Algorithms for Decision Support

NP-Completeness (1/3)

Turing Machine, **P**, and **NP**

Hilbert's 10th Problem

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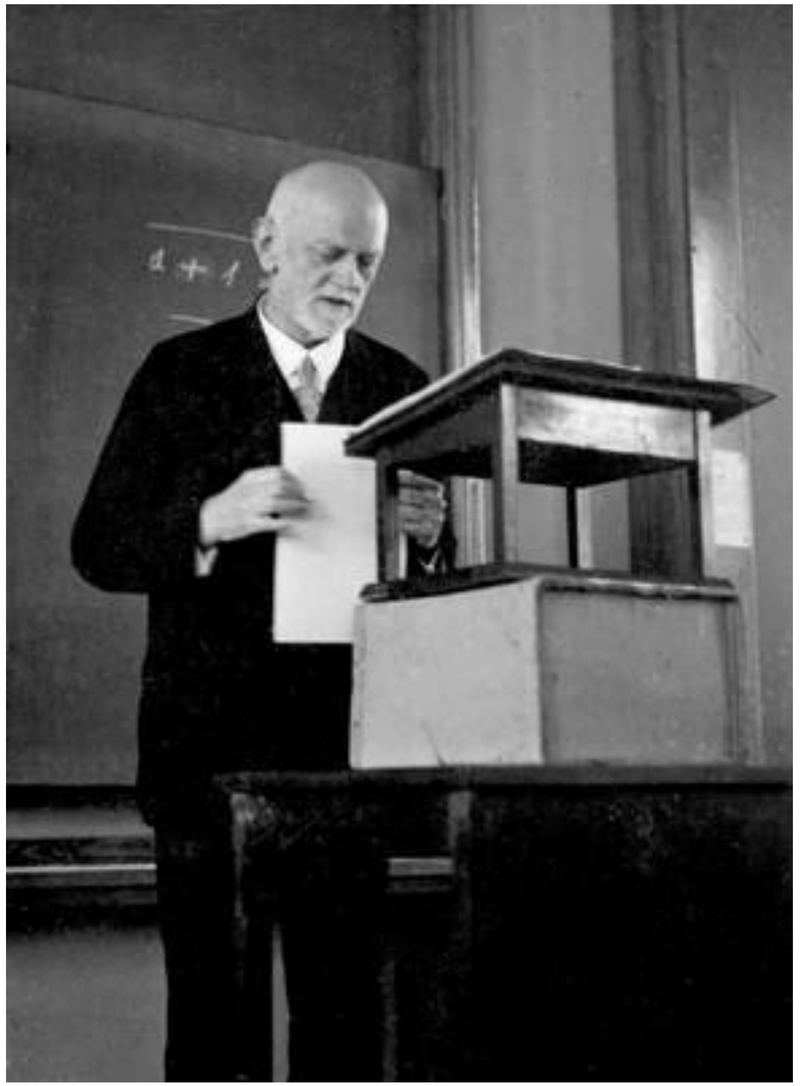

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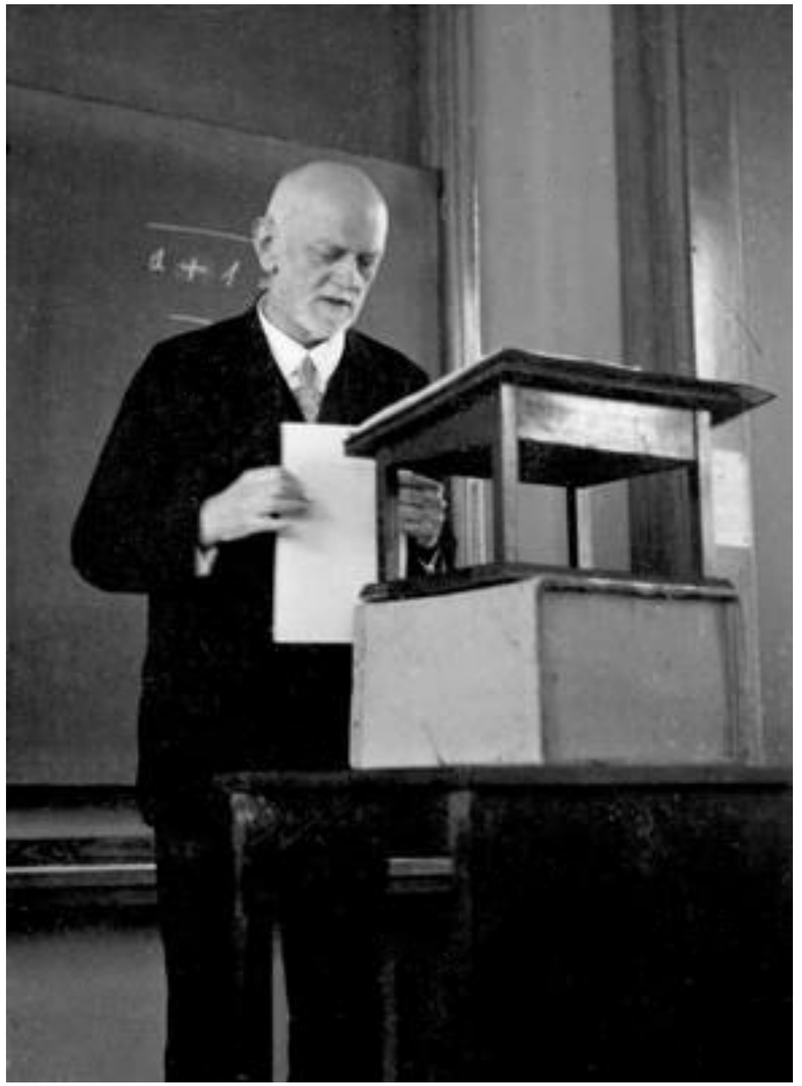

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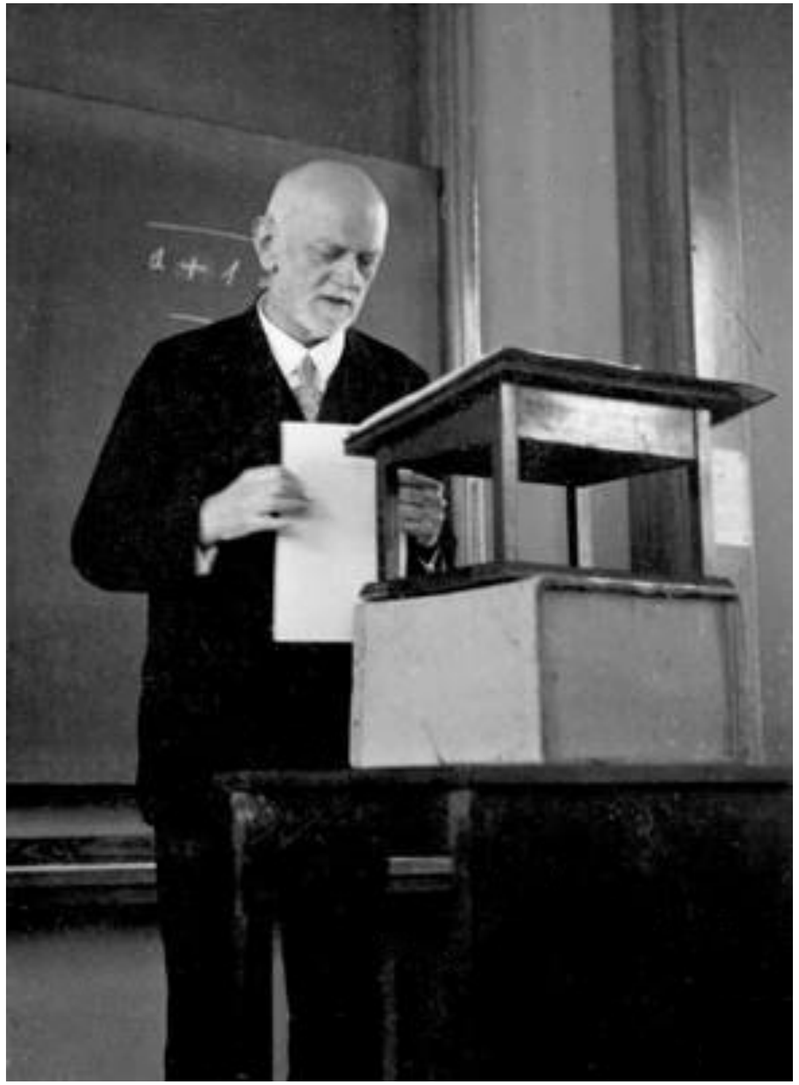

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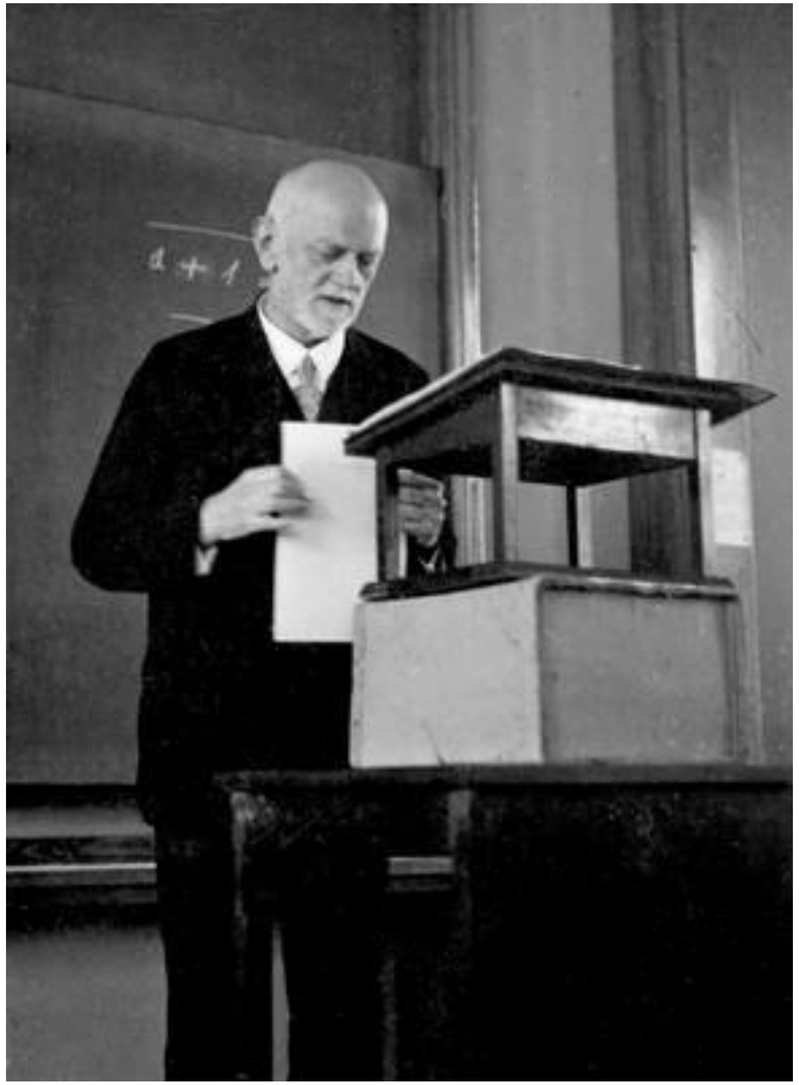

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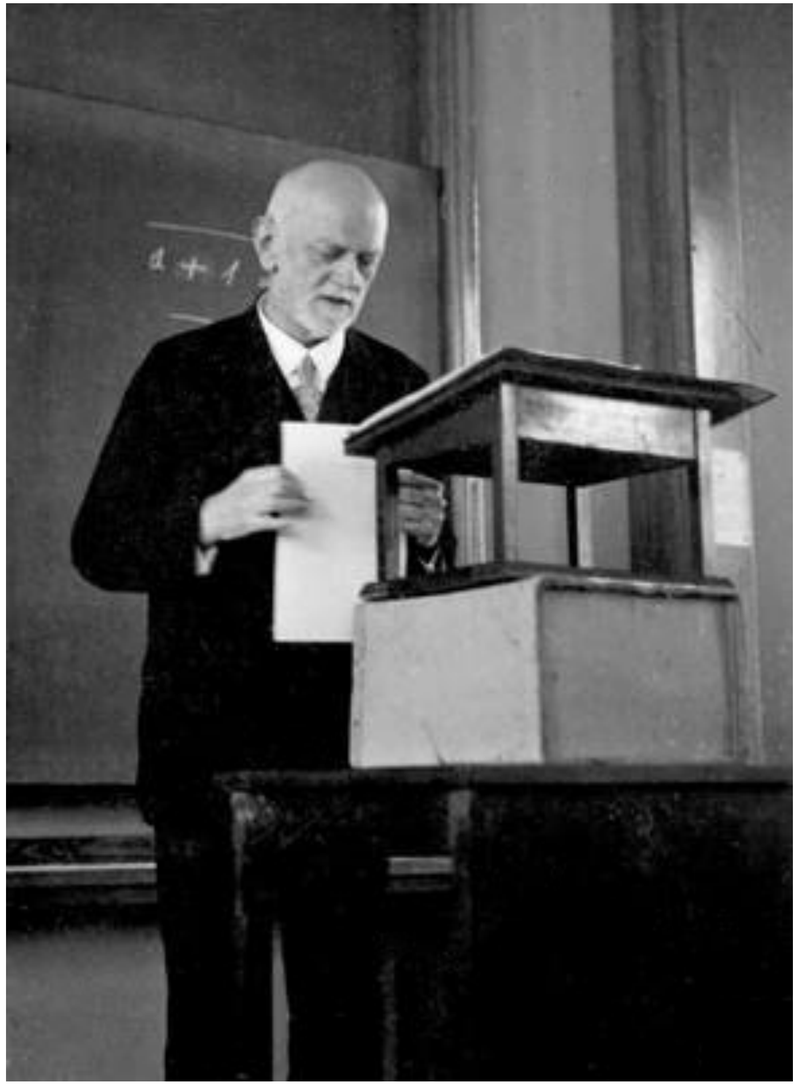

???

What is an algorithm/computer?

Overview

- Turing machine
  - Deterministic and non-deterministic
- String and language
- Classes **P** and **NP**
  - Verify
  - How to show that a problem is in **P** or in **NP**

Overview

- Turing machine
  - Alphabet, symbol, string, language
  - Turing machine halts (accept or reject) or loops
- Turing-**decidable** and Turing-**recognizable** languages
- **Undecidable** languages

Turing Machine

- Proposed by Alan Turing in 1936

Alan Turing 1912~1954

Turing Machine

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- A model of a general purpose computer:
  a Turing machine can do every thing that a real computer can do

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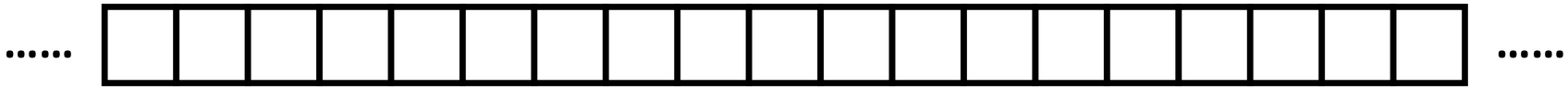

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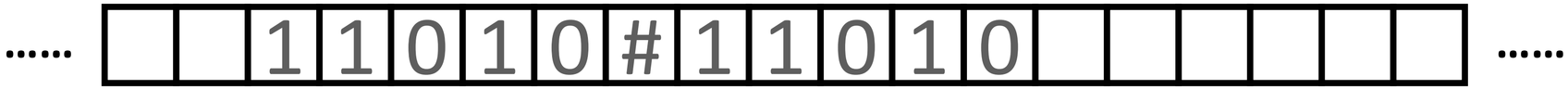

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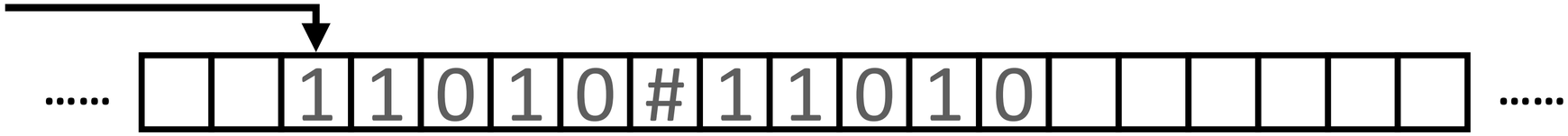

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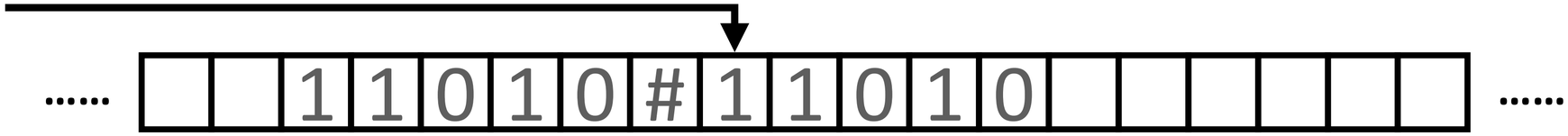

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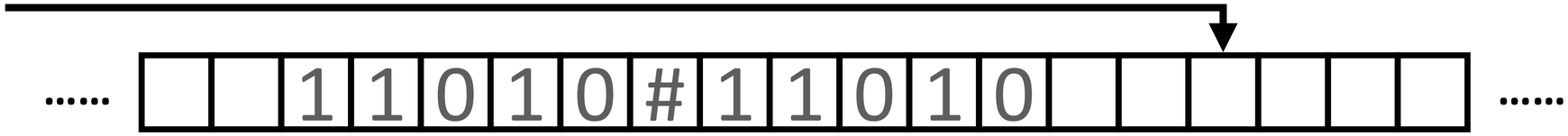

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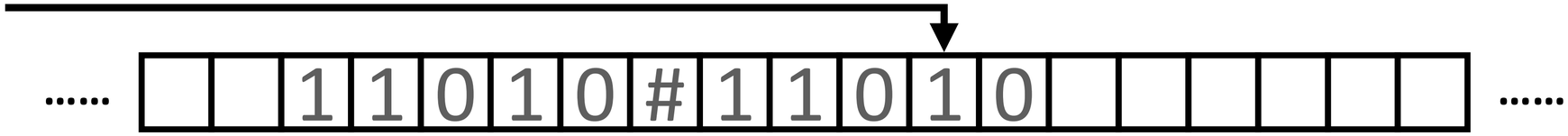

