x_2 \in A : f(x_1) = f(x_2) \Rightarrow x_1 = x_2]$$

It becomes now much more natural to prove injectivity. Look at the previous example: we need to show that $\forall x_1, x_2 \in \mathbb{R}$, such that $f(x_1) = f(x_2)$ we must obtain that $x_1 = x_2$.

$$\begin{array}{llll} f(x_1) = f(x_2) & \Leftrightarrow & 2x_1 + 3 = 2x_2 + 3 & \text{(definition of } f\text{)} \\ & \Leftrightarrow & 2x_1 = 2x_2 & \text{(subtract 3 from both sides)} \\ & \Leftrightarrow & x_1 = x_2 & \text{(devide by 2 on both sides)} \end{array}$$

This was of course quite a simple example, so let us see a more difficult one: Prove that $h : \mathbb{R} \to (0, \infty)$, $g(x) = e^{2x+3}$ is injective. We will use the same method: show that $\forall x_1, x_2 \in \mathbb{R}$, such that $h(x_1) = h(x_2)$ we must obtain that $x_1 = x_2$.

$$\begin{array}{rcll} h(x_1) = h(x_2) & \Leftrightarrow & e^{2x_1+3} = e^{2x_2+3} & \text{(definition of } h) \\ & \Rightarrow & \ln(e^{2x_1+3}) = \ln(e^{2x_2+3}) & \text{(apply } \ln \text{ on both sides)} \\ & \Rightarrow & 2x_1 + 3 = 2x_2 + 3 & \text{(property of the logarithms)} \\ & \Leftrightarrow & 2x_1 = 2x_2 & \text{(subtract 3 from both sides)} \\ & \Leftrightarrow & x_1 = x_2 & \text{(devide by 2 on both sides)} \end{array}$$

There is one problem at the red arrow: how do we know that the equation $\ln(e^{2x_1+3}) = \ln(e^{2x_2+3})$ has one single solution, resulting in $2x_1 + 3 = 2x_2 + 3$? For this result we actually need to know either that ln itself is an injective function or a result about inverse functions, which we know neither as yet. For this end we will also use a third way to prove injectivity, but we will only prove this result during block 3 at Analysis.

Theorem 11.2. ***Strictly increasing and strictly decreasing functions are injective.***

Knowing the ln function, we already know that it is strictly increasing, but we can also prove it by taking the derivative.

$f : (0, \infty) \to \mathbb{R}$, $f(x) = \ln(x)$ then $f'(x) = \frac{1}{x} > 0$, for all $x \in (0, \infty)$ (note, that we can only take x from the domain of f). Since the derivative is everywhere strictly positive, it follows that f is strictly increasing on its whole domain. By Theorem 11.2 the function f defined by $f(x) = \ln(x)$ is an injective function (injection).

11.2 Surjective functions (Surjective functies)

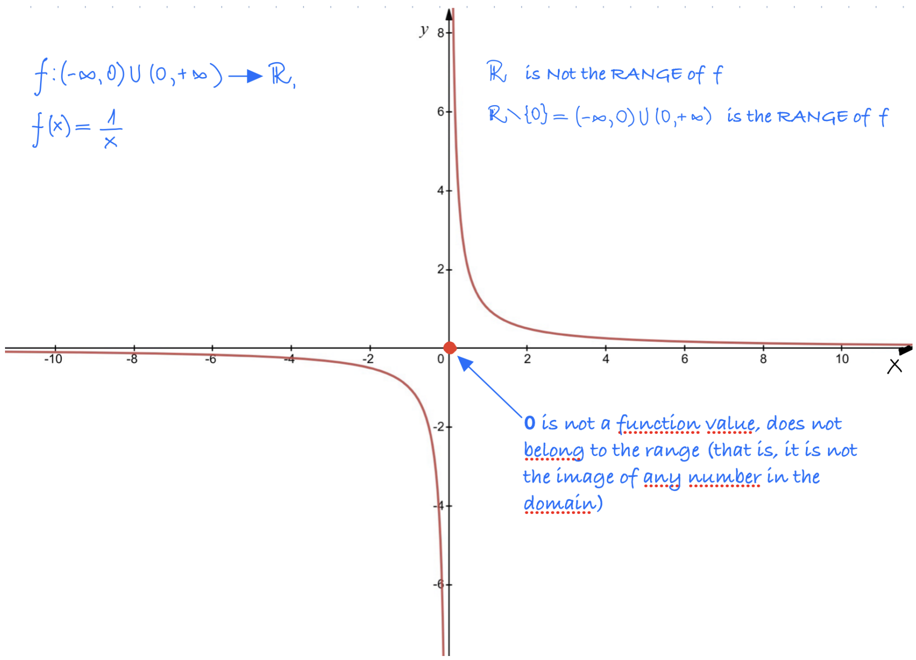
Definition 11.3 (**Surjectivity**). A function $f : A \to B$ is called an surjective function if for all $y \in B$ there exists an element in the domain $x \in A$ such that y is the image of x in f, that is, $f(x) = y$.

With short mathematical notation:

$$f : A \to B \text{ is surjective} \overset{\text{DEF}}{\Longleftrightarrow} \forall y \in B\ \exists x \in A \text{ s.t. } f(x) = y.$$

The definition says it already: all functions $f : AtoB$ for which B is the range (and not just any codomain) is surjective. Hence, every function can be made surjective by restricting its codomain to its range.

For example $f : (-\infty, 0) \cup (0, \infty) \to \mathbb{R}$, $f(x) = \frac{1}{x}$ is NOT a surjective function, since $0 \in \mathbb{R}$ is not the image of any element in the domain (in other words, the function can never return the value 0). $\mathbb{R}$ is here a codomain, not the range. However, we can MAKE this function surjective by considering $f : (-\infty, 0) \cup (0, \infty) \to (-\infty, 0) \cup (0, \infty)$, $f(x) = \frac{1}{x}$, since $(-\infty, 0) \cup (0, \infty)$ is the range of f.



Another example: Is the function $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2$ surjective? The answer is NO, because $\mathbb{R}$ is not the range of the function: none of the negative numbers is the image of any of the numbers from the domain. However, we can make this function surjective by considering it defined with its range: $g : \mathbb{R} \to [0, \infty)$, $g(x) = x^2$.

11.3 Bijective functions (Bijective functies)

A function $f : A \to B$ is bijective if it is injective AND it is surjective simultaneously.

The function $h : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$, $h(x) = \tan(x)$ is a bijective function. In order to prove this statement, we need to prove that this function is injective and it is surjective.

Proof:

Proof of injectivity: Proving injectivity here through the definition is difficult, so we will use Theorem 11.2 instead, and we prove that h is strictly increasing.

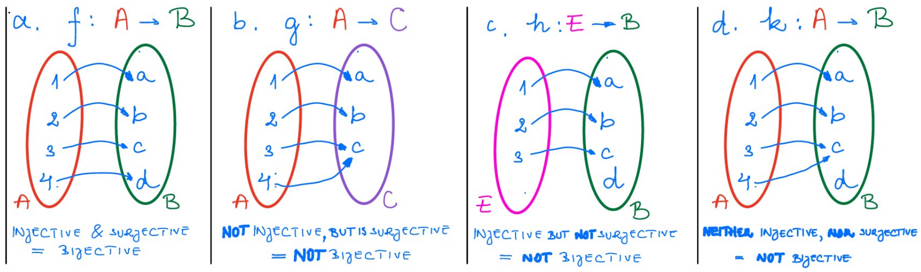
$$h'(x) = \frac{1}{\cos^2(x)} > 0,\ \forall x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

hence h is strictly increasing and consequently it is injective.

Proof of surjectivity: We know from the definition of the tan function that $\mathbb{R}$ is the range of this function, hence h is surjective.

Sice h is both injective and surjective, it is a bijective function.

Before I go on to discuss the composition of functions, let me give some ”baby-examples” of injective and surjective functions.



Chapter 12

Composition of functions (Samengestelde functies)

Let two functions f and g be given

$$f : A \to B \quad \text{and} \quad g : C \to D$$

and assume that $D \subseteq A$.

Composition (Samenstelling)
The composition $f \circ g : C \to B$ is given by

$$(f \circ g)(x) = f(g(x)) \quad \text{for all } x \in C.$$

Example: For $f : [0, \infty) \to [0, \infty), f(x) = \sqrt{x}$ and $g : \mathbb{R} \to [0, \infty), g(x) = x^2 + 4$, we have

$$f \circ g : \mathbb{R} \to [0, \infty), \ (f \circ g)(x) = \sqrt{x^2 + 4}.$$

The importance of $D \subseteq A$

- Consider three functions
  - $f : \mathbb{R} \to (-\infty, 9], \ f(x) = 9 - x^2$;
  - $g : (-3, 3) \to (0, 9], \ g(x) = 9 - x^2$;
  - $h : (0, \infty) \to \mathbb{R}, \ h(x) = \ln x$.
- Then the function $h \circ f$ is *not* well-defined:

  $$h \circ f : \mathbb{R} \to \mathbb{R}, \ h(f(x)) = \ln(9 - x^2)$$

  does not exist for $x \leq -3$ and $x \geq 3$. Problem: $(-\infty, 9] \not\subseteq (0, \infty)$.

- On the contrary, the function

$$(h \circ g) : (-3, 3) \to \mathbb{R},\ h(g(x)) = \ln(9 - x^2)$$

is well-defined.

- We say that g is the *restriction* of f to $(-3, 3)$.

Chapter 13

Inverse functions (Inverse functies)

For a given set A, we define the function $id_A : A \to A,\ id_A(a) = a$ as the *identity function* on A.

Definition 13.1 (**Inverse function (Inverse functie)**). Two functions $f : A \to B$ and $g : B \to A$ are each others inverse functions if

$$f \circ g = id_B \qquad \text{and} \qquad g \circ f = id_A.$$

This means that

$$f(g(b)) = b \text{ for all } b \in B \qquad \text{and} \qquad g(f(a)) = a \text{ for all } a \in A.$$

- A function is called *invertible* if it has an inverse function.
- If a function $f : A \to B$ is invertible, then the inverse function is uniquely defined.
- The inverse function of f is denoted by f^{-1}.
- The graph of f^{-1} is symmetric to the graph of f with resprect to the line $y = x$.

Notice that the domain of f^{-1} is the range of f and the range of f^{-1} is the domain of f:

$$f : A \to B \qquad \text{and} \qquad f^{-1} : B \to A.$$

Warning: Do not confuse $f^{-1}(x)$ with $f(x)^{-1} = \dfrac{1}{f(x)}$.

Examples of inverse functions The function $f^{-1} : \mathbb{R} \to \mathbb{R},\ f^{-1}(x) = 4x+12$ is the inverse of

$$f : \mathbb{R} \to \mathbb{R},\ f(x) = \frac{1}{4}x - 3.$$

Many well-known functions are defined as inverse functions.

- Natural logarithm $\ln : (0,\infty) \to \mathbb{R}$ is the inverse function of
$$\exp : \mathbb{R} \to (0,\infty),\ \exp(x) = e^x.$$
- In trigonometry $\arccos : [-1,1] \to [0,\pi]$ is the inverse function of
$$f : [0,\pi] \to [-1,1],\ f(x) = \cos(x).$$
- Root functions $\mathrm{sqrt} : \mathbb{R}_+ \to \mathbb{R}_+,\ \mathrm{sqrt}(x) = \sqrt{x}$ is the inverse function of
$$\mathrm{square} : \mathbb{R}_+ \to \mathbb{R}_+,\ \mathrm{square}(x) = x^2.$$

Theorem 13.2 (**Bijections are invertible functions**). *Let a function $f : A \to B$ be given. Then f is invertible if and only if it is bijective.*

We will prove this theorem during the Analysis course, we will now only use it: if we can show that a function is bijective then we know that it has an inverse.

