Exercise 6: Missing Cow

Consider you are a cow getting lost on a misty grassland. Nearby there is a very long fence. You know that somewhere at the fence, there is a hole so you can escape. However, you don't know where the hole is nor which direction the hole is in. Furthermore, because it is so foggy, you see the hole only when it is exactly in front of you. In such a situation, all you can do is walk straightly toward one direction and search for the hole, or turn around, walk toward the other direction, and search for the hole. You want to escape using as small a total walking distance as possible.

Answer the following questions:

1. Let your starting position be the origin and the fence is the x-axis. Assume that the hole is at position n (where n can be positive or negative), what is the optimal solution cost?

   The optimal solution knows where the hole is. It can walk directly toward the whole and the cost is $|n|$.

2. Consider the following algorithm ALG_1 that first go to position 1, then go to position -1, and then 2, -2, 4, -4, ... (Figure 1):

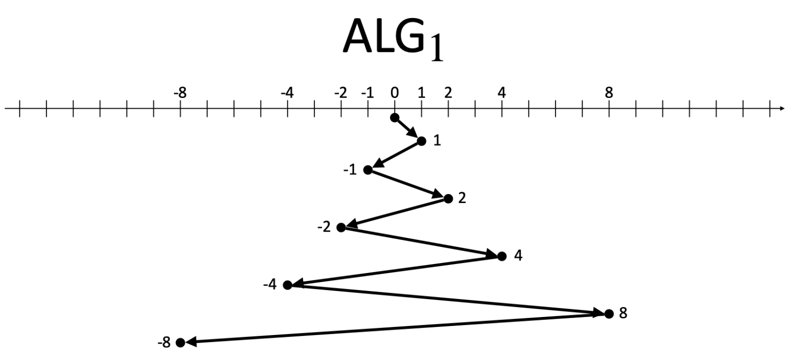

Figure 1: Cow path for ALG_1

   (a) Find an adversary for ALG_1 on the *right* hand side of the cow and give a lower bound of ALG_1's competitive ratio.

   Let n be $= 2^k + \epsilon$. The cost of ALG_1

$$
\begin{aligned}
&=1+1+1+1+2+2+2+2+4+4+4+4+\cdots+2^k+2^k+2^k+2^k+|n| \\
&>4\cdot\Sigma_{i=0}^{k}2^i+2^k \\
&\approx 4\cdot 2^{k+1}+2^k \\
&=9\cdot 2^k
\end{aligned}
$$

Hence, $\frac{\text{ALG}_1(n)}{\text{OPT}(n)} \geq \frac{9 \cdot 2^k}{n} \approx \frac{9 \cdot 2^k}{2^k} = 9$ when ϵ is very small, and ALG_1 is at least 9-competitive.

(b) Find an adversary for ALG_1 on the *left* hand side of the cow and give a lower bound of ALG_1's competitive ratio.

Let n be $= -(2^k + \epsilon)$. The cost of ALG_1

$$\begin{aligned}
=&1+1+1+1+2+2+2+2+4+4+4+4+\cdots+2^k+2^k+2^k+2^k \\
&+2^{k+1}+2^{k+1}+|n| \\
>&4 \cdot \Sigma_{i=0}^{k} 2^i + 2 \cdot 2^{k+1} + 2^k \\
\approx&4 \cdot 2^{k+1} + 2 \cdot 2^{k+1} + 2^k \\
=&13 \cdot 2^k
\end{aligned}$$

Hence, $\frac{\text{ALG}_1(n)}{\text{OPT}(n)} \geq \frac{13 \cdot 2^k}{n} \approx \frac{13 \cdot 2^k}{2^k} = 13$ when ϵ is very small, and ALG_1 is at least 13-competitive.

(c) From (a) and (b), which adversary gives a stronger lower bound?

(b) is a stronger lower bound as it is bigger.

(d) Prove that ALG_1 is 13-competitive.

