Algorithms for Decision Support

Online Algorithms (1/3)

Buy-or-Rent

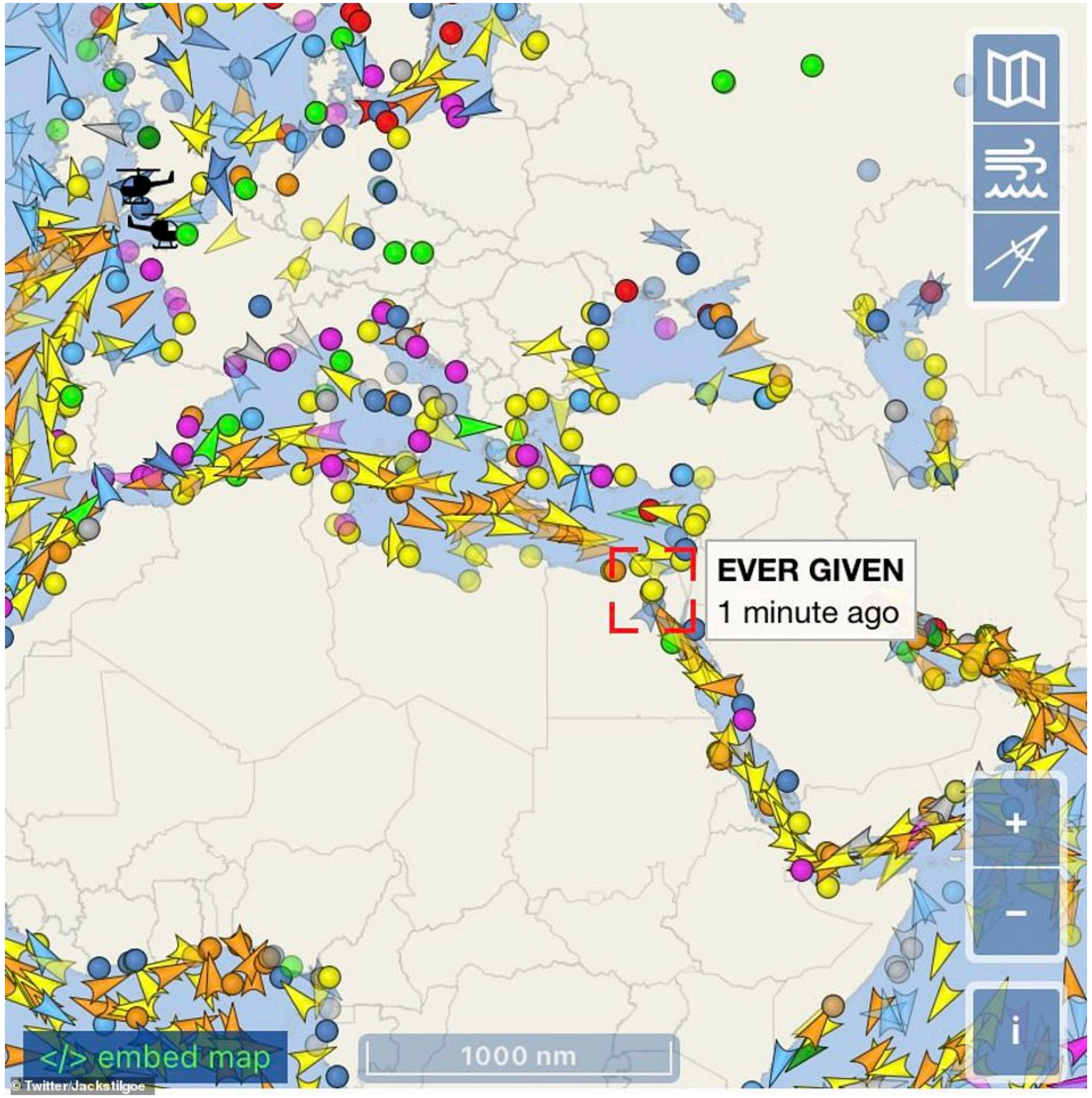
Outline

- **Online problems & online algorithms** — optimization with uncertainty
  - First example: **Ski-rental**
- Measure the performance: **Competitive ratio**
  - How good is an online algorithm?
- **Adversarial game**
  - How bad is an online algorithm?

Ever Given and Suez Canal

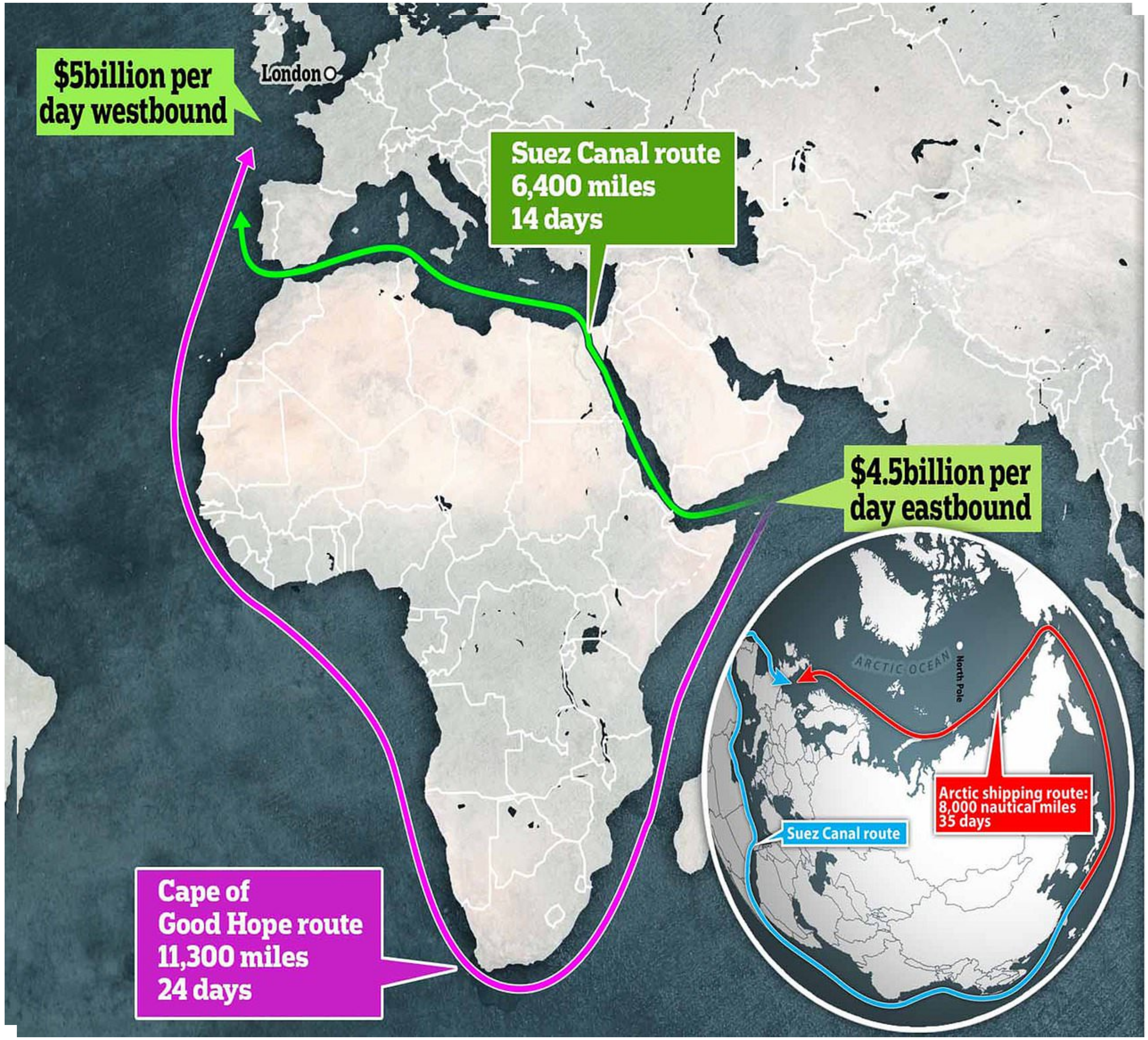
In March 2021, a container ship got stuck in the Suez Canal.

Ever Given and Suez Canal



Ever Given and Suez Canal

- To arrive the destination **as soon as possible**, what will you do?
  - Keep **waiting** until the Ever Given container ship is free, or
    - You may wait for a very long time
  - Turn around and **take the alternate route**, which cost extra 10 days
    - You may start turning around and find the Ever Given is free, and if you turn back again, it will cost you a even longer time



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Online Optimization

- Optimization under *uncertainty*
  - A strategy/algorithm has to respond to each event *without knowledge of future input*
  - A strategy/algorithm *cannot revoke any decision* that is already made
- A good strategy/algorithm should guarantee that even for the worst case, it performs not to bad compared to the optimal solution with hindsight

Online Problems and Online Algorithms

- Offline (optimization) problem:

