Online Computation with Untrusted Advice[*]

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Abstract

The advice model of online computation captures the setting in which the online algorithm is given some partial information concerning the request sequence. This paradigm allows to establish tradeoffs between the amount of this additional information and the performance of the online algorithm. However, unlike real life in which advice is a recommendation that we can choose to follow or to ignore based on trustworthiness, in the current advice model, the online algorithm treats it as infallible. This means that if the advice is corrupt or, worse, if it comes from a malicious source, the algorithm may perform poorly. In this work, we study online computation in a setting in which the advice is provided by an untrusted source. Our objective is to quantify the impact of untrusted advice so as to design and analyze online algorithms that are robust and perform well even when the advice is generated in a malicious, adversarial manner. To this end, we focus on well- studied online problems such as ski rental, online bidding, bin packing, and list update. For ski-rental and online bidding, we show how to obtain algorithms that are Pareto-optimal with respect to the competitive ratios achieved; this improves upon the framework of Purohit *et al.* [NeurIPS 2018] in which Pareto-optimality is not necessarily guaranteed. For bin packing and list update, we give online algorithms with worst-case tradeoffs in their competitiveness, depending on whether the advice is trusted or not; this is motivated by work of Lykouris and Vassilvitskii [ICML 2018] on the paging problem, but in which the competitiveness depends on the reliability of the advice. Furthermore, we demonstrate how to prove lower bounds, within this model, on the tradeoff between the number of advice bits and the competitiveness of any online algorithm. Last, we study the effect of randomization: here we show that for ski-rental there is a randomized algorithm that Pareto-dominates any deterministic algorithm with advice of any size. We also show that a single random bit is not always inferior to a single advice bit, as it happens in the standard model.

1   Introduction

Suppose that you have an investment account with a significant amount in it, and that your financial institution advises you periodically on investments. One day, your banker informs you that company X will soon receive a big boost, and advises to use the entire account to buy stocks. If you were to completely trust the bankers advice, there are naturally two possibilities: either the advice will prove correct (which would be great) or it will prove wrong (which would be catastrophic). A prudent customer would take this advice with a grain of salt, and would not be willing to risk everything. In general, our understanding of advice is that it entails *knowledge that is not foolproof.*

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In this work we focus on the online computation with advice. Our motivation stems from observing that, unlike the real world, the advice under the known models is often closer to "fiat" than "recommendation". Our objective is to propose a model which allows the possibility of incorrect advice, with the objective of obtaining more realistic and robust online algorithms.

Online computation and advice complexity. In the standard model of online computation that goes back to the seminal work of Sleator and Tarjan [25], an online algorithm receives as input a sequence of *requests*. For each request in this sequence, the algorithm must make an irrevocable decision concerning the item, without any knowledge of future requests. The performance of an online algorithm is usually evaluated by means of the competitive ratio, which is the worst-case ratio of the cost incurred by the algorithm (assuming a minimization problem) to the cost of an ideal solution that knows the entire sequence in advance.

In practice, however, online algorithms are often provided with some (limited) knowledge of the input, such as lookahead on some of the upcoming requests, or knowledge of the input size. While competitive analysis is still applicable, especially from the point of view of the analysis of a known, given algorithm, a new model was required to formally quantify the power and limitations of offline information. The term *advice complexity* was first coined by Dobrev *et al.* [11], and subsequent formal models were presented by Böckenhauer *et al.* [5] and Emek *et al.* [12], with this goal in mind. More precisely, in the advice setting, the online algorithm receives some bits that encode information concerning the sequence of input items. As expected, this additional information can boost the performance of the algorithm, which is often reflected in better competitive ratios.

Under the current models, the advice bits can encode any information about the input sequence; indeed, defining the "right" information to be conveyed to the algorithm plays an important role in obtaining better online algorithms. Clearly, the performance of the online algorithm can only improve with larger number of advice bits. The objective is thus to identify the exact trade-offs between the size of the advice and the performance of the algorithm. This is meant to provide a smooth transition between the purely online world (nothing is known about the input) and the purely "offline" world (everything is known about the input). In the last decade, a substantial number of online optimization problems have been studied in the advice model; we refer the reader to the survey of Boyar *et al.* [6] for an in-depth discussion of developments in this field.

As argued in detail in [6], there are compelling reasons to study the advice complexity of online computation. Lower bounds establish strict limitations on the power of any online algorithm; there are strong connections between randomized online algorithms and online algorithms with advice (see, e.g., [15]); online algorithms with advice can be of practical interest in settings in which it is feasible to run multiple algorithms and output the best solution (see [16] about obtaining improved data compression algorithms by means of list update algorithms with advice); and the first complexity classes for online computation have been based on advice complexity [7].

Notwithstanding such interesting attributes, the known advice model has certain drawbacks. The advice is always assumed to be some error-free information that may be used to encode some property often explicitly connected to the optimal solution. In many settings, one can argue that such information cannot be readily available, which implies that the resulting algorithms are often impractical.

Online computation with untrusted advice. In this work, we address what is a significant drawback in the online advice model. Namely, all previous works assume that advice is, in all circumstances, completely trustworthy, and precisely as defined by the algorithm. Since the advice is infallible, no reasonable online algorithm with advice would choose to ignore the advice.

It should be fairly clear that such assumptions are very unrealistic or undesirable. Advice bits, as all information, are prone to transmission errors. In addition, the known advice models often require that the information encodes some information about the input, which, realistically, cannot be known exactly (e.g., some bits of the optimal, offline solution). Last, and perhaps more significantly, a malicious entity that takes control of the advice oracle can have a catastrophic impact. For a very simple example, consider the well-known ski rental problem: this is a simple, yet fundamental resource allocation, in which we have to decide ahead of time whether to rent or buy equipment without knowing the time horizon in advance. In the traditional advice model, one bit suffices to be optimal: 0 for renting throughout the horizon, 1 for buying right away. However, if this bit is wrong, then the online algorithm has unbounded competitive ratio, i.e., can perform extremely badly. In contrast, an online algorithm that does not use advice at all has competitive ratio at most 2, i.e., its output can be at most twice as costly as the optimal one.

The above observations were recently made in the context of online algorithms with machine-learned predictions. Lykouris and Vassilvitskii [20] and Purohit *et al.* [22] show how to use predictors to design and analyze algorithms with two properties: (i) if the predictor is good, then the online algorithm should perform close to the best offline algorithm (what is called *consistency*); and (ii) if the predictor is bad, then the online algorithm should gracefully degrade, i.e., its performance should be close to that of the online algorithm without predictions (what is called *robustness*).

Motivated by these definitions from machine learning, in this work we analyze online algorithms based on their performance in both settings of trusted and untrusted advice. In particular, we will characterize the performance of an online algorithm A by a pair of competitive ratios, denoted by (r_A, w_A), respectively. Here, r_A is the competitive ratio achieved assuming that the advice encodes precisely what it is meant to capture; we call this ratio the competitive ratio with *trusted* (thus, always correct) advice. In contrast, w_A is the competitive ratio of A when the advice is untrusted (thus, potentially wrong). More precisely, in accordance with the worst-case nature of competitive analysis, we allow the incorrect advice to be chosen *adversarially*. Namely, assuming a deterministic online algorithm A, the incorrect advice string is generated by a malicious, adversarial entity.

To formalize the above concept, assume the standard advice model, in which a deterministic online algorithm A processes a sequence of requests $\sigma = (\sigma[i])_{i \in [1,n]}$ using an advice tape. At each time t, A serves request $\sigma[t]$, and its output is a function of $\sigma[1. \ldots . t-1]$ and $\phi \in \{0,1\}^*$. Let $A(\sigma, \phi)$ denote the cost incurred by A on input σ, using an advice string ϕ. Denote by r_A, w_A as

$$r_A = \sup_\sigma \inf_\phi \frac{A(\sigma, \phi)}{\text{OPT}(\sigma)}, \quad \text{and} \quad w_A = \sup_\sigma \sup_\phi \frac{A(\sigma, \phi)}{\text{OPT}(\sigma)}, \tag{1}$$

where $\text{OPT}(\sigma)$ denotes the optimal offline cost for σ. Then we say that algorithm A is (r, w)-competitive for every $r \geq r_A$ and $w \geq w_A$. In addition, we say that A has advice complexity $s(n)$ if for every request sequence σ of length n, the algorithm A depends only on the first $s(n)$ bits of the advice string ϕ. To illustrate this definition, the opportunistic 1-bit advice algorithm for ski rental that was described above is $(1, \infty)$-competitive, whereas the standard competitively optimal algorithm without advice is $(2, 2)$-competitive. In general, every online algorithm A without advice or ignoring its advice is trivially (w, w)-competitive, where w is the competitive ratio of A.

Hence, we can associate every algorithm A to a point in the 2-dimensional space with coordinates (r_A, w_A). These points are in general incomparable, e.g., it is difficult to argue that a $(2, 10)$-competitive algorithm is better than a $(4, 8)$-competitive algorithm. However, one can appeal to the notion of *dominance*, by saying that algorithm A dominates algorithm B if $r_A \leq r_B$ and $w_A \leq w_B$. More precisely, we are interested in finding the Pareto frontier in this representation of all online algorithms. For the ski rental example, the two above mentioned algorithms belong to the Pareto set.

A natural goal is to describe this Pareto frontier, which in general, may be comprised of several algorithms with vastly different statements. Ideally, however, one would like to characterize it by a single *family* $\mathcal{A}$ of algorithms, with similar statements (e.g., algorithms in $\mathcal{A}$ are obtained by appropriately selecting a parameter). We say that $\mathcal{A}$ is *Pareto-optimal* if it consists of pairwise incomparable algorithms, and for every algorithm B, there exists $A \in \mathcal{A}$ such that A dominates B. Regardless of optimality, given $\mathcal{A}$, we will describe its competitiveness by means of a function $f : \mathbb{R}_{\geq 1} \to \mathbb{R}_{\geq 1}$ such that for every ratio r there is an $(r, f(r))$-competitive algorithm in $\mathcal{A}$. This function will in general depend on parameters of the problem, such as, for example, the buying cost B in the ski rental problem.

Contribution. We study various online problems in the setting of untrusted advice. We also demonstrate that it is possible to establish both upper and lower bounds on the tradeoff between the size of the advice and the competitiveness in this new advice model. We begin in Section 2 with a simple, yet illustrative online problem as a case study, namely the *ski-rental* problem. Here, we give a Pareto-optimal algorithm with only one bit of advice. We also show that this algorithm is Pareto-optimal even in the space of all (deterministic) algorithms with advice of *any* size.

In Section 3 we study the *online bidding* problem, in which the objective is to guess an unknown, hidden value, using a sequence of bids. This problem was introduced in [10] as a vehicle for formalizing efficient doubling, and has applications in several important online and offline optimization problems. As with ski rental, this is another problem for which a trivial online algorithm is $(1, \infty)$-competitive. We first show how to find a Pareto-optimal strategy, when the advice encodes the hidden value, and thus can have unbounded size. Moreover, we study the competitiveness of the problem with only k bits of advice, for some fixed k, and show both upper and lower bounds on the achieved competitive ratios. The results illustrate that is is possible to obtain non-trivial lower bounds on the competitive ratios, in terms of the advice size. In particular, the lower bound implies that, unlike the ski rental problem, Pareto-optimality is not possible with a bounded number of advice bits.

In Sections 4 and 5, we study the *bin packing* and *list update* problems; these problems are central in the analysis of online problems and competitiveness, and have numerous applications in practice. For these problems, an efficient advice scheme should address the issues of "what constitutes good advice" as well as "how the advice should be used by the algorithm". We observe that the existing algorithms with advice perform poorly in the case the advice is untrusted. To address this, we give algorithms that can be "tuned" based on how much we are willing to trust the advice. This enables us to show guarantees in the form $(r, f(r))$-competitiveness, where r is strictly better than the competitive ratio of all deterministic online algorithms and $f(r)$ smoothly decreases as r grows, while still being close to the worst-case competitive ratio. To illustrate this, consider the bin packing problem. Our $(r, f(r))$-competitive algorithm has $f(r) = \max\{33 - 18r, 7/4\}$ for any $r \geq 1.5$. If $r = 1.5$, our algorithm is $(1.5, 6)$-competitive, and matches the performance of a known algorithm [9]. However, with a slight increase of r, one can improve competitiveness in the event the advice is untrusted. For instance, choosing $r = 1.55$, we obtain $f(r) = 5.1$. In other words, the algorithm designer can hedge against untrusted advice, by a small sacrifice in the trusted performance. Thus we can interpret r as the "risk" for trusting the advice: the smaller the r, the bigger the risk. Likewise, for the list update problem, our $(r, f(r))$-competitive algorithm has $f(r) = 2 + \frac{10 - 3r}{9r - 5}$ for $r \in [5/3, 2]$. If the algorithm takes maximum risk, i.e., if r is smallest, the algorithm is equivalent to an existing $(5/3, 5/2)$-competitive algorithm [8]. Again, by increasing r, we better safeguard against the event of untrusted advice.