Example: Prove that the function $f : \mathbb{R}\setminus\{0\} \to \mathbb{R}\setminus\{0\}$, $f(x) = \frac{1}{x}$ is invertible, without first computing its inverse. Determine afterwards its inverse and check whether you have found the correct inverse function.
$f'(x) = -\dfrac{1}{x^2} < 0$, for all $x \in \mathbb{R}\setminus\{0\}$, that is, f is strictly decreasing on its domain. By theorem 11.2 we can conclude that f is injective. Since $\mathbb{R}\setminus\{0\}$ is the range of the function, f is also surjective, hence f is a bijection. This means that f is invertible, that is, its inverse exists: $f^{-1} : \mathbb{R}\setminus\{0\} \to \mathbb{R}\setminus\{0\}$.

Let us compute the inverse. It needs to hold that $(f \circ f^{-1})(x) = x$ for all $x \in \mathbb{R}\setminus\{0\}$.

$$f(f^{-1}(x)) = x \ \Leftrightarrow\ \frac{1}{f^{-1}(x)} = x \ \Leftrightarrow\ f^{-1}(x) = \frac{1}{x}.$$

Now we check whether this is indeed the correct inverse function by checking whether the identity $(f^{-1} \circ f)(x) = x$ holds for all $x \in \mathbb{R}\setminus\{0\}$.

$$f^{-1}(f(x)) = \frac{1}{\frac{1}{x}} = x,\ \ \forall x \in \mathbb{R}\setminus\{0\},$$

so indeed we have found the correct inverse function.

Quiz: Is the graph of f^{-1} indeed symmetric to the graph of f w.r.t. the line $y = x$?

Part VI

Limits (Limieten)

Chapter 14

Limits of sequences (Limieten van rijen)

Please read now sections 12.1, 12.2 and 12.3 (pages 842 − 867) from your Precalculus book before you proceed reading this part of the syllabus.

A **limit** always implies that there are infinitely many terms (of a sequence) or infinitely many function values (in case of function) in any neighbourhood of the limit.

In the case of a sequence, the variable, that is, the input to the function is clearly the index n (read again the definition of a sequence on page 842 carefully!). Since the indeces are natuaral numbers, for a sequence we can only obtain infinitely many terms as n tends to infinity. Therefore, when we talk about the limit of a sequence we will always solely consider the case when n tends to infinity: $n \to \infty$. The limit notation for a sequence is therefore

$$\lim_{n\to\infty} a_n,$$

meaning that we are investigating the behaviour of the terms a_n, by looking at infinitely many terms of the sequence. We distinguish among four different situations.

1. $\lim_{n\to\infty} a_n = L$, where $L \in \mathbb{R}$,

   that is, the sequence converges to a real number L, when infinitely many terms of the sequence are very close to this number L. In fact, however small interval we choose around L, absolutely all terms of the sequence need to be in that interval after a certain index $n = N$. With this definition we allow the first finitely many terms to be far away from L, but all the rest will need to be concentrated aribitrarily close by around L. You will learn the precise mathematical definition with mathematical notation in block 3 at Analysis, but you may already look it up in your Calculus book (Chapter 2).

Example: Consider the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$, then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0,$$

thus $L = 0$. Let us choose a fairly tight interval around $L = 0$, of length 0.002; then, if the number a is in the interval then $|L-a| = |a| < 0.001$. We want to show that infinitely many terms of a_n are in this interval, except a finitely many first terms. If a_n is in this interval, then $|a_n| < 0.001$, that is, $|\frac{1}{n}| < 0.001$, as you see, this will happen after $N = 1000$, that is after the $1000th$ term infinitely many terms will be concentrated around L in the interval of length 0.002, only the first 1000 terms will be outside of the interval.

As long as L is a real number, the sequence is called **convergent**.

2. $\lim_{n\to\infty} a_n = \infty$

   that is, the terms of the sequence keep growing unboundedly, larger and larger with every single term. In this case, the function (read again the definition of a sequence on page 842 carefully!) $a(n)$ is strictly increasing. The sequence is called **divergent**, and we say that the sequence tends to infinity.

   Example: Consider the sequence $1, 2, 3, \ldots, n, \ldots$, this sequence keeps growing infinitely without bounds towards ∞.

3. $\lim_{n\to\infty} a_n = -\infty$

   which case can be deduced to the second case, since $\lim_{n\to\infty}(-a_n) = \infty$, the sequence is thus also divergent.

   Example: Consider the sequence $-1, -2, -3, \ldots, -n, \ldots$, this sequence keeps growing infinitely without bounds towards $-\infty$. Obviously,

   $\lim_{n\to\infty}(-a_n) = \lim_{n\to\infty}(-(-n)) = \lim_{n\to\infty} n = \infty$.

4. There is no limit at all.

   Consider the sequence $-1, 1, -1, 1, \ldots, (-1)^n, \ldots$. This sequence has no finite or infinite limit, since the terms alternate between -1 and 1: there will be infinitely many terms concentrated around -1 but also infinitely many terms concentrated around 1. If we take an interval of length for instance 1 centered at 1, then however we choose N, not all terms after the Nth term will be concentrated in this interval around 1: take for instance $N = 1000$, then $a_{1000} = 1$, and thus $|a_n - L| = |1-1| = 0 < 0.5$, so a_{1000} is in the interval, but $a_{1001} = -1$ and $|a_n - L| = |-1-1| = 2 > 0.5$, hence a_{1001} falls outside of the interval. In fact, all odd indexed terms will fall outside the interval, 1 thus cannot be a limit. In a similar fashion, -1 cannot be a limit either.

14.1 Limit laws (Limietregels)

The limit laws for sequences and for functions are exactly the same, please compare them with the limit laws on page 906 in your Precalculus book.

Suppose that c is a constant, then:

1. $\lim_{n\to\infty}[a_n + b_n] = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$.

2. $\lim_{n\to\infty}[c \cdot a_n] = c \cdot \lim_{n\to\infty} a_n$.

3. $\lim_{n\to\infty}[a_n \cdot b_n] = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$.

4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$.

Important note to the 4th law: With limits we do allow the denominator to be 0. We are working with limits, which are most of the time different from real numbers. Think of it as a mere symbol, as infinity is not a number but a symbol. The denominator being (the symbol) 0, that is $\lim_{n\to\infty} b_n = 0$, will mean one of the three following cases.

1. If $\lim_{n\to\infty} a_n = L$, $L \in \mathbb{R}$, then
   $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{L}{\mathbf{0}} = \infty$.
2. If $\lim_{n\to\infty} a_n = \infty$, then
   $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{\infty}{\mathbf{0}} = \infty$.
3. If $\lim_{n\to\infty} a_n = 0$, then
   $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{\mathbf{0}}{\mathbf{0}}$ **indeterminate form**.

We will return to Indeterminate forms of limits in section 15.2. Shortly, indeterminate form means that the limit could be any real number, but could also turn out to be infinity, one needs to investigate further to be able to determine the limit.

The limit laws enable us to compute limits.
Example:

$$\lim_{n\to\infty}(3n^2 + n + 1) \overset{\text{lim.law1\&3}}{=} 3[\lim_{n\to\infty} n]^2 + \lim_{n\to\infty} n + 1 = \infty$$

Exercise: Prove by using limit laws 1, 2 and 3 the following laws:

5. $\lim_{n\to\infty}[a_n - b_n] = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$.

6. $\lim_{n\to\infty} c = c$.

Besides the limit laws, which are essential for limit computation, we also need some special limits for more complex limits.

14.2 Special limits for sequences (Speciale limieten voor rijen)

Let $p > 0$ be an arbutrary positive real number, then

1. $\lim_{n\to\infty} n^p = \infty$
2. $\lim_{n\to\infty} \frac{1}{n^p} = 0$
3. $\lim_{n\to\infty} \sqrt[n]{p} = \lim_{n\to\infty} p^{\frac{1}{n}} = 1$
4. $\lim_{n\to\infty} r^n$, with $r \in \mathbb{R}$

   4a. $\lim_{n\to\infty} r^n = \infty$ if $r > 1$

   4b. $\lim_{n\to\infty} r^n = 1$, if $r = 1$

   4c. $\lim_{n\to\infty} r^n$, does not exist if $r = -1$

   4d. $\lim_{n\to\infty} r^n = 0$, if $-1 < r < 1$

14.3 Limits of fractions (Limiten van breuken)

In order to compute the limit of a fraction, we can use the 4th limit law. However, this might not always be sufficient for determining the limit:

$$\lim_{n\to\infty} \frac{2n^2 - 3n + 4}{3n^2 + 4n + 9} \overset{\text{lim.laws}}{=} \frac{2\lim_{n\to\infty} n^2 - 3\lim_{n\to\infty} n + 4}{3\lim_{n\to\infty} n^2 + 4\lim_{n\to\infty} n + 9} = \frac{\infty - \infty + 4}{\infty + \infty + 9}$$

Since both $\frac{\infty}{\infty}$ as $\infty - \infty$ are **indeterminate forms** (see section 15.2) we are simply said stuck. How can we circumvent these indeterminate forms? In this case it is relatively easy: just devide both the numerator and the denominator with the "**dominant term**"; here: the highest power of n, that is, n^2, and only after that use the limit laws:

$$\lim_{n\to\infty} \frac{2n^2 - 3n + 4}{3n^2 + 4n + 9} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{2 - \frac{3}{n} + \frac{4}{n^2}}{3 + \frac{4}{n} + \frac{9}{n^2}},$$

now use the limit laws and obtain

$$\overset{\text{lim.laws}}{=} \frac{\lim_{n\to\infty} 2 - 3\lim_{n\to\infty} \frac{1}{n} + 4\lim_{n\to\infty} \frac{1}{n^2}}{\lim_{n\to\infty} 3 + 4\lim_{n\to\infty} \frac{1}{n} + 9\lim_{n\to\infty} \frac{1}{n^2}} = \frac{2 - 0 + 0}{3 + 0 + 0} = \frac{2}{3}$$

One question remains. In the following fraction

$$\lim_{n\to\infty} \frac{2n^{20} - 3^n + 4}{3n! + 4n + 9},$$

how do we know with absolute security which term is dominant, that is, with which term we should devide? Is the dominant term $n^{20}, 3^n$, or $3n!$?

14.4 Dominant terms: fast growing sequences (Snelle stijgers)

Let us compare the following sequences:

1. $1, 2^{20}, 3^{20}, 4^{20}, \ldots, n^{20}, \ldots$
2. $1, 20, 20^2, 20^3, \ldots, 20^n, \ldots$
3. $1, 2, 6, 24, \ldots, n!, \ldots$
4. $1, 2^2, 3^3, 4^4, \ldots, n^n, \ldots$

The first terms are identical for all four sequences, the second term of the first sequence is clearly the highest of all. But does this mean anything for how the sequence keeps growing as n gets larger and larger, tending to infinity? As it turns out: not at all. If you take a calculator you'll find out that as you select a larger and larger n, the 4th sequence yields far larger terms than the 3d one, which is on its turn larger then the second and then the first. In general, we need to remember the following rules, for $a > 1$ and $p > 0$ real numbers:

$$\lim_{n\to\infty} \frac{n^p}{a^n} = 0, \quad \lim_{n\to\infty} \frac{a^n}{n!} = 0, \text{ and } \lim_{n\to\infty} \frac{n!}{n^n} = 0,$$

which means, that $(n^n)_{n\in\mathbb{N}}$ is the fastest growing sequence, followed by $(n!)_{n\in\mathbb{N}}$, then $(a^n)_{n\in\mathbb{N}}$ and, as last $(n^p)_{n\in\mathbb{N}}$. This answers the question of the previous section: if there is a term in the form n^n then that will be the dominant term, otherwise $n!$, and so on. So the solution of the limit of the fraction from the previous section is:

$$\lim_{n\to\infty} \frac{2n^{20} - 3^n + 4}{3n! + 4n + 9} \cdot \frac{\frac{1}{n!}}{\frac{1}{n!}} = \lim_{n\to\infty} \frac{2\frac{n^{20}}{n!} - \frac{3^n}{n!} + \frac{4}{n!}}{3 + 4\frac{n}{n!} + \frac{9}{n!}} =$$

applying now the limit laws

$$\frac{2\lim_{n\to\infty} \frac{n^{20}}{n!} - \lim_{n\to\infty} \frac{3^n}{n!} + \lim_{n\to\infty} \frac{4}{n!}}{3 + 4\lim_{n\to\infty} \frac{n}{n!} + \lim_{n\to\infty} \frac{9}{n!}} =$$

and applying now the rules of fastest growing sequences

$$= \frac{2 \cdot 0 - 0 + 0}{3 + 4 \cdot 0 + 0} = 0.$$

Exercises

1. $\lim_{n\longrightarrow\infty} \frac{2^n - 2^{-n}}{2^n - 1}$ (answer is 1)
2. $\lim_{n\longrightarrow\infty} \frac{2^{n+1} + 1}{2^n + 1}$ (answer is 2)

3. $\lim_{n\longrightarrow\infty} \frac{2^{3n}-1}{2^{3n}+3^{2n}}$ (answer is 0; Hint: $2^{3n} = 8^n$, $3^{2n} = 9^n$)

4. $\lim_{n\longrightarrow\infty} \frac{n^2+n!}{3^n-n!}$ (answer is -1)

5. $\lim_{n\longrightarrow\infty} \frac{n^n+3n!}{n^n+(3n)!}$ (answer is 0; Hint: investigate $(3n)!$, it is not what it seems...)

