Exercise 1: Ever Given

1 Ever Given

In March 2021, a container ship *Ever Given* blocked Suez Canal. Consider your are the captain of one of the ships that is waiting for passing through Suez Canal. You want to arrive your destination as soon as possible. However, because of the obstruction, your ship is idle near by Suez Canal and you have no idea when the canal will be available again. You have two options:

- to wait until the canal is available, and you can pass through the canal in F unit of time; or
- to go around Africa via the Cape of Good Hope, and you can arrive your destination in S units of time.

Note that F stands for fast, S stands for slow, and $F < S$.

Answer the following questions:

1. What is an *instance* in this problem?
   S, F, and the time when the canal is available.
2. What is the *cost* in this problem?
   The total units of time it takes to arrive the destination.
3. Consider the following strategy ALG_{S-F} that keeps waiting until the $(S-F)$-th time unit:

Algorithm 1 ALG_{S-F}

$t \leftarrow$ the current time
while $t < S - F$ **do**
  if The canal is available **then**
    `Pass through the canal`
  else
    `Wait`
`Turn around and take the Cape of Good Hope route` ▷ the canal is still blocked at the $S-F$-th unit of time

(a) Prove that ALG_{S-F} is $(2 - \frac{F}{S})$-competitive.
(Hint:

i. Let a be the actual time when the canal is available. How much is the optimal cost when $a < S - F$? How much is the cost of ALG_{S-F}?
ii. Let a be the actual time when the canal is available. How much is the optimal cost when $a \geq S - F$? How much is the cost of ALG_{S-F}?

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Proof. Assume that the canal becomes available actual at the a-unit of time. There are two cases: $a < S - F$ or $a \geq S - F$.

- If $a < S - F$, $\text{OPT} = a + F$ and $\text{ALG}_{S-F} = a + F$.
- If $a \geq S - F$, $\text{OPT} = S$ and $\text{ALG}_{S-F} = (S - F) + S$.

Hence,

$$\begin{aligned}\text{The competitive ratio of the algorithm } \mathrm{ALG}_{S-F} =& \max\{\frac{a+F}{a+F}, \frac{(S-F)+S}{S}\} \\ \leq& \frac{(S-F)+S}{S} \\ =& 2-\frac{F}{S}\end{aligned}$$

□

(b) Show that your analysis is tight.

We show the analysis is tight by designing an instance such that the ratio of the algorithm cost to the optimal cost on the same instance matches the competitive ratio upper bound.

Consider the instance that the canal becomes available at the $(S-F+\epsilon)$-th unit of time, where $\epsilon > 0$ but tends to 0. On this instance, the algorithm cost is $(S-F)+S$, while the optimal cost is S. The ratio of the algorithm cost on the input to the optimal cost on the same input is $\frac{2S-F}{S} = 2-\frac{F}{S}$, which matches the upper bound of the algorithm's competitive ratio. Therefore, the analysis is tight.

4. Prove that for this problem, there is no deterministic online algorithm better than $(2-\frac{F}{S})$-competitive.
(Hint: Any deterministic online algorithm must turns around to the Cape of Good Hope route at some time T. Show that for any of these algorithms, there is an adversarial instance such that the ratio of the algorithm's cost to the optimal cost is at least $2-\frac{F}{S}$.)

Proof. Consider any deterministic online algorithm, it must must turns around to the Cape of Good Hope route at some time T. For algorithm ALG_T which turns around at the T-th unit of time, we design the adversarial input I_T that the canal is available at time $T+\epsilon$, where $\epsilon > 0$ but very small. With this instance, the cost of algorithm ALG_T is $T+S$, while the optimal cost is $\min\{S, T+\epsilon+F\}$.

There are two cases of T: $T \geq S-F$ or $T < S-F$.