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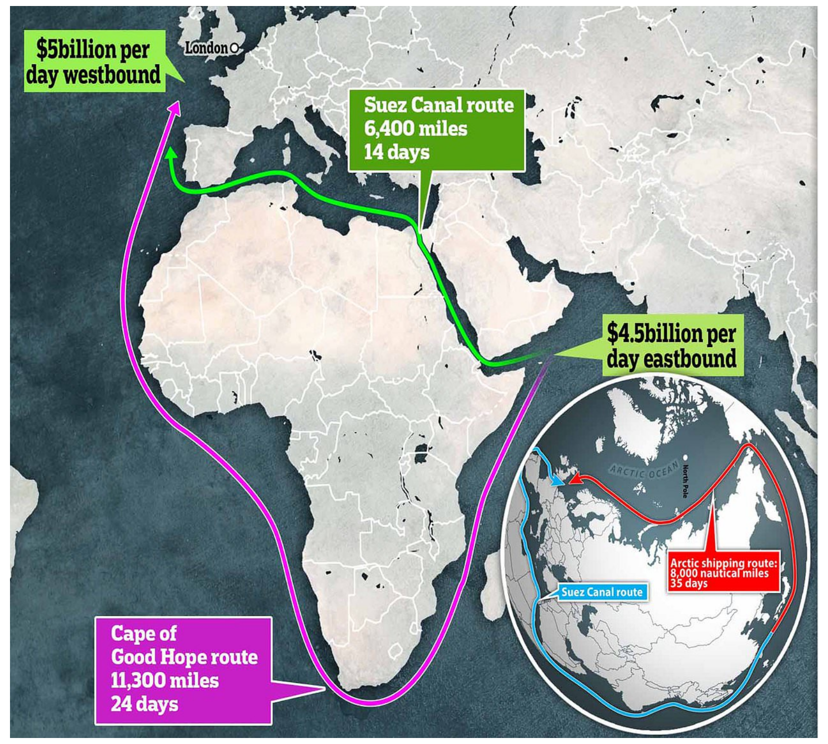

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Instance: the status of the canal at any time (When will it be free)

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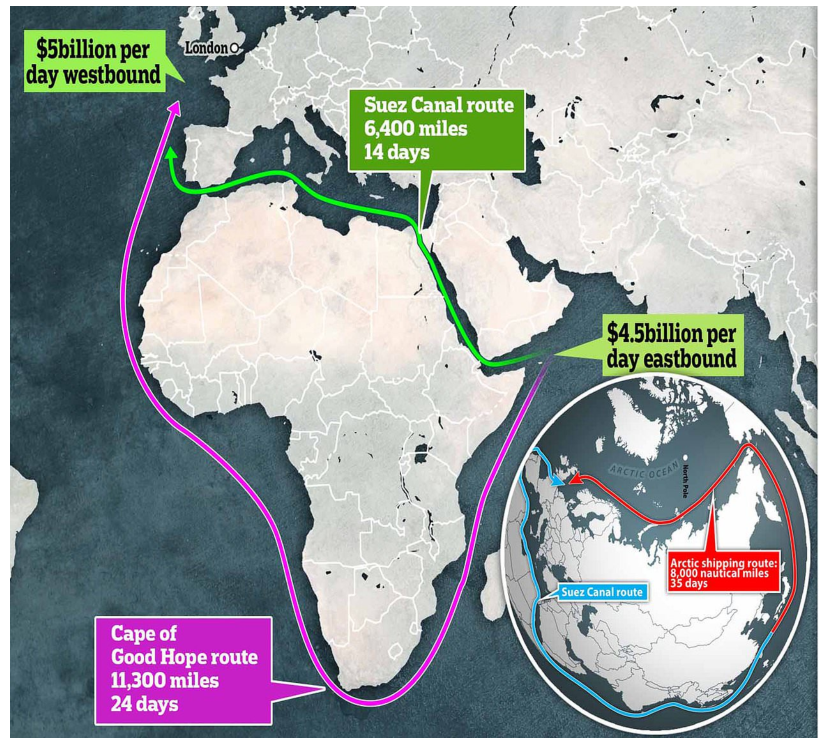

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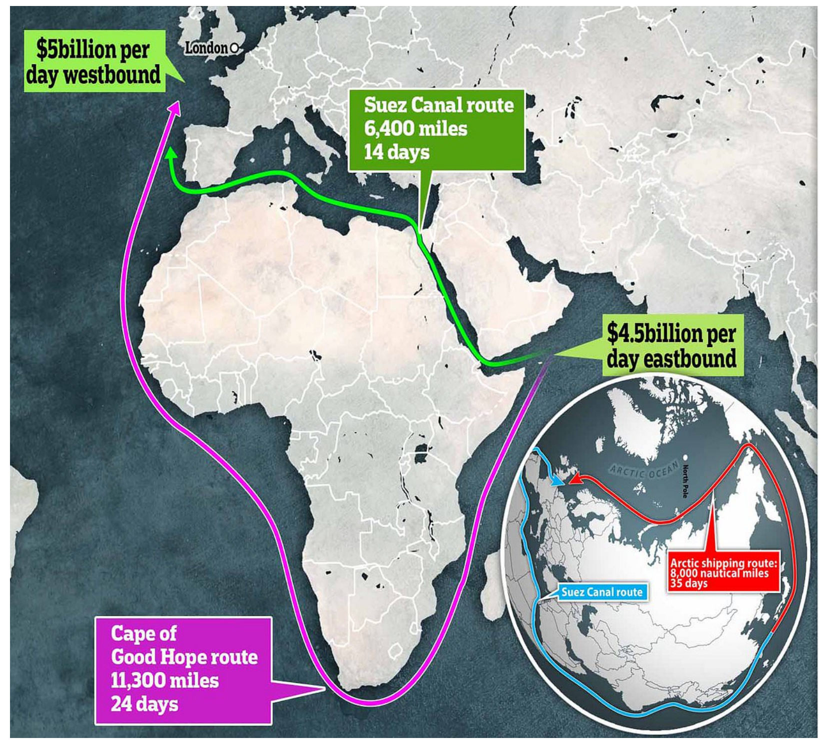

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We only know the canal is CURRENTLY free or not

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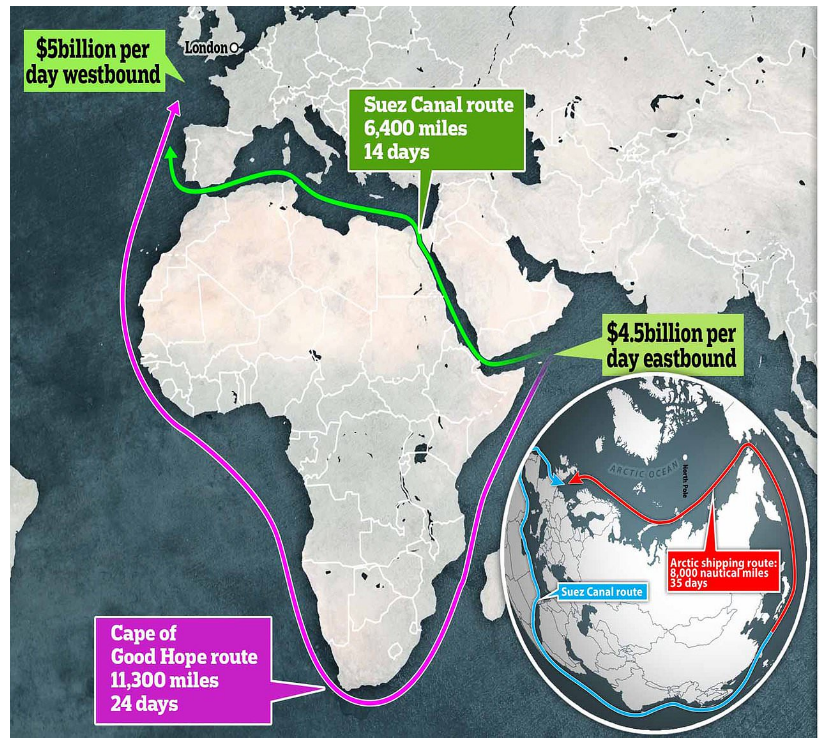

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  - The online algorithm has to respond and make an irrevocable decision once a piece of instance is presented, without knowing the future input

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- Set of images and set of unavailable time intervals
  - $n, s_1, s_2, s_3, \cdots, s_n$
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    - ***Cost***: the objective that we want to minimize or maximize
  - Usually we don’t care about the time complexity

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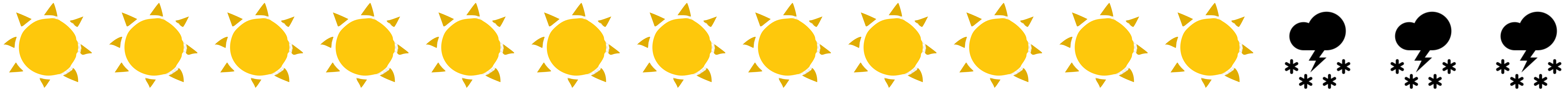
- Instance: B and the number of skiing days d

Buy or Rent? — Offline

- Rent: 1
- Buy: $B = 10$

An example

- Goal: minimize the total cost over the sky holiday



At least 10 skiing days

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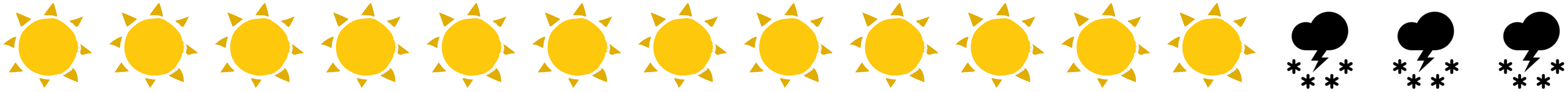

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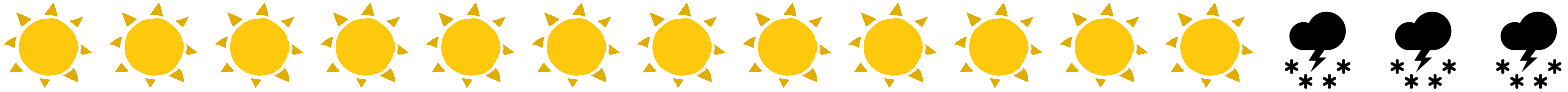

OPT: Buy

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OPT:	Rent	Rent	Rent
3	1	1	1

Less than 10 skiing days

Buy or Rent? — Offline

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OPT: Rent Rent Rent

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Optimal (offline) strategy: Buy the ski iff there are at least B skying days