6. $\lim_{n\longrightarrow\infty} n^{10}0.9999^n$ (answer is 0)

7. $\lim_{n\longrightarrow\infty} \frac{(n+3)!}{n!+2^n}$ (answer is ∞)

8. $\lim_{n\longrightarrow\infty} \frac{1.002^n}{n^{1000}+1.001^n}$ (answer is ∞)

9. $\lim_{n\longrightarrow\infty}(n^2+3n-7)\left(\frac{1}{2}\right)^{n-1}$ (answer is 0)

10. $\lim_{n\longrightarrow\infty} \sqrt[n]{\frac{8}{9}} = \lim_{n\longrightarrow\infty} \sqrt[n]{\frac{9}{8}}$ (answer is 1; Hint: you cannot bring the limit under the root!).

Chapter 15

Limits of functions (Limieten van functies)

For functions we have a similar type of definition as for sequences, with the obvious difference that instead of the discrete index n, now the input x can be continuous. As a consequence of this distinction, the limit of a function $f : D \to \mathbb{R}$ (with domain $D \subset \mathbb{R}$ and co-domain $\mathbb{R}$)can be described according to the following situations. Let $a \in \mathbb{R}$ (not necessarily in D), then

1. $\lim_{x\to a} f(x) = L$, where $L \in \mathbb{R}$,

   we say that a function is converging to L as x tends to a. An intuitively explanation of the definition of this limit is, that the function values need to be concentrated around the limit L in the following sense: any aritrary neighborhood of f should contain absolutely all function values $f(x)$, for all x from a neighborhood of a. More precisely, if one chooses an interval centered at L arbitrarily, then all the function values $f(x)$ should be contained in this interval for x from a corresponding interval centered around a (but not containing a). a can be either in the domain or just outside the domain. We can compute the limit $\lim_{x\to 3}(x+1) = 4$, $x+1$ is clearly defined in 3, but also the limit $\lim_{x\to 0} \arctan(\frac{1}{x}) = \frac{\pi}{2}$, where 0 is not in the domain of $\arctan(\frac{1}{x})$. (Look up the definition of the arctan function if you do not know this limit by heart!)

2. $\lim_{x\to\pm\infty} f(x) = L$, where $L \in \mathbb{R}$,

   we say that the function is converging to L as x tends to infinity. Again, the function values should be concentrated around L as x gets larger and larger in absolute value. For example, $\lim_{x\to\infty} \arctan(x) = \frac{\pi}{2}$.

3. $\lim_{x\to a} f(x) = \infty$,

   the function diverges to infinity as x tends to a, that is $f(x)$ gets larger and larger as x tends to a. For example, $\lim_{x\to 0} \frac{1}{x^2} = \infty$. Similarly,

4. $\lim_{x\to\infty} f(x) = \infty$,

   the function diverges to infinity as x tends to infinity, that is, the function values $f(x)$ get larger and larger as x gets larger and larger. For example, $\lim_{x\to\infty} \ln(x) = \infty$.

5. $\lim_{x\to a} f(x)$, or $\lim_{x\to\infty} f(x)$ do not exist.

   that is, the function is again divergent as x tends to a or to infinity, as for instance $\lim_{x\to\infty} \sin(x)$ assumes periodic values but there is no one single value the function tends to, as it keeps changing infinitely often. Also, $\lim_{x\to 0} \ln(x)$ does not exist as the ln is not defined for values less then 0, only the one sided limit, $\lim_{x\to 0^+} \ln(x) = -\infty$. Which also leads us to the notion of one sided limits.

6. $\lim_{x\to a^-} f(x)$, **the left hand limit (linker limiet)**, and $\lim_{x\to a^+} f(x)$ **the right hand limit (rechter limiet)**.

   In the case of a left hand limit we only consider function values $f(x)$ for x from an interval to the left of a, and for the right hand limit an interval to the right of a, not containing a.

15.1 Limit laws (Limiet regels)

We have the exact same limit laws for functions as for sequences. Suppose that f and g are two real functions, c is a constant, and a is either a number or infinity, then:

1. $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.

2. $\lim_{x\to a}[c \cdot f(x)] = c \cdot \lim_{x\to a} f(x)$.

3. $\lim_{x\to a}[f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$.

4. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$.

The same important note to the 4th law: With limits we do allow the denominator to be 0. We are working with limits, which are most of the time different from real numbers. Think of it as a mere symbol, as infinity is not a number but a symbol. The denominator being (the symbol) 0, that is $\lim_{x\to a} g(x) = 0$, means again that either we have an indeterminate form, or the limit will diverge to infinity. Let us see now which indeterminate forms we exactly have.

15.2 Indeterminate forms of limits (Onbepaalde vormen van limieten)

Computing a limit as $\lim_{x\to 0}\sin(x) = 0$ is quite easy, because we can just substitute the value 0 in the function and we obtain $\sin(0)$, which is 0. However, as we have already seen it happening with sequences, we need to be very careful when drawing conclusions about certain forms of limits, as

$$\lim_{x\to\infty}\frac{\ln(x)}{x} \overset{\text{lim.laws}}{=} \frac{\lim_{x\to\infty}\ln(x)}{\lim_{x\to\infty}x} = \frac{\infty}{\infty}.$$

Do remember that we are dealing with symbols and not with numbers! As both functions tend to infinity, does this mean that we can just draw the conclusion that the limit is 1? Absolutely not! We have already seen with sequences that some functions grow faster than others, so we need to investigate further. This case is again a special limit, the ln function is the slowest grower of all exponential functions, so this limit is actuall y 0. Other limits also with indeterminate form $\frac{\infty}{\infty}$ could result in infinity, any real number, or even could turn out not to exist. We will also list a number of special limits, but let us first list all indeterminate forms of limits. lease remember all these forms by heart, it is very important that in whichever situation you encounter them you tread carefully and investigate further!

1. $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\infty}{\infty}$

   Example: As we have seen $\lim_{x\to\infty}\frac{\ln(x)}{x} = \frac{\infty}{\infty} = 0$,

   but $\lim_{x\to\infty}\frac{e^x}{x} = \frac{\infty}{\infty} = \infty$, the exponential grows much fater than any power of x,

   or $\lim_{x\to\infty}\frac{2x+1}{x-3} = \frac{\infty}{\infty} = \lim_{x\to\infty}\frac{2+\frac{1}{x}}{1-\frac{3}{x}} \overset{\text{lim.laws}}{=} \frac{2+\lim_{x\to\infty}\frac{1}{x}}{1-\lim_{x\to\infty}\frac{3}{x}} = 2.$

2. $\lim_{x\to a}[f(x)-g(x)] = \infty-\infty$

   Example: $\lim_{x\to\infty}(\ln(x)-x) = \infty-\infty = \lim_{x\to\infty}x(\frac{\ln(x)}{x}-1) \overset{\text{lim.laws}}{=} \lim_{x\to\infty}x\cdot\left(\lim_{x\to\infty}\frac{\ln(x)}{x}-1\right) = -\infty$,

   on the other hand,

   $\lim_{x\to 0}\left(\frac{1}{\sin(x)}-\frac{1}{\tan(x)}\right) = \infty-\infty = \lim_{x\to 0}\left(\frac{1}{\sin(x)}-\frac{\cos(x)}{\sin(x)}\right) = \lim_{x\to 0}\frac{1-\cos(x)}{\sin(x)}\cdot\frac{1+\cos(x)}{1+\cos(x)} = \lim_{x\to 0}\frac{\sin^2(x)}{\sin(x)(1+\cos(x))} \overset{\text{lim.laws}}{=} \frac{\lim_{x\to 0}\sin(x)}{1+\lim_{x\to 0}\cos(x)} = \frac{0}{2} = 0.$

3. $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{0}{0}$

   Example: See the previous one,

   $\lim_{x\to 0}\frac{1-\cos(x)}{\sin(x)} = \frac{0}{0} = \lim_{x\to 0}\frac{\sin^2(x)}{\sin(x)(1+\cos(x))} = \frac{0}{2} = 0,$

and the limit

$\lim_{x\to 0}\frac{\sin(x)}{x} = 1$, is a special limit and will be explained in the next section.

4. $\lim_{x\to a}[f(x)\cdot g(x)] = 0\cdot\infty$

   Example: $\lim_{x\to 0^+} x\ln(x) = 0\cdot(-\infty) = 0$ we will show the computation of this limit by substitution in the next section.

   $\lim_{x\to\infty}\frac{1}{x}e^x = 0\cdot\infty = \lim_{x\to\infty}\frac{e^x}{x} = \infty$

5. $\lim_{x\to a}[f(x)]^{g(x)} = 0^0$

6. $\lim_{x\to a}[f(x)]^{g(x)} = \infty^0$

7. $\lim_{x\to a}[f(x)]^{g(x)} = 1^\infty$

Since we cannot compute all limits with the techniques we learn during this course, we will only see examples for the last three indeterminate forms during the course Analysis.

15.3 Special limits (Speciale limieten)

The special limits which we state here without proof, the last three will be explained during Week 5, by means of the definition of the derivative.

Consider the real numbers a, p and q such that $a > 1$ and $q > 0$. The following relations hold true.

1. $\lim_{x\to\infty}\frac{x^p}{a^x} = 0$,

   that is a^x grows faster than x^p no matter how large p is; please compare this result with fast growing sequences.

2. $\lim_{x\to\infty}\frac{\log_a x}{x^q} = 0$.

   Please note that q is only required to be positive, which means that $\sqrt[n]{x}$ grows faster than $\log_a x$, no matter how large n is.

3. $\lim_{x\to 0}\frac{\sin(x)}{x} = 1$.

4. $\lim_{x\to 0}\frac{e^x-1}{x} = 1$.

5. $\lim_{x\to 1}\frac{\ln(x)}{x-1} = 1$, or equivalently, $\lim_{x\to 0}\frac{\ln(1+x)}{x} = 1$.

These last three special limits are nothing else than the definition of the derivative of the respective functions at the point 0 - or the point 1 in the case of the ln function.

15.4 Solving limits by substitution (Limieten met substitutie)

Please note that during this course you are not allowed to use the Rule of L'Hospital, you will need to use the special limits and substitution - also at the exam! (Do not worry if you do not know what the the Rule of L'Hospital is, you will learn it in block 3 during the course Analysis).

How can we compute a limit as $\lim_{x\to 0^+} x\ln(x)$? We will use a variable substitution. In this case the logarithm causes us the headaches, that is what we aim to get rid of. We will define a new variable $y = -\ln(x)$ and we aim to rewrite the original limit in terms of y instead of x. The new variable y is actually a function of x, we can see it as $y = y(x) = -\ln(x)$. Now, as in the original limit, the variable x tends to 0^+, we need to find out where y tends to in our new limit: $\lim_{y\to ?}$. In order to find this out, we need to know the behaviour of the function $y(x)$ as x tends to 0^+, or with other words, we need to compute $\lim_{x\to 0^+} y(x)$. Since $\lim_{x\to 0^+} y(x) = \lim_{x\to 0^+} -\ln(x) = -(-\infty) = \infty$, we know that we will have to consider the new limit as $y \to \infty$.