All the above results pertain to deterministic online algorithms. In Section 6, we study the power of randomization in online computation with untrusted advice. First, we show that the random-

ized algorithm of Purohit et al. [22] for the ski rental problem Pareto-dominates any deterministic algorithm, even when the latter is allowed unbounded advice. Furthermore, we show an interesting difference between the standard advice model and the model we introduce: in the former, an advice bit can be at least as powerful as a random bit, since an advice bit can effectively simulate any efficient choice of a random bit. In contrast, we show that in our model, there are situations in which a randomized algorithm with L advice bits and one random bit is Pareto-incomparable to the Pareto-optimal deterministic algorithm with $L+1$ advice bits. This confirms the intuition that a random bit is considered trusted, and thus not obviously inferior to an advice bit.

While our work addresses issues similar to [20] and [22], in that trusted advice is related to consistency whereas untrusted advice is related to robustness, it differs in two significant aspects: First, our ideal objective is to identify an optimal family of algorithms, and we show that in some cases (ski rental, online bidding), this is indeed possible; when this is not easy or possible, we can still provide approximations. Note that finding a Pareto-optimal family of algorithms presupposes that the exact competitiveness of the online problem with no advice is known. For problems such as bin packing, the exact optimal competitive ratios are not known. Hence, a certain degree of approximation is unavoidable in such cases. In contrast, [20, 22] focus on "smooth" tradeoffs between the trusted and untrusted competitive ratios, but do not address the issues related to optimality and approximability of these tradeoffs.

Second, our model considers the size of advice and its impact on the algorithm's performance, which is the main focus of the advice complexity field. For all problems we study, we parameterize advice by its size, i.e., we allow advice of a certain size k. Specifically, the advice need not necessarily encode the optimal solution or the request sequence itself. This opens up more possibilities to the algorithm designer in regards to the choice of an appropriate advice oracle, which may have further practical applications in machine learning.

2 A warm-up: the ski rental problem

Background. The ski rental problem is a canonical example in online rent-or-buy problems. Here, the request sequence can be seen as vacation days, and on each day the vacationer (that is, the algorithm) must decide whether to continue renting skis, or buy them. Without loss of generality we assume that renting costs a unit per day, and buying costs $B \in \mathbb{N}^+$. The number of skiing days, which we denote by D, is unknown to the algorithm, and we observe that the optimal offline cost is $\min\{D, B\}$. Generalizations of ski rental have been applied in many settings, such as dynamic TCP acknowledgment [18], the parking permit problem [21], and snoopy caching [17].

Consider the single-bit advice setting. Suppose that the advice encodes whether to buy on day 1, or always rent. An algorithm that blindly follows the advice is optimal if the advice is trusted, but, if the advice is untrusted, the competitive ratio is as high as D/B, if $D > B$. Hence, this algorithm is $(1, \infty)$-competitive, for $D \to \infty$.

Ski rental with untrusted advice. We define the family of algorithms A_k, with parameter $0 < k \leq B$ as follows. There is a single bit of advice, which is the indicator of the event $D < B$. If the advice bit is 1, then A_k rents until until day $B - 1$ and buys on day B. Otherwise, the algorithms buys on day k.

Proposition 1 (Appendix). *Algorithm A_k is $(1 + \frac{k-1}{B}, 1 + \frac{B-1}{k})$-competitive.*

Our algorithm A_k is slightly different from the one proposed in [22], which buys on day $\lceil B/k \rceil$ if the advice is 1 and is shown to be $(1 + k/B, 1 + B/k)$-competitive. More importantly, we show

that A_k is Pareto-optimal in the space of all deterministic online algorithms with advice of *any size*. This implies that more than a single bit of advice will not improve the tradeoff between the trusted and untrusted competitive ratios.

Theorem 2. *For any deterministic $(1 + \frac{k-1}{B}, w)$-competitive algorithm A, with $1 \leq k \leq B$, with advice of any size, it holds that $w \geq 1 + \frac{B-1}{k}$.*

Proof. Let A be an algorithm with trusted competitive ratio at most $1 + \frac{k-1}{B}$. First, note that if the advice is untrusted, the competitive ratio cannot be better than the competitive ratio of a purely online algorithm. For ski-rental, it is known that no online algorithm can achieve a competitive ratio better than $1 + (B-1)/B$ [17]. So, in the case $k = B$, the claim trivially holds. In the remainder of the proof, we assume $k < B$.

We use σ_D to denote the instance of the problem in which the number of skiing days is D, and use $A_t(\sigma_D)$ to denote the cost of A for σ_D in case of trusted advice.

Consider a situation in which the input is σ_{B+k} and the advice for A is trusted. Let j be the day the algorithm will buy under this advice. Since the advice is trusted and thus $\text{OPT}(\sigma_{B+k}) = B$, it must be that

$$A_t(\sigma_{B+k}) \leq \left(1 + \frac{k-1}{B}\right) \text{OPT}(\sigma_{B+k}),$$

which implies $j < B + k$. In other words, A indeed buys on day j. We conclude that $A_t(\sigma_{B+k}) = j - 1 + B$ which further implies $j \leq k$.

Let x be the trusted advice A receives on input σ_{B+k} and suppose A receives the same advice x on input σ_j. Note that x can be trusted or untrusted for σ_j. The important point is that A serves σ_j in the same way it serves σ_{B+k}, that is, it rents for $j - 1$ days and buys on day j. The cost of A for σ_j is then $j - 1 + B$, while $\text{OPT}(\sigma_j) = j$. The ratio between the cost of the algorithm and Opt is therefore $1 + \frac{B-1}{j}$, which is at least $1 + \frac{B-1}{k}$ since $j \leq k$. Note that $1 + \frac{B-1}{k} > 1 + \frac{k-1}{B}$ (since we assumed $k < B$) and therefore the advice in this situation has to be untrusted, by the assumption on the trusted competitive ratio of A. We conclude that the untrusted competitive ratio must be at least $1 + (B-1)/k$. $\qquad\square$

3 Online bidding

Background. In the *online bidding* problem, a player wants to guess a hidden, unknown real value $u \geq 1$. To this end, the player submits a sequence $X = (x_i)$ of increasing *bids*, until one of them is at least u. The strategy of the player is defined by this sequence of bids, and the cost of guessing the hidden value u is equal to $\sum_{i=1}^{j} x_i$, where j is such that $x_{j-1} < u \leq x_j$. Hence the following natural definition of the competitive ratio of the bidder's strategy.

$$w_X = \sup_u \frac{\sum_{i=1}^{j} x_i}{u}, \text{ where } j \text{ is such that } x_{j-1} < u \leq x_j.$$

The problem was introduced in [10] as a canonical problem for formalizing doubling-based strategies in online and offline optimization problems, such as searching for a target on the line, minimum latency, and hierarchical clustering. It is worth noting that online bidding is identical to the problem of minimizing the *acceleration ratio of interruptible algorithms* [24]; the latter and its generalizations are problems with many practical applications in AI (see, for instance [19]).

Without advice, the best competitive ratio is 4, and can be achieved using the doubling strategy $x_i = 2^i$. If the advice encodes[1] the value u, and assuming trusted advice, bidding $x_1 = u$ is a trivial optimal strategy. The above observations imply that there are simple strategies that are $(4, 4)$-competitive and $(1, \infty)$-competitive, respectively.

Online bidding with untrusted advice. Suppose that $w \geq 4$ is a fixed, given parameter. We will show a Pareto-optimal bidding strategy X_u^*, assuming that the advice encodes u, which is $(\frac{w - \sqrt{w^2 - 4w}}{2}, w)$-competitive (Theorem 5).

We begin with some definitions. Since the index of the bid which reveals the value will be important in the analysis, we define the class $S_{m,u}$, with $m \in \mathbb{N}^+$ as the set of bidding strategies with advice u which are w-competitive, and which, if the advice is trusted, succeed in finding the value with precisely the m-th bid. We say that a strategy $X \in S_{m,u}$ that is (r, w)-competitive dominates $S_{m,u}$ if for every $X' \in S_{m,u}$, such that X' is (r', w)-competitive, $r \leq r'$ holds.

The high-level idea is to identify, for any given m, a dominant strategy in $S_{m,u}$. Let $X_{m,u}^*$ denote such a strategy, and denote by $(r_{m,u}^*, w)$ its competitiveness. Then $X_{m,u}^*$ and $r_{m,u}^*$ are the solutions to an infinite linear program which we denote by $P_{m,u}$, and which is shown below. For convenience, for any strategy X, we will always define x_0 to be equal to 1.

$$
\begin{aligned}
\min \quad & r_{m,u} & (P_{m,u}) \\
\text{s.t.} \quad & x_i < x_{i+1}, \quad i \in \mathbb{N}^+ \\
& x_{m-1} < u \leq x_m \\
& \sum_{j=1}^{m} x_j \leq r_{m,u} \cdot u \\
& \sum_{j=1}^{i} x_j \leq w \cdot x_{i-1}, \quad i \in \mathbb{N}^+ \\
& x_i \geq 0, \quad i \in \mathbb{N}^+.
\end{aligned}
$$

$$
\begin{aligned}
\min \quad & \frac{1}{u} \cdot \sum_{i=1}^{m} x_i & (L_{m,u}) \\
\text{s.t.} \quad & x_i < x_{i+1}, \quad i \in \mathbb{N}^+ \\
& x_m = u \\
& \sum_{j=1}^{i} x_i \leq w \cdot x_{i-1}, \quad i \in \mathbb{N}^+ \quad (C_i) \\
& x_i \geq 0, \quad i \in \mathbb{N}^+.
\end{aligned}
$$

Note that in $P_{m,u}$ the constraints $\sum_{j=1}^{i} x_j \leq w \cdot x_{i-1}$ guarantee that the untrusted competitive ratio of X is at most w, whereas the constraints $\sum_{j=1}^{m} x_j \leq r_{m,u} \cdot u$ and $x_{m-1} < u \leq x_m$ guarantee that if the advice is trusted, then X succeeds in finding u precisely with its m-th bid, and in this case the competitive ratio is $r_{m,u}$.

We also observe that an optimal solution $X_{m,u}^* = (x_i^*)_{i \geq 1}$ for $P_{m,u}$ must be such that $x_m = u$, otherwise one could define a strategy $X_{m,u}'$ in which $x_i' = x_i^*/\alpha$, for all $i \geq 1$, with $\alpha = u/x_m^*$, which is still feasible for $P_{m,u}$, is such that $x_m' = u$, and has better objective value than $X_{m,u}^*$, a contradiction. Furthermore, in an optimal solution, the constraint $\sum_{i=1}^{m} x_i \leq r_{m,u} \cdot u$ must hold with equality. Therefore, $X_{m,u}^*$ and $r_{m,u}^*$ are also solutions to the linear program $L_{m,u}$.

Next, define $r_u^* = \inf_m r_{m,u}^*$, and $r^* = \sup_u r_u^*$. Informally, r_u^*, r^* are the optimal competitive ratios, assuming trusted advice. More precisely, the dominant strategy in the space of all w-competitive strategies is (r_u^*, w)-competitive, and r^* is an upper bound on r_u^*, assuming the worst-case choice of u.

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[1]We assume that the advice provides the exact value u to the algorithm. For practical considerations, it suffices to assume an oracle that provides an $(1 + \epsilon)$-approximation of the hidden value, for sufficiently small $\epsilon > 0$. This will only affect the competitive ratios by the same negligible factor.

We first argue how to compute $r^*_{m,u}$ and the corresponding strategy $X^*_{m,u}$, provided that $L_{m,u}$ is feasible. This is accomplished in Lemma 3. The main idea behind the technical proof is to show that in an optimal solution of $L_{m,u}$, all constraints C_i hold with equality. This allows us to describe the bids of the optimal strategy by means of a linear recurrence relation which we can solve so as to obtain an expression for the bids of $X^*_{m,u}$.

Define the sequences a_i and b_i as follows:

$$a_i = \frac{a_{i-1}}{w - 1 - b_{i-1}}, \text{ with } a_0 = 1, \text{ and } b_i = \frac{1 + b_{i-1}}{w - 1 - b_{i-1}}, \text{ with } b_0 = 0, \tag{2}$$

Moreover, for $w > 4$, let $\rho_1 = \frac{w - \sqrt{w^2 - 4w}}{2}$ and $\rho_2 = \frac{w + \sqrt{w^2 - 4w}}{2}$ denote the two roots of $x^2 - wx + w$, the characteristic polynomial of the above linear recurrence.

Lemma 3 (Appendix). *For every m define $X_{m,u}$ as follows:*

- *If $w > 4$, then $x_{m,u,i} = \alpha \cdot \rho_1^{i-1} + \beta \cdot \rho_2^{i-1}$, where $\alpha = \frac{a_{m-1}\rho_2^{m-1} - 1}{\rho_2^{m-1} - \rho_1^{m-1}} \cdot u$, and $\beta = \frac{a_{m-1}\rho_1^{m-1} - 1}{\rho_1^{m-1} - \rho_2^{m-1}} \cdot u$,*

- *If $w = 4$, then $x_{m,u,i} = (\alpha + \beta \cdot i) \cdot 2^i$, where $\alpha = \frac{2^{m-1} \cdot m \cdot a_{m-1} - 1}{2^m(m-1)} \cdot u$, and $\beta = \frac{1 - 2^{m-1} \cdot a_{m-1}}{2^m(m-1)} \cdot u$.*

Then, $X_{m,u}$ is an optimal feasible solution if and only if $a_{m-1} \cdot u \leq w$.