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ALG:

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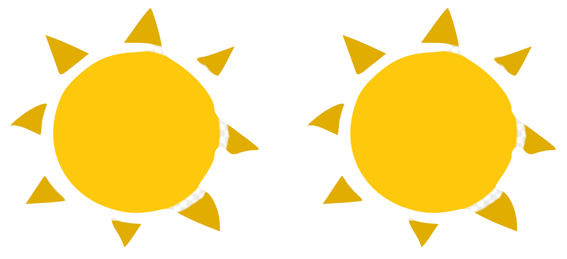

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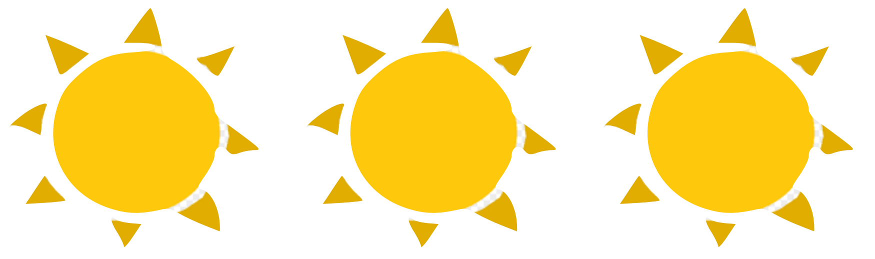

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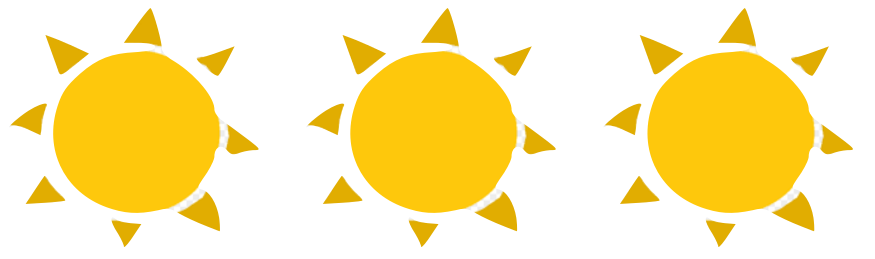

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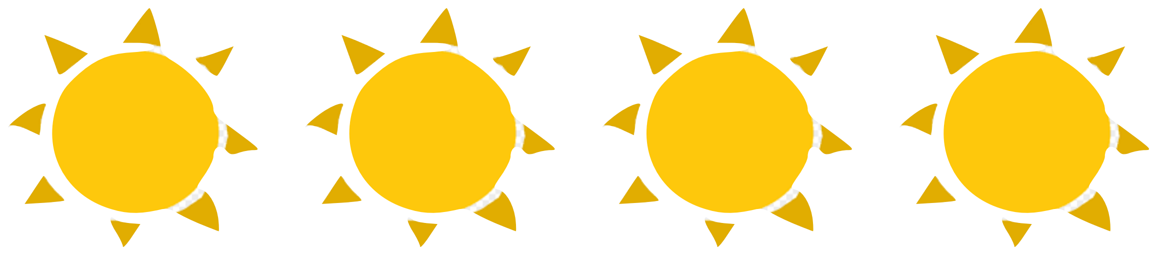

ALG:	Rent	Rent	Rent	Buy
	1	1	1	10

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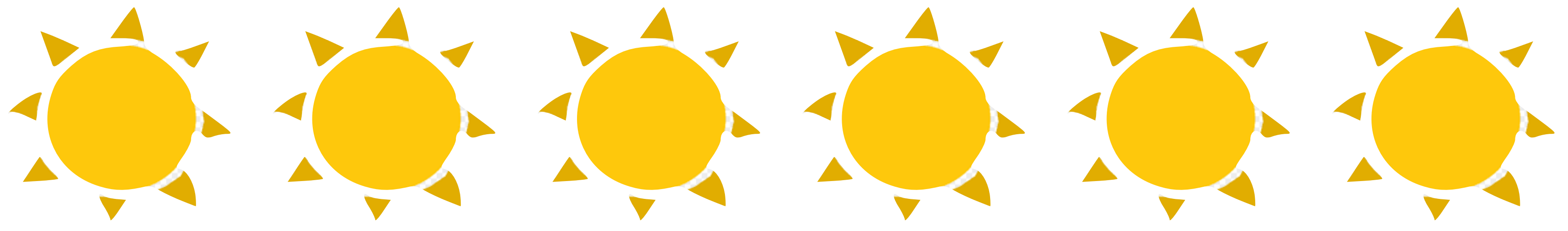

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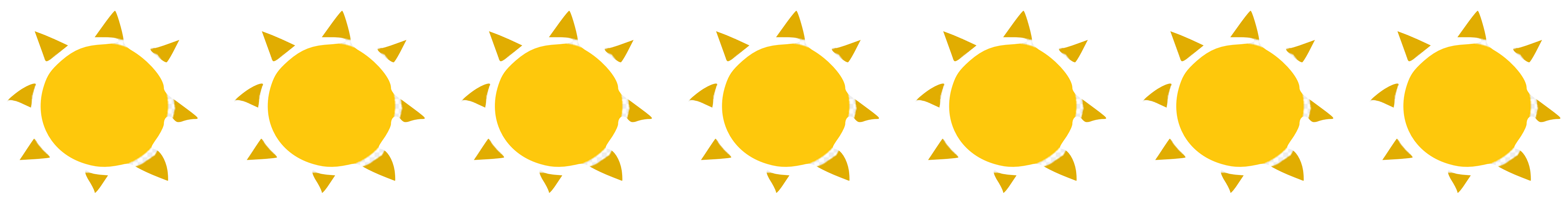

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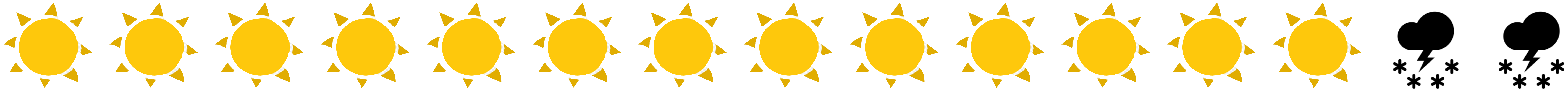

ALG:	Rent	Rent	Rent	Buy
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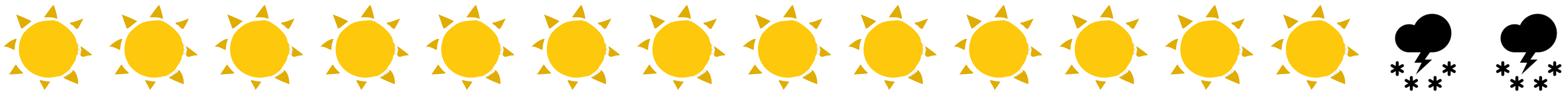

ALG:	Rent	Rent	Rent	Buy
13	1	1	1	10

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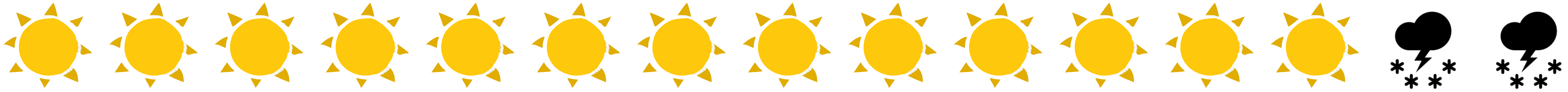
Optimal (offline) strategy: Buy the ski iff there are at least B skying days



ALG:	Rent	Rent	Rent	Buy
13	1	1	1	10
OPT:	Buy			
10	10			

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OPT:	Rent	Rent	Rent	Rent
4	1	1	1	1

What happened

- In an online problem, not every piece of information is known
- An online algorithm has to make decisions based on partial information
  - The decisions are not revokable
- Since the future input is unknown, the whole instance is uncertain
  - The currently good solution can become bad when more information is revealed

What will you do?

- Rent: 1
- Buy: 10
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On *which day of skiing* will you buy the ski?

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  - ALG attains a **competitive ratio** c
  - The closer c is to 1, the better ALG is (note that c is always at least 1)

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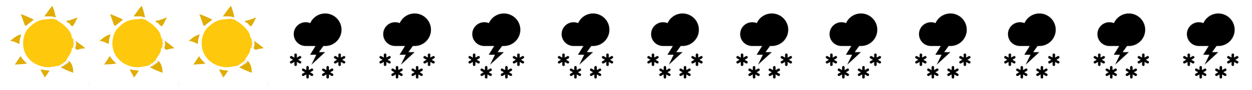
4, 

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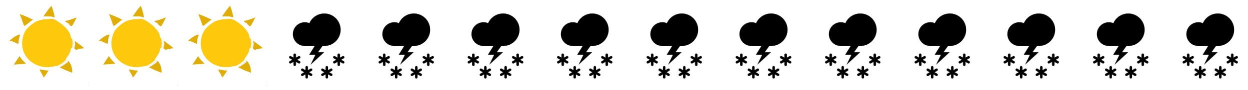
4, 

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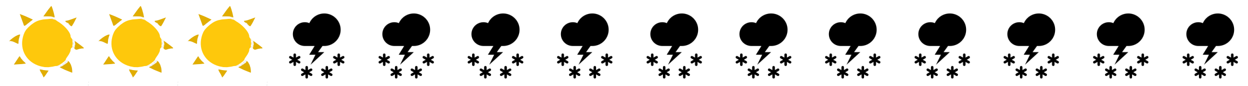
4, 