There is one more problem we need to solve: in the original limit, there is also a term x besides the $\ln(x)$. There is one important rule you should always remember: never mix the variable from the original limit with the newly defined variable! Our newly defined variable here is y, that means there should be only y in the new limit, and absolutely no x. This means that we need to express x in terms of y and substitute that instead of x in the new limit. In our case, since $y = -\ln(x)$, we have solve this equation for x and obtain $x = e^{-y}$. Summarizing,

$$\lim_{x\to 0^+} x\ln(x) = \lim_{y\to\infty} e^{-y}y = \lim_{y\to\infty} \frac{y}{e^y} = 0,$$

since we know that the exponencial grows much faster than any power of the variable.

Let us consider one more example. Compute the limit $\lim_{x\to 0} \frac{\arcsin(x)}{\arctan(x)}$.
The first thing we should look for, is what kind of tools we could use for solving this limit. This limit has the indeterminate form $\frac{0}{0}$, which means we need to work further, we cannot conclude anything yet. Although this limit might seem impossible at first sight, we have quite a lot of tools we might use. The two functions in this limit are trigonometric functions (goniometrische functies), so we should think of the one special limit we have for trigonometric functions, $\lim_{x\to 0} \frac{\sin(x)}{x}$. But how do we get to this special limit? Well, we can use substitution: $y = \arcsin(x)$ then $x = sin(y)$, which is exactly what we wanted. However, if we use the same substitution for the $\arctan(x)$ function, we obtain $\arctan(\sin(y))$, which is plainly scary. So how could we avoid this complication? This is exactly why we need the limit laws! With the fourth limit law, we first separate the limit into two limits, then we use different substitutions for the two limits. That is, $y = \arcsin(x)$, $z = \arctan(x)$, $y \to 0$ and $z \to 0$ as $x \to 0$,

yielding

$$\lim_{x\to 0}\frac{\arcsin(x)}{\arctan(x)} \overset{\text{lim.laws}}{=} \frac{\lim_{x\to 0}\arcsin(x)}{\lim_{x\to 0}\arctan(x)} = \frac{\lim_{y\to 0} y}{\lim_{z\to 0} z} = \frac{0}{0},$$

so we are stuck again. We did not get the sin in the picture as we wanted. We could get the sin into the limit, if we also had a free x in the expression. We will perform a trick: when we need an x, we create and x by multiplying and deviding by x, as $\frac{x}{x} = 1$, so it will not change the initial limit.

$$\lim_{x\to 0}\frac{\arcsin(x)}{\arctan(x)}\cdot\frac{x}{x} \overset{\text{lim.laws}}{=} \frac{\lim_{x\to 0}\frac{\arcsin(x)}{x}}{\lim_{x\to 0}\frac{\arctan(x)}{x}} = \frac{\lim_{y\to 0}\frac{y}{\sin(y)}}{\lim_{z\to 0}\frac{z}{\tan(z)}}.$$

The limit in the numerator becomes

$$\lim_{y\to 0}\frac{y}{\sin(y)} \overset{\text{lim.laws}}{=} \frac{1}{\lim_{y\to 0}\frac{\sin(y)}{y}} \overset{\text{sp.lim.}}{=} \frac{1}{1} = 1.$$

The limit in the denominator needs slightly more effort:

$$\lim_{z\to 0}\frac{z}{\tan(z)} = \lim_{z\to 0}\frac{z}{\frac{\sin(z)}{\cos(z)}} = \lim_{z\to 0}\frac{z}{\sin(z)}\cos(z),$$

splitting the limit now according to the limit laws, we obtain

$$\overset{\text{lim.laws}}{=} \frac{1}{\lim_{z\to 0}\frac{\sin(z)}{z}}\lim_{z\to 0}\cos(z) \overset{\text{sp.lim.}}{=} \frac{1}{1}\cdot 1 = 1.$$

In conclusion, the original limit equals 1.

Exercises

1. $$\lim_{x\longrightarrow 0}\frac{2x}{\sqrt[3]{x+27}-3} \qquad \text{(answer: 54)}$$

2. $$\lim_{x\longrightarrow 8}\frac{\sqrt[3]{x}-2}{x-8} \qquad \text{(answer: 1/12)}$$

3. $$\lim_{x\longrightarrow 0}\frac{\sin(5x)}{3x} \qquad \text{(answer: 5/3)}$$

4. $$\lim_{x\longrightarrow 0}\frac{\cos(2x)-1}{\cos(x)-1} \qquad \text{(answer: 4)}$$

5. $$\lim_{x\longrightarrow \frac{\pi}{2}}\frac{tan(2x)}{x-\frac{\pi}{2}} \qquad \text{(answer: 2)}$$

6.
$$\lim_{x \longrightarrow 1} \frac{\ln(x^2+1) - \ln(2)}{x-1} \qquad \text{(answer: 1)}$$

7.
$$\lim_{x \longrightarrow \frac{\pi}{2}^+} \frac{\cos(x)}{1 - \sin(x)} \qquad \text{(answer: } -\infty\text{)}$$

8.
$$\lim_{x \longrightarrow 0} \frac{e^{2x} - 1}{\sin(x)} \qquad \text{(answer: 2)}$$

9.
$$\lim_{x \longrightarrow 0} \frac{\tan(4x)}{x + \sin(2x)} \qquad \text{(answer: 4/3)}$$

10.
$$\lim_{x \longrightarrow 0} \frac{\tan(3x)}{\sin(2x)} \qquad \text{(answer: 3/2)}$$

Part VII

Differentiation and the Lagrange multiplier method (Differentiëren an de Lagrange multiplicatormethode)

Chapter 16

Standard derivatives of functions of a single variable (Standaardafgeleiden van functies met één variabele)

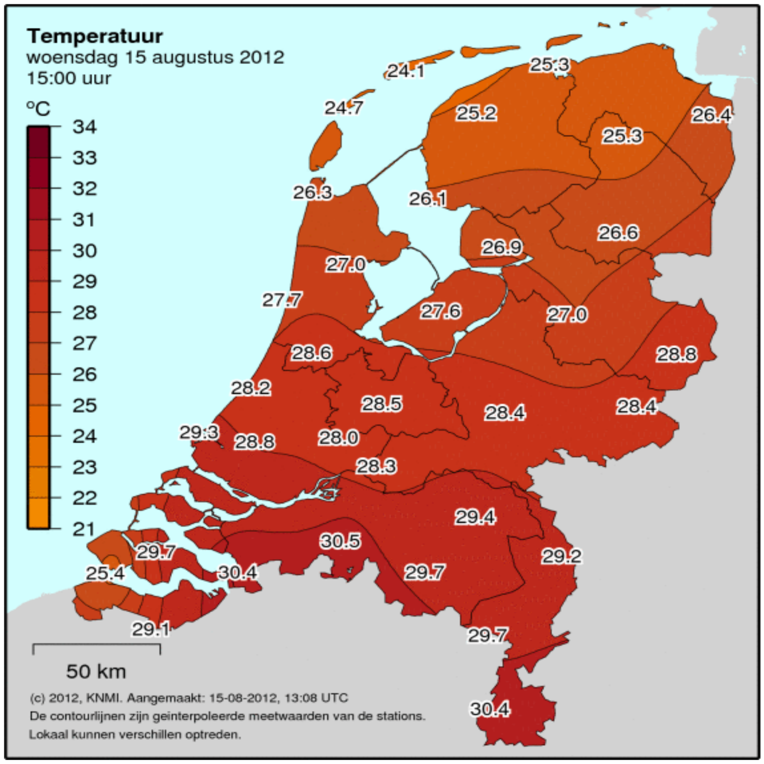
For the definition of the derivative for functions of a single variable, please see your Precalculus book, Chapter 13.3. The standard derivative formulae which you need to know by heart, are listed here:

$$
\begin{aligned}
\frac{d}{dx}\left(x^{p}\right) &= px^{p-1}, \quad p \in \mathbb{R} \\
\frac{d}{dx}\left(a^{x}\right) &= a^{x}\ln a, \quad a>0 \\
\frac{d}{dx}\left(e^{x}\right) &= e^{x} \\
\frac{d}{dx}\left(\log_a x\right) &= \frac{1}{x\ln a}, \quad a>0, a\neq 1 \\
\frac{d}{dx}\left(\ln x\right) &= \frac{1}{x} \\
\frac{d}{dx}\left(\sin x\right) &= \cos x \\
\frac{d}{dx}\left(\cos x\right) &= -\sin x \\
\frac{d}{dx}\left(\tan x\right) &= \frac{1}{\cos^2 x} \\
\frac{d}{dx}\left(\arcsin x\right) &= \frac{1}{\sqrt{1-x^2}} \\
\frac{d}{dx}\left(\arccos x\right) &= -\frac{1}{\sqrt{1-x^2}} \\
\frac{d}{dx}\left(\arctan x\right) &= \frac{1}{1+x^2}
\end{aligned}
$$

Chapter 17

Functions of two variables (Fucties van twee variabelen)

Consider the temperature map of The Netherlands. Such a map is called a contour map or contour diagram of a function. In this case the temperature is a function of two variables: longitude and latitude (north-south and east-west). The continuous lines indicate the regions with equal temperatures, these lines are called *isothermals*. Do not confuse them with the dotted lines indicating the provinces of the Netherlands.



For this function there is no closed-form formula, but there are plenty functions which can be defined by such a closed form formula. For example

$$z = f(x, y) = 6 - 3x - 2y.$$

As we have seen at the course of Week 3 about functions, we need to specify the domain and range of the function. For the example above the domain is $D = \{(x, y)|f(x, y) \text{ well defined } \} = \mathbb{R}^2$, and its range is $R = \{z = f(x, y)|(x, y) \in D\} = \mathbb{R}$.

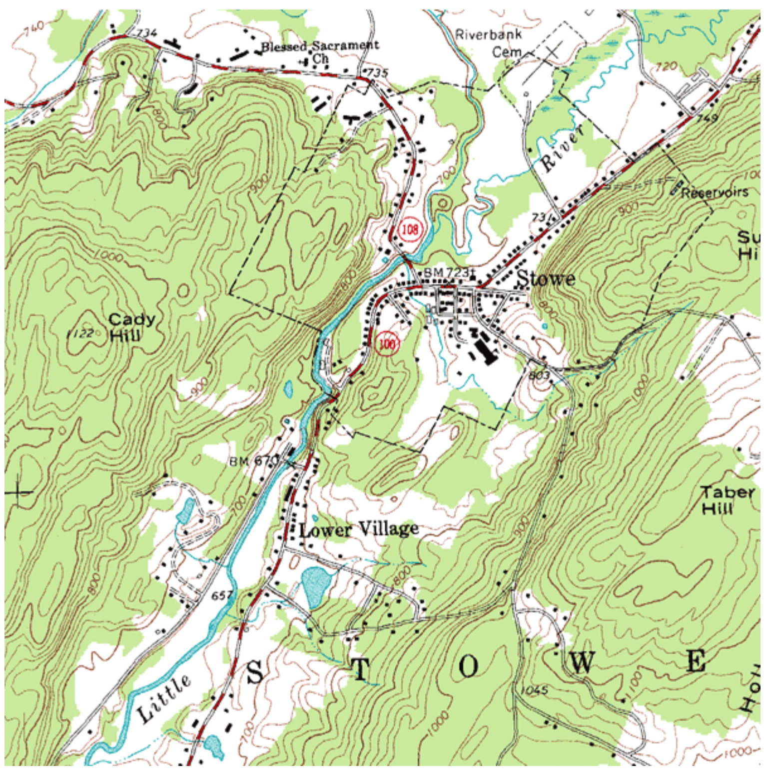
17.1 Level curves or contour lines (Hoogtelijnen of level kromme)