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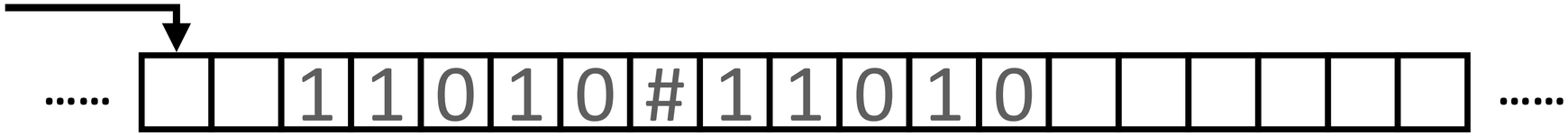

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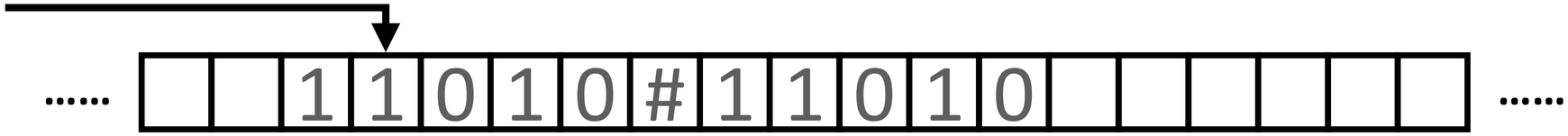

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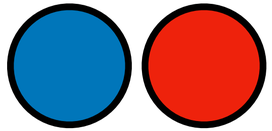

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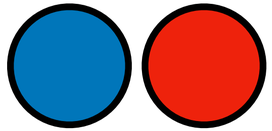

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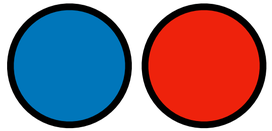

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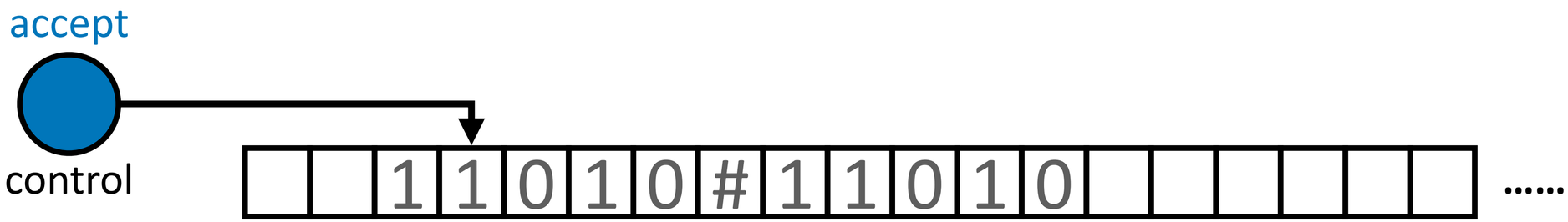

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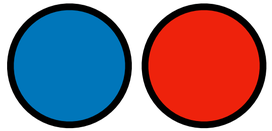

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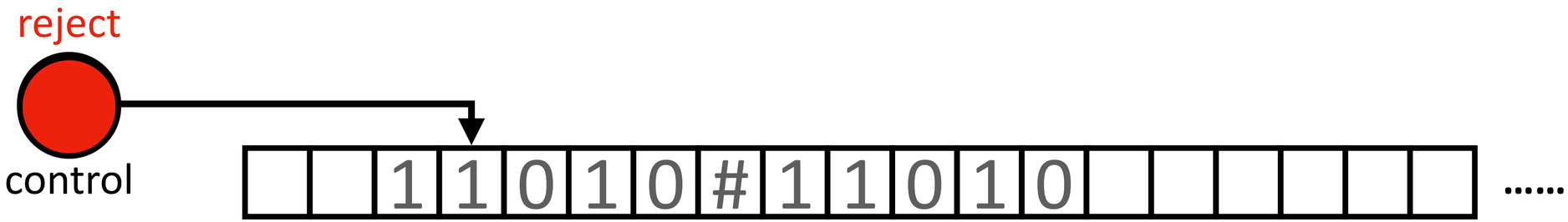

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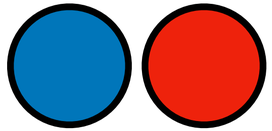

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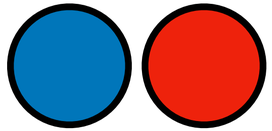

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  - If it does not enter the accept or reject states, TM will run forever (***loop***), and never halt

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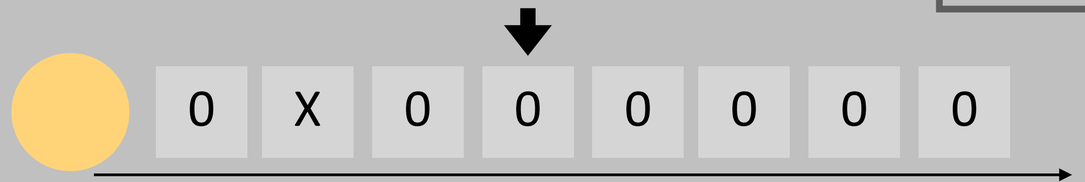

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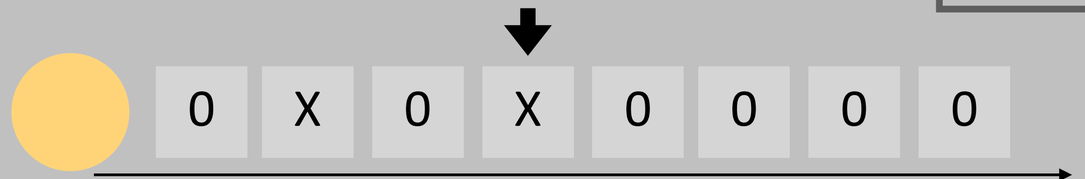

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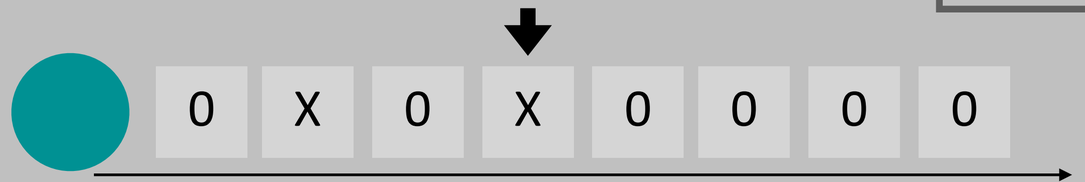

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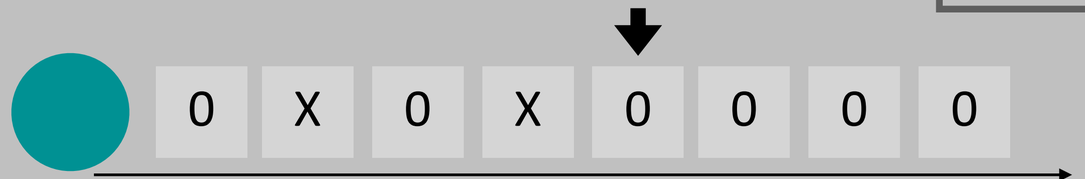

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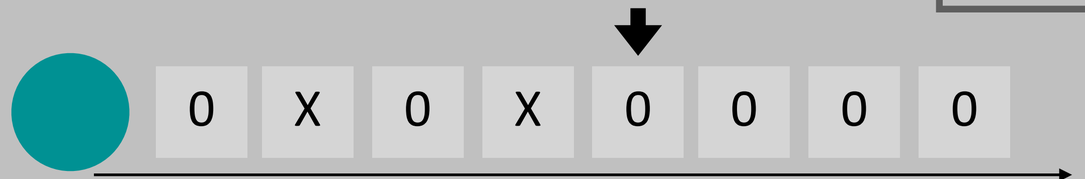

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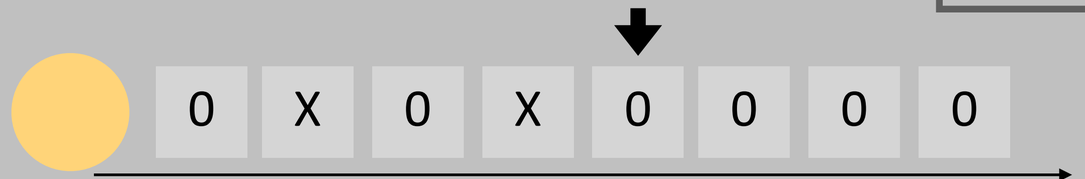

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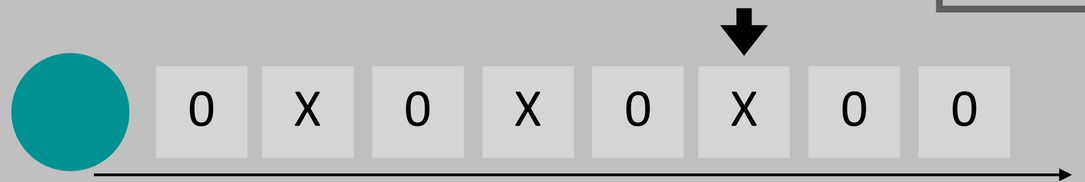

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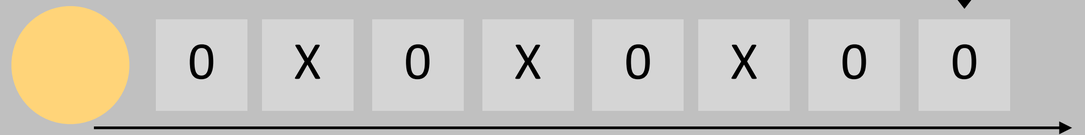

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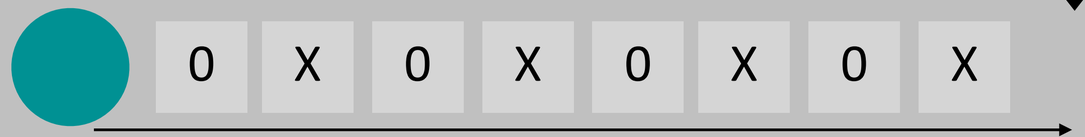

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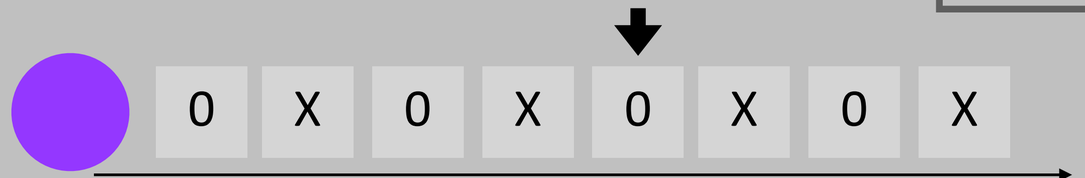

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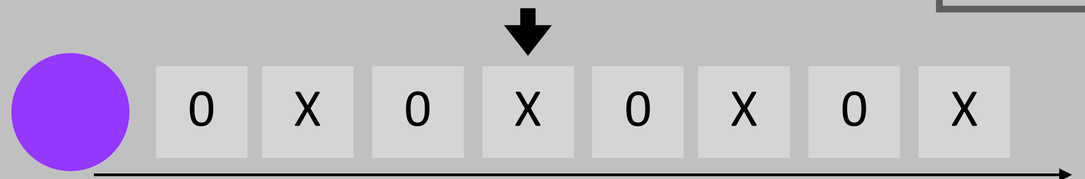

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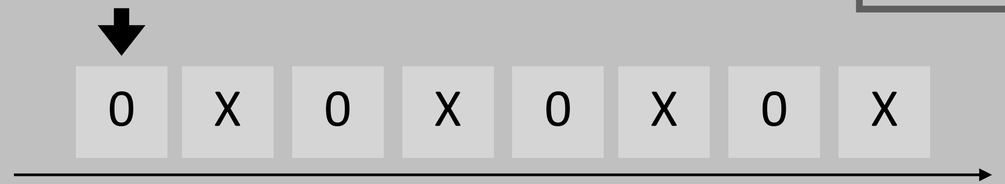

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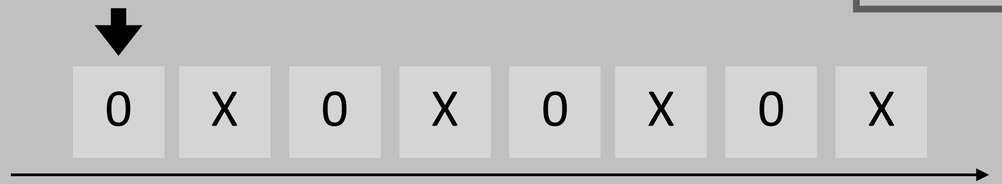

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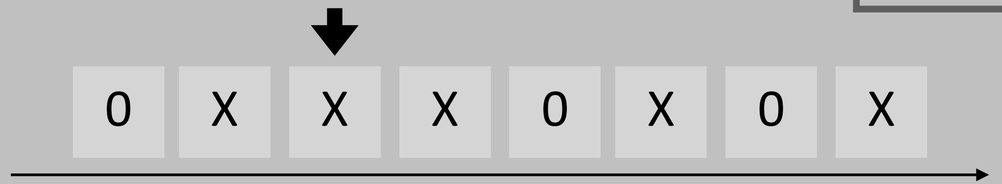

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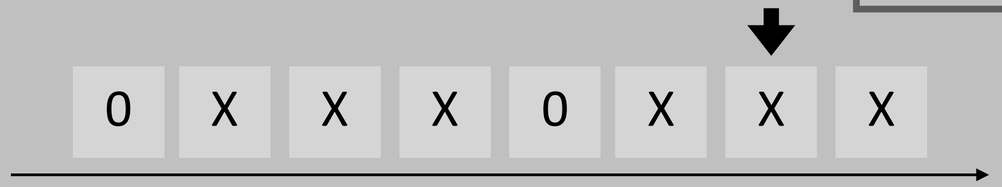

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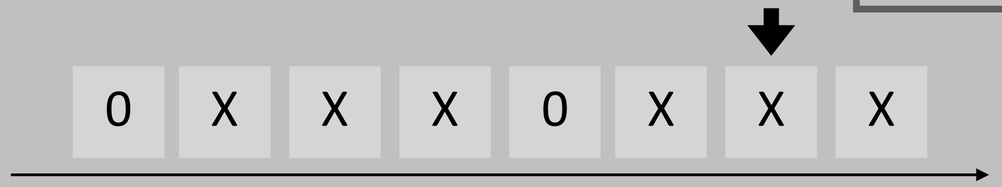

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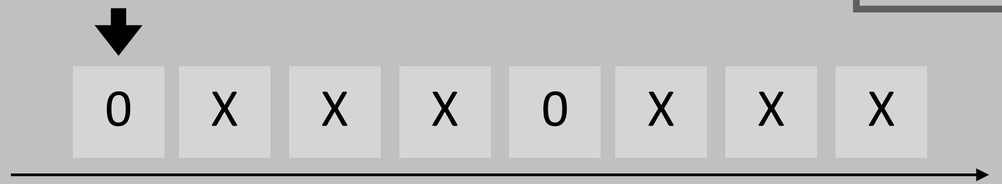

Turing Machine Example

- There is a Turing machine that decides if an input string has 2^k 0's:

  M = "On input string w:

  1. Sweep left to right across the tape, crossing off every other 0.

  2. If in stage 1 the tape contained a single 0, *accept*.

  3. If in stage 1 the tape contained more than a single 0 and the number of 0s was odd, *reject*.

  4. Return the head to the left-hand end of the tape.

  5. Go to stage 1."

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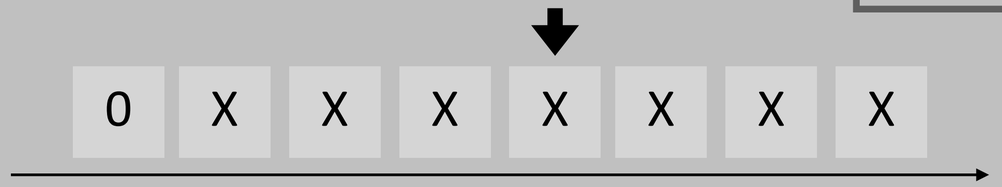

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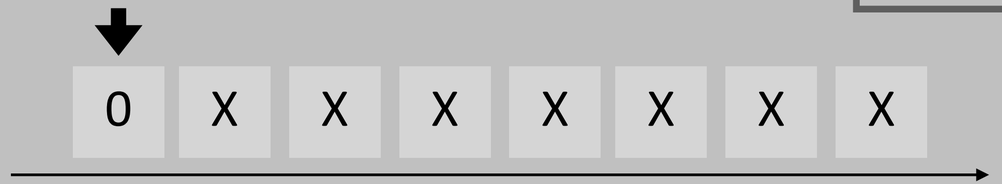

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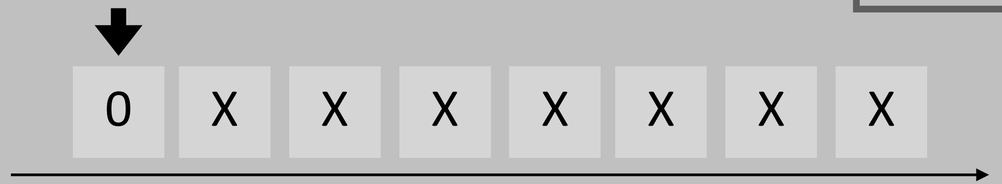