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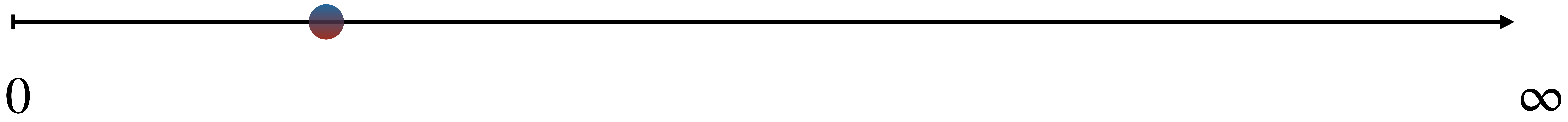

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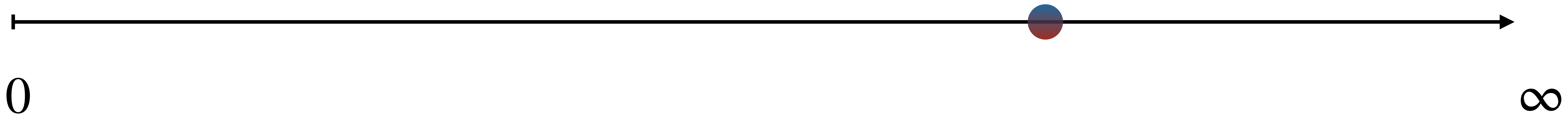

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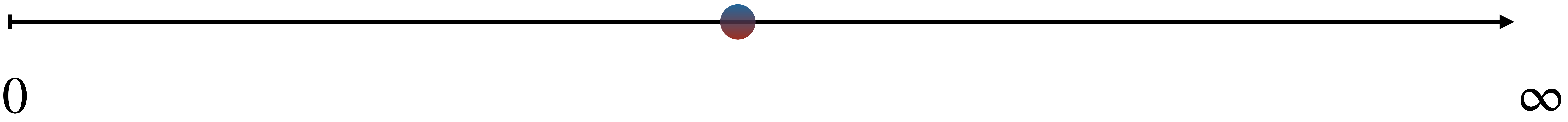

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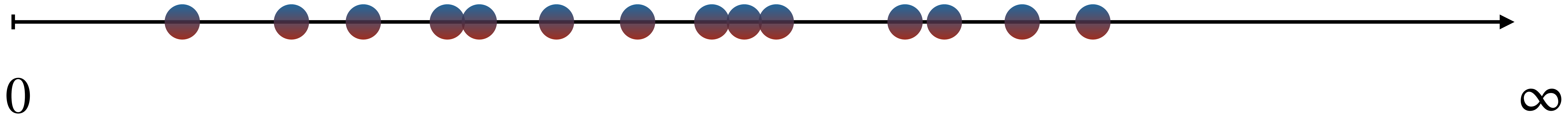

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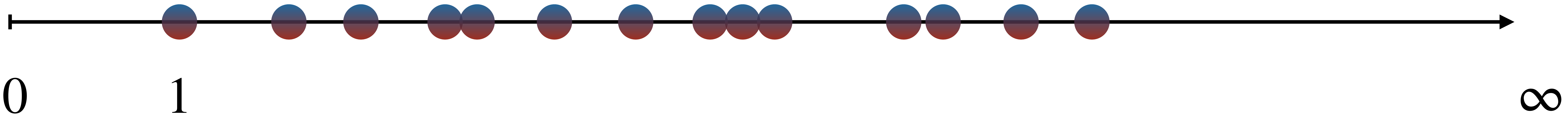

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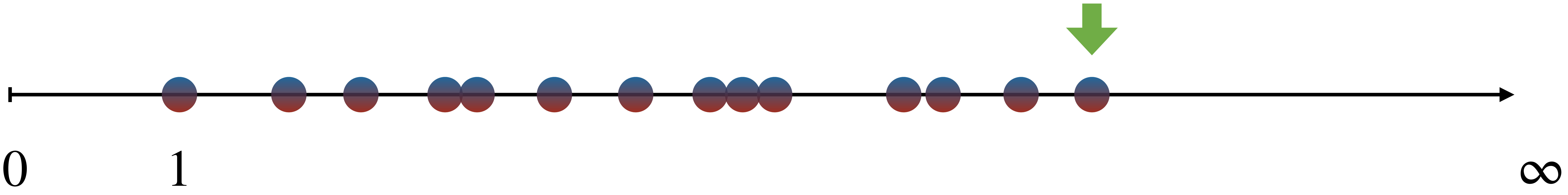

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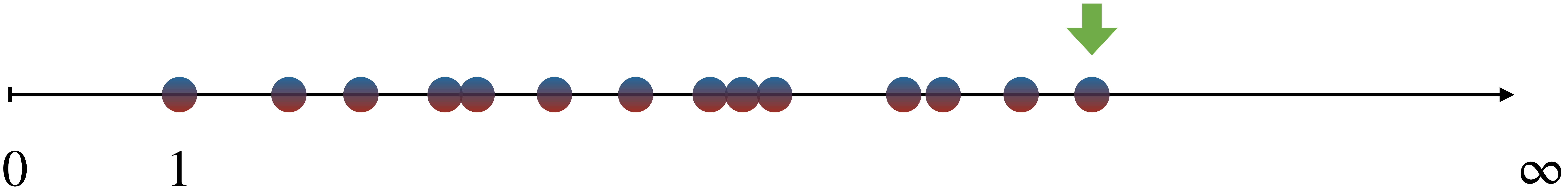
All the “balls” are at the left or at c



For minimization problems,

$ALG(I) \ge OPT(I)$

Competitive Ratio (Minimization)

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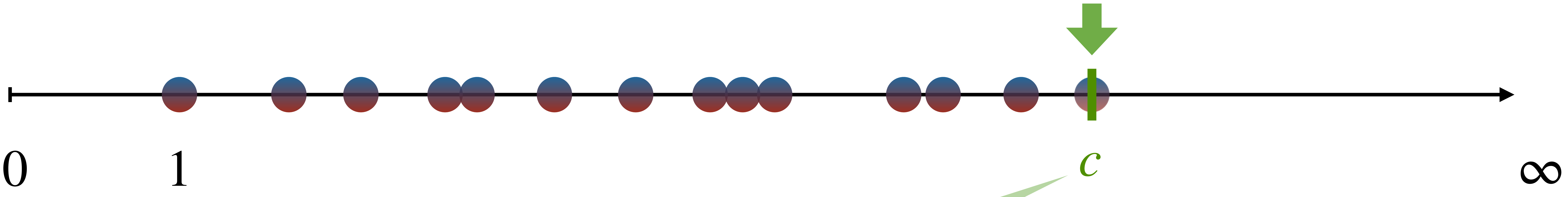

Our goal is
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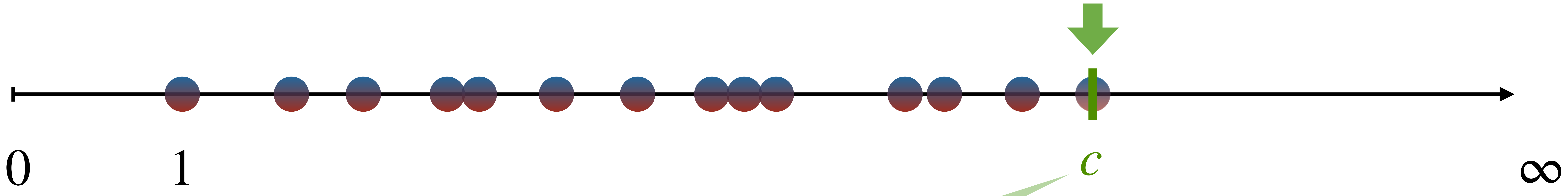


For minimization problems,

$ALG(I) \geq OPT(I)$

Our goal is
to find this c

Even in the worst case,
we will not pay more than
c times the optimal solution.

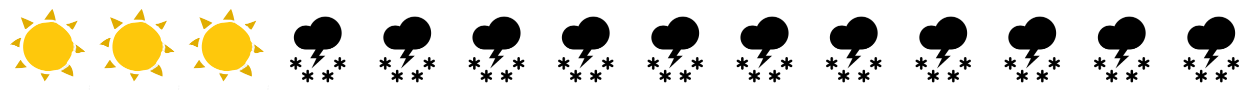
Competitive Ratio (Minimization)

- An online algorithm ALG is c-**competitive** if for all instance I,

$$\text{ALG}(I) \leq c \cdot \text{OPT}(I)$$

  - That is, for all instance I,

$$\frac{\text{ALG}(I)}{\text{OPT}(I)} \leq c$$

4, 

4, 

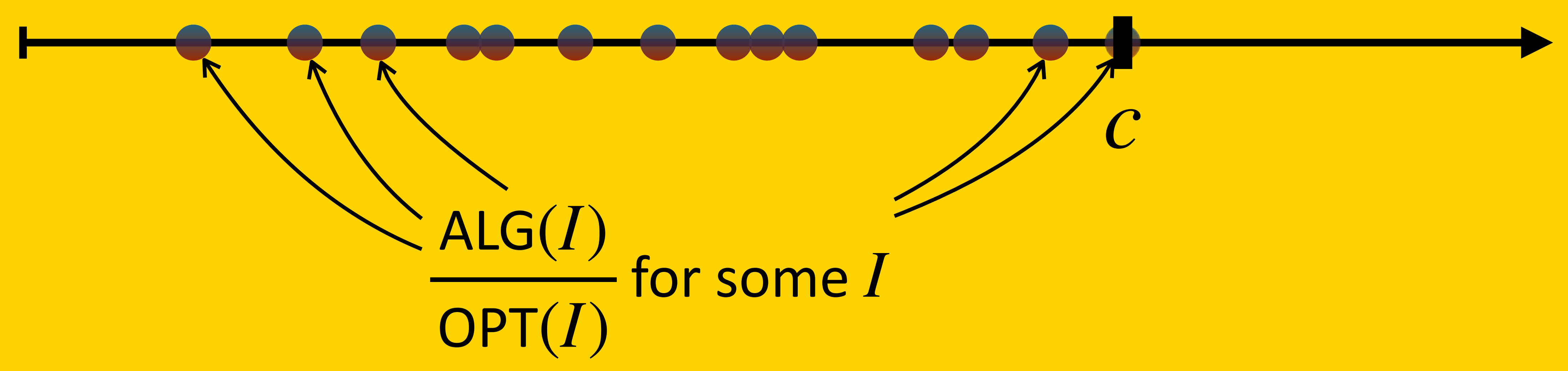
100, 

- The competitive ratio of an online algorithm is its performance guarantee: Even for the worst case, it is not too much worse than the optimal solution

What Happened?