A contour map is also a way of representing a function, just like arrow diagrams and graphs. By why would one bother with contour maps and not stick with a graph? In most of the cases the graph of $z = f(x, y)$ is a three dimensional surface, for example a mountain landscape with multiple peaks, saddle points, planes. That is, it can be very hard and time consuming to construct its graph. Contrary to such a graph, the contour lines allow an easy 2-dimensional representation of this complex 3-dimensional graph: points of the same value of elevation form *contour lines* or *level curves*, that is for any $k \in R$, with R the range of f,

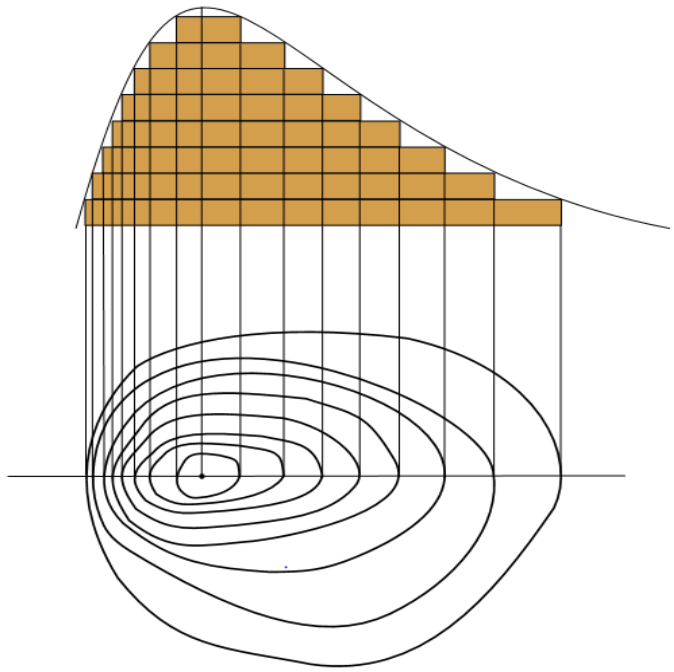
$$\{(x, y) \in D | f(x, y) = k\} \text{ is the level curve of level } k.$$

A collection of level curves of f we call a contour map. As we will see later on, with the help of a simple contour map we can analyze f quite extensively, given that the level curves are chosen cleverly. Needless to mention that one never plots a contour map for every $k \in R$ (from the range of f).

Exercise 1.: *What is the contour map obtained for all $k \in R$?*



You can see here the level curves $f(x, y) = k$.



The closer the level curves are together the steeper the surface is. One can estimate function values $f(x, y)$ from the contour map.

Graphs of linear functions in 2-dimensions are planes. If a plane has slope m in the x direction and slope n in the y direction and passes through the point (x_0, y_0, z_0) then its equation is

$$f(x, y) = z_0 + m(x - x_0) + n(y - y_0),$$

or, with $c = z_0 - mx_0 - ny_0$

$$f(x, y) = c + mx + ny.$$

Exercise 2: *What do the contours of a linear function look like?*

17.2 Functions of three variables (Fucties van drie variabelen)

The function f assigns to each ordered triple $(x, y, z) \in D$ a unique real number denoted by $f(x, y, z)$, where D is called as before the domain of f. For example, the domain of $f(x, y, z) = \ln(z - y) + xy \sin z$ is $D = \{(x, y, z) \in \mathbb{R}^3 | z > y\}$.

Take now the temperature example before. The temperature T at a location in the Netherlands depends on the longitude x, the latitude y and on time t, so we could write $T = f(x, y, t)$.

Visualization of a function of three variables by a graph is, to say the least, difficult, one would need a $4D$-space for the purpose. However, these functions have *level surfaces* in $3D$ (instead of the level curves for functions with 2 variables), defined by

$$\{(x, y, z) \in D | f(x, y, z) = k\} \subset \mathbb{R}^3 \text{ the level surface of level } k,$$

which give a $3D$ representation of these functions. For example, the level surfaces of the function $f(x, y, z) = x^2 + y^2 + z^2$ are concentric spheres with radius $\sqrt{k}$.

17.2.1 Distance (Afstand)

In $2D$-space the distance between (x, y) and (a, b) is given by

$$d = \sqrt{(x-a)^2 + (y-b)^2}.$$

In $3D$-space the distance between (x, y, z) and (a, b, c) is (draw a cube and use Pythagoras' theorem):

$$d = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$$

17.3 Limit and continuity informally (Informele introductie van limiet en continuïteit)

The function $f(x, y)$ has a limit L at the point (a, b), written

$$\lim_{(x,y) \longrightarrow (a,b)} f(x, y) = L,$$

if $f(x, y)$ is arbitrarily close to L whenever (x, y) is sufficiently close to (a, b) but not 0. A function f is continuous at the point (a, b) if

$$\lim_{(x,y) \longrightarrow (a,b)} f(x, y) = f(a, b).$$

Chapter 18

Partial derivatives (Partiële afgeleiden)

Image the following temperature function $T(x, y)$, given by the table

$y \setminus x$	0	1	2	3	4	5
0	85	90	110	135	155	180
1	100	110	120	145	190	170
2	125	128	135	160	175	160
3	120	135	155	160	160	150

We want to examine how T varies near the point $(x, y) = (2, 1)$. First we examine $u(x) := T(x, 1)$, the cross-section along the line $y = 1$. The meaning of $u'(2)$ is the following: *the rate of change of the temperature T in the x-direction at the point* $(2, 1)$*, **keeping** y **fixed*** (and $y = 1$ in this case). We denote this rate of change by $T_x(2, 1)$

$$T_x(2,1) = u'(2) = \lim_{h \longrightarrow 0} \frac{u(2+h) - u(2)}{h} = \lim_{h \longrightarrow 0} \frac{T(2+h,1) - T(2,1)}{h}.$$

$T_x(2, 1)$ is the *partial derivative of T with respect to x at the point* $(2, 1)$.

In order to accomplish our goal and estimate $T_x(2, 1)$, taking $h = 1$ and substituting it in the formula above: $T_x(2, 1) \approx 25$ (degree per meter). Since this value is positive, the temperature is increasing as we move past $(2, 1)$ in the direction of increasing x (i.e. horizontally from left to right).

Now let us estimate the rate of change in the y-direction. We take the cross-section of T with $x = 2$, that is $v(y) = T(2, y)$ and we estimate $v'(1)$. Taking again $h = 1$ we obtain the approximation $T_y(2, 1) = v'(1) \approx -15$ degrees per meter (temperature decreases as y increases).

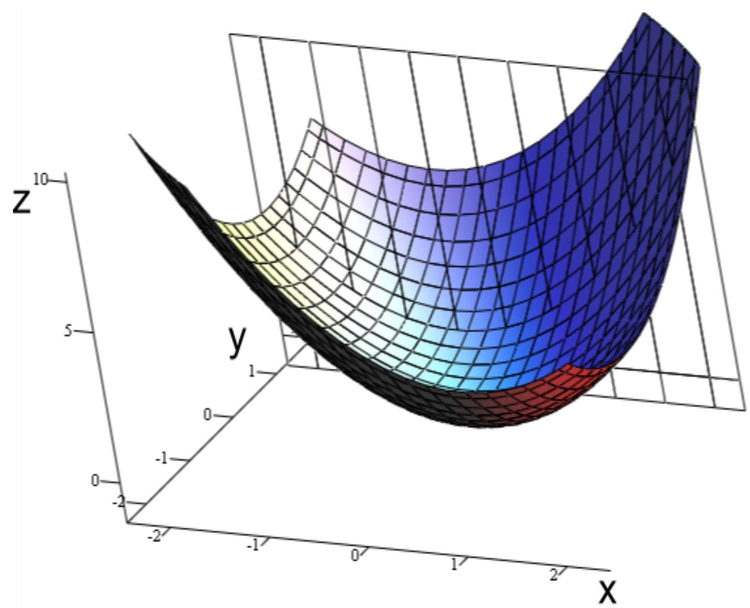
Formal definition of partial derivatives:

For $f(x, y)$ for all points at which the limits below exist and are finite, *the* *partial derivatives at the point* (a, b) are given by

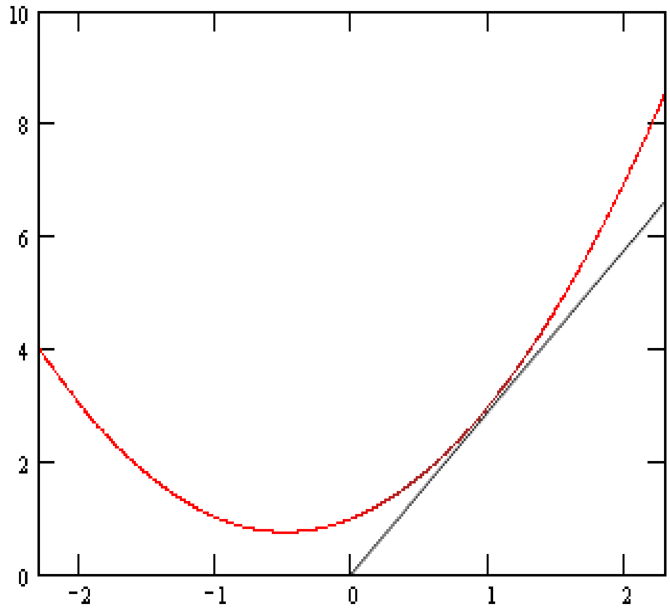
$$f_x(a, b) = \lim_{h \longrightarrow 0} \frac{f(a+h, b) - f(a, b)}{h},$$

$$f_y(a, b) = \lim_{h \longrightarrow 0} \frac{f(a, b+h) - f(a, b)}{h}.$$

Visualizing partial derivatives on a graph: Suppose $f(x, y) = x^2 + xy + y^2$ which defines the surface in the figure.



The graph of the single variable function $f(x, b)$ is the curve where the vertical plane $y = b$ cuts the graph of $f(x, y)$. Thus $f_x(a, b)$ is the slope of the tangent line to this curve at $x = a$, as illustrated on the $2D$ graph.



Similar reasoning holds for $f_y(a, b)$.

18.1 Exercise 3: Estimating partial derivatives from contour diagrams

Solve now exercise 10 *of Chapter* 14.3*, page* 924*.*

Hint: $f_x(2, 1)$ is the rate of change of f at $(2, 1)$ in the x direction. Looking at the contour map you find that the point $(2, 1)$ lies left of the 10 contour (almost on it). The change in x which brings us to the next contour 12 is approximately 0.6.
Pay careful attention to the sign of the partial derivatives!

18.2 Tangent planes (Raakvlakken)

Suppose f has continuous partial derivatives, then the equation of the tangent plane to the surface $z = f(x, y)$ at (a, b) is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Thus the **local linearization** of f near (a, b) is the tangent plane approximation

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b),$$

for (x, y) near (a, b).

For a three variable function we have similarly, the local linearization of f near (a, b, c) is

$$f(x, y, z) \approx f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c),$$

for (x, y, z) near (a, b, c).

18.3 The differential (De differentiaal)

Rewrite the tangent plane approximation with the following notation:
$\triangle\ f = f(x, y) - f(a, b)$, $\triangle\ x = x - a$, and $\triangle\ y = y - b$:

$$\triangle\ f \approx f_x(a, b) \triangle\ x + f_y(a, b) \triangle\ y.$$

Denoting now with dx instead of $\triangle\ x$ and dy instead of $\triangle\ y$ the infinitesimal change in x and y respectively we obtain

$$df = f_x(a, b)dx + f_y(a, b)dy,$$

the ***differential*** of f at the point (a, b); the differential at a general point is written

$$df = f_x dx + f_y dy.$$

Example: $f = x^2e^{5y}$, then $f_x(x, y) = 2xe^{5y}$, $f_y(x, y) = 5x^2e^{5y}$ and

$$df = 2xe^{5y}dx + 5x^2e^{5y}dy.$$

18.4 Gradients and directional derivatives (Gradiënten en richtingsafgeleiden)

Take another look at the temperature map of the NL: Rotterdam is at the point above the Maas withe temperature 28.8 degree. The partial derivative T_x at Rotterdam is the rate of change of temperature w.r.t. distance if we travel east from Rotterdam in a horizontal direction, and T_y is the rate of change of temperature when we travel north. But what is the rate of change if we want to travel north west to Monster for instance? The tool we need for answering this question is called the *directional derivative*.