For any instance $n \in (2^k, 2^{k+1}]$,
$\text{ALG}_1 = 4 \cdot \Sigma_{i=0}^{k} 2^i + n \leq 4 \cdot 2^{k+1} + n$.
Hence, $\frac{\text{ALG}_1(n)}{\text{OPT}(n)} \leq \frac{8 \cdot 2^k + n}{n}$. The ratio increases as n decreases. However, $|n| > 2^k$ and $\frac{\text{ALG}_1(n)}{\text{OPT}(n)} \leq \frac{8 \cdot 2^k + |n|}{|n|} < \frac{8 \cdot 2^k + 2^k}{2^k} = 9$.

For any instance $n \in [-2^{k+1}, -2^k)$,
$\text{ALG}_1 = 4 \cdot \Sigma_{i=0}^{k} 2^i + 2 \cdot 2^{k+1} + |n| \leq 4 \cdot 2^{k+1} + 2 \cdot 2^{k+1} + |n|$.
Hence, $\frac{\text{ALG}_1(n)}{\text{OPT}(n)} \leq \frac{12 \cdot 2^k + |n|}{|n|}$. The ratio increases as $|n|$ decreases. However, $|n| > 2^k$ and $\frac{\text{ALG}_1(n)}{\text{OPT}(n)} \leq \frac{12 \cdot 2^k + |n|}{|n|} < \frac{12 \cdot 2^k + 2^k}{2^k} = 13$.

The competitive ratio of ALG_1 is $\max\{9, 13\} = 13$.

3. Consider the following algorithm ALG_2: First go to 1, then go to -2, 4, -8, 16, $\cdots$ (Figure 2.

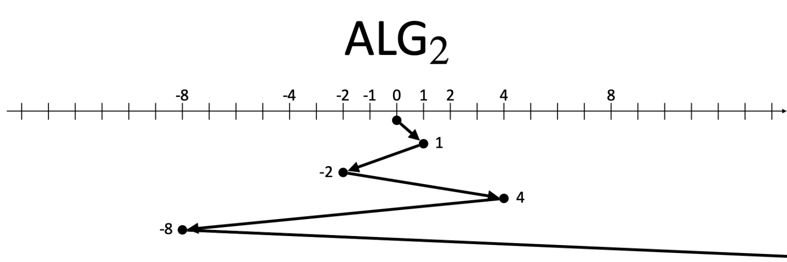

Figure 2: Cow path for ALG_1

(a) By intuition, do you think this ALG_2 is better than ALG_1 or worse?

ALG_2 is better. To get the same position, ALG_2 passes through smaller number of zig-zags compared to ALG_1.

(b) Show that this algorithm is 9-competitive.

For any instance $n \in (2^{2k}, 2^{2k+2}]$,
$\text{ALG}_2 = 2 \cdot \Sigma_{i=0}^{2k+1} 2^i + n \le 2 \cdot 2^{2k+2} + n$.
Hence, $\frac{\text{ALG}_2(n)}{\text{OPT}(n)} \le \frac{8 \cdot 2^{2k}+n}{n}$. The ratio increases as n decreases. However, $|n| > 2^{2k}$ and $\frac{\text{ALG}_2(n)}{\text{OPT}(n)} \le \frac{8 \cdot 2^{2k}+|n|}{|n|} < \frac{8 \cdot 2^{2k}+2^{2k}}{2^{2k}} = 9$.

For any instance $n \in [-2^{2k+3}, -2^{2k+1})$,
$\text{ALG}_2 = 2 \cdot \Sigma_{i=0}^{2k+2} 2^i + |n| \le 2 \cdot 2^{2k+3} + |n|$.
Hence, $\frac{\text{ALG}_2(n)}{\text{OPT}(n)} \le \frac{8 \cdot 2^{2k+1}+|n|}{|n|}$. The ratio increases as $|n|$ decreases. However, $|n| > 2^{2k+1}$ and $\frac{\text{ALG}_2(n)}{\text{OPT}(n)} \le \frac{8 \cdot 2^{2k+1}+|n|}{|n|} < \frac{8 \cdot 2^{2k+1}+2^{2k+1}}{2^{2k+1}} = 9$.

The competitive ratio of ALG_2 is $\max\{9, 9\} = 9$.