- The performance measurement of an online algorithm is by **competitive analysis**
  - Against an optimal offline algorithm (which knows the future and has unlimited computation power)

- The ultimate goal of competitive analysis on an online algorithm ALG is to show that there is some c such that for any instance, the cost of ALG is no more than c times the optimal cost
  - Algorithms with a smaller c are better



What is the goal for the project

- If you want to have some theoretical results for the online setting, you have to
  - Design an online algorithm
  - Prove that for any set of images and any set of unavailable time intervals, the cost (transmission completed time) of the online algorithm is at most c times of the optimal cost for some c

- **It is okay to target on some special cases!**

Outline

- **Online problems & online algorithms** — optimization with uncertainty
  - First example: **Ski-rental**
- Measure the performance: **Competitive ratio**
  - How good is an online algorithm?
- **Adversarial game**
  - How bad is an online algorithm?

What will you do?

- Rent: 1
- Buy: 10
- Goal: minimize the total cost over the sky holiday

On ***which day*** *of skiing* will you buy the ski?

What will you do?

- Rent: 1
- Buy: 10
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On ***which day*** *of skiing* will you buy the ski?

Keep renting the ski until the B-th skiing day

A 2-Competitive Online Algorithm

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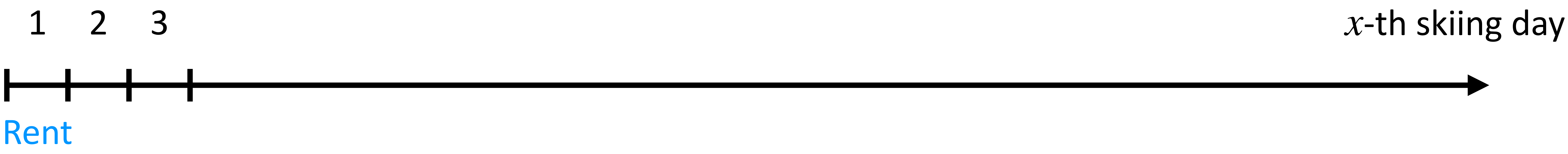

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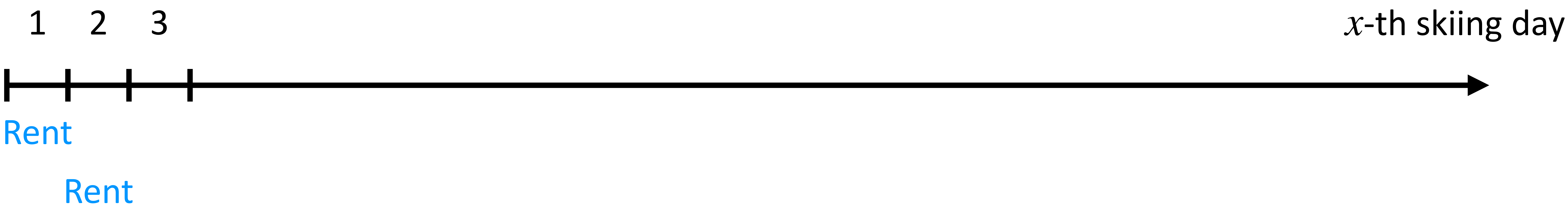

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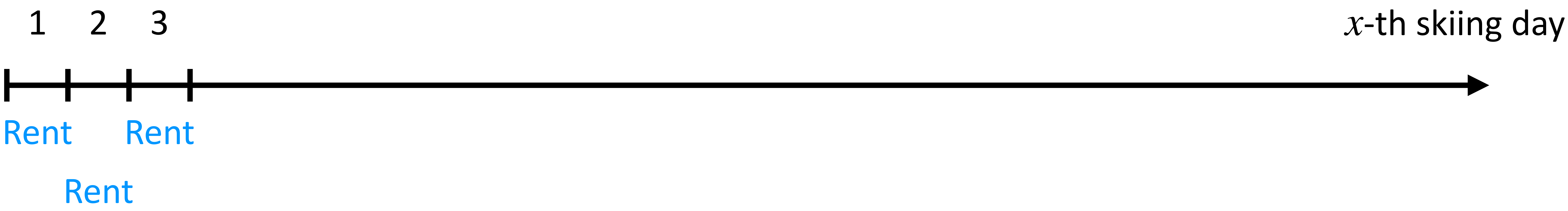

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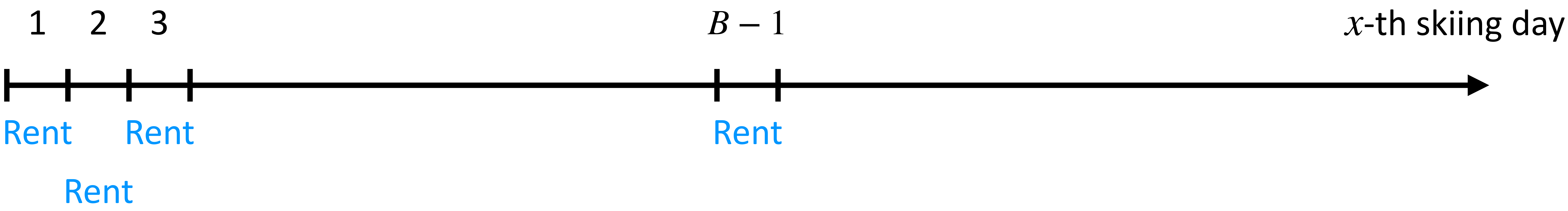

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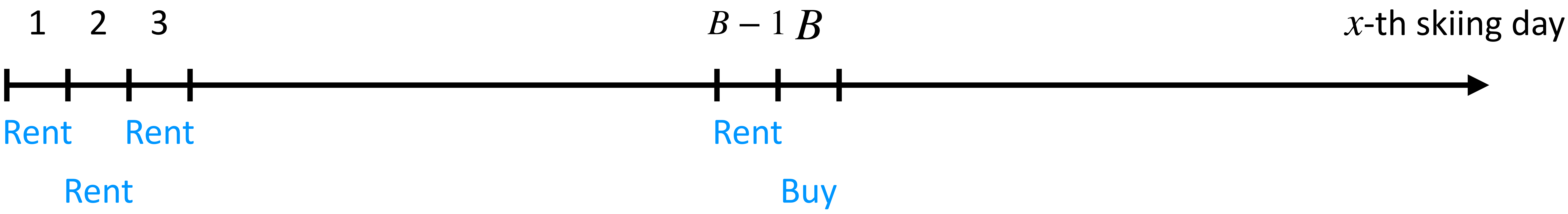

A 2-Competitive Online Algorithm

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- Theorem: For the Buy-or-Rent problem, this algorithm is $(2 - \frac{1}{B})$-competitive.

<Proof idea>

We prove this theorem by showing that no matter how many skiing days there are, the algorithm cost is no more than twice the optimal cost.

A 2-Competitive Online Algorithm

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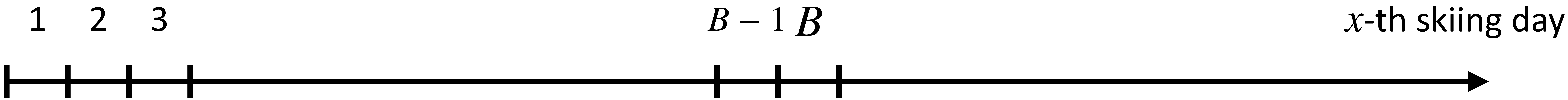

d: the number of actual total skiing days

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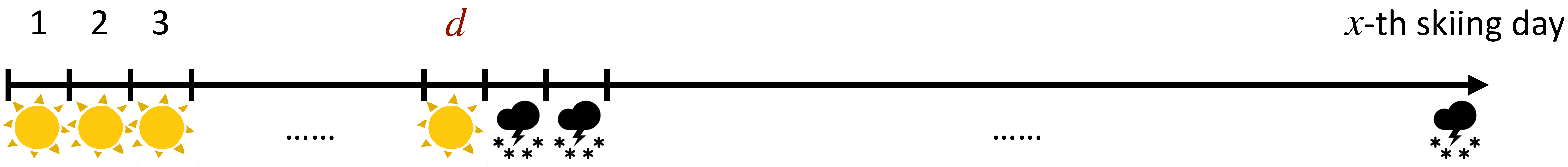

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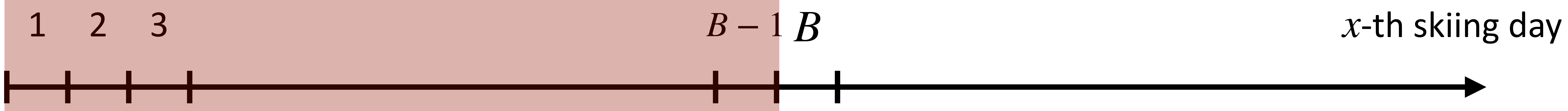

d: the number of actual total skiing days

If d is in this range, the optimal strategy is to buy the ski for each of the d days.