The partial derivatives f_x and f_y represent the rate of change of f in the x- and y-directions respectively, that is, in the direction of the unit vectors i and j. Suppose we now want to find the rate of change in the direction of the arbitrary unit vector $\mathbf{u} =< u_1, u_2 >$.

If we travel in northwesterly direction from Rotterdam, the unit vector directed to the northwest is $\mathbf{u} = (-\mathbf{i}+\mathbf{j})/\sqrt{2}$. Let's say that following this direction for 30km we'll be arriving in Monster. Monster lies approximately on the level curve of 28 degrees. We want to know the rate of change of temperature. Then

$$D_u T = \frac{28.0 - 28.8}{30} \approx -0.027 \text{ C/km},$$

meaning that by traveling in northwesterly direction, on average, the temperature drops 0.027 degrees each kilometer we travel towards Monster.

Now, in general, the directional derivative of f at (a, b) in the direction of the unit vector $\mathbf{u} =< u_1, u_2 >$ is

$$D_u f(a, b) = \lim_{h \longrightarrow 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h},$$

if f is differentiable we have

$$D_u f(x, y) = f_x(x, y)u_1 + f_y(x, y)u_2;$$

Now, if the unit vector $\mathbf{u}$ makes an angle θ with the positive x-axis, then $\mathbf{u} =< \cos\theta, \sin\theta >$ and

$$D_u f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta.$$

Notice that these identities are dot products, hence we can use the notation

$$D_u f(x, y) =< f_x(x, y), f_y(x, y) >< u_1, u_2 >=< f_x(x, y), f_y(x, y) > \cdot \mathbf{u}.$$

The first vector is called the **gradient** of f:

$$\text{grad} f = \nabla f(x, y) =< f_x(x, y), f_y(x, y) >= f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j},$$

$$D_u f(x, y) = \nabla f(x, y) \cdot \mathbf{u}.$$

Read Example 4 from Chapter 14.6, page 950, and pay careful attention to figure 6. Find the directional derivative of $f(x, y) = x^2y^3 - 4y$ at the point $(2, -1)$ in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$.

From the last identity we have

$$D_u f(x, y) = \nabla f(x, y) \cdot \mathbf{u} = \|\nabla f(x, y)\|\|\mathbf{u}\| \cos\theta = \|\nabla f(x, y)\| \cos\theta.$$

We can see that the maximum of $D_u f$ over θ occurs when $\cos\theta = 1$, that is, $\theta = 0$; equivalently the minimum occurs for $\theta = \pi$. The interpretation is as follows:

- **The gradient vector points in the direction of the greatest rate of change at a point and the magnitude is that rate of change.** It is large when the contours are close together and small when they are far apart.

- **The gradient vector is perpendicular to (also called orthogonal to or at right angles with) the contour of f through** (a, b).

Exercise 4: Explain why the gradient vector is large when the contours are close together and conversely, it is small when the contours are far apart?

18.5 Detecting local extrema (Locale extreme waarden)

From the properties of the gradient vector above we conclude the following: ***Points where the gradient is*** $\overrightarrow{0}$ ***are called*** **critical points** ***of the function. In fact, the gradient must be zero at a local minimum or local maximum (given that point is not on the boundary)***.

An informal, intuitive proof by reductio ad absurdum:
Suppose that f has a local maximum at a point P_0, which is *not on the boundary* of the domain, such that $\nabla f(P_0) \neq \overrightarrow{0}$. Since the gradient always poins in the direction of the greatest rate of change, then we can increase f by moving in the direction of $\nabla f(P_0)$ but that is a contradiction since $f(P_0)$ is a local maximum (we cannot increase f in a neighborhood of P_0). Hence, if $\nabla f(P_0)$ is defined, it must be zero.
In exactly the same fashion, suppose that f has a local minimum at a point P_0, which is *not on the boundary* of the domain, such that $\nabla f(P_0) \neq \overrightarrow{0}$. From the properties of the gradient we saw that we could decrease f by moving in the direction $-\nabla f(P_0)$ (for $\theta = \pi$), and that is again a contradiction since $f(P_0)$ is a local minimum (we cannot decrease f in a neighborhood of P_0).

To find critical points, we set $\nabla f = f_x \overrightarrow{\mathbf{i}} + f_y \overrightarrow{\mathbf{j}} = \overrightarrow{0}$, that is, we search for points (a, b) such that $f_x(a, b) = 0$ **and** $f_y(a, b) = 0$.

There is also a second derivative test for functions of several variables which helps one decide whether a critical point one has found is a minimum, maximum, saddle point or none of the above. This will however will only be a topic of the Calculus 2 course later on.

Chapter 19

Constrained optimization with Lagrange multipliers (Optimizatie met beperkende voorwaarden)

19.1 Geometrical approach (Meetkundige interpretatie)

Suppose we want to maximize the production function

$$f(x, y) = x^{2/3}y^{1/3}$$

of two raw materials x and y, with price p_1 and p_2 per unit, respectively. One might find logical to increase x and y to infinity, however, we also have a budget constrain c, that is, we only have an amount of money c we can spend on buying a certain amount of x and y. Formulating this mathematically we have

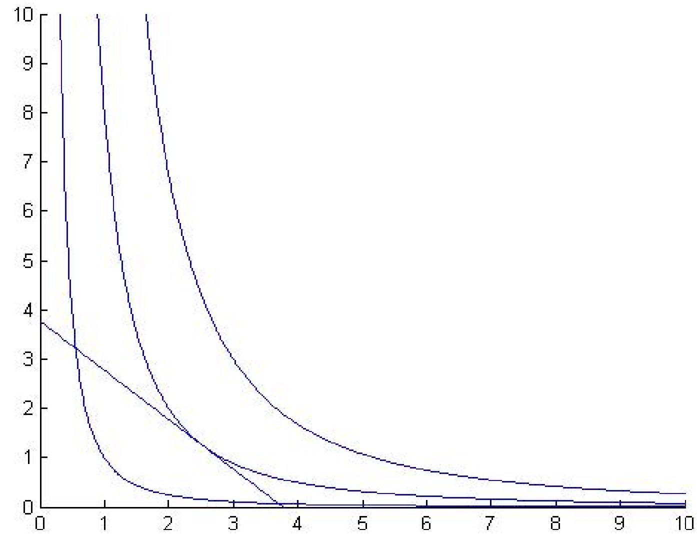
$$g(x, y) = p_1x + p_2y \leq c.$$

For the sake of keeping this numerical example simple, consider $p_1 = p_2 = 1$ and $c = 3.78$, then we have the optimization problem

$$\text{maximize } f(x, y) = x^{2/3}y^{1/3},$$

$$\text{subject to } g(x, y) = x + y \leq 3,78.$$

We have to find the largest possible value of f which satisfies the budget constraint. Plotting some of the contours of f (for $k = 1$, $k = 2$ and $k = 3$) and the line $g = 3,78$ we observe that all (x, y) which lie *under* or *on* the line $g = 3.78$ satisfy the budget constraint.



However, for values which lie completely under the budget constraint-line we can always increase f such that it will still be below the line. We suspect therefore that the optimal solution must lie ON the line $g = 3.78$. We are getting close. Suppose now we choose the (x, y) for which the contour $f = 1$ intersects $g = 3.78$, name it Q. If we move to right from this point ON the line $g = 3.78$, then f increases, this means that Q is not an optimum. Following this reasoning for any point on $g = 3.78$ we see that the optimum must be the point where $g = 3.78$ only touches the contour of f, at this point the f-value is seen to be 2, name that point P. In P $g = 3.78$ is a tangent to the contour $f = 2$, this means that the gradient vector $\nabla f(P)$ and the normal vector of g in P - which is nothing else than $\nabla g(P)$ - are parallel. Thus, for some scalar λ, called the Lagrange multiplier,

$$\nabla f = \lambda \nabla g.$$

Writing this out gives the vector equation:

$$\left(\frac{2}{3}x^{-1/3}y^{1/3}\right)\overrightarrow{\mathbf{i}} + \left(\frac{1}{3}x^{2/3}y^{-2/3}\right)\overrightarrow{\mathbf{j}} = \lambda\overrightarrow{\mathbf{i}} + \lambda\overrightarrow{\mathbf{j}},$$

which yields two equations:

$$\frac{2}{3}x^{-1/3}y^{1/3} = \lambda \quad \text{and} \quad \frac{1}{3}x^{2/3}y^{-2/3} = \lambda.$$

Eliminating λ from these equations gives $2y = x$. Substituting this into the budget constraint $g = 3.78$ yields $x = 2.52$ and $y = 1.26$, and $f(2.52, 1.26) \approx 2$.

Lagrange multiplier method in general, for functions of three variables:

To find the maximum or minimum values of $f(x,y,z)$ (assuming they exist) subject to the constraint $g(x,y,z)=c$ (and assuming that $\nabla g \neq \overrightarrow{0}$ on the surface $g(x,y,z)=c$):

1. Find all values x, y, z and λ such that

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z),$$

and

$$g(x,y,z) = c.$$

2. Evaluate f at all these points (x,y,z) from the first step. The largest of these value is the maximum of f and the smallest is the minimum of f (subject to the constraint).

19.2 Recipe for the Lagrange multiplier method (Recept voor de Lagrange multiplicatoormethode)

19.2.1 Problems with one constraint (met één beperkende voorwaarde):

step 1. Problem:

$$\begin{cases} \qquad\qquad f(x,y) & \longrightarrow \max(\text{or } \min) \\ \text{subject to: } g(x,y) & = 0. \end{cases}$$

step 2. Compute the following partial derivatives:

$$\frac{\partial f}{\partial x}(x,y); \quad \frac{\partial f}{\partial y}(x,y); \quad \frac{\partial g}{\partial x}(x,y); \quad \frac{\partial g}{\partial y}(x,y).$$

step 3. Solve the following system of equations:

$$\begin{cases} \frac{\partial f}{\partial x} - \lambda \cdot \frac{\partial g}{\partial x} &= 0 \\ \frac{\partial f}{\partial y} - \lambda \cdot \frac{\partial g}{\partial y} &= 0 \\ g(x,y) &= 0 \end{cases} \tag{19.1}$$

step 4.

i) If there are more than just one solutions to (19.1) then evaluate the function in all of the solutions and compare these function values to each other: the largest value is the maximum and the least the minimum. (Careful: you need to compare the function values in these points NOT the solutions themselves.)

ii) If there is only one single solution to (19.1) then evaluate the function in this point and compare it to ANY other function value *satisfying the restriction* g.

19.2.2 Problems with two constraints (met twee beperkende voorwaarden):

step 1. Problem:

$$\begin{cases} f(x,y) & \longrightarrow \max(\text{or } \min) \\ \text{subject to: } g(x,y) & = 0 \\ \text{and } h(x,y) & = 0. \end{cases}$$

step 2. Compute the following partial derivatives:

$$\frac{\partial f}{\partial x}(x,y); \quad \frac{\partial f}{\partial y}(x,y); \quad \frac{\partial g}{\partial x}(x,y); \quad \frac{\partial g}{\partial y}(x,y); \quad \frac{\partial h}{\partial x}(x,y); \quad \frac{\partial h}{\partial y}(x,y).$$

step 3. Solve the following system of equations:

$$\begin{cases} \frac{\partial f}{\partial x} - \lambda \cdot \frac{\partial g}{\partial x} - \mu \cdot \frac{\partial h}{\partial x} & = 0 \\ \frac{\partial f}{\partial y} - \lambda \cdot \frac{\partial g}{\partial y} - \mu \cdot \frac{\partial h}{\partial y} & = 0 \\ g(x, y) & = 0 \\ h(x, y) & = 0. \end{cases} \tag{19.2}$$

step 4.

i) If there are more than just one solutions to (19.2) then evaluate the function in all of the solutions and compare these function values to each other: the largest value is the maximum and the least the minimum. (Careful: you need to compare the function values in these points NOT the solutions themselves.)

ii) If there is only one single solution to (19.2) then evaluate the function in this point and compare it to ANY other function value *satisfying the restriction g*.