- If $T \geq S-F$, the optimal cost is S and the ratio $\frac{\mathrm{ALG}_T(I_T)}{\mathrm{OPT}(I_T)} = \frac{T+S}{S} \geq \frac{(S-F)+S}{S} = 2-\frac{F}{S}$.
- If $T < S-F$, the optimal cost is less than or equal to $T+F+\epsilon$. Therefore, the ratio $\frac{\mathrm{ALG}_T(I_T)}{\mathrm{OPT}(I_T)} \geq \frac{T+S}{T+\epsilon+F}$. The ratio decreases as T increases. Hence, since $T < S-F$, the ratio is lower bounded by $\frac{(S-F)+S}{(S-F)+\epsilon+F} = 2-\frac{F}{S}$ when ϵ tends to 0.

Therefore, for both cases, the ratio $\frac{\mathrm{ALG}_T(I_T)}{\mathrm{OPT}(I_T)} \geq 2-\frac{F}{S}$.

□

2 Buy on the $\frac{B}{2}$-th day

Recall the Ski Rental problem mentioned in the lecture. Now, assume that $B \geq 1$ is even and consider the algorithm $\mathrm{ALG}_{\frac{B}{2}}$ which buys the ski on the $\frac{B}{2}$-th day. Answer the following questions:

1. Prove that $\mathrm{ALG}_{\frac{B}{2}}$ is at least $(3-\frac{2}{B})$-competitive.
Consider the adversary where there are $\frac{B}{2}$ number of skiing days. The optimal strategy is to rent for $\frac{B}{2}$ days, and the optimal cost is $\frac{B}{2}$. The algorithm $\mathrm{ALG}_{\frac{B}{2}}$ rents for $\frac{B}{2}-1$ days and buys on the $\frac{B}{2}$-th day. Therefore, the algorithm pays $\frac{B}{2}-1+B$. The ratio between the algorithm cost and the optimal cost is $\frac{\frac{B}{2}-1+B}{\frac{B}{2}} = 3-\frac{2}{B}$.

2. Prove that $\text{ALG}_{\frac{B}{2}}$ is $(3-\frac{2}{B})$-competitive.
(Hint:
Consider the case where the number of skiing days $d < \frac{B}{2}$, $d \geq B$, and $\frac{B}{2} \leq d < B$. The last case is the most tricky one, where the algorithm buys the ski but the optimal keeps renting the ski. The ratio between the algorithm cost and the optimal cost in this case is $\frac{\frac{B}{2}-1+B}{d}$. You need to find a good value of d so you can get rid of it from the ratio while still having a valid upper bound.
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- If $d < \frac{B}{2}$, both the algorithm and the optimal algorithm do not buy the ski. The ratio between the algorithm cost and the optimal cost is 1.
- If $d \geq B$, the algorithm buys the ski on the $\frac{B}{2}$-th day and the optimal strategy is buying the ski on the first day. The ratio between the algorithm cost and the optimal cost is $\frac{\frac{B}{2}-1+B}{B} = \frac{3}{2} - \frac{1}{B}$.
- If $\frac{B}{2} \leq d < B$, the algorithm buys the ski on the $\frac{B}{2}$-th day and the optimal strategy keeps renting the ski for d days. In this case, the ratio is $\frac{\frac{B}{2}-1+B}{d}$ where $\frac{B}{2} \leq d < B$. The formula is maximized when $d = \frac{B}{2}$.[1] Therefore, $\frac{\frac{B}{2}-1+B}{d} \leq \frac{\frac{B}{2}-1+B}{\frac{B}{2}} = 3 - \frac{2}{B}$.

Therefore, for any d and even B, the ratio $\frac{\text{ALG}(B,d)}{\text{OPT}(B,d)} \leq \max\{1, \frac{3}{2} - \frac{1}{B}, 3 - \frac{2}{B}\} = 3 - \frac{2}{B}$.

[1] Intuitively, if d is larger, the optimal strategy pays more and the algorithm is punished less for its wrong decision. That's a simple explanation of why a small d gives a larger upper bound for this case.