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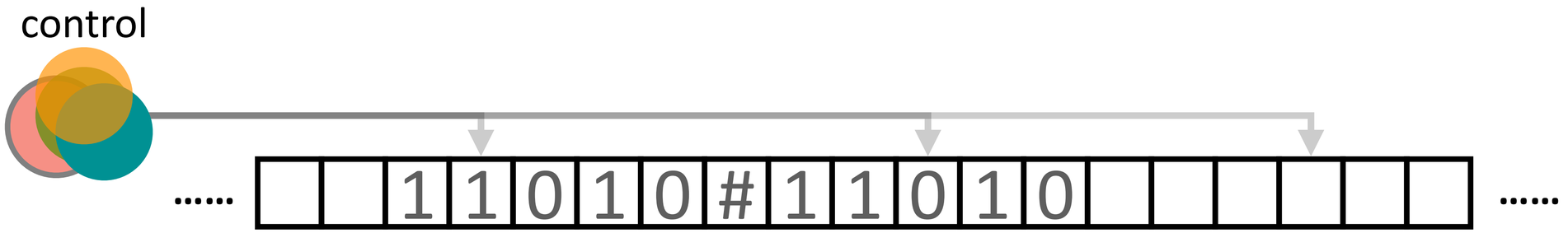
Turing machine

- An infinitely long tape/memory
  - Initially contains the (finite) input sequence and is blank everywhere else
- A tape head that can read and write symbols and move around on the tape
- Finite-state control

  - The Turing machine may end up with an *accept* state or *reject* state
  - It *accepts* the input or *rejects* the input

Nondeterministic Turing Machine

Nondeterministic Turing Machine



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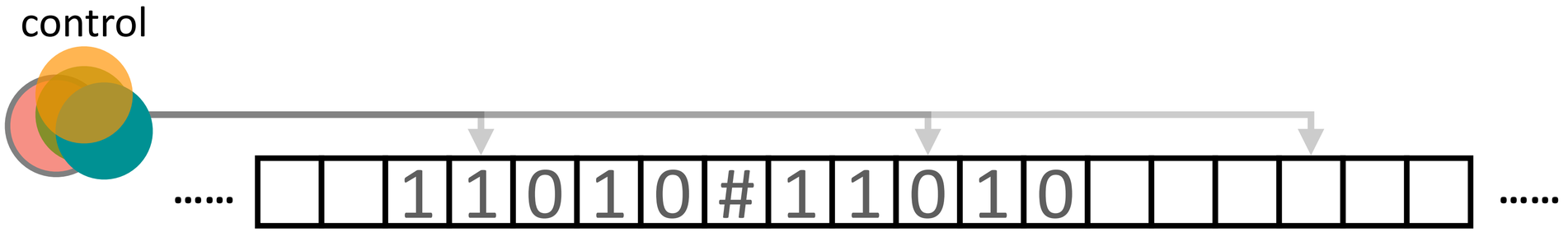


- It is like a (deterministic) Turing machine, but with non-deterministic control
- For an input w, we can describe all possible computations of nondeterministic Turing machine by a *computation tree*

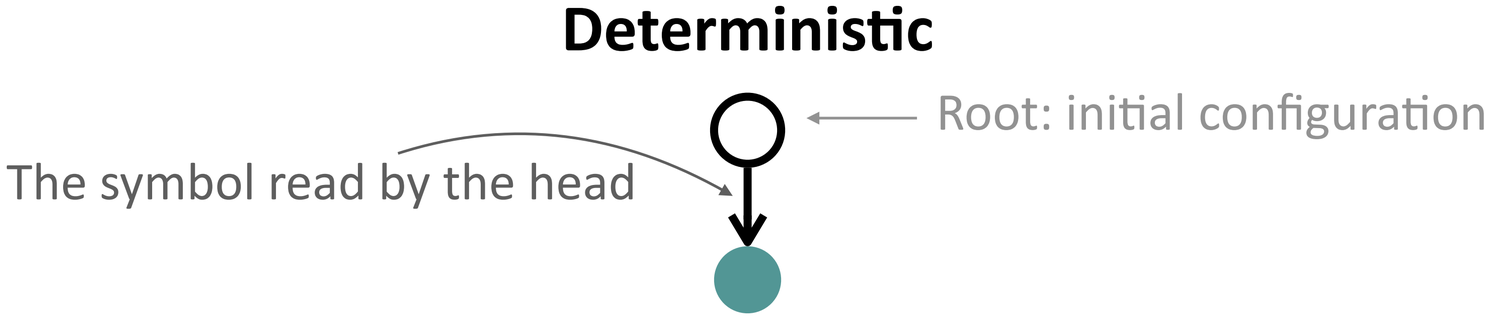
Nondeterministic Turing Machine

Deterministic

● : Configuration

(The current control state and
what the read-write head reads

Nondeterministic Turing Machine



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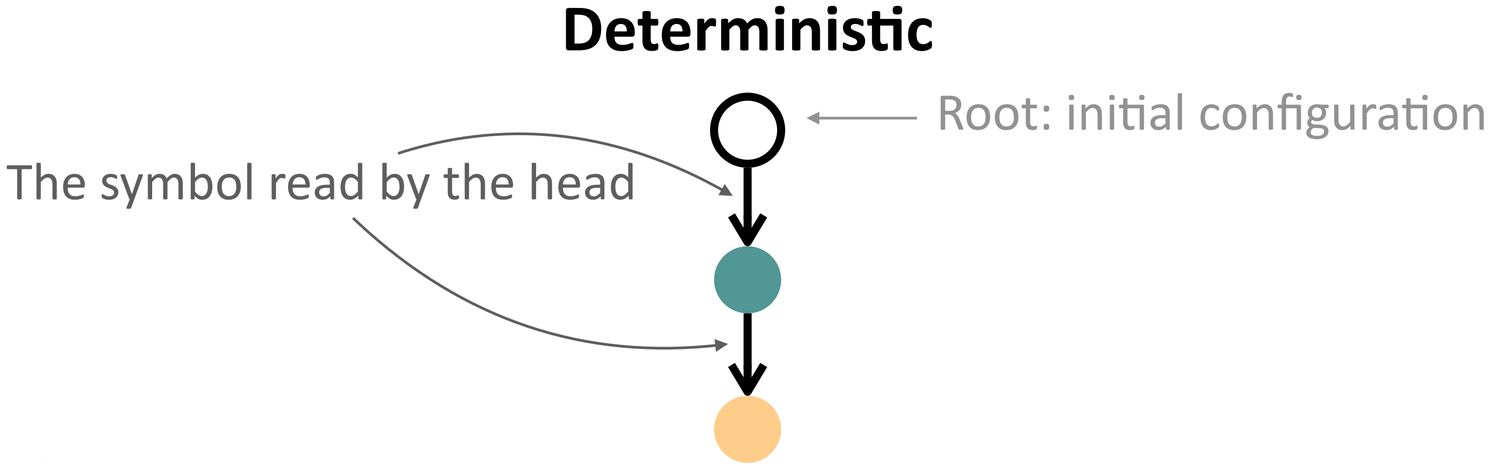

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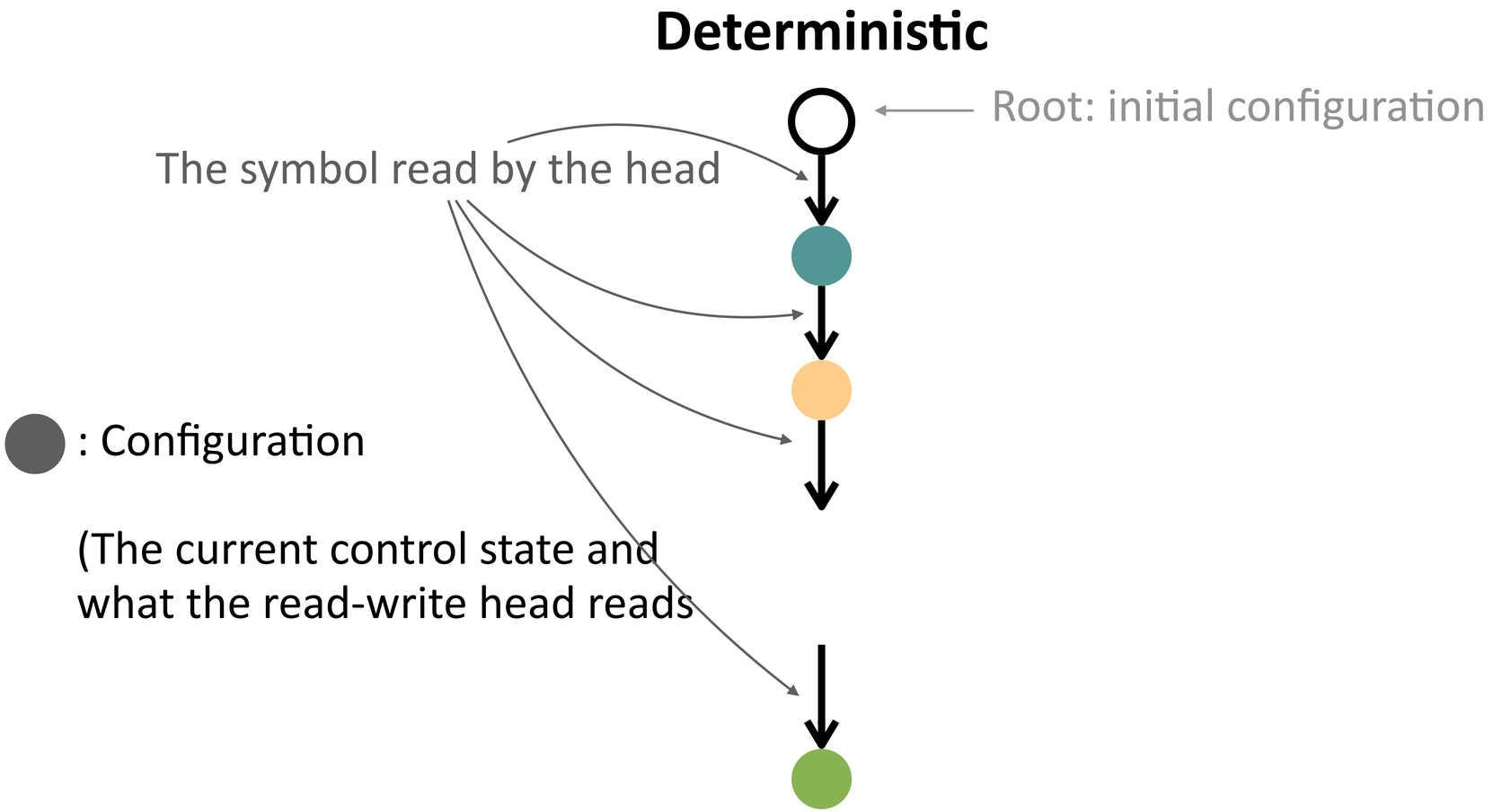

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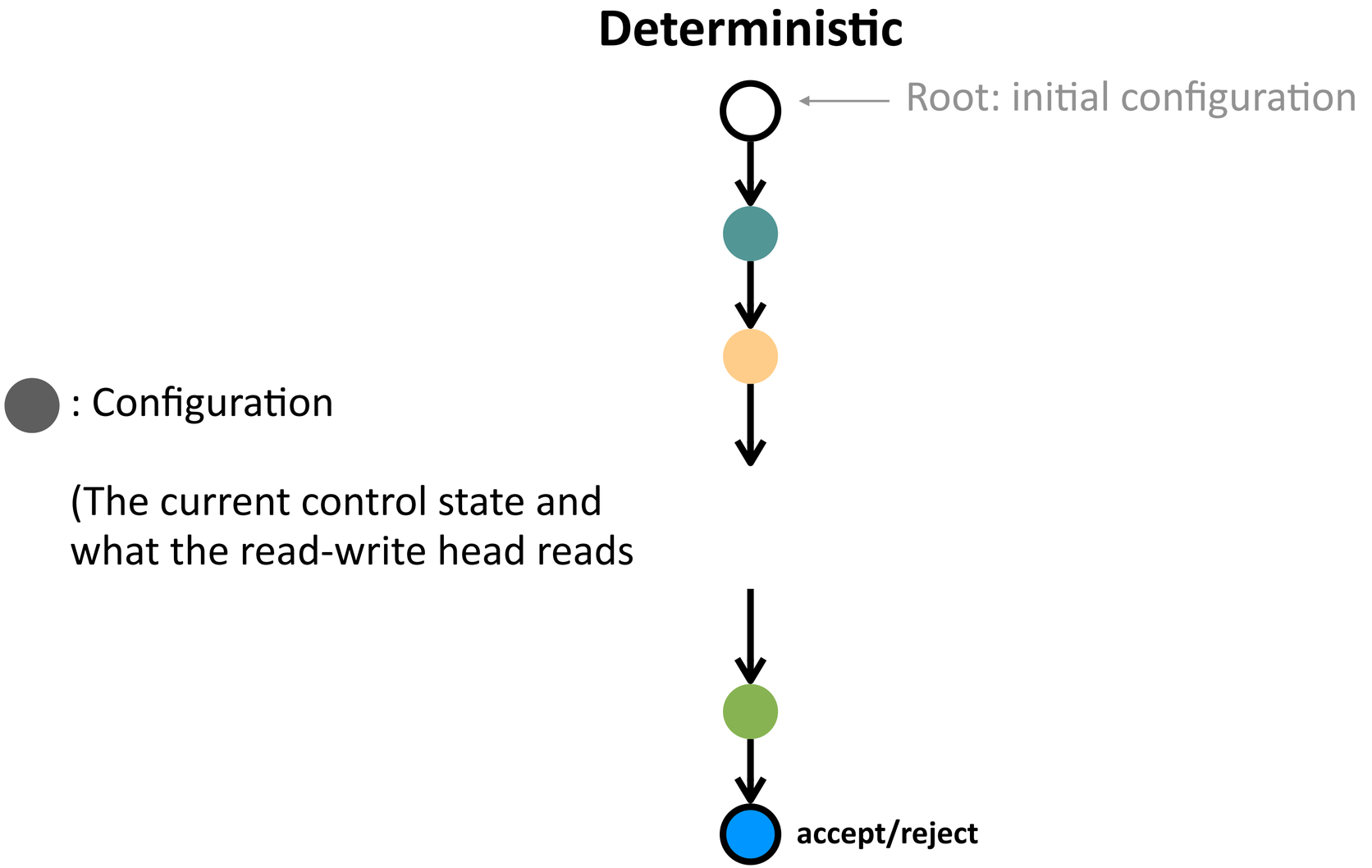

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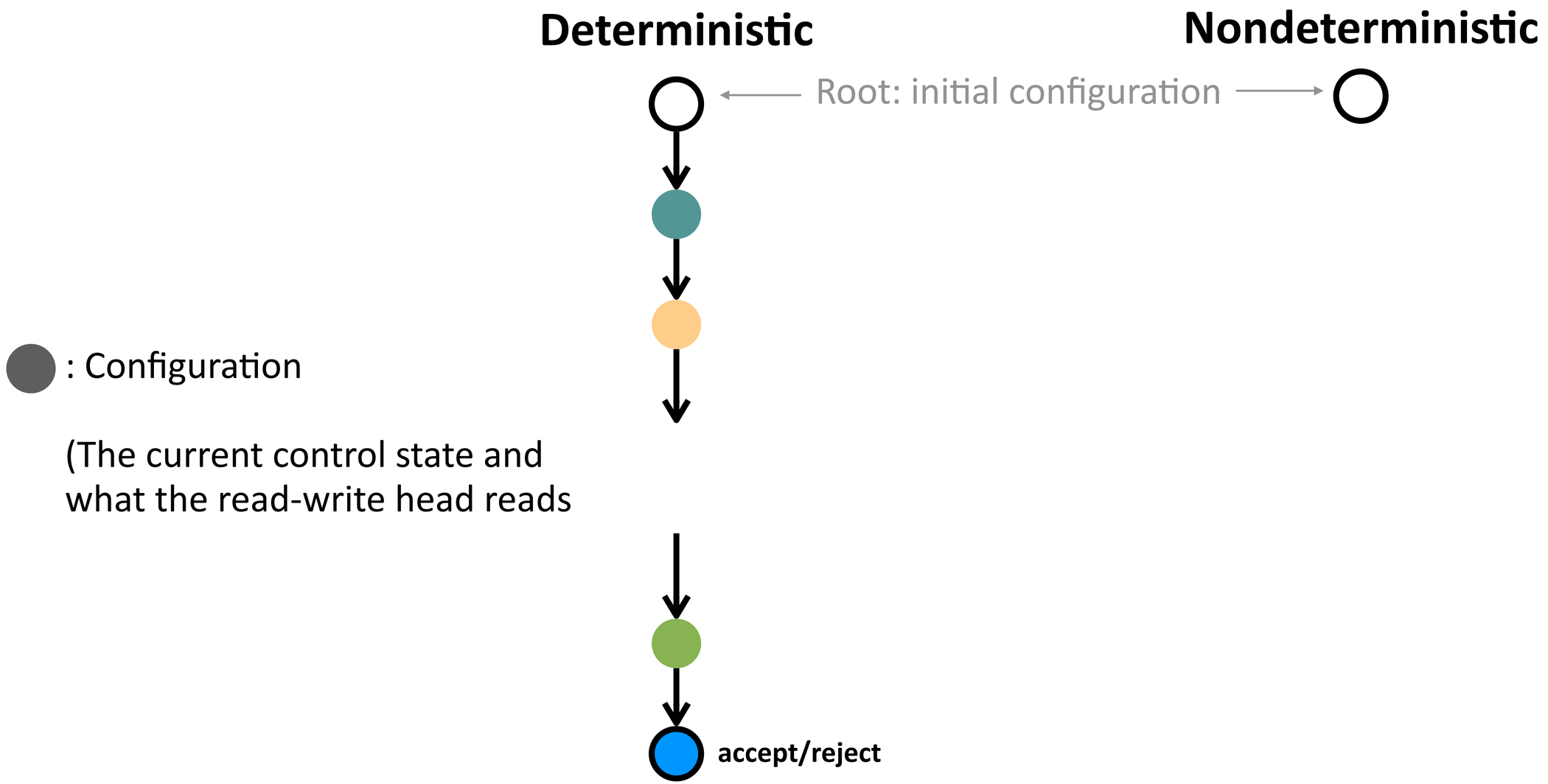