We can now give the statement of the optimal strategy X^*_u. First, we can argue that the optimal objective value of $L_{m,u}$ is monotone increasing in m, thus it suffices to find the objective value of the smallest m^* for which $L_{m^*,u}$ is feasible; This can be accomplished with a binary search in the interval $[1, \lceil \log u \rceil]$, since we know that the doubling strategy in which the i-th bid equals 2^i is w-competitive for all $w \geq 4$; hence $m^* \leq \lceil \log u \rceil$. Then X^*_u is derived as in the statement of Lemma 3. The time complexity of the algorithm is $O(\log \log u)$, since we can describe each a_i, b_i, and hence a_{m-1} in closed form, avoiding the recurrence which would add a $O(\log u)$ factor. The technical details can be found in the Appendix.

Last, the following lemma allows us to express r^* as a function of the values of the sequence b, which we can further exploit so as to obtain the exact value of r^*_u.

Lemma 4 (Appendix). *It holds that $r^* = 1 + \sum_{i=1}^{\infty} \prod_{j=1}^{i-1} b_j$. Furthermore, $r^* = \frac{w - \sqrt{w^2 - 4w}}{2}$.*

Combining Lemmas 3 and 4 we obtain following result:

Theorem 5. *Strategy X^*_u is Pareto-optimal and is $(\frac{w - \sqrt{w^2 - 4w}}{2}, w)$-competitive.*

Strategy X^*_u requires u as advice, which can be unbounded. A natural question is what competitiveness can one achieve with k advice bits, for some fixed k. We address this question both from the point of view of upper and lower bounds. Concerning upper bounds, we show the following:

Theorem 6 (Appendix). *For every $w \geq 4$, there exists a bidding strategy with k bits of advice which is (r, w)-competitive, where*

$$r = \begin{cases} \frac{\left(w + \sqrt{w^2 - 4w}\right)^{1 + 1/K}}{2^{1/K}(w + \sqrt{w^2 - 4w} - 2)} & \text{if } w \leq (1 + K)^2/K \\ \frac{(1 + K)^{1 + 1/K}}{K} & \text{if } w \geq (1 + K)^2/K. \end{cases}$$

and where $K = 2^k$.

In particular, for $w = 4$, the strategy of Theorem 6 is $(2^{1+\frac{1}{2^k}}, 4)$-competitive, whereas X_u^* is $(2, 4)$-competitive. The following theorem gives a lower bound on the competitiveness of any bidding strategy with k bits. The result shows that one needs unbounded number of bids to achieve $(2, 4)$-competitiveness.

Theorem 7. *For any bidding strategy with k advice bits that is $(r, 4)$-competitive it holds that $r \geq 2 + \frac{1}{3 \cdot 2^k}$.*

Proof sketch. We present only an outline; the full proof can be found in the Appendix. With k bits of advice, the online algorithm can differentiate only between $K = 2^k$ online bidding sequences, denoted by $X_1, \ldots, X_K$, each of which must have (untrusted) competitive ratio at least 4. Suppose, by way of contradiction, that the algorithm has trusted competitive ratio less than $2 + \frac{1}{3 \cdot 2^k}$. We reach a contradiction, by applying a game between the algorithm and the adversary, which proceeds in rounds. The adversary fixes a sufficiently large index $i \geq i_0$, for some i_0. In the first round, u is chosen by the adversary so as to be infinitesimally larger than $x_{K,i-1}$, namely the $(i-1)$-th bid of X_K. For the algorithm to guarantee the claimed r, we show that it will have to use the advice so as to "choose" one of the sequences $X_1, \ldots X_{K-1}$, say X_j. Then in the next round, the adversary will choose an appropriate u that is adversarial for X_j. The crux of the proof is to show that the algorithm's only response is to choose a sequence of index higher than j. Eventually, the only remaining choice for the algorithm is strategy X_K; moreover, we can show that throughout the execution of the algorithm the adversarial u is comparable to $x_{K,i-1}$, in particular, we show that $u \leq e^{1/3} x_{K,i-1}$. To conclude, the above argument shows that

$$r \geq \sup_{i \geq i_0} \frac{\sum_{j=1}^{i} x_{K,j}}{e^{\frac{1}{3}} x_{K,i-1}} = \frac{1}{e^{\frac{1}{3}}} \sup_{i \geq 1} \frac{\sum_{j=1}^{i} x_{K,j}}{x_{K,i-1}} \geq \frac{4}{e^{\frac{1}{3}}} > 2 + \frac{1}{3K}.$$

where we used the fact that $\sup_{i \geq 1} \frac{\sum_{j=1}^{i} x_{K,j}}{x_{K,i-1}} = 4$, since X_K is 4-competitive. $\square$

4   Online bin packing

In this section, we study the online bin packing problem under the untrusted advice model. An instance of the online bin packing problem consists of a sequence of items with different *sizes* in the range $(0, 1]$, and the objective is to pack these items into a minimum number of bins, each with a capacity of 1. For each arriving item, the algorithm must place it in one of the current bins or open a new bin for the item. We say that algorithm A has an asymptotic competitive ratio r if, on every sequence σ, the number of opened bins satisfies $A(\sigma) \leq r \cdot \text{OPT}(\sigma) + c$, where c is a constant. As standard in the analysis of bin packing problems, throughout this section, by "competitive ratio" we mean "asymptotic competitive ratio". The First Fit [14] algorithm maintains bins in the same order that they have been opened, and places an item into the first bin with enough free space; if no such bin exists, it opens a new bin. First Fit has a competitive ratio of 1.7 [14] while the best online algorithm has a competitive ratio of at least 1.54278 [4] and at most 1.5783 [3]. Online bin packing has also been studied in the advice setting [9, 23, 2]. In particular, it is possible to achieve a competitive ratio of 1.4702 with only a constant number of (trusted) advice bits [2].

In this section, we introduce an algorithm named Robust-Reserve-Critical (RRC) which has a parameter $\alpha \in [0, 1]$, indicating how much the algorithm relies on the advice. Provided with $O(1)$ bits of advice, the algorithm is asymptotically $(r_{\text{RRC}}, w_{\text{RRC}})$-competitive for $r_{\text{RRC}} = 1.5 + \frac{1-\alpha}{4-3\alpha}$ and $w_{\text{RRC}} = 1.5 + \max\{\frac{1}{4}, \frac{9\alpha}{8-6\alpha}\}$. If the advice is reliable, we set $\alpha = 1$ and the algorithm is asymptotically $(1.5, 6)$-competitive; otherwise, we set α to a smaller value.

The Reserve-Critical algorithm.   Our solution uses an algorithm introduced by Boyar et al. [9] which achieves a competitive ratio of 1.5 using $O(\log n)$ bits of advice [9]. We refer to this algorithm as Reserve-Critical in this paper and describe it briefly. The algorithm classifies items according to their size. Tiny items have their size in the range $(0, 1/3]$, small items in $(1/3, 1/2]$, critical items in $(1/2, 2/3]$, and large items in $(2/3, 1]$. In addition, the algorithm has four kinds of bins, called tiny, small, critical and large bins. Large items are placed alone in large bins, which are opened at each arrival. Small items are placed in pairs in small bins, which are opened every other arrival. Critical bins contain a single critical item, and tiny items up to a total size of $1/3$ per bin, while tiny bins contain only tiny items. The algorithm receives as advice the number of critical items, denoted by c, and opens c *critical bins* at the beginning. Inside each critical bin, a space of $2/3$ is reserved for a critical item, and tiny items are placed using First-Fit into the remaining space of these bins possibly opening new bins dedicated to tiny items. Each critical item is placed in one of the critical bins. Note that the algorithm is heavily dependent on the advice being trusted. Imagine that the advice is strictly larger than the real number of critical items. This results in critical bins which contain only tiny items. The worst case is reached when all tiny items have size slightly more than $1/6$ while there is no critical item. In this case, all critical bins are filled up to a level slightly more than $1/6$. Hence, untrusted advice can result in a competitive ratio as bad as 6.

The Robust-Reserve-Critical (RRC) algorithm.   Let t be the number of tiny bins opened by the Reserved-Critical algorithm. Recall that c is the number of critical bins. We call the fraction $c/(c + t)$ the *critical ratio*. The advice for RRC is a fraction γ, integer multiple of $1/2^k$, that is encoded in k bits such that if the advice is trusted then $\gamma \leq c/(c+t) \leq \gamma + 1/2^k$. In case $c/(c+t)$ is a positive integer multiple of $1/2^k$, we break the tie towards $\gamma < c/(c+t)$. Note that for sufficiently large, yet constant, number of bits, γ provides a good approximation of the critical ratio. Indeed having γ as advice is sufficient to achieve a competitive ratio that approaches 1.5 in the trusted advice model, as shown in [2].

The RRC algorithm has a parameter $0 \leq \alpha \leq 1$, which together with the advice γ can be used to define a fraction $\beta = \min\{\alpha, \gamma\}$. The algorithm maintains a proportion close to β of critical bins among critical and tiny bins. Formally, on the arrival of a critical item, the algorithm places it in a critical bin, opening a new one if necessary. Each arriving tiny item x is packed in the first critical bin which has enough space, with the restriction that the tiny items don't exceed a fraction $1/3$ in these bins. If this fails, the algorithm tries to pack x in a tiny bin using First-Fit strategy (this time on tiny bins). If this fails as well, a new bin B is opened for x. Now, B should be declared as a critical or a tiny bin. Let c' and t' denote the number of critical and tiny bins before opening B. If $c' + t' > 0$ and $\frac{c'}{c'+t'} < \beta$, then B is declared a critical bin; otherwise, B is declared a tiny bin. Large and small items are placed similarly to the Reserved-Critical algorithm (one large item in each large bin and two small items in each small bin).

Analysis.   Intuitively, RRC works similarly to Reserved-Critical except that it might not open as many critical bins as suggested in the advice. The algorithm is more "conservative" in the sense that it does not keep two-third of many (critical) bins open for critical items that might never arrive. The smaller the value of α is, the more conservative the algorithm is. Our analysis is based on two possibilities in the final packing of the algorithm. In the first case (case I), all critical bins receive a critical item, while in the second case (case II) some of them have their reserved space empty. In case I, we show the number of bins in the packing of RRC is within a factor $1.5 + \frac{1-\beta}{4-3\beta}$ of the number of bins in the optimal packing (Lemma 21). Note that this ratio decreases as the value of α (and β) grows. This implies a less conservative algorithm would be better packing in this case. Case II happens only if the advice is untrusted. In this case, the number of bins in the RRC packing

is within a factor $1.5 + \frac{9\beta}{8-6\beta}$ of the number of bins in an optimal packing (Lemma 22). This ratio increases with α (and β). This implies a more conservative algorithm would be better in this case as it would open less critical bins and, thus, would have fewer without critical items.

Assume the advice is trusted. Then either $\gamma \leq \alpha$ or $\gamma > \alpha$. In the former case, the algorithm maintains the same ratio as suggested by advice, and a result from [2] (Lemma 19) indicates that the competitive ratio is at most $1.5 + \frac{15}{2^{k/2+1}}$. In the former case, the algorithm maintains a smaller number of critical items than what the advice suggested; all these bins receive critical items and the final packing will be in Case I. Consequently, when the advice is trusted, the competitive ratio is at most $1.5 + \max\{\frac{1-\alpha}{4-3\alpha}, \frac{15}{2^{k/2+1}}\}$ (see Lemma 23). If the advice is untrusted, both case I and case II can be realized for the final packing. The competitive ratio will be at most $1.5 + \max\{\frac{1}{4}, \frac{9\alpha}{8-6\alpha}\}$ (Lemma 24). The statements and proofs of all the above lemmas can be found in the Appendix. We can conclude with the following theorem:

Theorem 8. *Algorithm Robust-Reserve-Critical with parameter $\alpha \in [0,1]$ and k bits of advice achieves a competitive ratio of $r_{\mathrm{RRC}} \leq 1.5 + \max\{\frac{1-\alpha}{4-3\alpha}, \frac{15}{2^{k/2+1}}\}$ when the advice is trusted and a competitive ratio of $w_{\mathrm{RRC}} \leq 1.5 + \max\{\frac{1}{4}, \frac{9\alpha}{8-6\alpha}\}$ when the advice is untrusted.*

Assuming the size k of the advice is a sufficiently large constant, we conclude the following.

Corollary 9. *For bin packing with untrusted advice, there is a $(r, f(r))$-competitive algorithm where $r \geq 1.5$ and $f(r) = \max\{33 - 18r, 7/4\}$.*

5 List update

The list update problem consists of a list of items of length m, and a sequence of n requests that should be served with minimum total cost. Every request corresponds to an 'access' to an item in the list. If the item is at position i of the list then its access cost is i. After accessing the item, the algorithm can move it closer to the front of the list with no cost using a 'free exchange'. In addition, at any point, the algorithm can swap the position of any two consecutive items in the list using a 'paid exchange' which has a cost of 1. Throughout this section, we adopt the standard assumption that m is a large integer but still a constant with respect to n.