OPT = d

A 2-Competitive Online Algorithm

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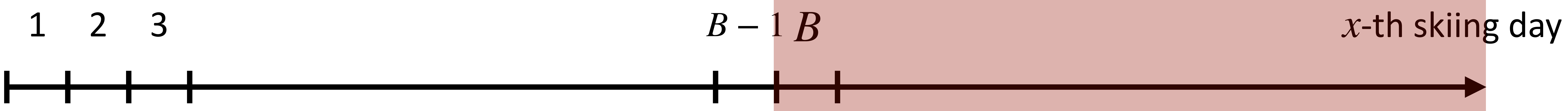

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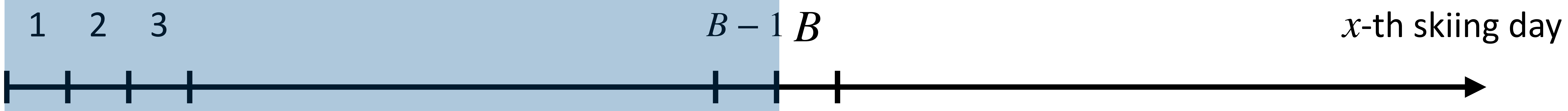

d: the number of actual total skiing days

If d is in this range, the algorithm rents for each of the d days.

ALG = d

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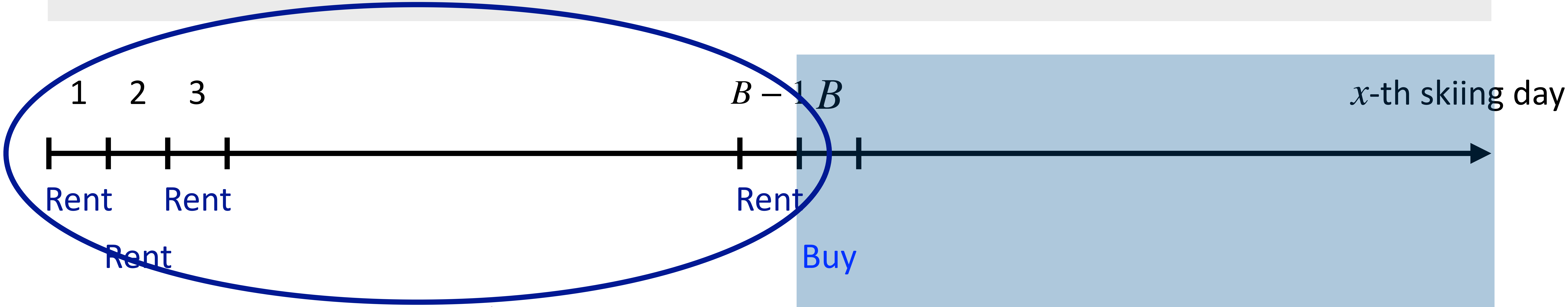

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If d is in this range, the algorithm buys the ski on the B-th day.

ALG $= (B-1) \cdot 1 + B$

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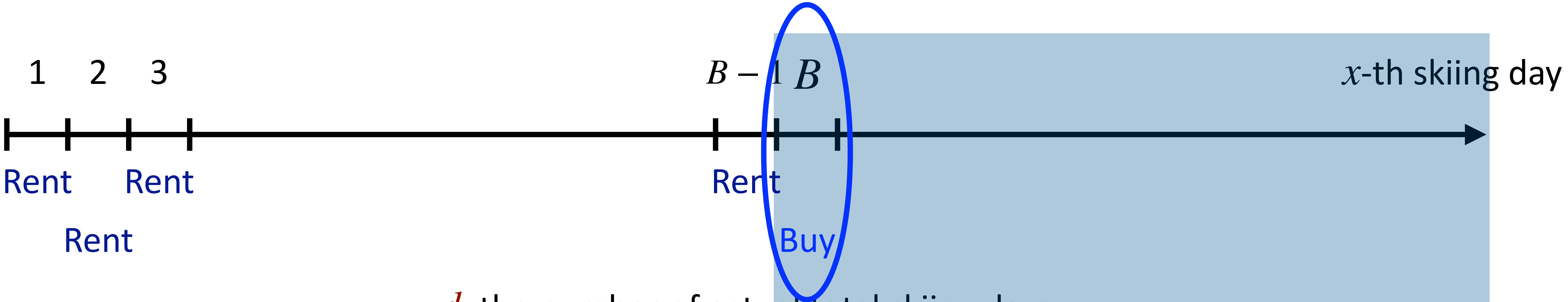

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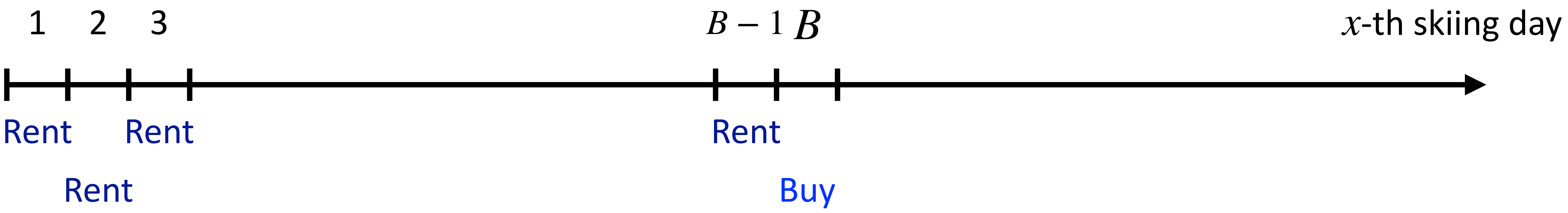

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If $d < B$

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ALG = d

$$\frac{\text{ALG}(I)}{\text{OPT}(I)} = \frac{d}{d}$$

If $d \geq B$

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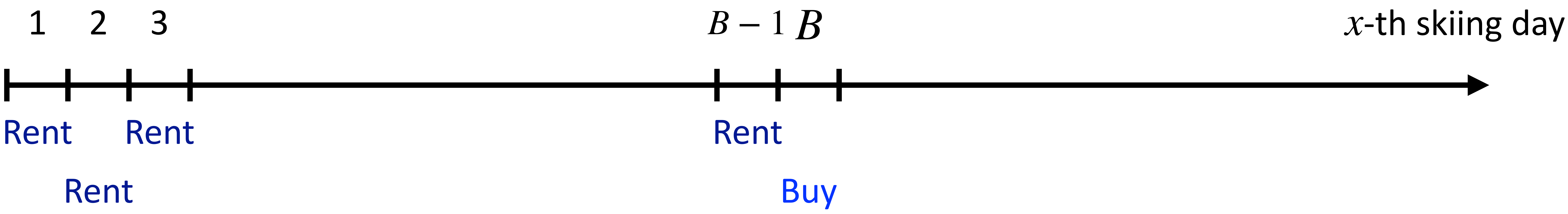

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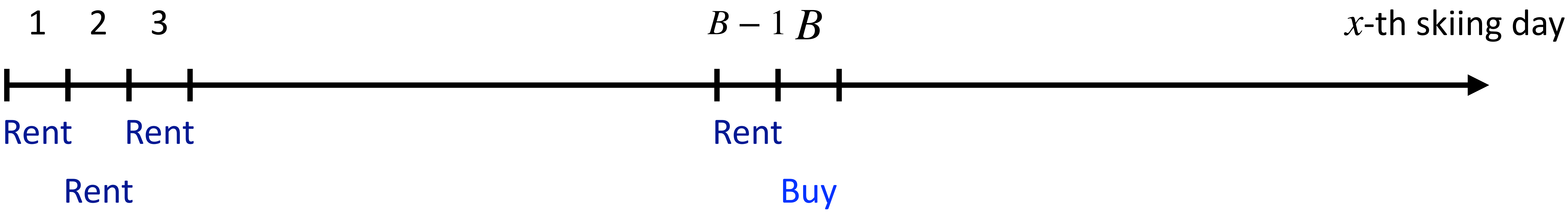

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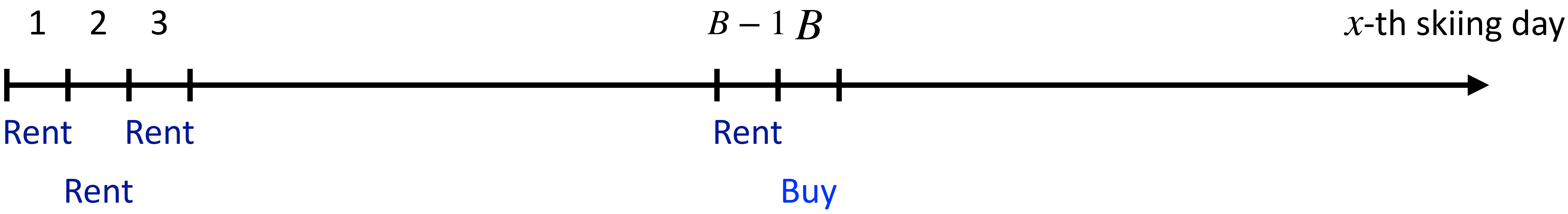