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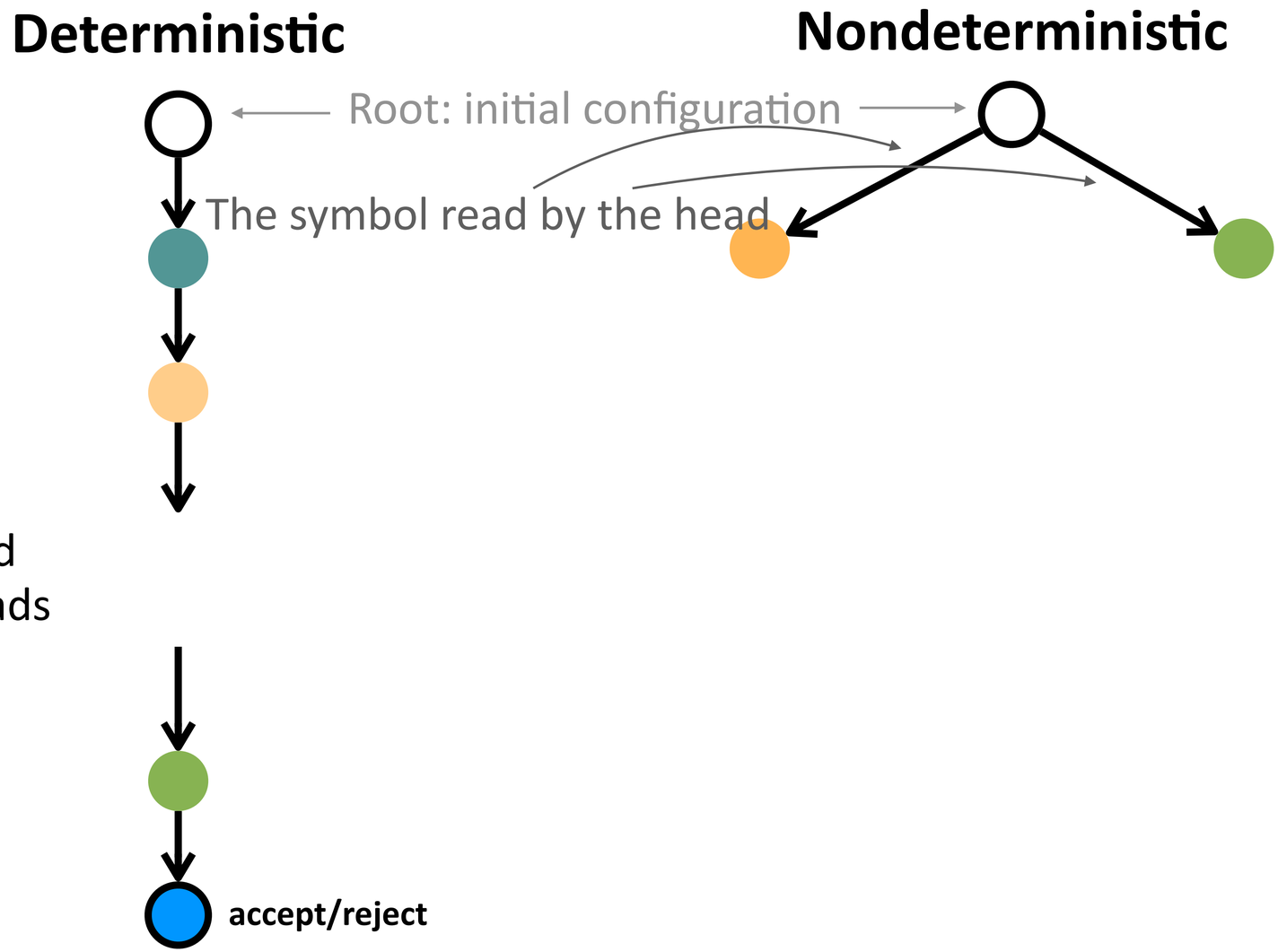

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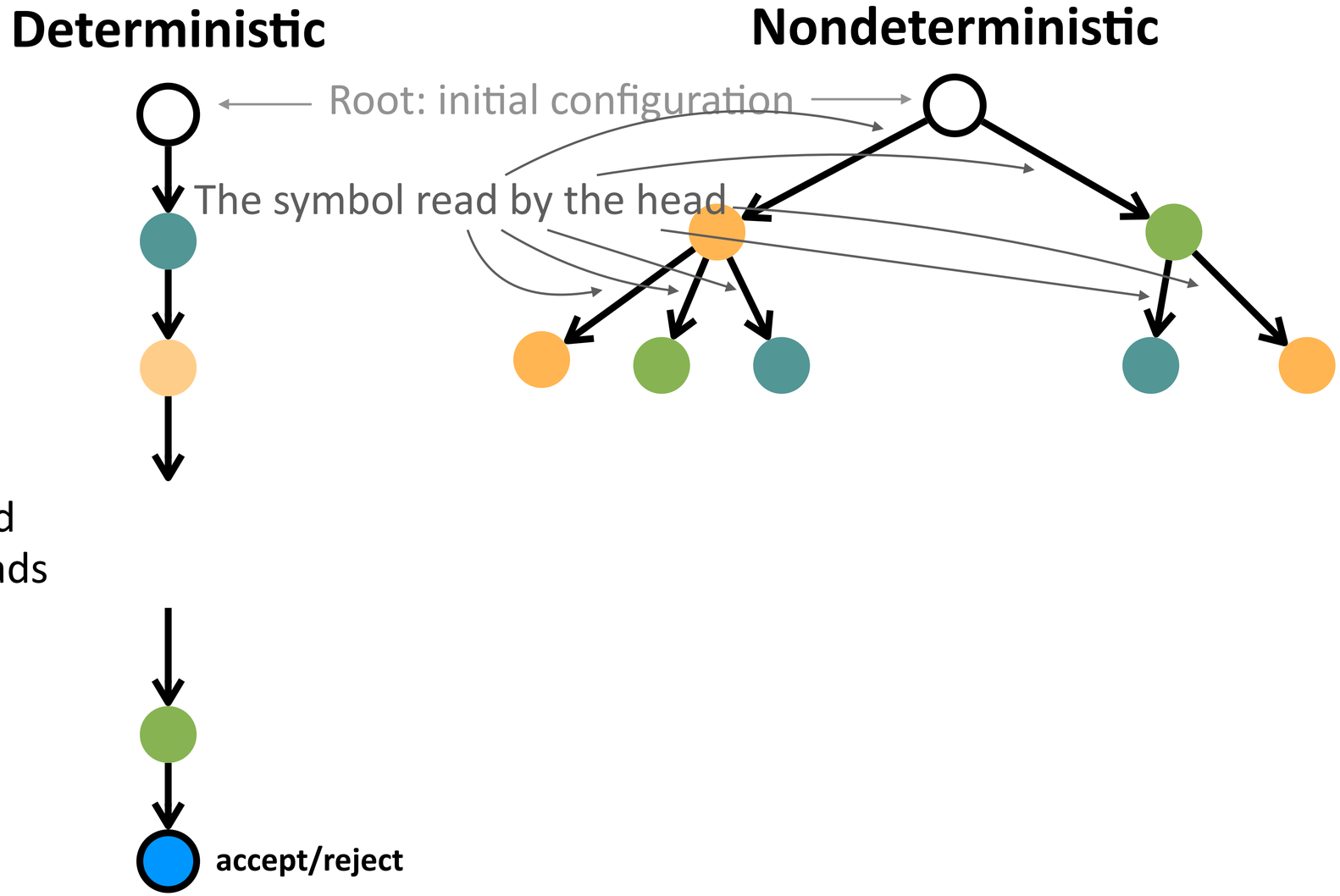

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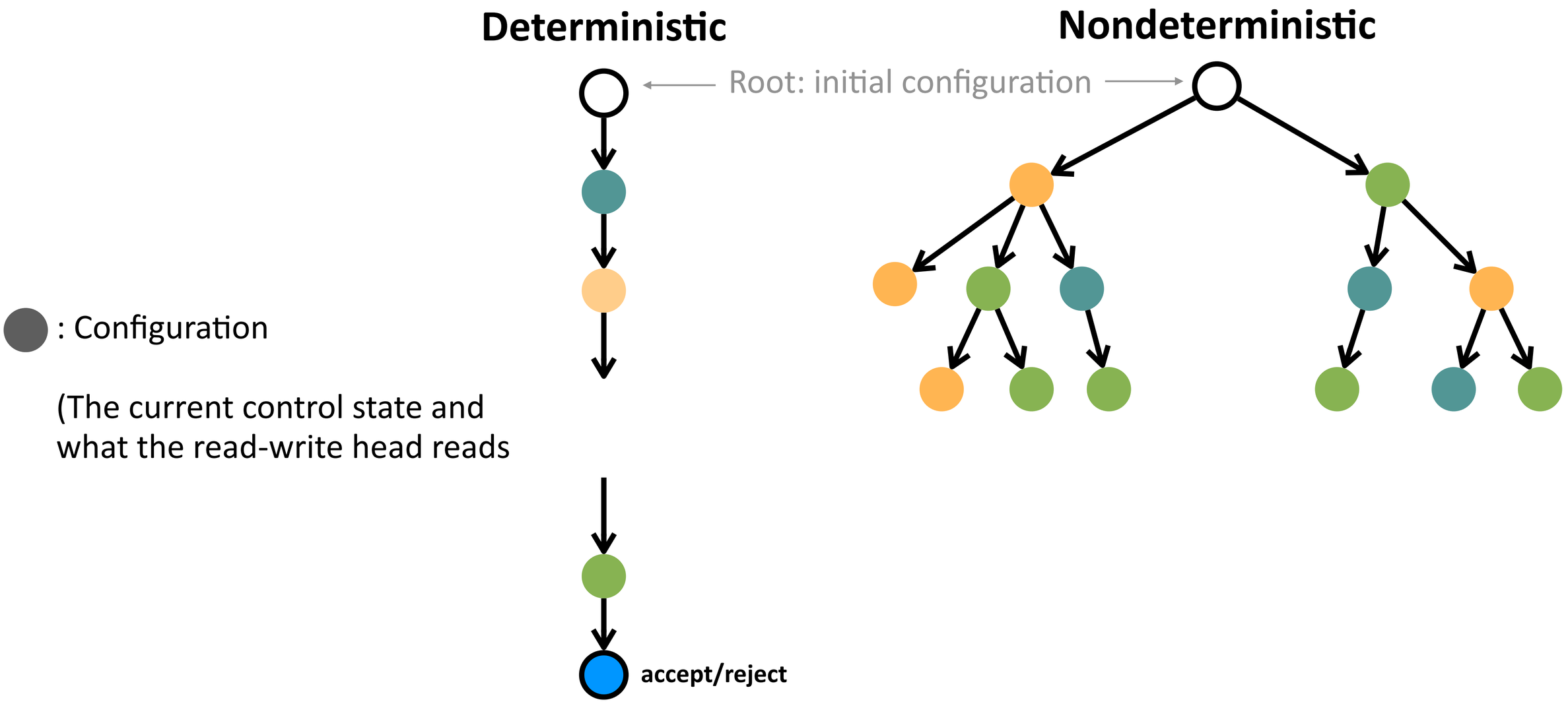

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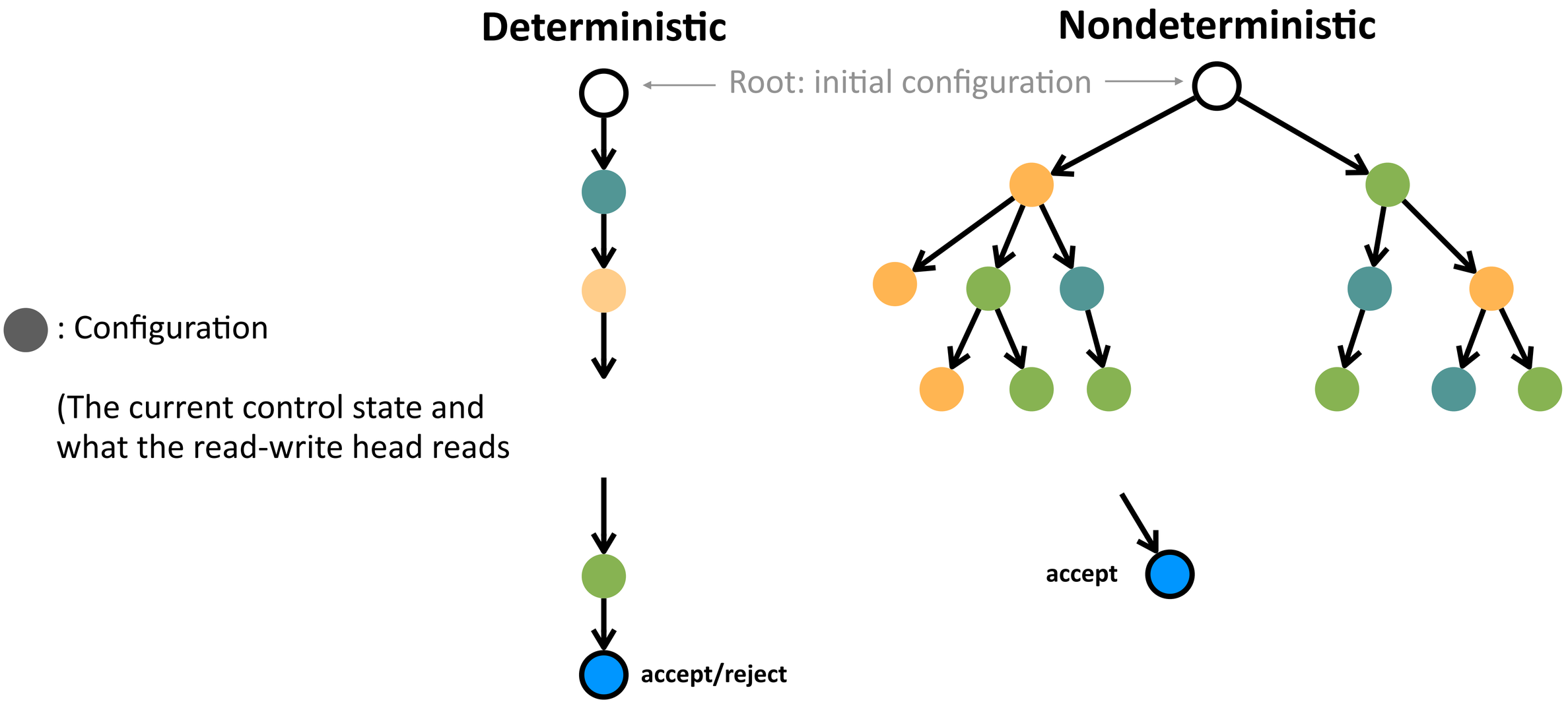

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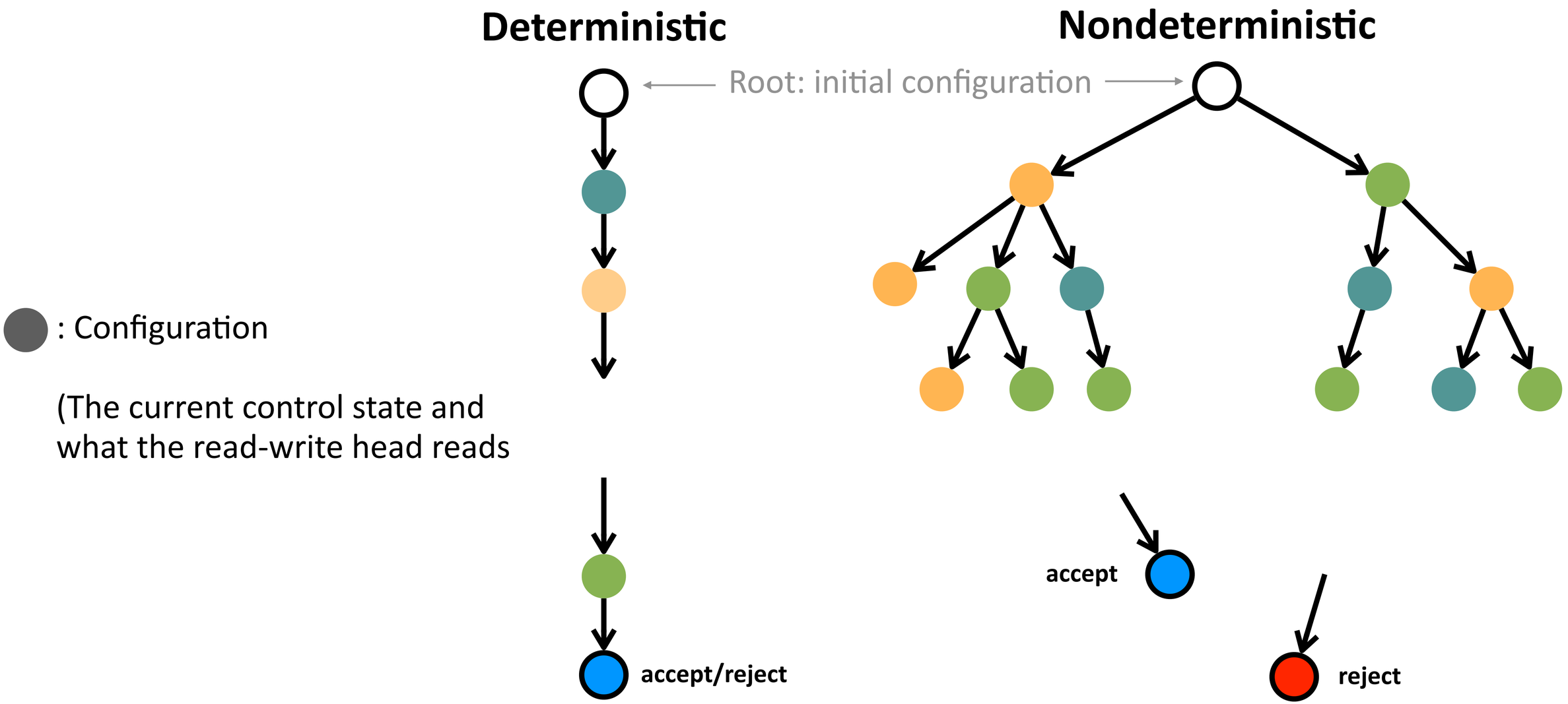