Move-to-Front (Mtf) is an algorithm that moves every accessed item to the front of the list using a free exchange. Mtf has a competitive ratio of at most 2 [26], which is the best that a deterministic algorithm can achieve [13]. Timestamp [1] is another algorithm that achieves the optimal competitive ratio of 2. This algorithm uses a free exchange to move an accessed item x to the front of the first item that has been accessed at most once since the last access to x. Move-To-Front-Every-Other-Access (Mtf2) is a class of algorithms which maintain a bit for each item in the list. Upon accessing an item x, the bit of x is flipped, and x is moved to front if its bit is 0 after the flip (otherwise the list is not updated). If all bits are 0 at the beginning, Mtf2 is called called Move-To-Front-Even (Mtfe), and if all bits are 1 at the beginning, Mtf2 is called Move-To-Front-Odd (Mtfo). Both Mtfe and Mtfo algorithms have a competitive ratio of 5/2 [8]. In [8] it is shown that, for any request sequence, at least one of Timestamp, Mtfo, and Mtfe has a competitive ratio of at most 5/3. For a given request sequence, the best option among the three algorithms can be indicated with two bits of advice, giving a 5/3-competitive algorithm. However, if the advice is untrusted, the competitive ratio can be as bad as 5/2.

To address this issue, we introduce an algorithm named Toggle (Tog) that has a parameter $\beta \in [0, 1/2]$, and uses 2 advice bits to select one of the algorithms Timestamp, Mtfe or Mtfo. This algorithm achieves a competitive ratio of $r_{\mathrm{Tog}} = 5/3 + \frac{5\beta}{6+3\beta}$ when the advice is trusted and a competitive ratio of at most $w_{\mathrm{Tog}} = 2 + 2/(4 + 5\beta)$ when the advice is untrusted. The parameter

β can be tuned and should be smaller when the advice is more reliable. In particular, when $\beta = 0$, we get a $(5/3, 2.5)$-competitive algorithm.

The Toggle algorithm. Given the parameter β, the Toggle algorithm (TOG) works as follows. If the advice indicates Timestamp, the algorithm runs Timestamp. If the advice indicates either MTFO or MTFE, the algorithm will proceed in phases (the length of which partially depend on β) alternating ("toggling") between running MTFE or MTFO, and MTF. In what follows, we use MTF2 to represent the algorithm indicated by the advice. The algorithm TOG will initially begin with MTF2 until the cost of the accesses of the phase reaches a certain threshold, then a new phase begins and TOG switches to MTF. This new phase ends when the access cost of the phase reaches a certain threshold, and TOG switches back to MTF2. This alternating pattern continues as TOG serves the requests. As such, TOG will use MTF2 for the odd phases which we will call *trusting phases*, and MTF for the even phases which we will call *ignoring phases*. The actions during each phase are formally defined below.

Trusting phase: In a trusting phase, TOG will use MTF2 to serve the requests. Let σ_i be the first request of some trusting phase j for $1 \le i \le n$ and an odd $j \ge 1$. Before serving σ_i, TOG modifies the list with paid exchanges to match the list configuration that would result from running MTF2 on the request sequence $\sigma_1, \ldots, \sigma_i$. The number of paid exchanges will be less than m^2. In addition, TOG will set the bits of items in the list to the same value as at the end of this hypothetical run. As such, during a trusting phase, TOG incurs the same access cost as MTF2. The trusting phase continues until the cost to access a request σ_ℓ, $i < \ell \le n$, for TOG would cause the total access cost for the phase to become at least m^3 (or the request sequence ends). The next phase, which will be an ignoring phase, begins with request $\sigma_{\ell+1}$.

Ignoring phase: In an ignoring phase, TOG will use the MTF rule to serve the request. Unlike the trusting phase, TOG does not use paid exchanges to match another list configuration. Let σ_i be the first request of some ignoring phase j for $1 \le i \le n$ and an even $j \ge 1$. The ignoring phase continues until the cost to access a request σ_ℓ, $i < \ell \le n$, for TOG would cause the total access cost for the phase to exceed $\beta \cdot m^3$ (or the request sequence ends). The next phase, which will be a trusting phase, begins with request $\sigma_{\ell+1}$.

Analysis. The access cost in each phase of TOG is $\Theta(m^3)$, while the cost that it might pay at the beginning of each phase is $O(m^2)$. Consequently, the cost of the algorithm in a trusting phase is $m^3 + o(m^3)$ (see Lemma 26 for the exact cost) and $\beta m^3 + o(m^3)$ for an ignoring phase (see Lemma 27). Moreover, the optimal algorithm incurs a cost of at least $m^3/2.5 - m^2$ in a trusting phase and at least $\beta m^3/2 - m^2$ in an ignoring phase; these results follow from the upper bounds of respectively 2.5 and 2 for the competitive ratios of MTF and MTF2, and the fact that the discrepancy in the initial configuration of each phase changes the cost of OPT in that phase by at most m^2 (Lemma 25). We can extend these results to show that, for sufficiently long lists, the competitive ratio of TOG (regardless of the advice being trusted or not) converges to at most $2 + \frac{2}{4+5\beta}$ (see Lemma 29). We conclude that, when the advice is untrusted, the competitive ratio of TOG is at most $2 + \frac{2}{4+5\beta}$ (Lemma 32).

Now, assume the advice is trusted. If the advice indicates Timestamp as the best algorithm among MTFE, MTFO, and Timestamp, the algorithm uses Timestamp to serve the entire sequence, and since the advice is right, the competitive ratio will be at most 5/3 [8]. If the advice indicates MTF2 (either MTFE or MTFO), we compare the cost of TOG with that of MTF2 in each phase. A careful phase analysis, similar to the one for the competitive ratio, shows that the ratio between the costs of the two algorithms converges to at most $2 + \frac{2}{4+5\beta}$ (see Lemma 29). We conclude that,

when the advice is trusted, the competitive ratio of the ToG algorithm converges to $5/3 + \frac{5\beta}{6+3\beta}$ for sufficiently long lists (Lemma 31). The statements and proofs of all the above lemmas can be found in the Appendix. We can state the following theorem:

Theorem 10. *Algorithm* ToG *with parameter $\beta \in [0, 1/2]$ and k bits of advice achieves a competitive ratio of at most $5/3 + \frac{5\beta}{6+3\beta}$ when the advice is trusted and a competitive ratio of at most $2 + \frac{2}{4+5\beta}$ when the advice is untrusted.*

Corollary 11. *For list update with untrusted advice, there is a $(r, f(r))$-competitive algorithm where $r \in [5/3, 2]$ and $f(r) = 2 + \frac{10-3r}{9r-5}$.*

6   Randomized online algorithms with untrusted advice

The discussion in all previous sections pertains to deterministic online algorithms. In this section we focus on randomization and its impact on online computation with untrusted advice. We will assume, as standard in the analysis of randomized algorithms, that the source of randomness is trusted (unlike the advice). Given a randomized algorithm A, its trusted and untrusted competitive ratios are defined as in (1), with the difference that the cost $A(\sigma, \phi)$ is now replaced by the expected cost $\mathbb{E}(A(\sigma, \phi))$.

First, we will argue that randomization can improve the competitiveness of the ski rental problem. For this, we note that [22] gave a randomized algorithm with a single advice bit for this problem which is $\left(\frac{\lambda}{1-e^{-\lambda}}, \frac{1}{1-e^{-(\lambda-1/B)}}\right)$-competitive, where $\lambda \in (1/B, 1)$ is a parameter of the algorithm. For simplicity, we may assume that B is large, hence this algorithm is $\left(\frac{\lambda}{1-e^{-\lambda}}, \frac{1}{1-e^{-\lambda}}\right)$-competitive, which we can write in the equivalent form $(w \ln \frac{w}{w-1}, w)$. In contrast, Theorem 2 shows that any deterministic Pareto-optimal algorithm with advice of any size is $(1+\lambda, 1+1/\lambda)$-competitive, or equivalently $\left(\frac{w}{w-1}, w\right)$-competitive. Standard calculus shows that $w \ln \frac{w}{w-1} < \frac{w}{w-1}$; therefore we conclude that the randomized algorithm Pareto-dominates any deterministic algorithm, even when the latter is allowed unbounded advice.

A second issue we address in this section is related to the comparison of random bits and advice bits as resource. More specifically, in the standard model in which advice is always trustworthy, an advice bit can be at least as powerful as a random bit since the former can simulate the efficient choice of the latter, and thus provide a "no-loss" derandomization. However, in the setting of untrusted advice, the interplay between advice and randomization is much more intricate. This is because random bits, unlike advice bits, are assumed to be trusted.

We show, using online bidding as an example, that there are situations in which a deterministic algorithm with $L+1$ advice bits is Pareto-incomparable to a randomized algorithm with 1 random bit and L advice bits. In particular we focus on the *bounded online bidding* problem, in which $u \le B$, for some given B.

Theorem 12 (Appendix). *For every $\epsilon > 0$ there exist sufficiently large B and L such that there is a randomized algorithm for bounded online bidding with L advice bits and 1 random bit, and which is $\left(\frac{1+\rho_1}{2}\rho_1 + \epsilon, \frac{1+\rho_1}{2\rho_1}w + \epsilon\right)$-competitive for all $w > 4$, where $\rho_1 = \frac{w-\sqrt{w^2-4w}}{2}$.*

Note that when $B, L \to \infty$, the competitiveness of the best deterministic algorithm with L advice bits approaches the one of X_u^*, as expressed in Theorem 5, namely (ρ_1, w). Thus, Theorem 12 shows that randomization improves upon the deterministic untrusted ratio w by a factor $\frac{1+\rho_1}{2\rho_1} > 1$, at the expense of a degradation of the trusted competitive ratio by a factor $\frac{1+\rho_1}{2} > 1$. For instance, if w is close to 4, then the randomized algorithm has untrusted competitive ratio less than 4, and thus better than any deterministic strategy.

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Appendix

A   Proofs from the ski rental section

Proof of Proposition 1. Table 1 summarizes the competitive ratios of the algorithm, for the four different settings, depending on the value and the trustworthiness of the advice.

advice	trusted	untrusted
$0\ (D \geq B\ )$	$\frac{k-1+B}{B} = 1 + \frac{k-1}{B}$	$\frac{k-1+B}{k} = 1 + \frac{B-1}{k}$
$1\ (D < B)$	1	$2 - \frac{1}{B}$

Table 1: The competitive ratios of the family of algorithms A_k.

It follows that the trusted competitive ratio of A_k is at most $1 + \frac{k-1}{B}$ and its untrusted competitive ratio is at most $\max\{2 - \frac{1}{B}, 1 + \frac{B-1}{k}\} = 1 + \frac{B-1}{k}$. $\qquad\square$

B   Proofs from the online bidding section

B.1   Details from the analysis of X_u^*

In this appendix, we provide a detailed proof of Lemmas 3 and 4.

Given $u, m \geq 1$, assuming that $(L_{m,u})$ is feasible, we will first show how to compute the optimal objective value of $(L_{m,u})$. Let Obj denote the numerator of objective value of $(L_{m,u})$ (namely, $\sum_{j=1}^{i} x_j$). For convenience, we denote $x_{m,u,i}$ by x_i and let T_i denote $\sum_{j=1}^{i} x_j$, with $T_0 = 0$.

Define the sequences c_i and d_i as follows:

$$a_i = \frac{a_{i-1}}{w - 1 - b_{i-1}}, \ \text{ with } a_0 = 1, \tag{3}$$

$$b_i = \frac{1 + b_{i-1}}{w - 1 - b_{i-1}}, \ \text{ with } b_0 = 0, \tag{4}$$

$$c_i = c_{i-1} + d_{i-1} \cdot a_{i-1}, \ \text{ with } c_0 = 0, \tag{5}$$

$$d_i = d_{i-1} \cdot (1 + b_{i-1}), \ \text{ with } d_0 = 1. \tag{6}$$

The sequences a_i, b_i, c_i and d_i satisfy the following technical properties.

Lemma 13. *For $i \geq 0$, we have*

$$a_i = \begin{cases} \frac{2}{i+2} \cdot \frac{1}{2^i}, & w = 4 \\ \frac{p^2-1}{p^{i+2}-1} \cdot \left(\frac{p}{w}\right)^{\frac{i}{2}}, & w > 4 \end{cases} \quad , \quad b_i = \begin{cases} \frac{i}{i+2}, & w = 4 \\ p \cdot \frac{p^i-1}{p^{i+2}-1}, & w > 4 \end{cases} \ ,$$

$$c_i = \begin{cases} 2 - \frac{2}{i+1}, & w = 4 \\ 1 + p - \frac{p^i(p^2-1)}{p^{i+1}-1}, & w > 4 \end{cases} \quad and \quad d_i = \begin{cases} \frac{2^i}{i+1}, & w = 4 \\ \frac{p-1}{p^{i+1}-1} \cdot (pw)^{\frac{i}{2}}, & w > 4 \end{cases} \ ,$$

with $p = \frac{w-2-\sqrt{w^2-4w}}{2}$.