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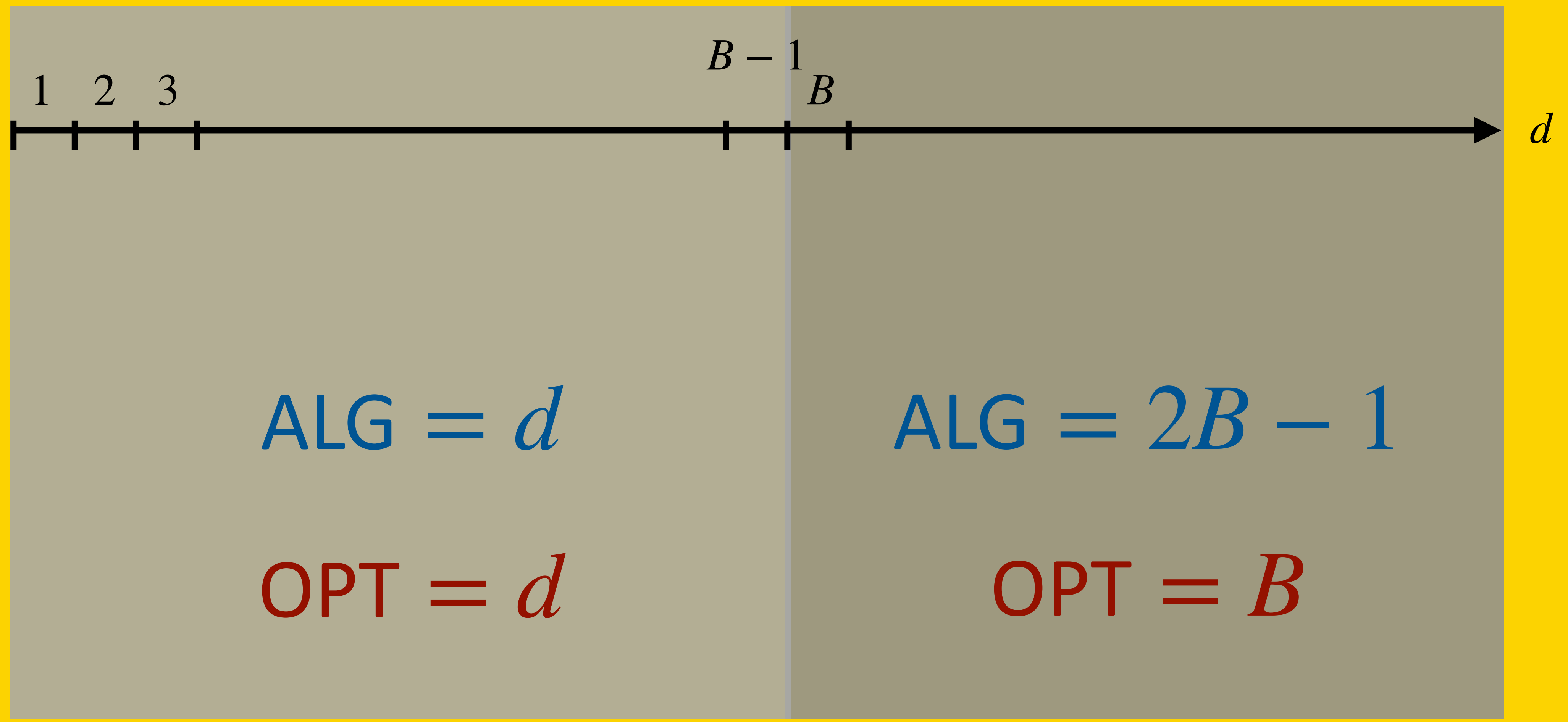
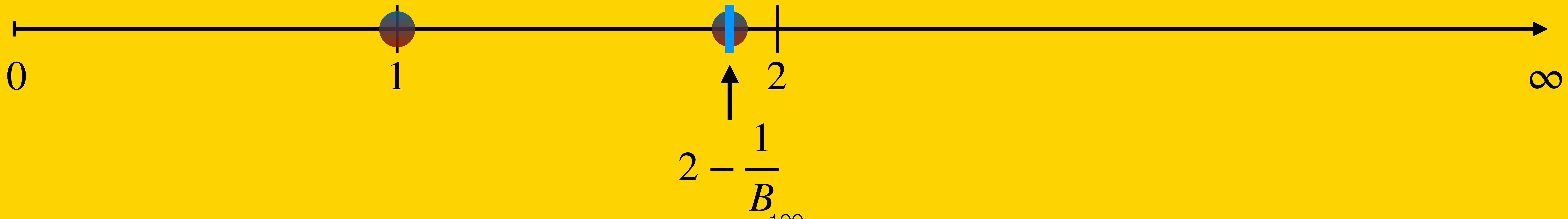
$$= \frac{2B-1}{B} = 2 - \frac{1}{B}$$

Competitive Ratio Proof Review

- We partition the set of instances into to cases: $d \leq B - 1$ or $d \geq B$.
  - For the case $d \leq B - 1$, $\dfrac{\text{ALG}(B, d)}{\text{OPT}(B, d)} = 1$
  - For the case $d \geq B$, $\dfrac{\text{ALG}(B, d)}{\text{OPT}(B, d)} = 2 - \dfrac{1}{B}$

➡ The algorithm **ALG** is $(2 - \dfrac{1}{B})$-competitive





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Bad person

Online algorithm

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Online Player - Adversary Game

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Bad person

I want to make the algorithm fail
to guarantee a small c!

Online algorithm

I want to guarantee a small c
for any I!

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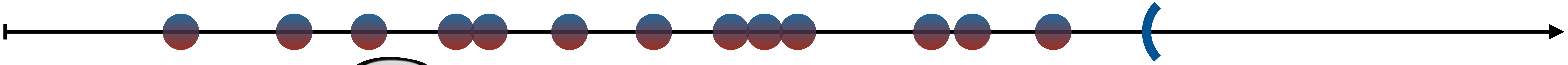
Bad person

Online algorithm

I want to find some instance I' such that $\frac{\text{ALG}(I')}{\text{OPT}(I')} > c$!

I want to guarantee that for any I, $\frac{\text{ALG}(I)}{\text{OPT}(I)} \le c$!

Online Player - Adversary Game



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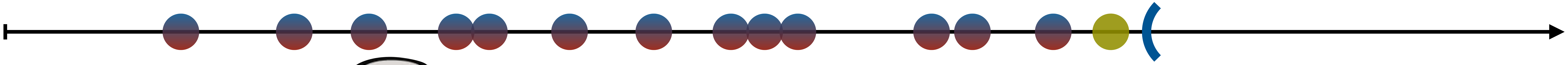

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Online Player - Adversary Game

- One way to view the problem of analyzing online algorithms is to view it as a game between an **online player** (algorithm) and a malicious **adversary**

  - The online player runs its online algorithm on an input created by the adversary

  - The adversary, based on the knowledge of the algorithm used by the online player, constructs the worst possible input so as to maximize the competitive ratio

Adversary

ALG_1: Buy the ski on the first day 

Adversary

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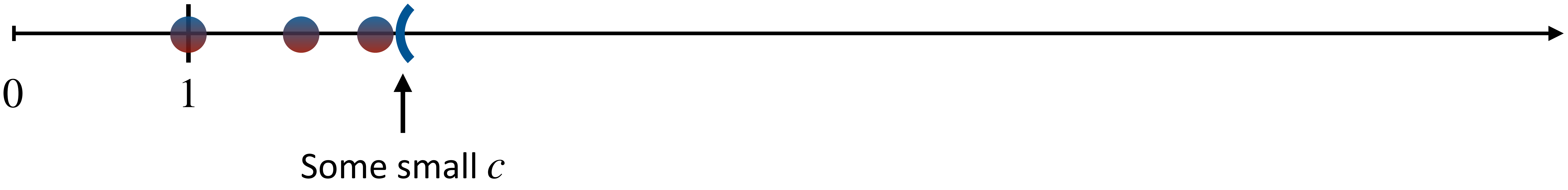
Adversary

ALG$_1$: Buy the ski on the first day

It starts raining since
the 4-th day!



Adversary

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$$\frac{\text{ALG}_1(d=3)}{\text{OPT}(d=3)} = \frac{B}{3}$$

Adversary

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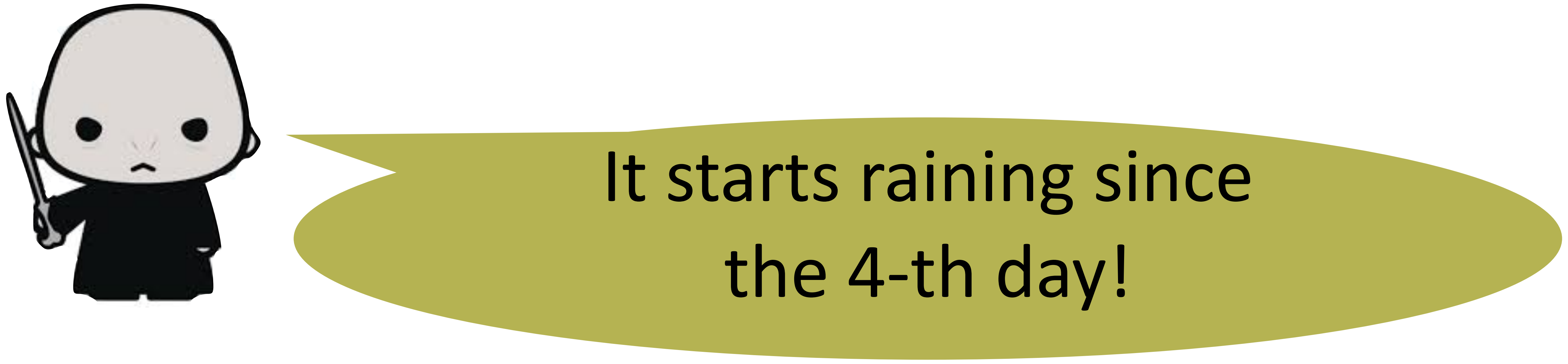