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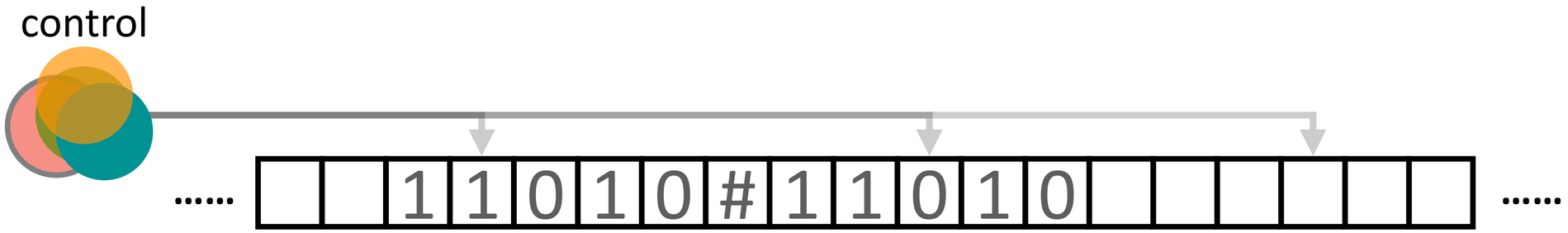

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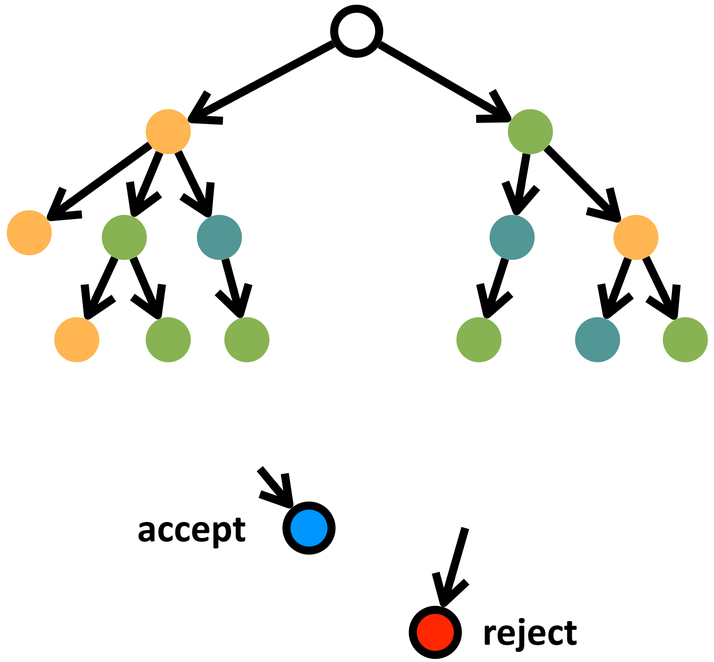

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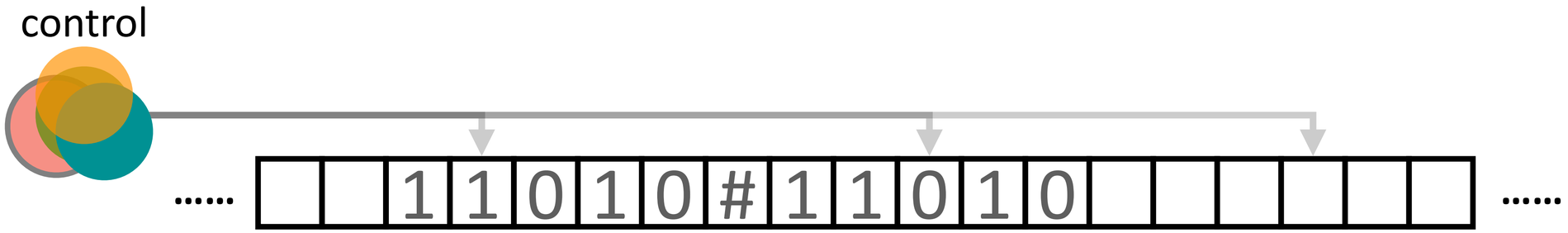

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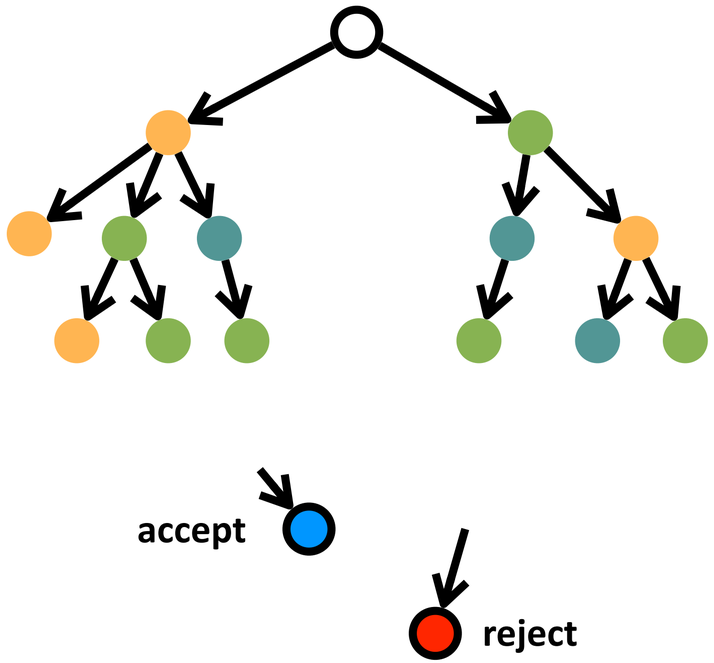
  - Given an state and read a symbol, the non-deterministic Turing machine may enter different states



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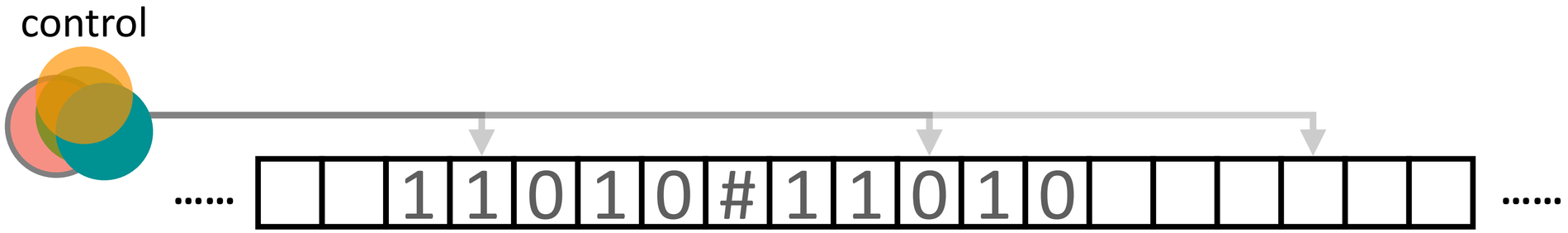
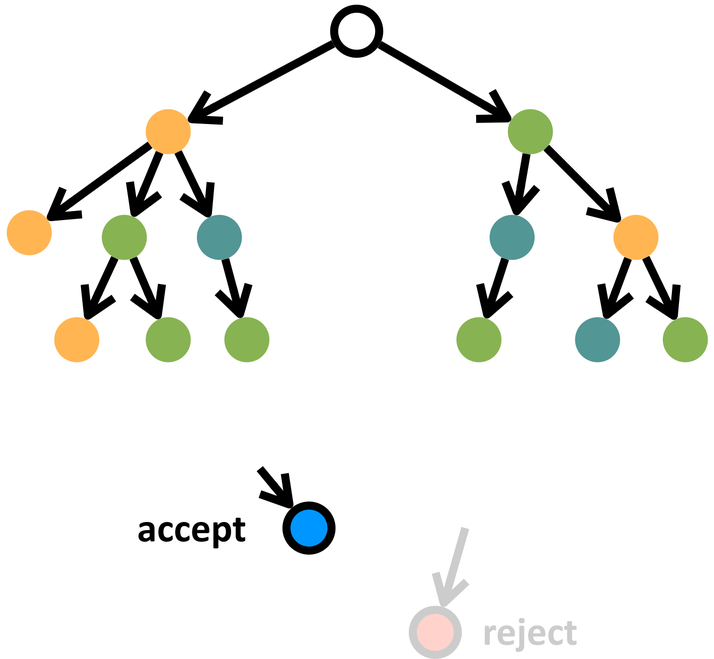
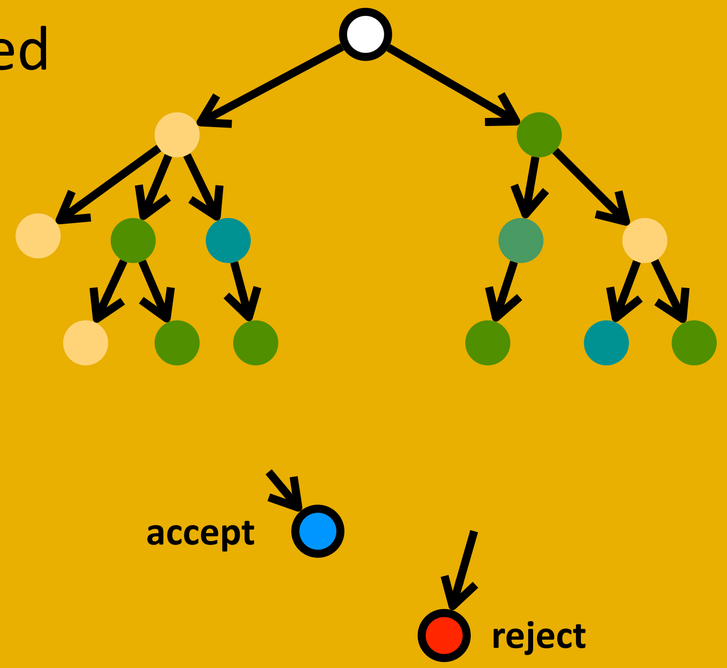


- It is like a (deterministic) Turing machine, but with non-deterministic control
  - Given an state and read a symbol, the non-deterministic Turing machine may enter different states
- The nondeterministic Turing machine ***accepts*** the input w if some branch of computation (i.e., a path from root to some node) leads to the **accept** state



Non-Deterministic Turing machine

- Like the (deterministic) Turing machine, but have non-deterministic behavior
- If there is a path ends at an accept state, the input is accepted



Overview

- Turing machine
  - Deterministic and non-deterministic
- String and language
- Classes **P** and **NP**
  - Verify
  - How to show that a problem is in **P** or in **NP**

A Formal Language Framework for *Problems*

- Initially, there is a string of symbols on the Turing machine tape

A Formal Language Framework for *Problems*

- Definition: A ***language*** is a set of strings
  - $\{1, 01, 001, 0001, 00001, \ldots, 0^*1\}$
  - $\{0, 1\}^*$ Example: 010, 0, 1, 11111, ⋯

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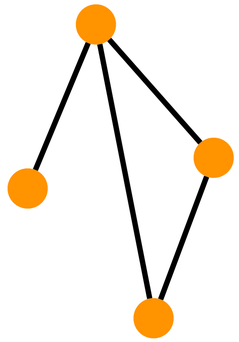

We use $\langle \cdot \rangle$ to represent an encoding of $\cdot$

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  - $\{\langle G \rangle \mid G$ is connected graph$\}$
  - $\{\langle p \rangle \mid p$ is a polynomial with an integral root$\}$

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A Formal Language Framework for *Problems*

- Definition: A ***language*** is a set of strings

- Given a language L = $\{1, 01, 001, 0001, 00001, \ldots, 0^*1\}$ and a string w = 10110, is w in L?

A Formal Language Framework for *Problems*

- Definition: A ***language*** is a set of strings
  - Language ⇔ problem
  - String ⇔ instance

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- Given a language L = {prime number} and a string w = 15, is w in L?

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  ⇔ Is w a prime number?

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  ⇔ Is f a polynomial with an integral root?

Yes-Instance and No-Instance

- Consider a language A. For any instance w, either $w \in A$ or $w \notin A$.
  - If $w \in A$, we call w a **yes-instance** (that is, a correct algorithm should return *accept* or *yes*)
  - If $w \notin A$, we call w a **no-instance** (that is, a correct algorithm should return *reject* or *no*)

What Happened

- Following the vein of Turing machine concept, a **language** is a set of **strings**
  - Language ⇔ **problem**
  - String ⇔ **instance**
  - Asking if a string is in a language
    ⇔ if the instance satisfies the property that the problem asks

- Given a problem/language, a instance/string is a
  - yes instance: an instance that satisfies the property that the problem asks
  - no instance: an instance that does not satisfy the property that the problem asks

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Time Complexity

Time Complexity

- Definition: Let M be a deterministic Turing machine that accepts or rejects all inputs. The ***running time*** or ***time complexity*** of M is the function $f : \mathcal{N} \to \mathcal{N}$, where $f(n)$ is the maximum number of steps that M uses on any input of length n.

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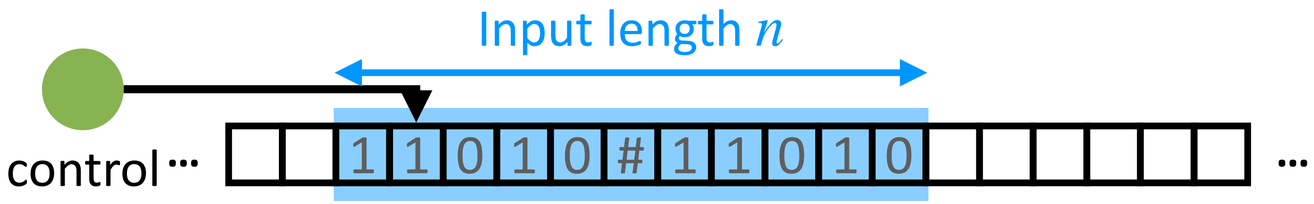
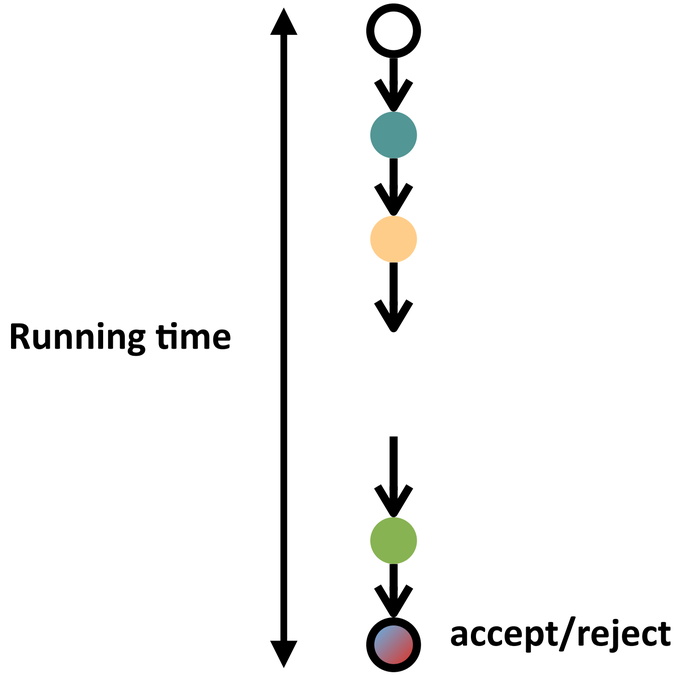
  - Ex: $f(n) = O(n^2)$
  - Ex: $f(n) = O(2^n)$





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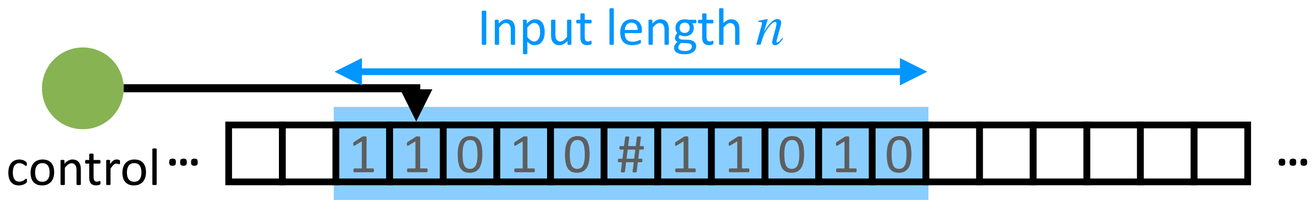
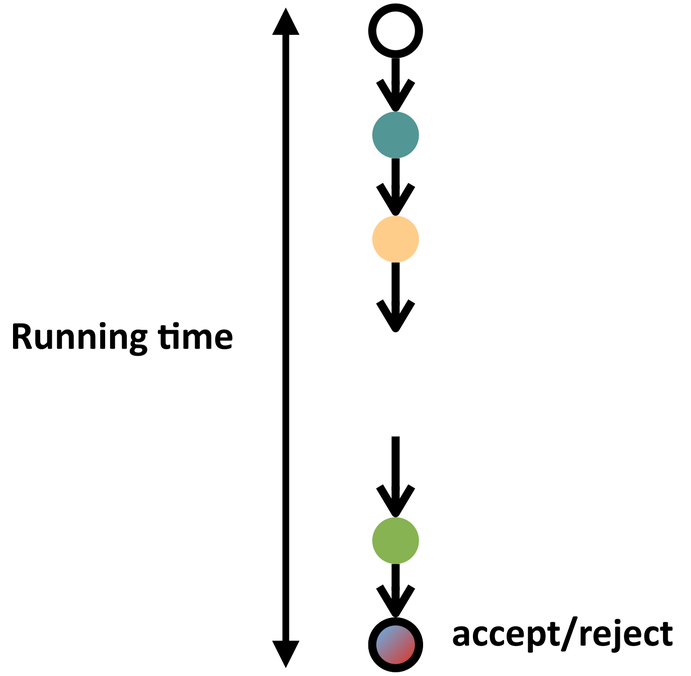
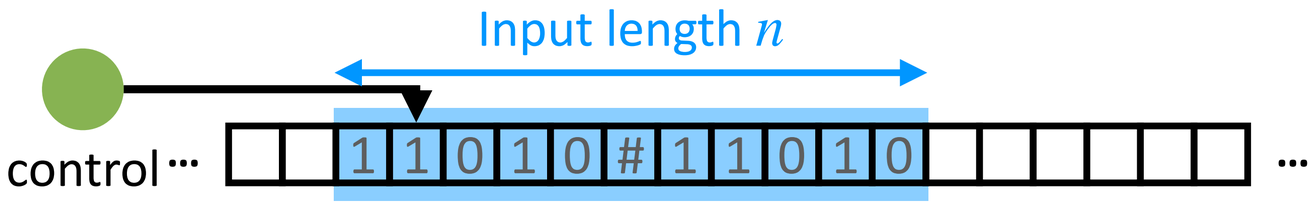
  - If $f(n)$ is the running time of M, we say that M is an $f(n)$ *time Turing machine*