Proof. Choose $p < 1$ such that

$$p = \frac{1+p}{w-1-p}. \tag{7}$$

In other words, $p = \frac{w-2-\sqrt{w^2-4w}}{2}$. From (4), we have

$$b_i - p = \frac{(p+1)(b_{i-1}-p)}{w-1-p-(b_{i-1}-p)},$$

which implies that

$$\frac{1}{b_i - p} = \frac{w-1-p}{p+1} \cdot \frac{1}{b_{i-1}-p} - \frac{1}{p+1}.$$

Define the sequence $(u_i)_{i\geq 0}$ as $u_i = \frac{1}{b_i - p}$ for $i \geq 0$, then

$$u_i = \frac{1}{p} \cdot u_{i-1} - \frac{1}{p+1}, \quad \text{with} \quad u_0 = \frac{-1}{p}.$$

Thus,

$$u_i = \begin{cases} -\frac{i+2}{2}, & w = 4 \\ -\frac{p^{i+2}-1}{(p^2-1)p^{i+1}}, & w > 4 \end{cases},$$

which implies that

$$b_i = \begin{cases} \frac{i}{i+2}, & w = 4 \\ p \cdot \frac{p^i - 1}{p^{i+2}-1}, & w > 4 \end{cases}.$$

Then

$$\prod_{j=1}^{i} b_j = \begin{cases} \frac{2}{(i+1)(i+2)}, & w = 4 \\ p^i \cdot \frac{(p-1)(p^2-1)}{(p^{i+1}-1)(p^{i+2}-1)}, & w > 4 \end{cases}, \tag{8}$$

In addition, from (3) and (6), for $i \geq 1$, we have

$$a_i = \prod_{j=1}^{i} \frac{1}{w-1-b_{j-1}} \quad \text{and} \quad d_i = \prod_{j=1}^{i}(1+b_{j-1}).$$

Then, for $i \geq 2$,

$$a_i d_i = \prod_{j=1}^{i} \frac{(1+b_{j-1})}{w-1-b_{j-1}} = \prod_{j=1}^{i} b_j. \tag{9}$$

Moreover, from (4), we have

$$1 + b_i = \frac{w}{w-1-b_{i-1}},$$

then

$$\prod_{j=1}^{i}(1+b_j) = w^i \cdot \prod_{j=1}^{i} \frac{1}{w-1-b_{j-1}},$$

which implies that

$$d_{i+1} = w^i \cdot a_i. \tag{10}$$

Combining (6), (9) and (10), we have

$$a_i = \sqrt{\frac{(1+b_i) \cdot \prod_{j=1}^{i} b_j}{w^i}} \quad \text{and} \quad d_i = \sqrt{\frac{w^i \cdot \prod_{j=1}^{i} b_j}{1+b_i}}.$$

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Thus, if $w > 4$,

$$a_i = \sqrt{\frac{(1 + p \cdot \frac{p^i - 1}{p^{i+2} - 1}) \cdot p^i \cdot \frac{(p-1)(p^2-1)}{(p^{i+1}-1)(p^{i+2}-1)}}{w^i}} = \frac{p^2 - 1}{p^{i+2} - 1} \cdot (\frac{p}{w})^{\frac{i}{2}}$$

and

$$d_i = \sqrt{\frac{w^i \cdot p^i \cdot \frac{(p-1)(p^2-1)}{(p^{i+1}-1)(p^{i+2}-1)}}{(1 + p \cdot \frac{p^i - 1}{p^{i+2} - 1})}} = \frac{p - 1}{p^{i+1} - 1} \cdot (pw)^{\frac{i}{2}}.$$

If $w = 4$,

$$a_i = \sqrt{\frac{(1 + \frac{i}{i+2}) \cdot \frac{2}{(i+1)(i+2)}}{w^i}} = \frac{2}{i+2} \cdot \frac{1}{w^{\frac{i}{2}}}$$

and

$$d_i = \sqrt{\frac{w^i \cdot \frac{2}{(i+1)(i+2)}}{1 + \frac{i}{i+2}}} = \frac{1}{i+1} w^{\frac{i}{2}}$$

From (5), for $i \geq 1$, we have

$$c_i = \sum_{j=1}^{i} a_{j-1} d_{j-1} = 1 + \sum_{j=2}^{i} \prod_{k=1}^{j-1} b_k = 1 + \sum_{j=1}^{i-1} \prod_{k=1}^{j} b_k.$$

Then, by combining with (8), we have

$$c_i = \begin{cases} 1 + \sum_{j=1}^{i-1} \frac{2}{(j+1)(j+2)}, & w = 4 \\ 1 + \sum_{j=1}^{i-1} p^j \cdot \frac{(p-1)(p^2-1)}{(p^{j+1}-1)(p^{j+2}-1)}, & w > 4 \end{cases}$$

$$= \begin{cases} 1 + \sum_{j=1}^{i-1} \frac{2}{j+1} - \frac{2}{j+2}, & w = 4 \\ 1 + \sum_{j=1}^{i-1} (p^2 - 1) \left(\frac{p^j}{p^{j+1}-1} - \frac{p^{j+1}}{p^{j+2}-1} \right), & w > 4 \end{cases}$$

$$= \begin{cases} 2 - \frac{2}{i+1}, & w = 4 \\ 1 + p - \frac{p^i(p^2-1)}{p^{i+1}-1}, & w > 4 \end{cases}.$$

This concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The following lemma gives a lower bound on x_i for any feasible solution X of $(L_{m,u})$, for $i \in [1, m]$, as well as a lower bound on Obj.

Lemma 14. *For every feasible solution $X = (x_1, x_2, \ldots)$ of $(L_{m,u})$, it holds that, for $i \in [1, m]$,*

$$x_i \geq a_{m-i} \cdot u + b_{m-i} \cdot T_{i-1} \quad and \quad Obj \geq c_{m-i} \cdot u + d_{m-i} \cdot T_i.$$

In addition, $x_i = a_{m-i} \cdot u + b_{m-i} \cdot T_{i-1}$ and $Obj = c_{m-i} \cdot u + d_{m-i} \cdot T_i$ if constraints (C_j) for $j \in [i+1, m]$ are tight.

Proof. The proof is by induction on i, for $i \in [1, m]$. The base case, namely for $i = m$, can be readily verified. For the inductive step, suppose that for $i \in [1, m-1]$ it holds that $x_j \geq a_{m-j} \cdot u + b_{m-j} \cdot T_{j-1}$

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and $\text{Obj} \geq c_{m-j} \cdot u + d_{m-j} \cdot T_j$ with $j \in [i+1, m]$. We will show that $x_i \geq a_{m-i} \cdot u + b_{m-i} \cdot T_{i-1}$ and $\text{Obj} \geq c_{m-i} \cdot u + d_{m-i} \cdot T_i$. By (C_{i+1}), we have

$$
\begin{aligned}
w \cdot x_i &\geq T_{i+1} \\
&= x_{i+1} + T_i \\
&\geq a_{m-i-1} \cdot u + b_{m-i-1} \cdot T_i + T_i \\
&= a_{m-i-1} \cdot u + (1 + b_{m-i-1}) \cdot T_i \\
&= a_{m-i-1} \cdot u + (1 + b_{m-i-1}) \cdot (x_i + T_{i-1})
\end{aligned}
$$

It implies that

$$
x_i \geq \frac{a_{m-i-1}}{w - 1 - b_{m-i-i}} \cdot u + \frac{1 + b_{m-i-1}}{w - 1 - b_{m-i-i}} \cdot T_{i-1},
$$

which is equivalent to

$$
x_i \geq a_{m-i} \cdot u + b_{m-i} \cdot T_{i-1}.
$$

It is straightforward to see that the previous inequality holds with equality if constraints (C_j) are tight for $j \in [i+1, m]$. Moreover, from induction hypothesis, we have

$$
\begin{aligned}
\text{Obj} &\geq c_{m-i-1} \cdot u + d_{m-i-1} \cdot T_{i+1} \\
&= c_{m-i-1} \cdot u + d_{m-i-1} \cdot (x_{i+1} + T_i) \\
&\geq c_{m-i-1} \cdot u + d_{m-i-1} \cdot (a_{m-i-1} \cdot u + b_{m-i-1} \cdot T_i + T_i) \\
&= (c_{m-i-1} + d_{m-i-1} \cdot a_{m-i-1}) \cdot u + d_{m-i-1} \cdot (1 + b_{m-i-1}) T_i,
\end{aligned}
$$

which is equivalent to

$$
\text{Obj} \geq c_{m-i} \cdot u + d_{m-i} \cdot T_i.
$$

The inequality holds with equality if constraints (C_j) are tight for $j \in [i+1, m]$. This concludes the proof. $\qquad\square$

Corollary 15. If $(L_{m,u})$ is feasible, then $a_{m-1} \cdot u \leq w$.

Proof. From Lemma 14, for any feasible solution $X(x_1, x_2, \dots)$ of $(L_{m,u})$, it holds that $x_1 \geq a_{m-1} \cdot u$ and $x_1 \leq w$. Hence $a_{m-1} \cdot u \leq w$. $\qquad\square$

Define a sequence x_i^* as follows:

$$
x_i^* = w \cdot x_{i-1}^* - \sum_{j=1}^{i-1} x_j^*, \quad \text{with} \quad x_1^* = a_{m-1} \cdot u.
$$

Lemma 16. x_i^* has a closed formula as follows.

- If $w > 4$, then $x_i^* = \alpha \cdot \rho_1^{i-1} + \beta \cdot \rho_2^{i-1}$, where $\alpha = \frac{a_{m-1}\rho_2^{m-1} - 1}{\rho_2^{m-1} - \rho_1^{m-1}} \cdot u$, and $\beta = \frac{a_{m-1}\rho_1^{m-1} - 1}{\rho_1^{m-1} - \rho_2^{m-1}} \cdot u > 0$,

- If $w = 4$, then $x_i^* = (\alpha + \beta \cdot i) \cdot 2^i$, where $\alpha = \frac{2^{m-1} \cdot m \cdot a_{m-1} - 1}{2^m(m-1)} \cdot u$, and $\beta = \frac{1 - 2^{m-1} \cdot a_{m-1}}{2^m(m-1)} \cdot u > 0$,

with $\rho_1 = \frac{w - \sqrt{w^2 - 4w}}{2} > 1$ and $\rho_2 = \frac{w + \sqrt{w^2 - 4w}}{2} > 1$, two roots of the quadratic equation $x^2 - wx + w = 0$. In addition, x_i^* is monotone increasing in i and $x_i \to +\infty$ as $i \to +\infty$.

Proof. By definition of x_i^*, we have the linear recurrence relation

$$x_i^* = w(x_{i-1}^* - x_{i-2}^*) \text{ for } i \geq 3.$$

Its characteristic equation is $x^2 - wx + w = 0$. We distinguish between two cases, namely for $w = 4$ and $w > 4$.

If $w > 4$, then the characteristic equation has two roots $\rho_1 = \frac{w - \sqrt{w^2 - 4w}}{2} > 1$ and $\rho_2 = \frac{w + \sqrt{w^2 - 4w}}{2} > 1$. It implies that

$$x_i^* = \alpha \cdot \rho_1^{i-1} + \beta \cdot \rho_2^{i-1},$$

with some coefficients α, β. We can determine the value of α, β by using the fact $x_1^* = a_{m-1}u$ and $x_m = u$ (from Lemma 14). As a result, we have

$$\alpha = \frac{\rho_2^{m-1} a_{m-1} - 1}{\rho_2^{m-1} - \rho_1^{m-1}} \cdot u, \text{ and } \beta = \frac{a_{m-1}\rho_1^{m-1} - 1}{\rho_1^{m-1} - \rho_2^{m-1}} \cdot u.$$

We will argue that $\beta > 0$. Since $\rho_2 > \rho_1$, then it remains to show that $a_{m-1}\rho_1^{m-1} - 1 < 0$. From (3) and (7), we have

$$a_{m-1} = \prod_{j=1}^{m-1} \frac{1}{w - 1 - b_{j-1}} < \prod_{j=1}^{m-1} \frac{1}{w - 1 - p} = \prod_{j=1}^{m-1} \frac{p}{1 + p} = \left(\frac{p}{1 + p}\right)^{m-1}.$$

Since $\rho_1 = \frac{w - \sqrt{w^2 - 4w}}{2} = 1 + p$ and $p < 1$, then

$$a_{m-1}\rho_1^{m-1} - 1 < \left(\frac{p}{1 + p}\right)^{m-1} \cdot (1 + p)^{m-1} = p^{m-1} - 1 < 0,$$

which implies that $\beta > 0$. Now, we can argue that x_i^* is monotone increasing in i. We have

$$\alpha + \beta = \frac{\rho_2^{m-1} a_{m-1} - 1}{\rho_2^{m-1} - \rho_1^{m-1}} \cdot u + \frac{a_{m-1}\rho_1^{m-1} - 1}{\rho_1^{m-1} - \rho_2^{m-1}} \cdot u = \frac{a_{m-1}(\rho_2^{m-1} - \rho_1^{m-1})}{\rho_2^{m-1} - \rho_1^{m-1}} = a_{m-1} \cdot u.$$

By Lemma 13, we have $\alpha + \beta = a_{m-1} \cdot u > 0$, which implies that, for $i \geq 1$,

$$x_i^* = \alpha \cdot \rho_1^{i-1} + \beta \cdot \rho_2^{i-1} > (\alpha + \beta) \cdot \rho_1^{i-1} > 0.$$

Combining with $x_i^* = w(x_{i-1}^* - x_{i-2}^*)$, for $i \geq 3$, then x_i^* is monotone increasing in i. Moreover, since $\beta > 0$ and $1 < \rho_1 < \rho_2$, then $x_i^* \to +\infty$ as $i \to +\infty$. This concludes the proof of the case $w > 4$.