19.3 Equivalent solution method (Equivalente oplossingsmethode)

Although this method of solving restricted optimization problems with Lagrange multipliers seems different than the one presented before, it is completely equivalent to it. This equivalent form is presented here because this is the one will be used during the Microeconomics course.
ATTENTION! **Do not mix** the equivalent form with the original method. Choose the one you find the easiest and **stick to that one**, do not switch from one to the other within one problem!

step 1. Problem:

$$L(x, y, \lambda) = f(x, y) - \lambda g(x, y) \longrightarrow \max(\text{or } \min)$$

step 2. Compute the first order partial derivatives of L:

$$\frac{\partial L}{\partial x}(x, y, \lambda); \quad \frac{\partial L}{\partial y}(x, y.\lambda); \quad \frac{\partial L}{\partial \lambda}(x, y, \lambda).$$

step 3. Apply the first order conditions (first order partial derivatives equal 0), which results in solving the following system of linear equations:

$$\begin{cases} \frac{\partial L}{\partial x}(x, y, \lambda) = \frac{\partial f}{\partial x} - \lambda \cdot \frac{\partial g}{\partial x} & = 0 \\ \frac{\partial L}{\partial y}(x, y, \lambda) = \frac{\partial f}{\partial y} - \lambda \cdot \frac{\partial g}{\partial y} & = 0 \\ \frac{\partial L}{\partial \lambda}(x, y, \lambda) = g(x, y) & = 0 \end{cases}$$

Observe that this is exactly the system (19.1)!

step 4.

i) If there are more than just one solutions to (19.1) then evaluate the function in all of the solutions and compare these function values to each other: the largest value is the maximum and the least the minimum. (Careful: you need to compare the function values in these points NOT the solutions themselves.)

ii) If there is only one single solution to (19.1) then evaluate the function in this point and compare it to ANY other function value *satisfying the restriction* g.

Part VIII

Exam examples (Voorbeeldtentamens)

Chapter 20

Practice exam (Proeftentamen)

In this part you will find a practice exam which will be partly solved in the last week of the course. It is important that you first try to solve all questions of this exams first, before you look at the solutions. If you look at the solution first, you will not know which parts you would find a challenge during the exam.

1. Prove the following statement for the arbitrary sets A, B and C:

$$(A \cap C) \cup (B \cap D) \subseteq (A \cup B) \cap (C \cup D).$$

2. a. Let $\mathcal{P}(A)$ the set of all subsets of A, for an arbitrary set A. Determine the elements of set $\mathcal{P}\left(\mathcal{P}(\{1\})\right)$. Check afterwards in two steps whether you have indeed obtained all elements!

   b. 1_B denotes the indicator function of the set B. Determine the value $1_B(1)$ if $B = \mathcal{P}(\{1\})$.

3. Prove by mathematical or complete induction that for all $n \in \mathbb{N}$, $n \geq 3$

$$(n+1)^2 < 2n^2$$

   holds true.

4. Prove by mathematical or complete induction that $2^{2n+1} + 1$ is divisible by 3 for all $n \geq 1$, $n \in \mathbb{N}$.

5. Determine the following limit when $n \in \mathbb{N}$:

$$\lim_{n \longrightarrow +\infty} \frac{n^n + 2n!}{n^n + (2n)!}.$$

6. Determine the following limit when $n \in \mathbb{N}$:

$$\lim_{n \longrightarrow +\infty} \frac{n^{20} + 3^n}{2^n + 3^n}.$$

7. Find the fraction that represents the rational number $2.3\overline{51} = 2.351515151...$, by using the sum of the corresponding infinite geometric series.

8. The function f is determined by the rule

$$f(x) = 1 + \sqrt[4]{x}.$$

   a. Determine its domain D and range B.

   b. Determine on which intervals the function is increasing, decreasing, convex or concave, determine any asymptotes the function may have and make a sketch of its graph.

   c. Is f invertible? Motivate your answer. If it is, determine the inverse f^{-1} of f, its domain and range and make a sketch of its graph.

9. Determine the functions $f + g$, $g \circ f$ en $f \circ g$ and their domain and range when

$$f(x) = x^2 + 1, \text{ and } g(x) = x - 3.$$

10. Determine the following limits; you are NOT allowed to use L'Hospital's rule:

   a.
$$\lim_{x \to \sqrt{2}} \frac{x - \sqrt{2}}{x^2 - 2};$$

   b.
$$\lim_{x \to 0} \frac{\sin^2(2\sqrt{x})}{\tan(x)}.$$

   c.
$$\lim_{x \longrightarrow 0^+} x^2 \ln(x).$$

11. Let the function f be given by the rule $f(x, y) = \arctan(xy^2)$. Determine the directional derivative $D_u f(1, 1)$ at the point $(1, 1)$ when the angle θ between the vector $\mathbf{u}$ and the x-axes is $\frac{\pi}{6}$.

12. Given the function f of two variables determined by the rule $f(x, y) = \sin(2x + 3y)$.

   a. Determine its domain and range.

   b. Determine the differential $df(x, y)$ and the gradient $\nabla f(x, y)$ of f.

   c. In which direction does the maximal rate of change of f occur at the point $(-6, 4)$ and what is its magnitude? Determine the directional derivative in the direction $\mathbf{u} = \frac{1}{2}(\sqrt{3}i - j)$.

13. Use the Lagrange multiplier method to determine all extreme values of f, subject to the restrictions g and h, where f, g and h are given by the rules:

$$f(x,y,z) = yz + xy, \text{ and } g(x,y,z) = xy - 1, \ h(x,y,z) = y^2 + z^2 - 1.$$

14. Compute the following integral:

$$\int_0^{\frac{\pi}{2}} \sin^3 x \cos^2 x dx.$$

15. Compute the following indeterminate integral

a.
$$\int \frac{x}{1+x^4} dx$$

b.
$$\int x^3 \sqrt{1+x^2} dx$$

16. Compute the limit

$$\lim_{h \longrightarrow 0} \frac{1}{h} \int_2^{2+h} \sqrt{1+t^3} dt.$$

17. Compute the following improper integrals.

a.
$$\int_{-\infty}^{+\infty} \frac{x^2}{9+x^6} dx$$

b.
$$\int_0^1 \frac{dx}{\sqrt{1-x^2}}$$

Chapter 21

Introduction to Analysis Exam 2018-2019 (Tentamen Inleiding Analyse 2018-2019)

Here you can find the regular exam of the academic year 2018-2019 (first the English version, then the Dutch version). The answers to this exam will be released gradually, as the block advances, in the form of pencasts. It is of essential importance that you first try to solve all questions of these exams first, before you look at the solutions. If you look at the solution first, you will not know anymore which parts you would find a challenge during the exam.

Please note that complex numbers are not part of the material anymore, you do not need to be able to solve exercise 8 of this exam!

Questions

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Introduction to Analysis FEB21017X

FEB21017X Introduction to Analysis Exam 2018

26 October 2018 13:30 - 16:30

GENERAL INFORMATION
Lecturer: dr. E. Oldenkamp
Type of examination: Open questions (closed book)
Number of questions: **14 questions (question 15 is extra space)**
Number of pages: 24 pages (incl. cover sheet)

INSTRUCTIONS

- **Please use only a pen with black or blue ink**. At the top right of this page, **fill in your student number by coloring the correct blocks**. If you make a mistake, put a cross through the wrong block, and color the correct one.
- Answers have to be written in the block corresponding to the problem. In case you made a mistake in one such box and therefore you need more space, you can use the additional writing space: there are 6 "extra space" boxes at the end of the exam. Be careful to indicate clearly in the box where you made the mistake where we can find the right answer. The extra space boxes will not be checked otherwise. Because total writing space is limited (no additional exam paper will be provided), we strongly advise you to prepare your answers on scrap paper before writing them down in the boxes.

The instructions continue on the next page!

Instructions

- You are NOT allowed to use a calculator.
- You are NOT allowed to use notes.
- You are NOT allowed to use books.
- You are NOT allowed to use a dictionary.
- Scrap paper, examination assignments or other examination related documents must stay in the examination room during the examination and after the examination has finished.
- You are NOT allowed to take with you the answers nor may you copy answers in any way.
- It is NOT allowed to keep a watch within reach during the examination. Watches need to be stored in your coat or bag.
- Any kind of mobile devices or (potential) data media must be switched off during the examination. All devices need to be stored in your coat or bag.

Good luck!

If you have comments or suggestions, please go to the website of the ESSC: www.eur.nl/english/essc/.

Please notice: if you have not registered for this examination, you can only do so on the day of examination, against payment of €20.- in administrative charges. This registration can be done via the Webshop, till 24:00hrs on the day of exam (shop.ese.eur.nl) or at the Information Desk (during office hours) of the Education Service Center (Tinbergen Building H6-01). If the examination ends after 16:00hrs or takes place on a Friday or Saturday, registration has to take place no later than the next workday.

Question 1

Let A and B two subsets of the universal set $\mathcal{U}$.

1p **1a** Show that $\mathcal{P}(A)$ and $\mathcal{P}(B)$ are never disjoint.

2p **1b** Prove, using rigorous mathematical notation, that

$$\mathcal{P}(A) \cap \mathcal{P}(B) = \{\emptyset\} \iff A \cap B = \emptyset.$$

Question 2
Let the sets A and B be given with the following set-builder notation:

$$A = \{x \in \mathbb{Z} | \left|\frac{x+4}{3x-6}\right| > 1\}$$

$$B = \{x \in \mathbb{Z} || x| \geqslant 5 \text{ and } |x| < 7\}$$

1.5p **2a** Write these sets with list notation.

1.5p **2b** Determine the sets $A \cap B$, $\mathcal{P}(A)$, $\mathcal{P}(B)$ and $\mathcal{P}(A) \cap \mathcal{P}(B)$ also with list notation. Show afterwards that the statements in subquestions **1a** and **1b** also hold with these specific sets.

Question 3

3p **3a** Prove by mathematical or complete induction that $27 \cdot 23^n + 17 \cdot 10^{2n}$ is divisible by 11 for all $n \in \mathbb{N}, n \geqslant 0$.

1p **3b** Did you need mathematical or complete induction? Motivate your answer!

Question 4

3p **4a** Prove by mathematical or complete induction that every natural number $n \in \mathbb{N}$, $n \geqslant 2$ can be written as a product of i_n prime numbers ($i_n \in \mathbb{N}, i_n \geqslant 1$), that is

$$\exists p_{n_1}, p_{n_2}, \ldots, p_{n_{i_n}} \text{ prime numbers, such that } n = p_{n_1} \cdot p_{n_2} \cdot \ldots \cdot p_{n_{i_n}}$$

For example,
$6 = 2 \cdot 3$, so $i_6 = 2, p_{6_1} = 2, p_{6_2} = 3,$
$20 = 5 \cdot 2 \cdot 2$, so $i_{20} = 3, p_{20_1} = 5, p_{20_2} = 2, p_{20_3} = 2.$

1p **4b** Did you need mathematical or complete induction? Motivate your answer!

Question 5

Consider the function $f : D \longrightarrow B$ determined by the rule

$$f(x) = \arctan |x| \,.$$

1p **5a** Determine the domain D and the range B of f.

1p **5b** Determine, by analyzing the first derivative, on which intervals in its domain the function is increasing or decreasing. Please do write down the first derivative too!

2p **5c** Determine any horizontal and vertical asymptotes the function might have, and draw a sketch of its graph.