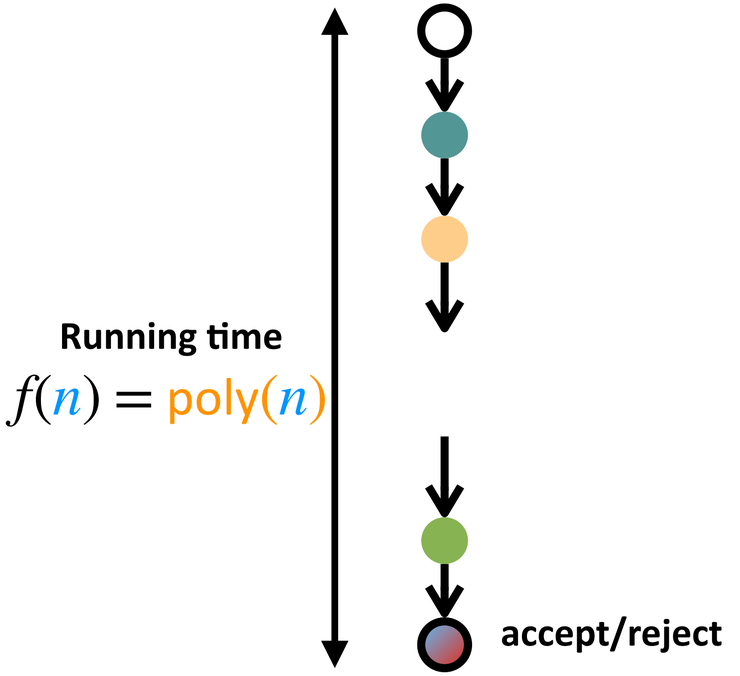




The Class **P**

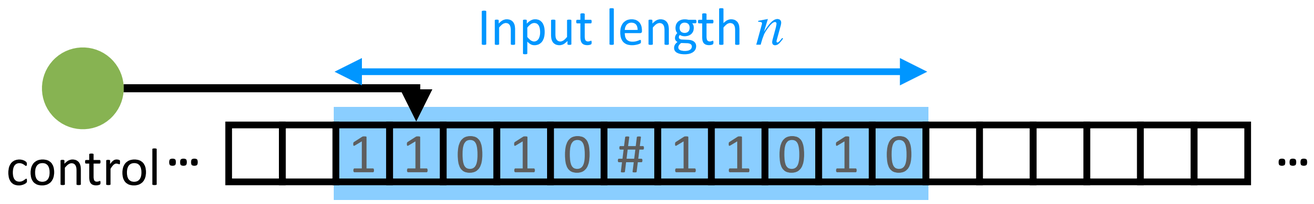
- Definition: **P** is the class of languages that are can be accepted or rejected in polynomial time by a deterministic single-tape Turing machine.

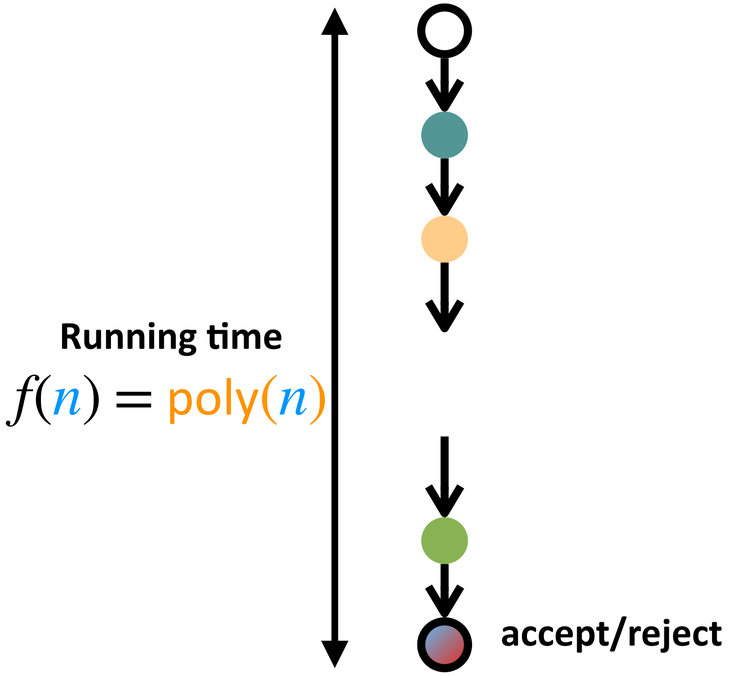




The Class **P**

- Definition: **P** is the class of languages that are can be accepted or rejected in polynomial time by a deterministic single-tape Turing machine.
  - **P** roughly corresponds to the class of problems that are realistically solvable on a computer





The Class **NP**

The Class **NP**

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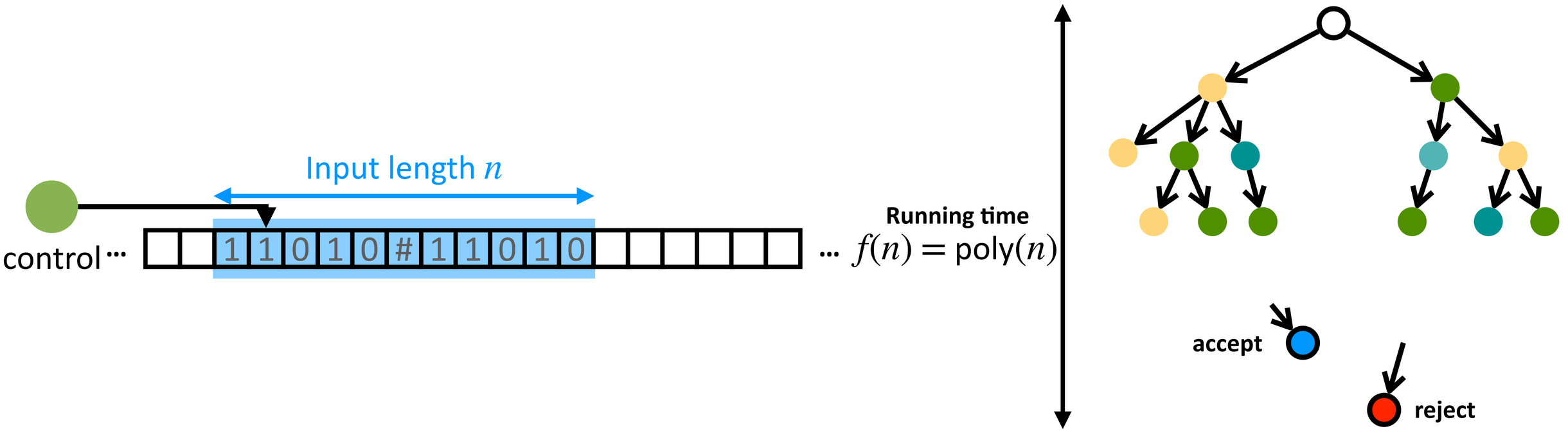
- Similarly, we can define the running time of a non-deterministic Turing machine N



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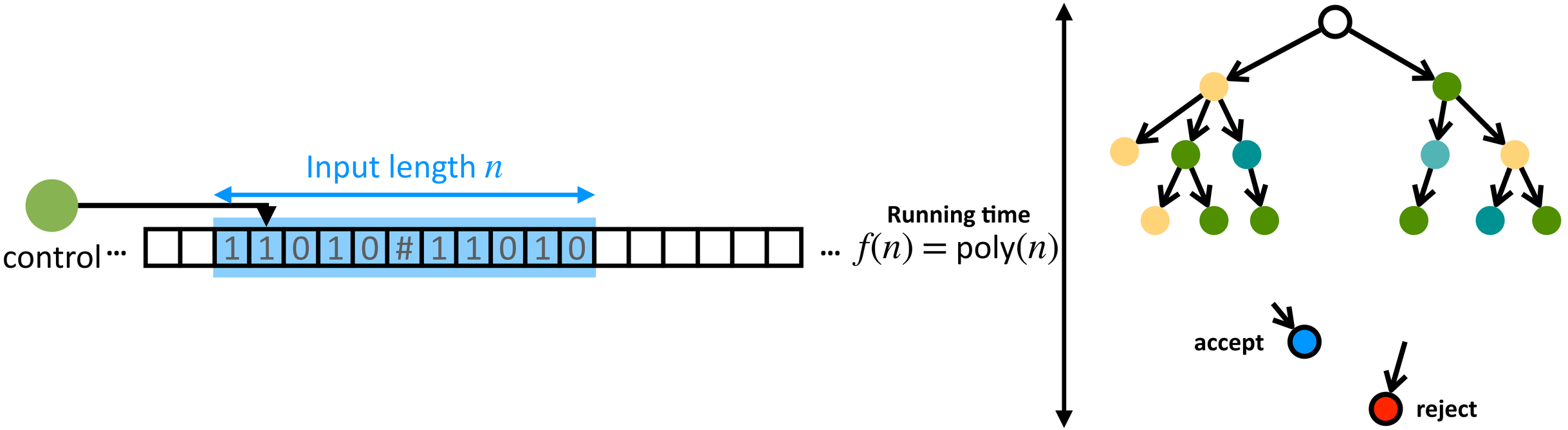
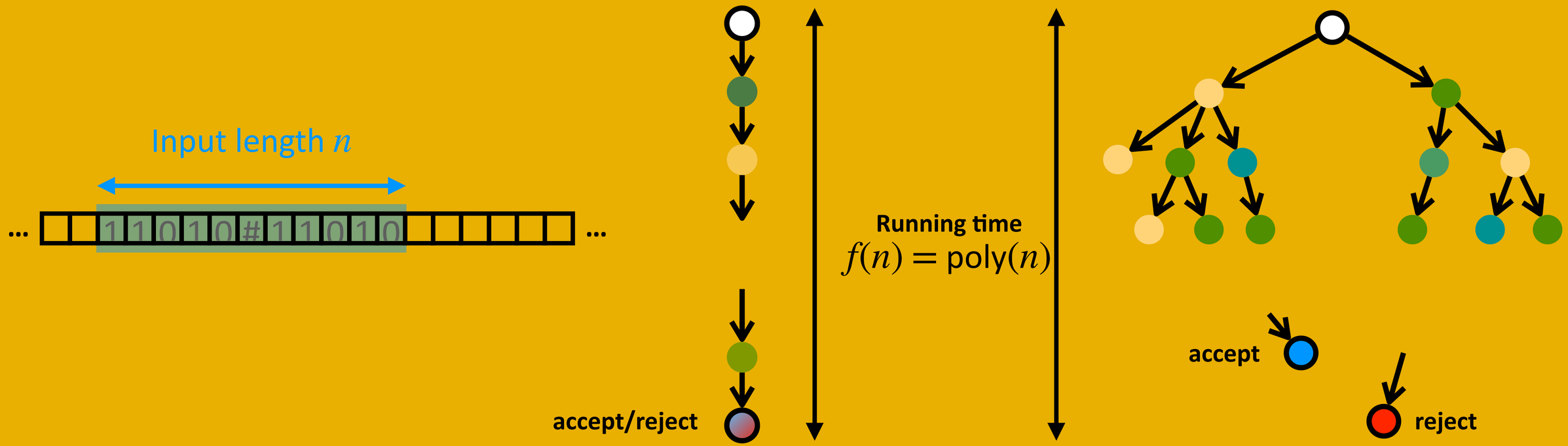
- Similarly, we can define the running time of a non-deterministic Turing machine N
- Definition: **NP** is the class of languages that are accepted or rejected in polynomial time by a nondeterministic Turing machine.



What Happened

- The class **P** is the class of languages that are accepted or rejected in polynomial time by a *deterministic* Turing machine
- The class **NP** is the class of languages that are accepted or rejected in polynomial time by a *non-deterministic* Turing machine.



Overview

- Turing machine
  - Deterministic and non-deterministic
- String and language
- Classes **P** and **NP**
  - Verify
  - How to show that a problem is in **P** or in **NP**

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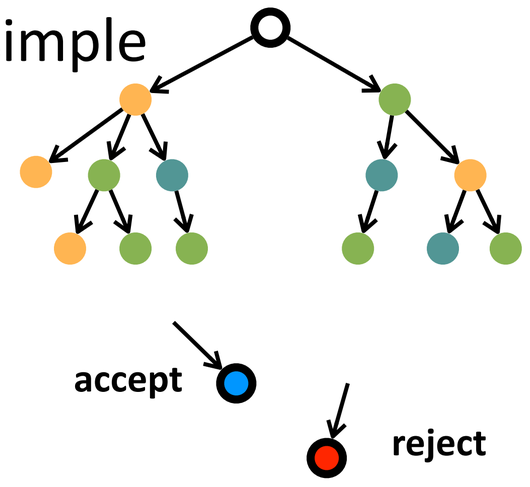

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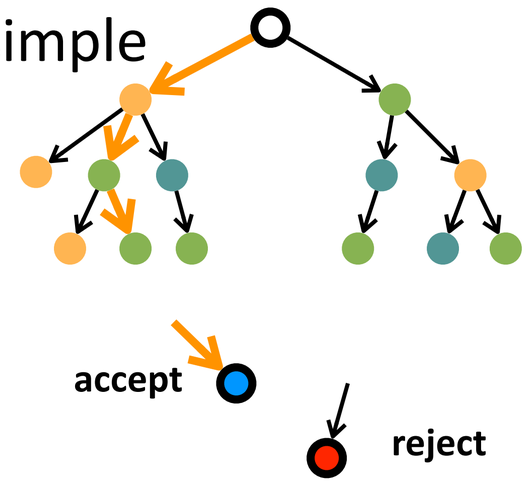

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A set of k vertices with edge between every pair of vertices

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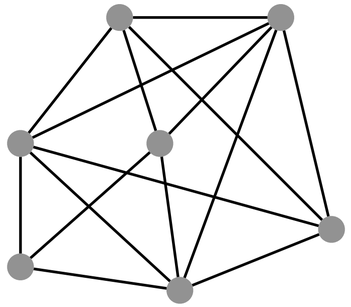

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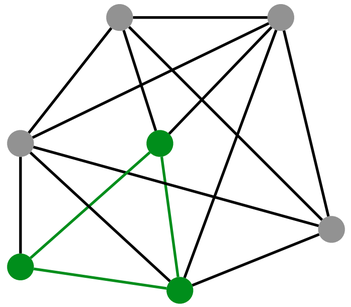

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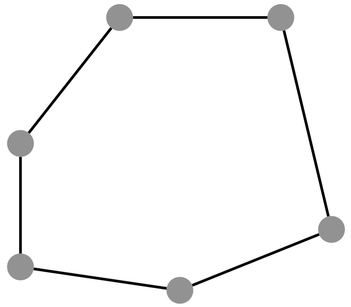

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    - A string c that encodes a clique with size k in G is a good certificate

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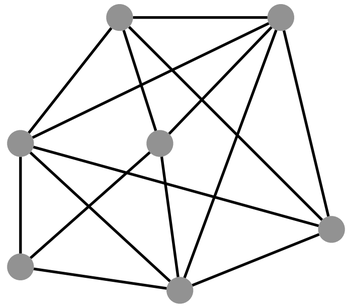

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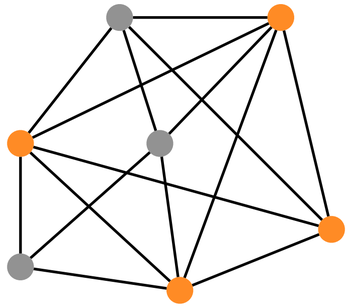

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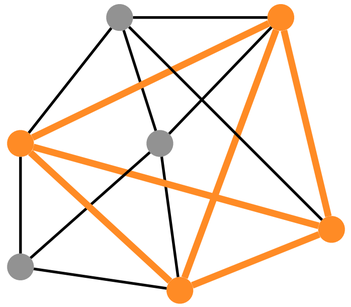

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  - An algorithm V verifies a language A if for any yes-instance $w \in A$, there is a certificate c that V can use to prove that $w \in A$
  - For any no-instance $w \notin A$, there must be no certificate proving that $w \in A$

Polynomial Time Verifier

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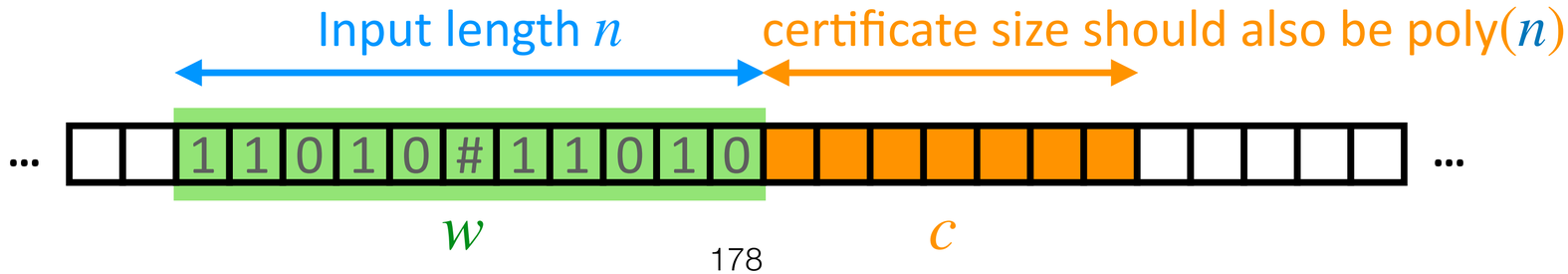
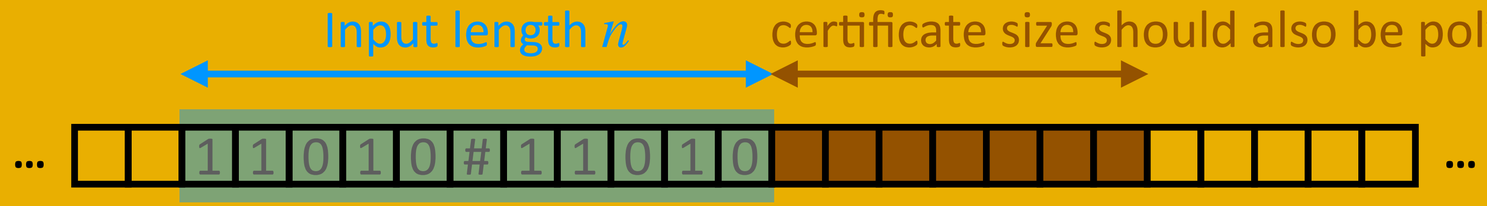
  - For polynomial verifiers, the certificate needs to have polynomial length (in the length of w)



What Happened

- A language A is **verifiable** if for any of its yes-instances w, there exists a piece of hint (certificate) c such that using this hint c, one can be convinced that w is indeed a yes-instance of A
  - Only yes-instances have certificates
- **Polynomial-time verifiable**: the verification can be done in time of polynomial in input length
  - The hint size should also be polynomial
  - It does NOT mean that the hint c should be constructed within polynomial time!