If $w = 4$, the proof is similar to the previous one. The characteristic equation has one double root $\rho = 2$. It implies that

$$x_i^* = (\alpha + \beta \cdot i)2^i,$$

with some coefficients α, β. We can determine the value of α, β by using the fact $x_1^* = a_{m-1}u$ and $x_m = u$ (from Lemma 14). As a result, we have

$$\alpha = \frac{2^{m-1} \cdot m \cdot a_{m-1} - 1}{2^m(m - 1)} \cdot u \quad \text{and} \quad \beta = \frac{1 - 2^{m-1} \cdot a_{m-1}}{2^m(m - 1)} \cdot u.$$

We will argue that $\beta > 0$. It suffices to show that $2^{m-1}a_{m-1} - 1 < 0$. By Lemma 13, we have

$$a_{m-1} = \frac{2}{m + 1} \cdot \frac{1}{2^{m-1}} < \frac{1}{2^{m-1}},$$

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which implies that $\beta > 0$. Moreover, since $\alpha + \beta = \frac{2^{m-1}(m-1)\cdot a_{m-1}}{2^m(m-1)} \cdot u = \frac{a_{m-1}\cdot u}{2} > 0$, then

$$x_i^* = (\alpha + \beta \cdot i)2^i > \alpha + \beta > 0.$$

Combining with $x_i^* = w(x_{i-1}^* - x_{i-2}^*)$, for $i \geq 3$, then x_i^* is monotone increasing in i. Moreover, since $\beta > 0$, then $x_i^* \to +\infty$ as $i \to +\infty$. This concludes the proof of the case $w = 4$. □

We are now able to prove the main lemmas.

Proof of Lemma 3. First, if X^* is an optimal feasible solution of $(L_{m,u})$, then by Corollary 15, we have $x_1^* = a_{m-1} \cdot u \leq w$.

In addition, we will show that X^* is an optimal feasible solution of $(L_{m,u})$, if $x_1^* = a_{m-1} \cdot u \leq w$. First we argue that X^* is a feasible solution of $(L_{m,u})$. By definition of X^*, it satisfies constraints (C_j) and $x_m = u$. Lemma 16 shows that x_i^* is monotone increasing in i. Moreover, from Lemma 14, X^* is optimal. □

Proof of Lemma 4. By Lemma 14 and Theorem 3, we have $r_{m,u}^* = c_m$. Combining with Lemma 13, we have

$$r_{m,u}^* = \begin{cases} 2 - \frac{2}{m+1}, & w = 4 \\ 1 + p - \frac{p^m(p^2-1)}{p^{m+1}-1}, & w > 4 \end{cases},$$

which implies that, the worst case ratio is

$$\begin{aligned} r^* &= \lim_{m \to +\infty} r_{m,u}^* = \lim_{m \to +\infty} 1 + p - \frac{p^m(p^2 - 1)}{p^{m+1} - 1} = 1 + p \\ &= \frac{w - \sqrt{w^2 - 4w}}{2}. \end{aligned}$$

□

B.2 Proof of Theorem 6

Proof. The strategy has a parameter $\rho > 1$ which will be specified later. The advice corresponding to the hidden value u is the number $a \in \{0, \dots, K-1\}$ defined as $a = \lceil u\rho^K \rceil \bmod K$, and our strategy A consists of the bidding sequence $x_i = \rho^{i+a/K}$ for all $i \geq 1$.

If the advice is untrusted, then the worst-case untrusted competitive ratio occurs when u is infinitesimally larger than some x_{i-1}. Thus,

$$w_A = \sup_i \frac{\sum_{j=1}^i x_j}{x_{i-1}} = \sup_i \frac{\sum_{j=1}^i x_j}{x_i/\rho} = \frac{\rho^2}{\rho - 1}.$$

This bound is convex in ρ and equals w for the following values

$$\rho_{1,2} = \frac{w \mp \sqrt{w^2 - 4w}}{2}.$$

Hence for any $\rho \in [\rho_1, \rho_2]$ the untrusted ratio is at most w.

To analyze the trusted competitive ratio r_A we observe that if the advice is correct, the ratio between the hidden value and the first successful bid is most $\rho^{1/K}$. Thus the trusted ratio is at most

$$r_A \leq \sup_i \frac{\sum_{j=1}^i x_j}{x_i/\rho^{1/K}} = \frac{\rho^{1+1/K}}{\rho - 1}.$$

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We now argue how to choose an appropriate base ρ. Using second order analysis we observe the above upper bound on r_A has an extreme point at $\rho = 1 + K$. The first derivative of the ratio is then

$$\frac{\rho^{1/K}(-K + \rho - 1)}{K(\rho - 1)^2}$$

which is negative for $1 < \rho < 1 + K$ and positive for $\rho > 1 + K$. Hence the ratio is minimum at its extreme point $\rho = 1 + K$.

However, the choice of ρ needs to satisfy $\rho \in [\rho_1, \rho_2]$. Since ρ_2 is increasing in w, there is a threshold w_0 such that $1 + K \le \rho_2$ iff $w \ge w_0$. This value is

$$w_0 = \frac{(1 + K)^2}{K}.$$

Choosing $\rho = \rho_2$ whenever $w \le w_0$ and $\rho = 1 + K$ yields the claimed bound on the right ratio r_A. Note that for both choices we have $\rho \ge \rho_1$ as required since $\rho_1 \le 2$ because ρ_1 is decreasing in w and equals 2 at $w = 4$. $\qquad\square$

B.3   Proof of Theorem 7

We begin with a useful property of all competitively optimal strategies (in the standard, no-advice model). The property essentially shows that competitively optimal strategies are small variants of the doubling strategy.

Lemma 17. *For any 4-competitive strategy $X = (x_i)$ for online bidding, it holds that*

$$x_i \le (2 + \frac{2}{i})x_{i-1},$$

for all $i \ge 1$, where x_0 is defined to be equal to 1.

Proof. For $i = 1$, the claim holds since x_1 must be at most 4, otherwise the strategy cannot be 4-competitive. We can then assume that $i \ge 2$, and prove the claim by induction on i. Suppose that the claim holds for all $j \le i$, that is $x_j \le (2 + \frac{2}{j})x_{j-1}$, for all $j \le i$. This implies that

$$x_{i-j} \ge \frac{1}{\prod_{k=0}^{j-1}(2 + \frac{2}{i-k})}x_i. \tag{11}$$

We will show that the claim holds for $i + 1$. From the 4-competitiveness of the strategy X we have that

$$\frac{\sum_{j=1}^{i+1} x_j}{x_i} \le 4 \Rightarrow x_{i+1} + \sum_{k=1}^{i-1} x_k \le 3x_i,$$

and substituting $x_1, , \ldots x_{i-1}$ using (11), we obtain that

$$x_{i+1} \le (3 - P_i)x_i, \quad \text{where } P_i = \sum_{k=1}^{i-1} \frac{1}{\prod_{j=0}^{k-1}(2 + \frac{2}{i-j})}.$$

It then suffices to show that

$$3 - P_i \le 2 + \frac{2}{i+1} \quad \text{or equivalently} \quad P_i \ge 1 - \frac{2}{i+1}. \tag{12}$$

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We will prove (12) by induction on i, in fact, even stronger, we will show that (12) holds with equality. For $i = 2$, this claim can be readily verified. Assuming that it holds for i, we will show that it holds for $i + 1$. Indeed we have that

$$P_{i+1} = \frac{1}{2 + \frac{2}{i+1}}(1 + P_i) = \frac{1}{2 + \frac{2}{i+1}}(2 - \frac{2}{i+1}) = 1 - \frac{2}{i+2},$$

where the second equality follows from the induction hypothesis, and the third equality can be readily verified. $\square$

Corollary 18. *For any 4-competitive strategy $X = (x_i)$, and every $\epsilon > 0$, there exists i_0 such that for all $i \geq i_0$, it holds that*

$$\sum_{j=1}^{i} x_j \geq (2 - \epsilon)x_i.$$

Proof. It holds that

$$\sum_{j=1}^{i} x_j = x_i + \sum_{j=1}^{i-1} x_j \geq x_i\left(1 + \sum_{k=1}^{i-2} \frac{1}{\prod_{j=0}^{k-1}(2 + \frac{2}{i-j})}\right) = x_i(1 + P_{i-1}) \geq x_i\left(2 - \frac{2}{i}\right),$$

where the inequality follows from Lemma 17, and the last inequality holds from the property on P_i that was shown in the proof of Lemma 17. Last, note that $\frac{2}{i} \leq \epsilon$, for sufficiently large i. $\square$

We can now proceed with the proof of Theorem 7. With k bits of advice, the online algorithm can only differentiate between $K = 2^k$ online bidding sequences, which we denote by $X_j = (x_{j,i})_{i \geq 1}$, with $j \in [1, K]$. (In words, the i-th bid in the j-th strategy is equal to $x_{j,i}$.) That is, the advice is a function that maps the hidden value u to one of these K strategies. We will say that the advice *chooses* X_j, or simply *chooses* j, with $j \in [1, K]$ to describe this action. Define

$$\delta = \frac{1}{3K},$$

and let r denote the trusted competitive ratio of a given strategy with k advice bits. We will prove that $r > 2 + \delta$. By way of contradiction, we will assume that $r \leq 2 + \delta$, and we will prove that the exists a choice of the hidden value which results in trusted competitive ratio larger than $2 + \delta$. Note that each X_j, with $j \in [1, K]$ must be 4-competitive, since the untrusted competitive ratio has to be 4.

For a given index i, and for any $j \in [1, K-1]$, let $l_{j,i}$ denote the smallest index such that $x_{j,l_{j,i}} > x_{K,i-1}$. To simplify notation, we will define $\overline{x}_j \equiv x_{j,l_{j,i}}$. Such $l_{j,i}$ must always exist; moreover, since i can be unbounded, so is $x_{K,i-1}$, and hence also $l_{j,i}$. Thus we can invoke Corollary 18, and obtain that for any $\epsilon > 0$, there exists i_0 such that

$$T_j(\overline{x}_j) \geq (2 - \epsilon)\overline{x}_j, \quad \text{for all } j \in [1, K-1],$$

where we use the notation $T_j(x)$ to denote the sum of all bids less than or equal to y in X_j, i.e., $\sum_{y \in X_j, y \leq x} y$.

We will assume, without loss of generality, that $\overline{x}_1 \leq \overline{x}_2 \ldots \leq \overline{x}_{K-1}$.

Outline of the proof To prove the lower bound, we will use a game between the online algorithm with advice, and an adversary, which proceeds in rounds. In each round, the adversary will consider a hidden value u from the following candidate set:

$$\{x_{K,i-1} + \epsilon, \overline{x}_1 + \epsilon, \ldots \overline{x}_{K-1} + \epsilon\},$$

where $\epsilon \to 0$ is an infinitesimally small value. These values are intuitive: The first value is a "bad" choice for X_K, and the remaining values are "bad" choices for $X_1, \ldots X_{K-1}$, respectively. For each choice, the online algorithm needs to maintain a trusted competitive ratio at most $2 + \delta$. To do so, we will argue that the algorithm will need to choose sequences of "progressively larger indexes". By this we mean that if in some round the advice chooses sequences X_j, then when presented with $u = \overline{x}_j + \epsilon$ in the next round, the advice will have to choose $X_{j'}$, with $j' > j$. Moreover, we will argue that the $\overline{x}_j$'s used as the hidden value by the adversary cannot be too big, in comparison to $x_{K,i-1}$, if the algorithm is to be $(2+\delta)$-competitive. At the end, the "top" strategy, namely strategy X_K, will be the best choice. Then r will be at least

$$\frac{\sum_{j=1}^{i} x_{K,i}}{\overline{x}_j},$$

where $\overline{x}_j$ is not too big in comparison to $x_{K,i-1}$, i.e., by a factor of at most $e^{1/3}$. This will imply that the trusted competitive ratio must be at least $1/e^{1/3}$ times 4 (since 4 is the untrusted competitive ratio of X_K), which is larger than $2 + \delta$ for all K, a contradiction.

Description of the game The adversarial game proceeds in rounds. In round 0, the adversary chooses u to be infinitesimally larger than $x_{K,i-1}$.

There are the following subcases:

1 *The advice chooses some $j \in [1, K - 1]$.* In this case it must be that

$$r \geq \frac{T_j(\overline{x}_j)}{x_{K,i-1}} \geq \frac{(2 - \epsilon)\overline{x}_j}{x_{K,i-1}},$$

and since $r = 2 + \delta$, we obtain that

$$\overline{x}_j \leq \frac{2 + \delta}{2 - \epsilon} x_{K,i-1}. \tag{13}$$

2 *The advice chooses K.* In this case we have that

$$r \geq \frac{T_K(x_{K,i})}{x_{K,i-1}}.$$

The remaining rounds are defined inductively. Suppose that rounds $0, \ldots \rho$ have been defined. We will now define round $\rho+1$. If in round ρ the advice chose X_K, the game is over, and no actions take place in any subsequent rounds. Otherwise, let j_ρ be such that the advice chose X_{j_ρ} in round ρ. For round $\rho + 1$, the adversary chooses u to be infinitesimally larger than $\overline{x}_{j_\rho}$. We consider the following cases concerning the algorithm.