Question 6

Consider the same function f you analyzed in the previous question, now restricted to the interval $(-\infty, 0]$. Call this function with domain $(-\infty, 0]$ g, and note that it is still determined by the rule

$$g(x) = \arctan|x|\,.$$

1p **6a** Determine the range C of g. Is $g : (-\infty, 0] \longrightarrow C$ invertible? Motivate your answer!

1.5p **6b** If the function g is invertible, determine the inverse g^{-1}, its domain and range, and the rule $g^{-1}(x)$.

1.5p **6c** Draw a sketch of the graph of g^{-1} in the same plot where you sketched f (please do NOT make a new plot). Do perform the check for the inverse function!

Question 7

Compute the following limit; please note that any solution making use of L'Hospital's rule will be completely disregarded.

3p **7**

$$\lim_{x \to 0} \frac{\ln(1-x)}{x}$$

Question 8

Consider the following quadratic equation

$$z^2 + (1+i)z + i = 0.$$

1p **8a** Solve the equation using the quadratic formula (also known as the "Δ" or the "abc"- formula).

1p **8b** Write the solutions in polar form, and also determine their complex conjugates in polar form.

1p **8c** Are all complex conjugates you determined for subquestion b also solutions of the same quadratic equation $z^2 + (1+i)z + i = 0$? Motivate your answer!

Question 9

2p **9** Evaluate the following limit by first recognizing the sum as a Riemann sum, and then evaluating the corresponding integral:

$$\lim_{n\to+\infty}\sum_{i=1}^{n}\left(\frac{i^4}{n^5}+\frac{i}{n^2}\right).$$

Question 10

4p **10** Evaluate the following integral in terms of areas, that is, write the Riemann sum corresponding to this integral and evaluate the limit:

$$\int_{-4}^{3} \left|\frac{1}{2}x\right| dx.$$

Question 11

2p **11** Compute the following integral:

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\sin(x)}{x^2+\cos(x)+1}dx.$$

Question 12

3p **12** Compute the following integral:

$$\int_1^2 \frac{x}{x^4 + x^2 + 1} dx.$$

Question 13

4p **13** Evaluate the following integral or show that it is divergent:

$$\int_1^{+\infty} \frac{\arctan(x)}{x^2} dx.$$

Question 14

2p **14a** Use the Lagrange multipliers method in order to find the extreme values of the function $f(x, y, z) = (x-4)^2 + (y-2)^2 + z^2$, subject to the constraint $x^2 + y^2 - z^2 = 0$.

1p **14b** Specify whether the extreme values of f you found are a minimum or a maximum.

Extra space

Only use this extra space if you have made a mistake: please identify at your wrong answer where you have exactly written the right answer, these places will otherwise NOT be checked.

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Questions

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FEB21017 Inleiding Analyse

FEB21017 Inleiding Analyse Tentamen 2018

26 October 2018 13:30 - 16:30

ALGEMENE INFORMATIE
Docent: dr. E. Oldenkamp
Soort tentamen: Open vragen (gesloten boek)
Aantal vragen: **14 vragen (vraag 15 is extra ruimte)**
Aantal pagina's: 24 pagina's (incl. voorblad)

INSTRUCTIES

• **Gebruik uitsluitend een pen met zwarte of blauwe inkt**. **Vul je studentnummer in in de rechter bovenhoek van deze pagina**, door de juiste blokjes te kleuren. Mocht je een fout maken, zet dan een kruis door het foute blokje en kleur het juiste.

• Antwoorden moeten in de daarvoor bestemde box geschreven worden. Aan het einde van dit tentamen zijn er 6 extra ruimte boxen te vinden. Je mag uitsluitend gebruik maken van deze ruimten in het geval dat je een fout hebt gemaakt in een van de antwoordenboxen waardoor je niet genoeg ruimte hebt om het correcte antwoord in de juiste antwoordendbox te schrijven. Deze ruimten zijn niet bedoeld voor te lange antwoorden. Geef altijd duidelijk aan in de antwoordenbox van de oorspronkelijke vraag waar wij je correcte antwoord kunnen vinden, anders worden de antwoorden in deze ruimten NIET gelezen! Omdat de totale schrijfruimte beperkt is (je krijgt geen extra tentamenbladen), raden we je met klem aan je antwoorden (deels) eerst op kladpapier te maken voordat je die in de antwoordenboxen schrijft.

De instructies gaan door op de volgende pagina! Z.O.Z

Instructies

- Je mag GEEN rekenmachine gebruiken.
- Je mag geen aantekeningen gebruiken.
- Je mag geen boeken gebruiken
- Het is niet toegestaan een woordenboek te gebruiken.
- Kladpapier, tentamenopgaven en andere tentamen gerelateerde documenten mogen de zaal tijdens en na het tentamen niet verlaten.
- Antwoorden, in welke vorm dan ook, mogen de zaal niet verlaten.
- Horloges zijn niet toegestaan gedurende het tentamen en moeten worden opgeborgen in jas of tas.
- Iedere vorm van mobiele (potentiële) datadragers is niet toegestaan gedurende het tentamen. Deze moeten uitgeschakeld zijn en worden opgeborgen in jas of tas.

Veel succes!

Indien je opmerkingen of suggesties hebt, bezoek dan de website van het ESSC: www.eur.nl/essc.

N.B.: indien u zich niet heeft ingeschreven voor dit tentamen, kan dit nog gedaan worden op de dag van dit tentamen zelf, tegen betaling van €20,- administratiekosten. Inschrijven kan via de webshop tot 24:00u., op de dag van het tentamen (shop.ese.eur.nl) of via de Informatiebalie (gedurende openingstijden) van het Onderwijs Service Centrum (Tinbergengebouw H6-02). Indien het tentamen eindigt na 16:00u. danwel plaatsvindt op een vrijdag of zaterdag, dient de inschrijving uiterlijk te geschieden op de eerstvolgende werkdag.

Vraag 1
Zij A en B twee deelverzamelingen van de universele verzameling $\mathcal{U}$.

1p **1a** Laat zien dat $\mathcal{P}(A)$ en $\mathcal{P}(B)$ nooit disjunct zijn.

2p **1b** Bewijs, met rigoureus wiskundige notatie, dat

$$\mathcal{P}(A) \cap \mathcal{P}(B) = \{\emptyset\} \iff A \cap B = \emptyset.$$

Vraag 2
Zij de verzamelingen A en B gegeven met de volgende set-builder notatie:

$$A = \{x \in \mathbb{Z} | \left|\frac{x+4}{3x-6}\right| > 1\}$$

$$B = \{x \in \mathbb{Z} || x| \geqslant 5 \text{ en } |x| < 7\}$$

1.5p **2a** Schrijf deze verzamelingen met lijst notatie.

1.5p **2b** Bepaal de verzamelingen $A \cap B$, $\mathcal{P}(A)$, $\mathcal{P}(B)$ en $\mathcal{P}(A) \cap \mathcal{P}(B)$ ook met lijst notatie. Laat vervolgens zien dat de beweringen in subvragen **1a** en **1b** ook met deze specifieke verzamelingen gelden.

Vraag 3

3p **3a** Bewijs door middel van wiskundige of volledige inductie dat $27 \cdot 23^n + 17 \cdot 10^{2n}$ deelbaar is door 11 voor alle $n \in \mathbb{N}, n \geqslant 0$.

1p **3b** Heb je wiskundige of volledige inductie gebruikt voor je bewijs? Motiveer je antwoord!

Vraag 4

3p **4a** Bewijs door middel van wiskundige of volledige inductie dat elk natuurlijk getal $n \in \mathbb{N}$, $n \geqslant 2$ geschreven kan worden als een product van i_n priemgetallen ($i_n \in \mathbb{N}, i_n \geqslant 1$), dat wil zeggen,

$$\exists p_{n_1}, p_{n_2}, \ldots, p_{n_{i_n}} \text{ priemgetallen, zodat } n = p_{n_1} \cdot p_{n_2} \cdot \ldots \cdot p_{n_{i_n}}$$

Bijvoorbeeld
$6 = 2 \cdot 3$, dus $i_6 = 2$, $p_{6_1} = 2, p_{6_2} = 3$,
$20 = 5 \cdot 2 \cdot 2$, dus $i_{20} = 3$, $p_{20_1} = 5, p_{20_2} = 2, p_{20_3} = 2$.

1p **4b** Heb je wiskundige of volledige inductie gebruikt voor je bewijs? Motiveer je antwoord!

Vraag 5

Zij de functie $f : D \longrightarrow B$ gegeven door de regel

$$f(x) = \arctan |x| \, .$$

1p **5a** Bepaal het domein D en het bereik B van f.

1p **5b** Bepaal, door de eerste afgeleide van f te onderzoeken, op welke intervallen van het domein de functie stijgend of dalend is. Schrijf ook de eerste afgeleide op.

2p **5c** Bepaal alle eventuele verticale en horizontale asymptoten en maak een schets van de grafiek van f.

Vraag 6
Beschouw dezelfde functie f onderzocht bij de vorige vraag, maar nu beperkt tot het interval $(-\infty, 0]$. Noem deze functie met domein $(-\infty, 0]$ g, en merk op dat g ook gegeven is door de regel

$$g(x) = \arctan |x| .$$

1p **6a** Bepaal het bereik C van g. Is $g : (-\infty, 0] \longrightarrow C$ inverteerbaar? Motiveer je antwoord!

1.5p **6b** Indien de functie g inverteerbaar is, bepaal dan de inverse g^{-1}, het domein en het bereik, en de regel van $g^{-1}(x)$.

1.5p **6c** Maak een schets van de grafiek van g^{-1} in dezelfde schets als die van f (maak aub geen nieuwe schets). Vergeet de controle voor de inverse niet!

Vraag 7

Bereken de volgende limiet; merk aub op dat het gebruik van de regel van L'Hospital geen punten oplevert.

3p **7**

$$\lim_{x \to 0} \frac{\ln(1-x)}{x}$$

Vraag 8
Beschouw de volgende kwadratische vergelijking

$$z^2 + (1+i)z + i = 0.$$

1p **8a** Los deze vergelijking op met de kwadratische formule (ook bekend onder de naam "Δ" of "abc"-formule).

1p **8b** Schrijf de oplossingen in polaire vorm, en bepaal ook de complexe geconjugeerden van deze oplossingen in polaire vorm.

1p **8c** Zijn alle complexe geconjugeerden die je bepaald hebt onder subvraag b ook oplossingen van dezelfde kwadratische vergelijking $z^2 + (1 + i)z + i = 0$? Motiveer je antwoord!

Vraag 9

2p **9** Bereken de volgende limiet door deze eerst als een Riemann som te herkennen en dan de bijbehorende integraal te berekenen:

$$\lim_{n\to+\infty}\sum_{i=1}^{n}\left(\frac{i^4}{n^5}+\frac{i}{n^2}\right).$$

Vraag 10

4p **10** Bereken de volgende integraal als oppervlakteberekening, dat wil zeggen, schrijf de bijbehorende Riemann som op en bereken de limiet:

$$\int_{-4}^{3} \left| \frac{1}{2}x \right| dx.$$

Vraag 11

2p **11** Bereken de volgende integraal:

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\sin(x)}{x^2+\cos(x)+1}dx.$$

Vraag 12

3p **12** Bereken de volgende integraal:

$$\int_1^2 \frac{x}{x^4 + x^2 + 1} dx.$$

Vraag 13

4p **13** Bereken de integraal of laat zien dat het divergent is:

$$\int_1^{+\infty} \frac{\arctan(x)}{x^2} dx.$$

Vraag 14

2p **14a** Gebruik de Lagrange multiplicator methode om de kritieke punten van de functie $f(x, y, z) = (x-4)^2 + (y-2)^2 + z^2$ te bepalen, onder de beperkende voorwaarde $x^2 + y^2 - z^2 = 0$.

1p **14b** Bepaal welke van deze kritieke punten van f een minimum en welke een maximumpunt zijn.

Extra space
Gebruik de extra ruimte alleen als je in een van de antwoorden een fout gemaakt hebt: geef in dat geval bij het foute antwoord aan waar we het juiste antwoord precies kunnen vinden, deze ruimtes worden anders niet nagekeken.

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