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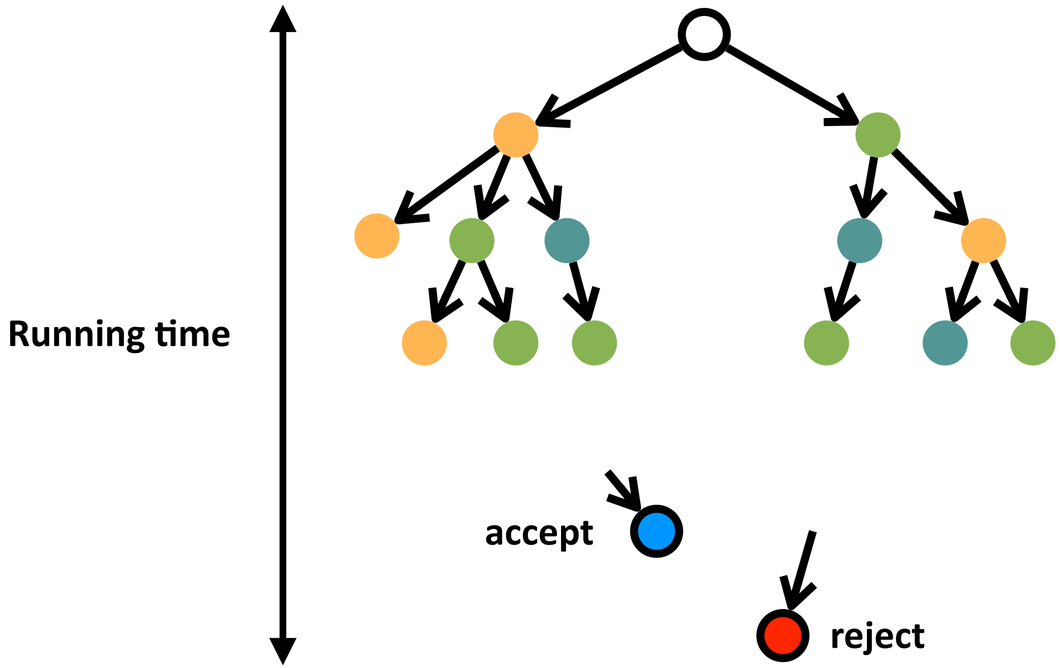

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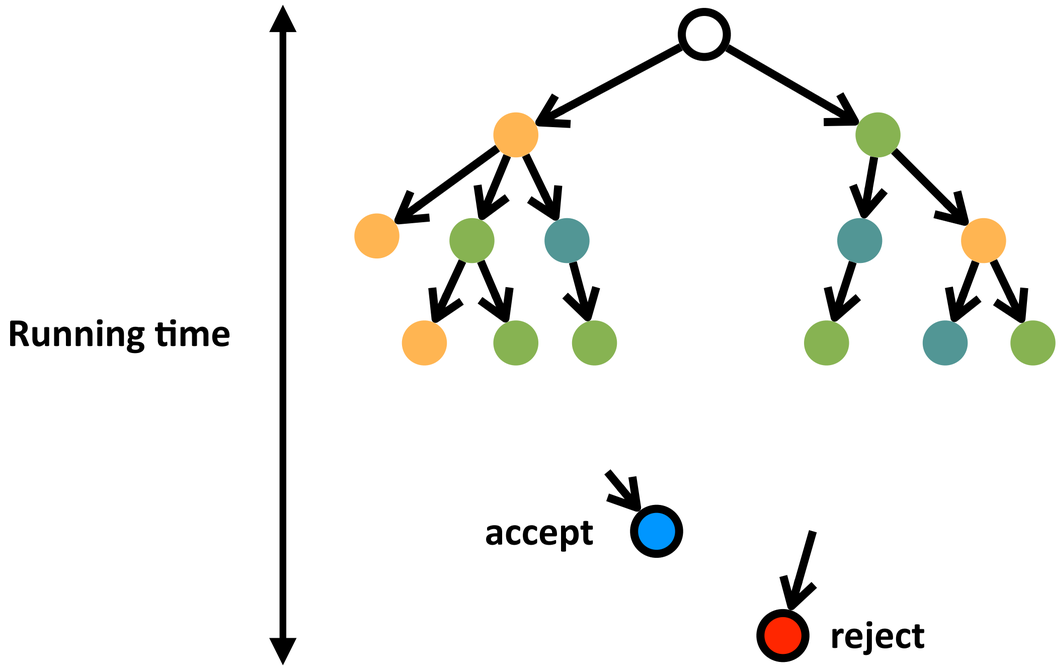

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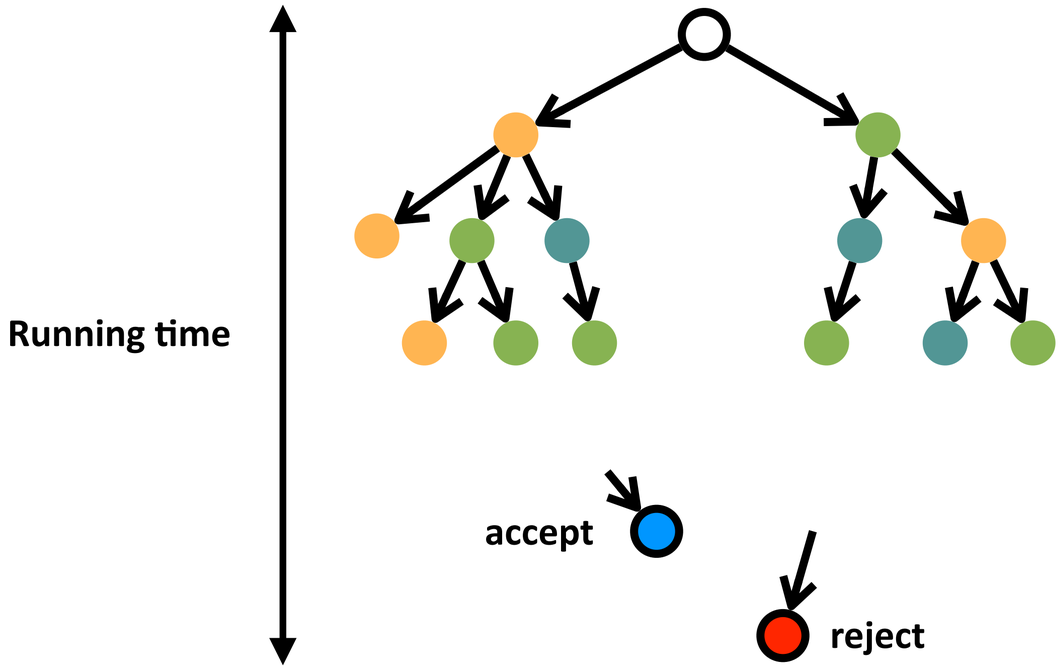

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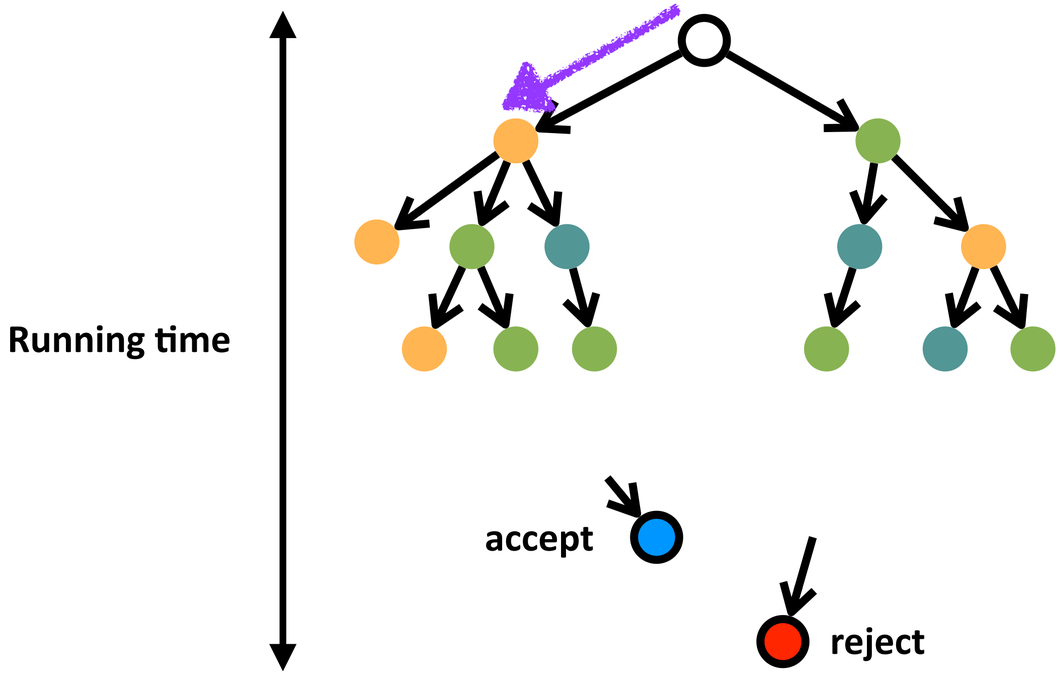

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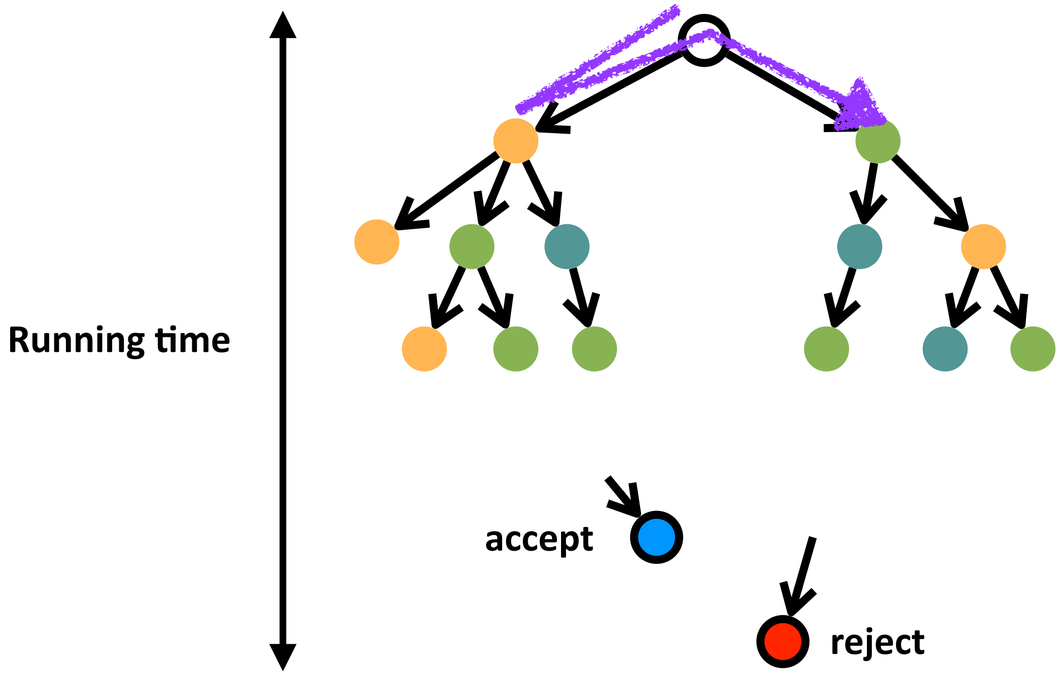

The Class **NP** — Alternative Definition

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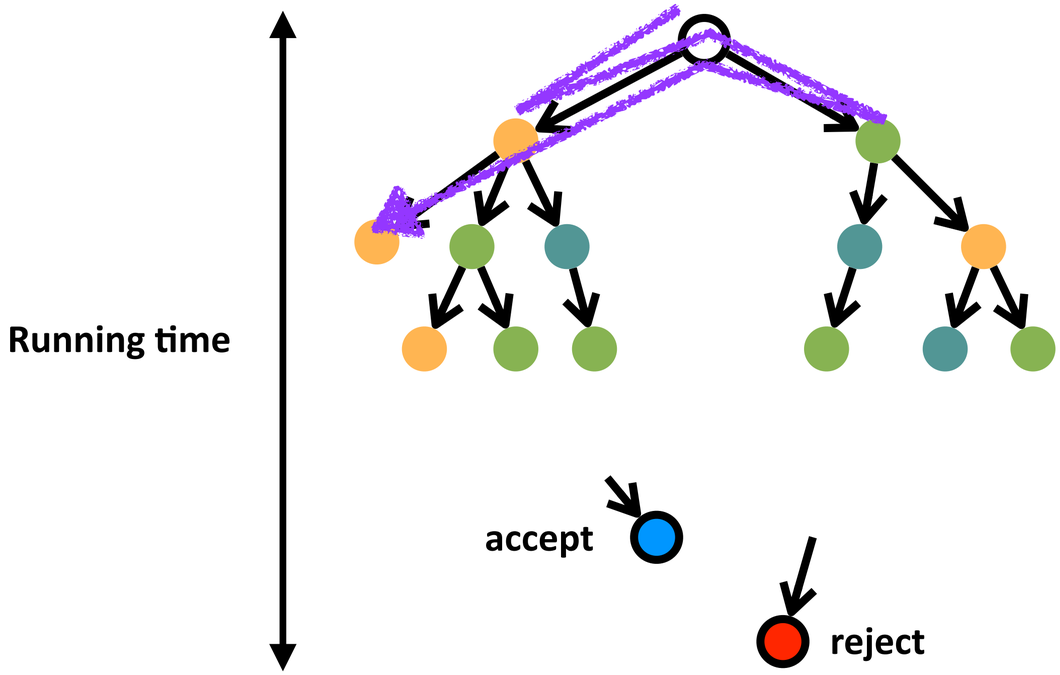

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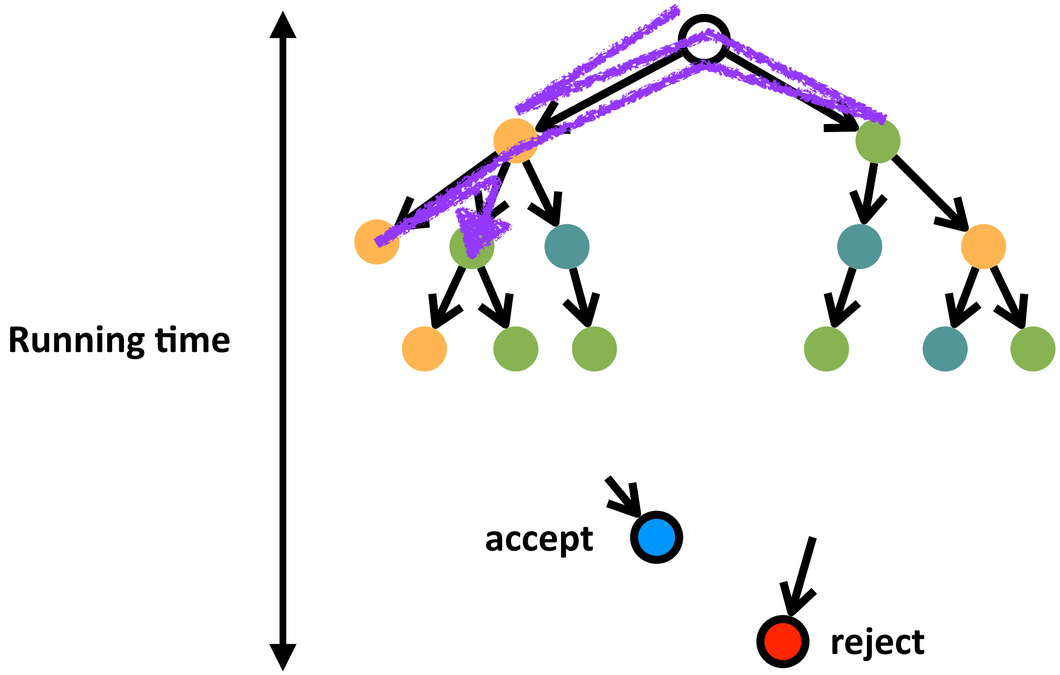