1 *The advice chooses some $j = j_{\rho+1} > j_\rho$, with $j < K$.* In this case there must be that

$$r \geq \frac{T_j(\overline{x}_j)}{\overline{x}_{j_\rho}} \geq \frac{(2 - \epsilon)\overline{x}_j}{\overline{x}_{j_\rho}},$$

and since $r = 2 + \delta$, we obtain, that

$$\overline{x}_j \leq \frac{2 + \delta}{2 - \epsilon} \overline{x}_{j_\rho}.$$

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2 *The advice chooses X_K.* In this case we have that

$$r \geq \frac{T_K(x_{K,i})}{\overline{x}_{j\rho}}.$$

3 *The advice chooses j with $j \leq j_\rho$.* We will argue that this case cannot occur, unless $r > 2 + \delta$.

We prove the claim concerning the last case. First, suppose that the algorithm chose $j_{\rho+1} = j_\rho$. Then we have that

$$r \geq \frac{T_j(\overline{x}_{j\rho}) + \overline{x}_{j\rho}}{\overline{x}_{j\rho}} \geq 3 - \epsilon > 2 + \delta.$$

Suppose now that the algorithm chose $j_{\rho+1} = j < j_\rho$. Then we have that

$$r \geq \frac{T_j(\overline{x}_j) + \overline{x}_{j\rho}}{\overline{x}_{j\rho}} \geq \frac{(2-\epsilon)\overline{x}_j + \overline{x}_{j\rho}}{\overline{x}_{j\rho}} \geq 1 + \frac{(2-\epsilon)x_{K,i-1}}{\overline{x}_{j\rho}}.$$

Since it must be that $r \leq 2 + \delta$, it follows that

$$\overline{x}_{j\rho} \geq \frac{2 - \epsilon}{1 + \delta} x_{K,i-1}.$$

However, from (13), we know that $\overline{x}_{j\rho}$ cannot exceed $(\frac{2+\delta}{2-\epsilon})^K \cdot x_{K,i-1}$, and since ϵ can be arbitrarily small, we have that

$$\overline{x}_{j\rho} \leq (1 + \frac{1}{3K})^K \cdot x_{K,i-1}.$$

It then follows that

$$(1 + \frac{1}{3K})^K \geq \frac{2}{1 + \frac{1}{3K}},$$

which is a contradiction, as it can readily be confirmed using standard calculus.

Analysis of the game Let R denote the last round of the game. From the arguments above it follows that either $r \geq 3 - \epsilon$, or in round R the advice chose algorithm X_K. As argued above, in this case we have that

$$\overline{x}_{j_R} \leq (1 + \frac{1}{3K})^K x_{K,i-1} \leq e^{\frac{1}{3}} x_{K,i-1}.$$

Thus, from the corresponding action in round R it must be that

$$r \geq \frac{T_K(x_{K,i})}{\overline{x}_{j_R}} \geq \frac{T_K(x_{K,i})}{e^{\frac{1}{3}} x_{K,i-1}}.$$

Note that, as discussed early in the proof, the above holds for all sufficiently large $i \geq i_0$. Therefore,

$$r \geq \sup_{i \geq i_0} \frac{\sum_{j=1}^i x_{K,j}}{e^{\frac{1}{3}} x_{K,i-1}} = \frac{1}{e^{\frac{1}{3}}} \sup_{i \geq 1} \frac{\sum_{j=1}^i x_{K,j}}{x_{K,i-1}} \geq \frac{4}{e^{\frac{1}{3}}} > 2 + \frac{1}{3K}.$$

We conclude that the online algorithm cannot be such that $r \leq 2 + \delta = 2 + \frac{1}{3K}$.

C   Analysis details of the online bin packing section

In this section, we provide details for the proof of Theorem 8. First, note that when $\gamma \leq \alpha$, then the algorithm works with the ratio γ as indicated in the advice. Consequently, if the advice is trusted, we have the same performance guarantee as stated in [2]:

Lemma 19. *[2] When $\gamma \leq \alpha$ and the advice is trusted, the competitive ratio of* RRC *is at most* $1.5 + \frac{15}{2^{k/2+1}}$.

The remaining cases are more interesting and involve scenarios when the advice is untrusted, or when the advice is trusted but the algorithm maintains a ratio of α instead of γ as indicated in the advice. Before discussing these cases in details, we prove the following lemma.

Lemma 20. *Let S denote the total size of tiny items in an input sequence and assume there are t tiny bins in the final packing of the* RRC *algorithm. We have $S > (t-1)\frac{4-3\beta}{6-6\beta} - 1/6$.*

Proof. Assume $t > 0$, otherwise the claim holds trivially. Let B denote the last tiny bin that is opened by the algorithm and let x be the tiny item which caused its opening. Let c' and t' respectively denote the number of critical and tiny bins before B was opened ($t' = t - 1$). Since B is declared a tiny bin, we have $\frac{c'}{c'+t'} \geq \beta$ which gives $c' \geq \frac{\beta}{1-\beta}t'$.

Since x is tiny and caused the opening of a new bin, all t' tiny bins have a level of at least $2/3$. Also we claim that all of the c' critical bins, except possibly one bin B', contain tiny items of total size at least $1/6$, we call it the *tiny level* of the bins. If there are two critical bins with a tiny level of at most $1/6$, then each of them must contain at least one tiny item, otherwise x could have fit. And this means that one tiny item of the second bin could have fit into the first bin, contradicting the First-Fit packing of the algorithm. In summary, the total size S of tiny items in the input sequence will be more than $t' \cdot 2/3$ (for tiny items in tiny bins) plus $(c' - 1) \cdot 1/6$ (for tiny items in critical bins). Since $c' > \frac{\beta}{1-\beta}t'$, we can write $S > t' \cdot 2/3 + \beta/(1-\beta)t' \cdot 1/6 - 1/6 > t'(2/3 + \frac{\beta}{6(1-\beta)}) - 1/6$. $\square$

To continue our analysis of the RRC algorithm, we consider two cases, captured by the following two lemmas. In the first case, the number of bins declared by RRC as critical is at most equal to the number of critical items (possibly much less). In this case, all bins declared as critical will receive a critical item. The second case is when the algorithm has declared too many bins as critical and some of them did not receive any critical item.

Lemma 21. *If all critical bins receive a critical item, then the number of bins in the final packing of* RRC *algorithm is within a ratio $1.5 + \frac{1-\beta}{4-3\beta}$ of the number of bins in the optimal packing.*

Proof. To prove the lemma, we use a weighting function argument. Define the *weight* of large and critical items to be 1, and the weight of small items to be $1/2$. The weight of a tiny item of size x is defined as $\frac{6-6\beta}{4-3\beta}x$. Note that the weight of x is less than $3x/2$ (and possibly less than x). Let W denote the total weight of all items in the sequence.

First we claim that the number of bins opened by RRC is at most $W + 3$. Large bins include 1 large item of weight 1, and small items include two items of weight $1/2$ (except possibly the last one) which gives a total weight of 1 for the bin. Critical bins all include a critical item of weight 1. So, if w_h, w_s, w_c respectively denote the total weight of large, small, and critical items, then the number of non-tiny bins opened by the algorithm is at most $w_h + w_s + w_c + 1$. Let S denote the total size of tiny items. By Lemma 20, we have $S > (t-1)\frac{4-3\beta}{6-6\beta} - 1/6$. The total weight of tiny bins is $\frac{6-6\beta}{4-3\beta} \cdot S \geq \frac{6-6\beta}{4-3\beta} \cdot ((t-1)\frac{4-3\beta}{6-6\beta} - 1/6) \geq t - 2$. So, tiny items have total weight of at least $t - 2$, that is, the number tiny bins is at most $w_t + 2$, where w_t is the total weight of tiny items. Consequently,

the total number of bins opened by the algorithm is at most $w_h + w_s + w_c + 1 + w_t + 2$, and the claim is established, i.e., $\text{RRC}(\sigma) \leq W + 3$.

Next, we show the number of bins in an optimal solution is at least $W(8 - 6\beta)/(14 - 11\beta)$. For that, it suffices to show the weight of any bin in the optimal solution (i.e., any collection of items with total size at most 1) is at most $(14 - 11\beta)/(8 - 6\beta)$. Define the *density* of an item as the ratio between its weight and size. To maximize the weight of a bin, it is desirable to place items of larger densities in it. This is achieved by placing a critical item of size $1/2 + \epsilon$, a small item of size $1/3 + \epsilon$ and a set of tiny items of total size $1/6 - 2\epsilon$ in the bin, where ϵ is an arbitrary small positive value. The weight of such a bin will be $1 + 1/2 + (1/6 - 2\epsilon)\frac{6 - 6\beta}{4 - 3\beta} < \frac{14 - 11\beta}{8 - 6\beta}$.

To summarize, we have $\text{RRC}(\sigma) \leq W + 3$ and $\text{OPT}(\sigma) \geq \frac{8 - 6\beta}{14 - 11\beta}W$. This gives an asymptotic competitive ratio of at most $\frac{14 - 11\beta}{8 - 6\beta} = 1.5 + \frac{1 - \beta}{4 - 3\beta}$ for RRC. $\qquad\square$

Lemma 22. *If some of the bins declared as critical do not receive a critical item, then the number of bins in the final packing of RRC algorithm within a ratio $1.5 + \frac{9\beta}{8 - 6\beta}$ of the number of bins in the optimal packing.*

Proof. Let C denote the last critical bin opened by RRC. Since there are critical bins without critical items at the final packing, C should be opened by a tiny item. Let x be the tiny open that opens C. Let c' and t' respectively denote the number of critical and tiny bins before C is opened. Since C is declared a critical bin, we have $\frac{c'}{c' + t'} < \beta$ which gives $c' < \frac{\beta}{1 - \beta}t'$. As before, let c and t respectively denote the number of critical and tiny bins in the final packing ($c' = c - 1$). We have $c < \frac{\beta}{1 - \beta}t + 1$.

In order to prove the lemma, we show that all bins on average have a level of at least $\frac{4 - 3\beta}{6}$. This clearly holds for bins opened by large and small items, except possibly for the bin opened for the last small item; these bins all have a level of at least $2/3 \geq \frac{4 - 3\beta}{6}$. Note that if $c + t$ is a constant, then all but a constant number of bins have level of at least $2/3$ and the algorithm has a competitive ratio of at most 1.5. In what follows, we assume $c + t$ is asymptotically large. By Lemma 20, the total size of tiny items is at least $(t - 1)\frac{4 - 3\beta}{6 - 6\beta} - 1/6$. These items are distributed between $t + c < t + \frac{\beta}{1 - \beta}t + 1 < (t - 1)\frac{1}{1 - \beta} + 2$ bins. So, if we ignore two bins, the average level of the remaining tiny/critical bins will be more than $\frac{(4 - 3\beta)/(6 - 6\beta)}{1/(1 - \beta)} - \frac{1}{6(t + c)} = \frac{4 - 3\beta}{6} - \frac{1}{6(t + c)}$. Since $t + c$ is asymptotically large, the average level of these bins converges to a value of size larger than $\frac{4 - 3\beta}{6}$.

Let S denote the total size of items in the input sequence σ. Clearly at least S bins are required to pack all items, i.e., $\text{OPT}(\sigma) \geq S$. On the other hand, since the average level of all bins (excluding 3 bins) is more than $\frac{4 - 3\beta}{6}$, for the cost of RRC we can write $\text{RRC}(\sigma) \leq \lceil \frac{6}{4 - 3\beta}S \rceil + 3 \leq \frac{6}{4 - 3\beta}S + 4 \leq (1.5 + \frac{9}{8 - 6\beta})\text{OPT}(\sigma) + 4$. $\qquad\square$

Provided with Lemmas 21 and 22, we are ready to prove an upper bound for the competitive ratio of RRC as a function of its parameter α. We consider two cases based on whether the advice is trusted.

Lemma 23. *If the advice is trusted, then the competitive ratio of the RRC algorithm is at most $1.5 + \max\{\frac{1 - \alpha}{4 - 3\alpha}, \frac{15}{2^{k/2 + 1}}\}$.*

Proof. First, if $\gamma \leq \alpha$, by Lemma 19, the competitive ratio will be at most $1.5 + \frac{15}{2^{k/2 + 1}}$. Next, assume $\alpha < \gamma$, that is $\beta = \alpha$. All critical bins receive a critical item in this case. This is because the algorithm maintains a critical ratio α which is smaller than γ. In other words, the algorithm declares a smaller ratio of its bins critical compared to the actual ratio in the Reserve-Critical algorithm. Hence, all critical bins receive a critical item. By Lemma 21, the competitive ratio is at most $1.5 + \frac{1 - \alpha}{4 - 3\alpha}$. $\qquad\square$

Lemma 24. *If the advice is untrusted, then the competitive ratio of* RRC *is at most* $1.5+\max\{\frac{1}{4}, \frac{9\alpha}{8-6\alpha}\}$.

Proof. Consider two cases. First, assume all bins declared as critical by the RRC algorithm receive a critical item. In this case, by Lemma 21, the competitive ratio of the algorithm will be bounded by $1.5 + \frac{1-\beta}{4-3\beta}$; this value decreases in β and hence is maximized at $\beta = 0$. Next, assume some of the bins declared as critical do not receive a critical item. By Lemma 22, the competitive ratio of RRC in this case is at most $1.5 + \frac{9\beta}{8-6\beta}$; this value however increases by β and is maximized at the upper bound $\beta = \alpha$. This completes the proof. $\square$

Theorem 8 directly follows from Lemmas 23 and 24.