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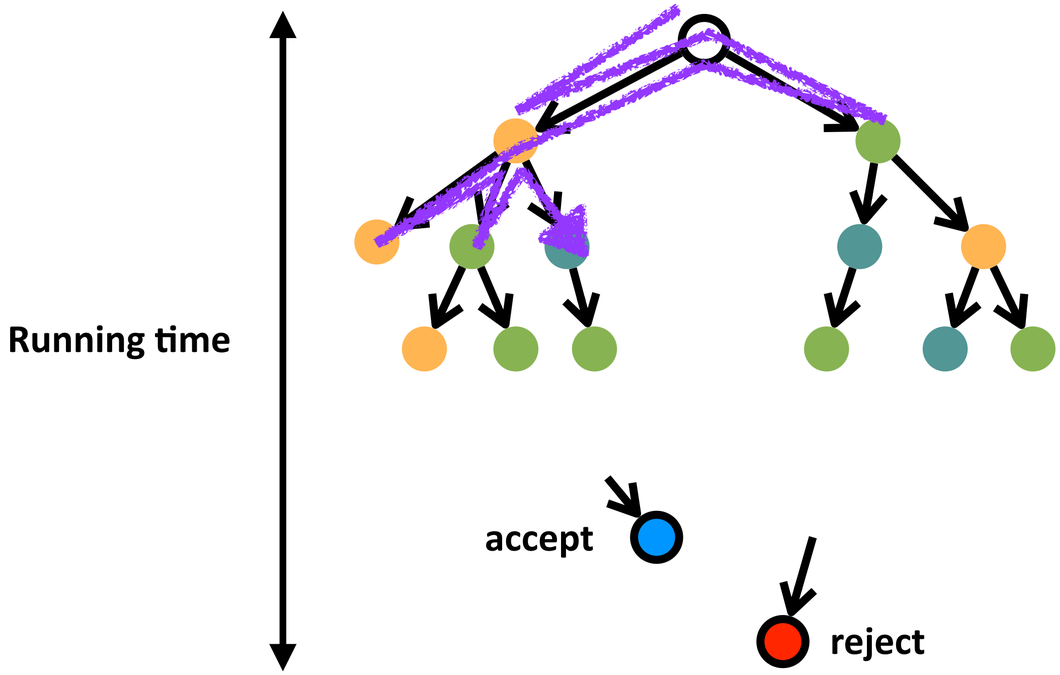

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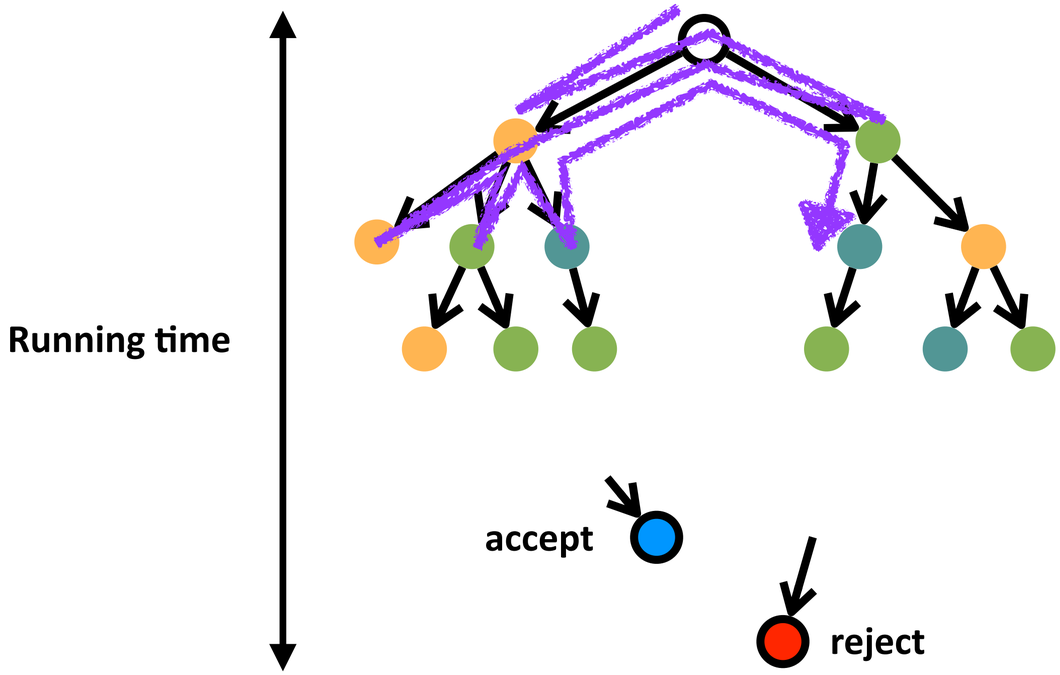

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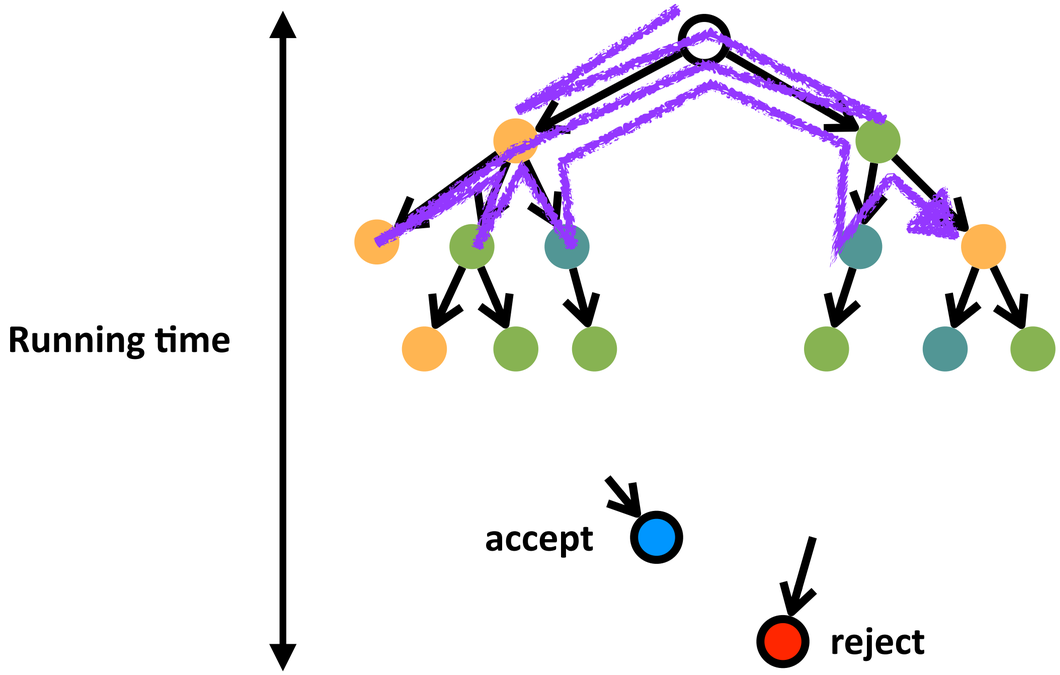

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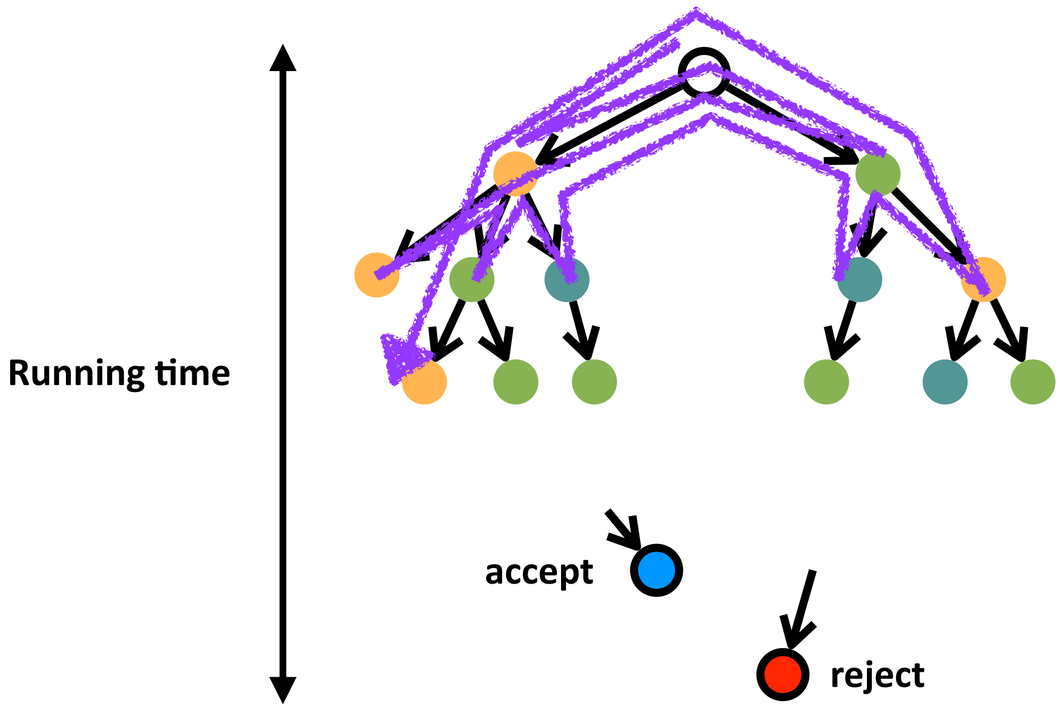

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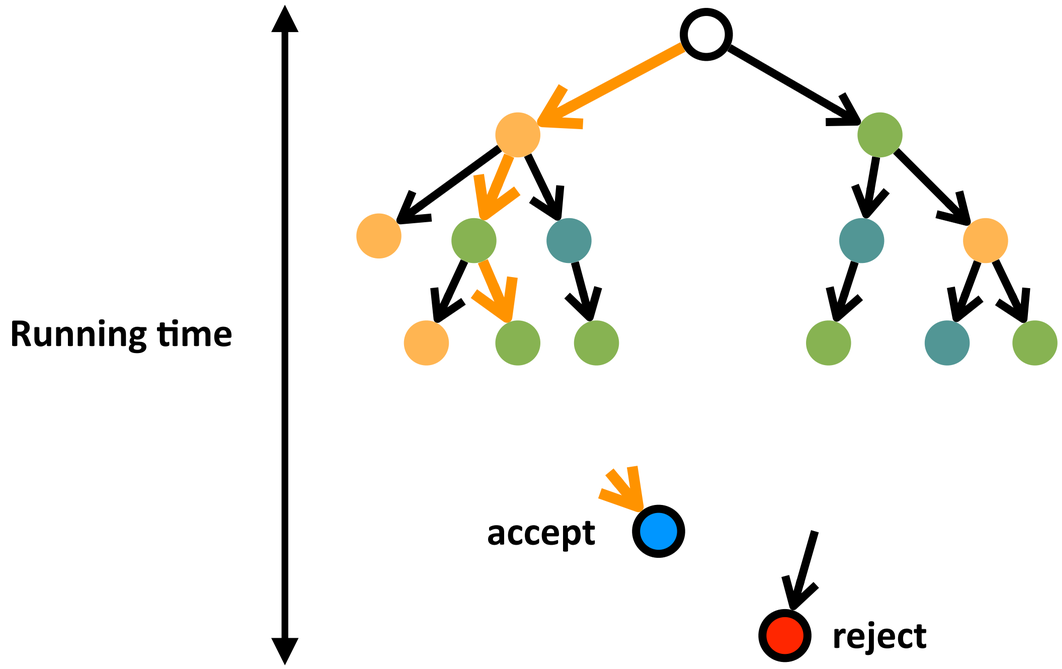
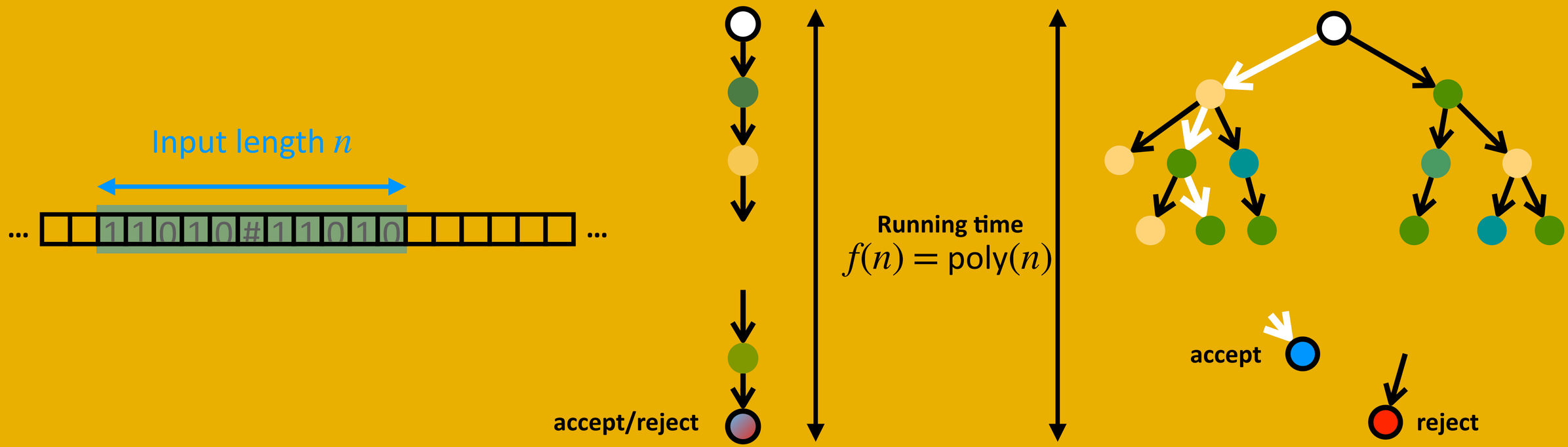
- Any non-deterministic Turing machine can be simulated by a deterministic Turing machine
  - It needs more than polynomial time
  - But if we know some **hint**, we know the path to an accept state with polynomial length



What Happened

- The class **P** is the class of languages that are *accepted* or *rejected* in polynomial time by a deterministic Turing machine
- The class **NP** is the class of languages that can be *verified* in polynomial time by a deterministic Turing machine.



Prove Language L is in **P**

- To prove that a language L is in **P**, we need to:
  - Design a Turing machine M
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Prove Language L is in **P**

- To prove that a language L is in **P**, we need to:
  - Design a Turing machine M — Design an algorithm
  - Show that M correctly accepts or rejects all input — Correctness proof
  - Show that M runs in polynomial time — Time complexity analysis

Use Polynomial Time Verifier to Prove that A is in **NP**

- Definition: A ***verifier*** for a language A is an algorithm V, where

$$A = \{w \mid V \text{ accepts } \langle w, c \rangle \text{ for some string } c\}.$$

Prove A is in **NP** $\Leftrightarrow$ Design a **polynomial time verifier** to decide A (with help from some c)

<Proof Idea>

1. Assume that there is a certificate c.
2. Design a verifier V on input $\langle w, c \rangle$ that *accepts* all $w \in A$ and *rejects* all $w \notin A$
3. Show that V runs in polynomial time (in the length of w)

CLIQUE is in **NP**

- CLIQUE $= \{\langle G, k\rangle \mid G$ is an undirected graph with a k-clique$\}$
  - A string c that encodes a clique with size k in G is a good certificate

Prove CLIQUE is in **NP** ⇔ Design a **polynomial time verifier** to decide CLIQUE (with help from some c)

<Proof Idea>

1. Assume that there is a certificate c that encodes a clique with size k in G.
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Let string c that encodes a clique with size k in G as a certificate

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Let string c that encodes a clique with size k in G as a certificate

V = "On input $\langle\langle G, k\rangle, c\rangle$:

1. Test whether c is a set of k nodes in G
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3. If both 1 and 2 pass, *accept*; otherwise, *reject*."

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SUBSET-SUM is in **NP**

- SUBSET-SUM $= \{\langle S, t\rangle \mid S = \{x_1, \cdots, x_k\}$ and for some $\{y_1, \cdots, y_m\} \subseteq \{x_1, \cdots, x_k\}$, we have $\Sigma y_i = t\}$
- Prove SUBSET-SUM is in **NP** ⇔ Design a **polynomial time verifier** to decide CLIQUE (with help from some c)

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What Happened

- To show that a problem is in **NP**, we can show that it is polynomial-time verifiable

<Proof Idea>

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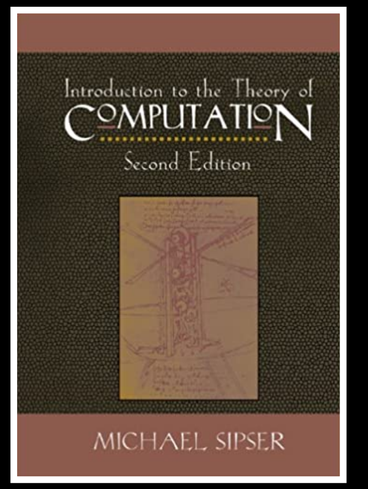
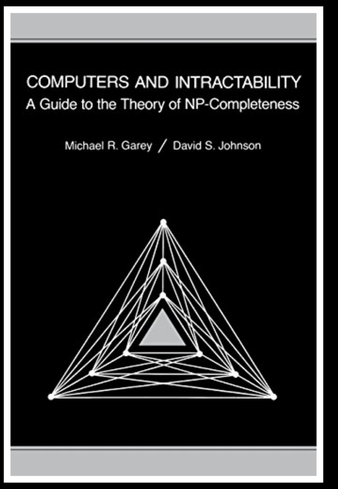
3. Show that V runs in polynomial time (in the length of w)

Why **P** and **NP**?

- There are many $f(n)$ time Turing machine variations that have an equivalent poly($f(n)$) time single-tape Turing machine
- Every $f(n)$ time non-deterministic Turing machine has an equivalent $2^{O(f(n))}$ time deterministic Turing machine
- There is at most a square or polynomial difference between the time complexity of problems measured on deterministic single-tape and many Turing machine variations
- There is at most an exponential difference between the time complexity of problems on deterministic and nondeterministic Turing machines

Reference





- *Introduction to the Theory of Computation* by Michael Sipser
- *Computers and Intractability - A Guide to the Theory of NP-Completeness* by Michael R. Garey and David S. Johnson
- *Introduction to Algorithms* by Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein
- *Computational Complexity* by Christos h. Papadimitriou