D   Analysis details of the online list update section

In this section, we provide details for the proof of Theorem 8. In our analysis, in the case of untrusted advice, we will focus on analyzing MTF2. The reason for this is that, based on the competitive ratio, Timestamp has a competitive ratio of at most 2 which is better than the worst case of $2+2/(4+5\beta)$ that we will show when the untrusted advice indicates one of MTFO or MTFE.

Throughout the analysis, we fix a sequence σ and use k to denote the number of trusting phases of TOG for serving σ. Note that the number of ignoring phases is either $k-1$ or k. For each request, any algorithm incurs an access cost of at least 1 and hence each phase has length at most m^3. Since m^3 is a constant independent of the length of the input, k grows with n. This observation will be used in the proof of the following two lemmas that help us bound the cost of TOG in the case of untrusted and trusted advice, respectively.

For the analysis, we will break the sequence into subsequences and analyse the cost over the subsequences. Let σ' be a subsequence of σ and ALG be any algorithm serving serving the sequence σ. We will will denote the cost of ALG over the subsequence σ' with $A(\sigma')$, where it is implicit that A has served the requests preceeding σ' in σ, and will serve the requests following σ' in σ. The following lemma bounds the cost for an optimal algorithm over a subsequence as compared to a c-competitive online algorithm.

Lemma 25. *Let A be online algorithm for the List Update problem such that, for all σ, $A(\sigma) \leq c \cdot \text{OPT}(\sigma) + \alpha$. For any σ' that is a subsequence of σ,*

$$\text{OPT}(\sigma') \geq \frac{A(\sigma')}{c} - \frac{\alpha}{c} - \frac{m^2}{2} + \frac{m}{2} \ .$$

Proof. Let r_i be the first request of σ'. Let $L_{r_j}^{\text{ALG}}$ be the list configuration of any algorithm ALG immediately before serving the request r_j. Define OPT$'$ to be an optimal algorithm for the subsequence σ' with an initial list configuration of $L_{r_i}^A$. That is, OPT$'$ is only serving σ', starting from the configuration of A.

Fix an optimal algorithm OPT for σ. Define another algorithm B that will only serve σ'. For a cost of at most $m(m-1)/2$ paid exchanges, B will change its initial configuration of $L_{r_i}^A$ to $L_{r_i}^{\text{OPT}}$, serve σ' as OPT serves the subsequence in σ. The total cost of B for σ' cannot be less than OPT$'$ without contradicting the optimality of OPT$'$ for σ'. Hence, we have that,

$$B(\sigma) = \text{OPT}(\sigma') + m(m-1)/2 \geq \text{OPT}'(\sigma') \ .$$

Using this and the competitive ratio of A, we get

$$A(\sigma') \leq c \cdot \text{OPT}'(\sigma') + \alpha \leq c \cdot (\text{OPT}(\sigma') + m(m-1)/2) + \alpha$$

and the claim follows. $\square$

Lemma 26. *For a trusting phase, the cost of* MO *is in the range* $(m^3(1+1/m), m^3(1+1/m+1/m^2))$ *(excluding the last phase).*

Proof. For paid exchanges at the beginning of the phase, TOG incurs a cost that is less than m^2. Before serving the last request σ_ℓ of the phase, the access cost of TOG is less than m^3 by definition, and the access cost to σ_ℓ is at most m. □

Similar arguments apply for an ignoring phase with the exception that the threshold is $\beta \cdot m^2$ and there are no paid exchanges performed by TOG. So, we can observe the following.

Observation 27. *In an ignoring phase, the cost of* TOG *for the phase is in the range* $(\beta m^3, \beta m^3(1+ 1/m^2))$.

The proof for the following lemma is direct from Lemma 26 and Observation 27, noting that there are k trusting phases and at most k ignoring phases.

Lemma 28. *The cost of* TOG *is is upper bound by* $k \cdot m^3 \cdot \left(1 + \beta + \frac{3}{m}\right)$.

Lemma 29. *For sufficiently long lists, and hence long request sequences the competitive ratio of* TOG *(regardless of the advice being trusted or not) converges to at most* $2 + \frac{2}{4+5\beta}$.

Proof. Consider an arbitrary trusting phase and let σ_t denote the subsequence of σ formed by requests in that phase. Recall that TOG uses the MTFO strategy during a trusting phase. We know that $\text{MTFO}(\sigma_t) \leq 2.5 \cdot \text{OPT}(\sigma_t)$ [8], and that MTFO incurs a cost of more than m^3 during the phase (Lemma 26). So, from Lemma 25, we conclude that OPT incurs a cost of at least $m^3/2.5 - m^2$ during the phase. Note that this lower bound for the cost of OPT applies for all trusting phases.

Next, consider an arbitrary ignoring phase and let σ' denote the subsequence of requests served by TOG during that phase. Recall that TOG applies MTF during an ignoring phase. We know $\text{MTF}(\sigma') \leq 2\text{OPT}(\sigma')$ [25], and MTF incurs an access cost of at least βm^3 during the phase (Lemma 27). So, from Lemma 25, we conclude that OPT incurs a cost of at least $\beta m^3/2 - m^2$ during the phase. Note that this lower bound for the cost of OPT applies for all ignoring phases.

Since we have at least $k-1$ of each trusting and ignoring phases, the total cost of OPT is at least $(k-1)(m^3/2.5 - m^2) + (k-1)(\beta m^3/2 - m^2) = (k-1)(m^3\frac{4+5\beta}{10} - 2m^2) > (k-1)m^3(\frac{4+5\beta}{10} - 2/m)$.

To summarize, the cost of OPT is larger than $(k-1)m^3(\frac{4+5\beta}{10} - 2/m)$ and, by Lemma 28, the cost of TOG is at most $km^3(1+\beta+3/m)$. The competitive ratio will be at most $\frac{km^3(1+\beta+3/m)}{(k-1)m^3(\frac{4+5\beta}{10}-2/m)}$ which converges to $\frac{10+10\beta}{4+5\beta} = 2 + 2/(4+5\beta)$ for long lists (as m grows) and long input sequences (as k grows). □

Lemma 30. *For sufficiently long lists, the ratio between the cost of* TOG *and that of* MTF2 *converges to* $1 + \frac{\beta}{2+\beta}$.

Proof. Note that TOG and MTF2 incur the same access cost of m^3 in any trusting phases. We use an argument similar to the previous lemma for analyzing ignoring phases. Consider an arbitrary ignoring phase and let σ' denote the subsequence of requests served by TOG during that phase. We know $\text{MTF}(\sigma') \leq 2 \cdot \text{OPT}(\sigma') + O(m^2)$ [8], and MTF incurs a cost of at least βm^3 during the phase. So, from Lemma 25, we conclude that OPT, and consequently MTF2, incur a cost of at least $\beta m^3/2 - m^2$ during the phase. Note that this lower bound for the cost of MTF2 applies for all ignoring phases.

The worst-case ratio between the costs of TOG and MTF2 is maximized when the last phase is an ignoring phase. In this case, we have k trusting phases and k ignoring phases. The total cost of

MTF2 is at least $km^3 + k(\beta m^3/2 - m^2) = km^3(1 + \beta/2 - 1/m)$. By Lemma 28, the cost of TOG is at most $km^3(1 + \beta + 3/m)$. The ratio between the two algorithms will be less than $\frac{km^3(1+\beta+3/m)}{km^3(1+\beta/2-1/m)}$ which converges to $1 + \frac{\beta}{2+\beta}$ for long lists. $\qquad\square$

Given the above lemmas, we can find upper bounds for competitive ratio of TOG.

Lemma 31. *If the advice is trusted, then the competitive ratio of the* TOG *algorithm converges to* $5/3 + \frac{5\beta}{6+3\beta}$ *for sufficiently long lists.*

Proof. If the advice indicates Timestamp as the best algorithm among MTFE, MTFO, and Timestamp, the algorithm uses Timestamp to serve the entire sequence, and since the advice is right, the competitive ratio will be at most $5/3$ [8]. If the advice indicates MTFE or MTFO as the best algorithm, the TOG algorithm uses the phasing scheme described above by alternating between the indicated algorithm and MTF. If the advice is right, by Lemma 30, the cost of the algorithm will be within a ratio $1 + \frac{\beta}{2+\beta}$ of the algorithm indicated by the advice, and consequently has a competitive ratio of at most $5/3(1 + \frac{\beta}{2+\beta}) = 5/3 + \frac{5\beta}{6+3\beta}$. $\qquad\square$

Lemma 32. *If the advice is untrusted, then the competitive ratio of the* TOG *algorithm converges to* $2 + \frac{2}{4+5\beta}$ *for sufficiently long lists.*

Proof. If the advice indicates Timestamp as the best algorithm, the algorithm trusts it and the competitive ratio will be at most 2 [1]. If the advice indicates MTFE or MTFO as the best algorithm, the TOG algorithm uses the phasing scheme described above by alternating between the indicated algorithm and Move-To-Front, and by Lemma 29, the competitive ratio of the algorithm will be at most $2 + \frac{2}{4+5\beta}$. $\qquad\square$

Theorem 8 directly follows from Lemmas 31 and 32.

E   Proofs from the randomization section

Details in the proof of Theorem 12. Recall that the strategy $X_u^* = (x_i)$ of Theorem 5 is $(\frac{w-\sqrt{w^2-4w}}{2}, w)$-competitive, and that for $w > 4$, its bids are as in the statement of Lemma 3. Namely, $x_i = \alpha\rho_1^{i-1} + \beta\rho_2^{i-1}$, with $1 < \rho_1 < \rho_2$. Define strategy $Y = (y_i)$ in which $Y = \rho_1 x_i$. Last, we define a randomized strategy R with 1 random bit, which mixes equiprobably between X and Y.

There is a subtlety concerning the advice bits required to encode u. First, we note that for every $\epsilon > 0$, there is sufficiently large B with the following property: if a deterministic algorithm has competitive ratio w for unbounded u, then it has competitive ratio at least $w - \epsilon$ when u is constrained to be at most B. Furthermore, if u is bounded, then with a sufficiently large number of bits, say b_u, we can approximate u to any required precision. This in turn implies that if u is sufficiently large, then an encoding with b_u and an encoding with $b_u - 1$ bits can differ only by ϵ, and therefore so do the competitive ratios of the algorithms $X_{u_1}^*$ and $X_{u_2}^*$, where u_1 and u_2 are the values encoded with b_u and $b_u - 1$ bits, respectively. Summarizing, we can assume that, excluding negligible effects to the trusted and untrusted competitive ratios, X_u^* and Y receive the precise value of u as advice, and that X_u^* requires one bit more than Y.

Note that $y_i > x_i$, and that

$$y_i = \rho_1 x_i = \rho_1(\alpha\rho_1^{i-1} + \beta\rho_2^{i-1}) < \alpha\rho_1^i + \beta\rho_2^i = x_{i+1}.$$

The worst case choices for the hidden value u are values infinitesimally larger than the ones in the sets $\cup_i\{x_i\}$ and $\cup_i\{y_i\}$. We thus consider two cases:

Case 1: u is infinitesimally larger than x_i. Clearly, $x_{i+1} > u$, and $y_i = \rho_1 x_i > u$. Thus the expected cost of R is at most

$$\frac{1}{2} \sum_{j=1}^{i+1} x_j + \frac{1}{2} \sum_{j=1}^{i} y_j.$$

Case 2: u is infinitesimally larger than y_i. Clearly, $y_{i+1} > u$; moreover, as shown above, it must be that $x_{i+1} > y_i$. Thus the expected cost of R is at most

$$\frac{1}{2} \sum_{j=1}^{i+1} (x_i + y_i) = \frac{1 + \rho_1}{2} x_i.$$

We can now bound the expected untrusted competitive ratio of R. If u is from the set $\cup_i \{x_i\}$, then

$$\frac{\mathbb{E}(cost(R))}{\text{OPT}} = \frac{1}{2} \cdot \sup_i \frac{\sum_{j=1}^{i+1} x_j + \sum_{j=1}^{i} y_j}{x_i}$$

$$= \frac{1}{2} \cdot \sup_i \{\frac{2 \sum_{j=1}^{i} x_j}{x_i} - \frac{y_{i+1}}{x_i}\}$$

$$= \frac{1}{2}(2w - \rho_1) = w - \frac{\rho_1}{2}.$$

If u is from the set $\cup_i \{y_i\}$, then

$$\frac{\mathbb{E}(cost(R))}{\text{OPT}} = \frac{1 + \rho_1}{2} \cdot \sup_i \frac{\sum_{j=1}^{i+1} x_j}{y_i}$$

$$= \frac{1 + \rho_1}{2} \cdot \sup_i \frac{\sum_{j=1}^{i+1} x_j}{\rho_1 x_i} = \frac{1 + \rho_1}{2\rho_1} w.$$

Summarizing, we have

$$w_R \le \max\{w - \frac{\rho_1}{2}, \frac{1 + \rho_1}{2\rho_1} w\} = \frac{1 + \rho_1}{2\rho_1} w,$$

where the equality follows from standard calculus.

Concerning the expected trusted competitive ratio of R, we observe that with probability $1/2$ it is equal to ρ_1, (if X_u^* is chosen), or equal to at most $\rho_1 \rho_1 = \rho_1^2$, if Y is chosen. Thus,

$$r_R \le \rho_1 \frac{1 + \rho_1}{2}.$$

$\square$