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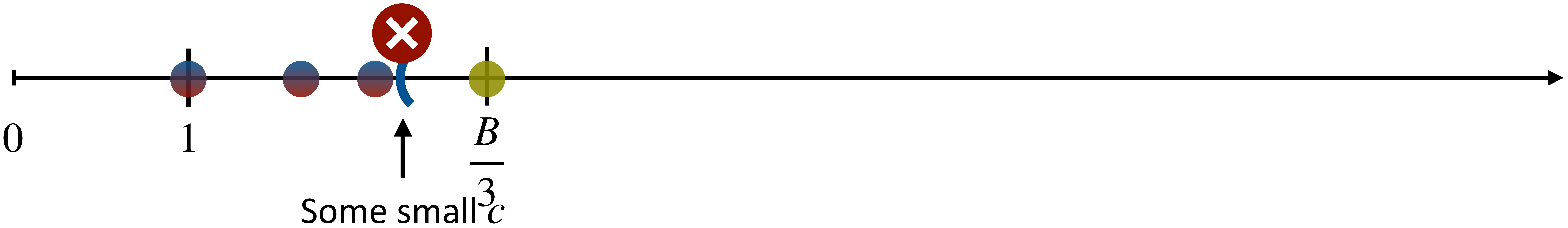

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ALG$_1$ is **at least** $\frac{B}{2}$-competitive

The competitive ratio (the point where all balls are at the left) can not be smaller than $\frac{B}{2}$

Adversary

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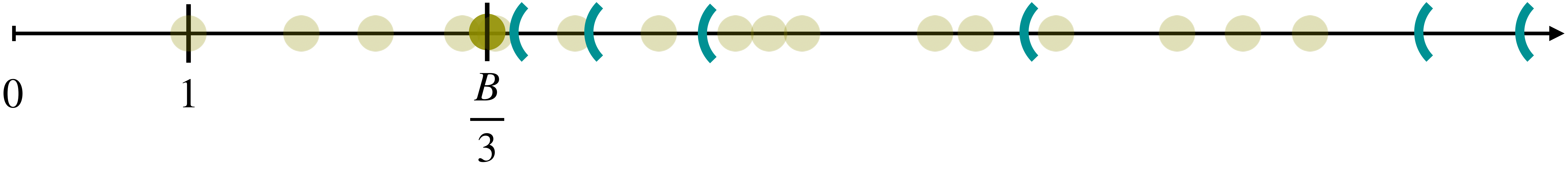

Adversary

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Adversary

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$$\frac{\text{ALG}_1(d = 1)}{\text{OPT}(d = 1)} = \frac{B}{1} = B$$

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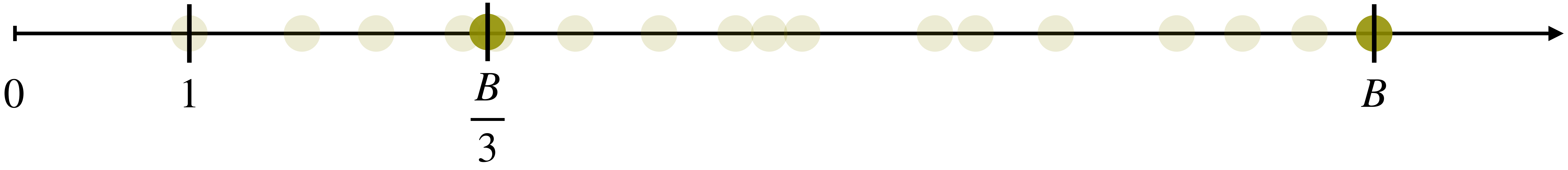

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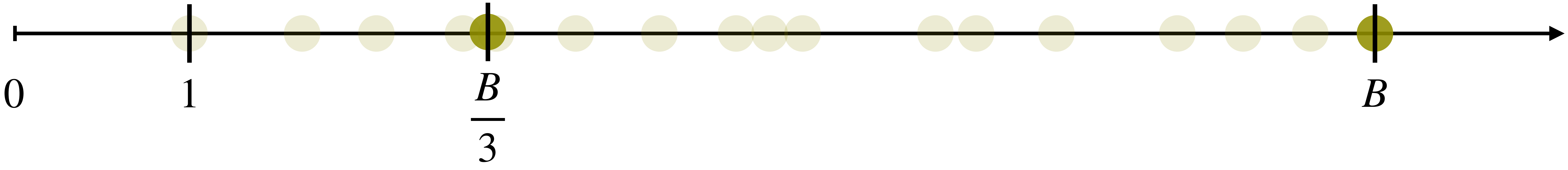

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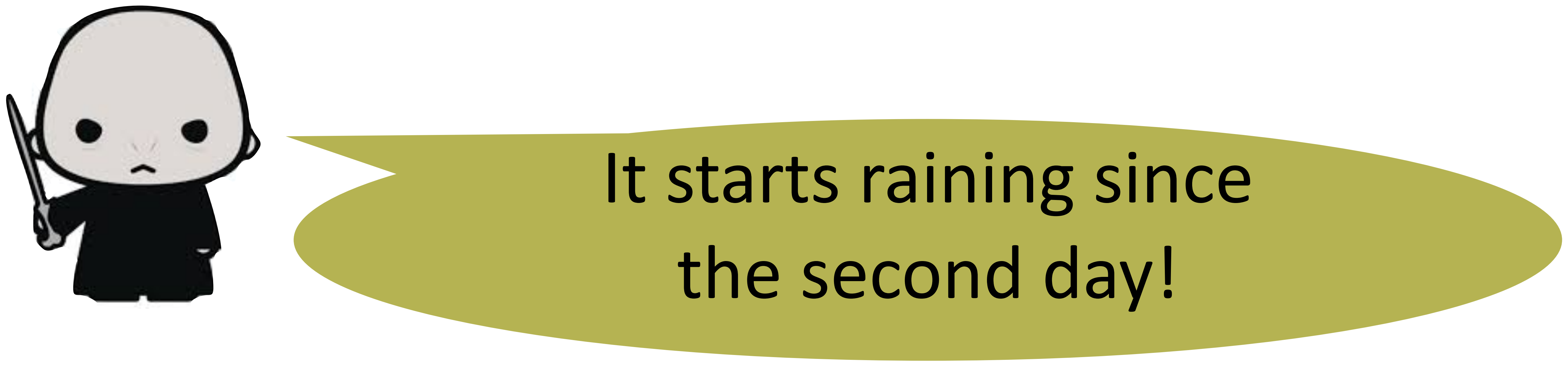

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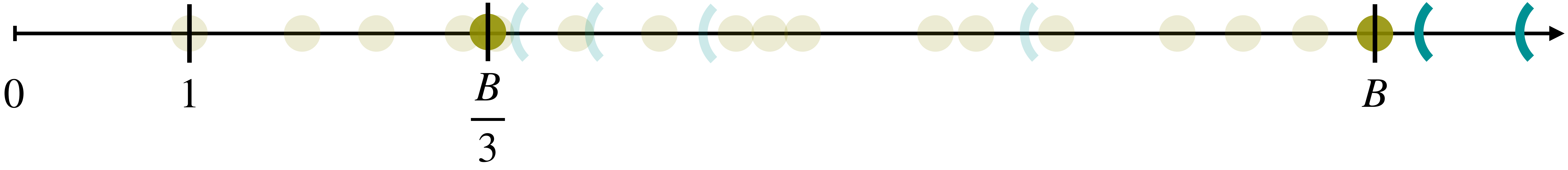

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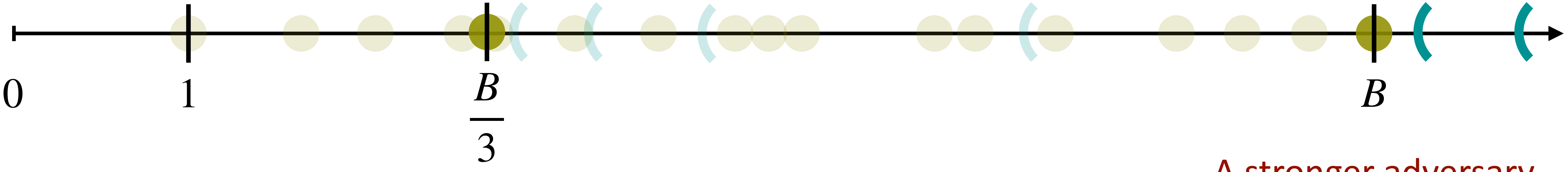

A stronger adversary

Adversary

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Adversary

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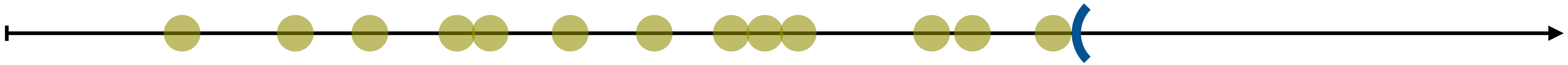
  - There is an input I' such that $\mathrm{ALG}(I') \geq c' \cdot \mathrm{OPT}(I')$

  $\iff$ NOT $\left\{ \text{for all instance } I,\ \mathrm{ALG}(I) \leq c \cdot \mathrm{OPT}(I) \right\}$ for some $c < c'$



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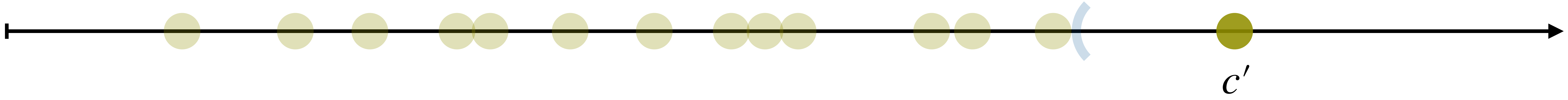

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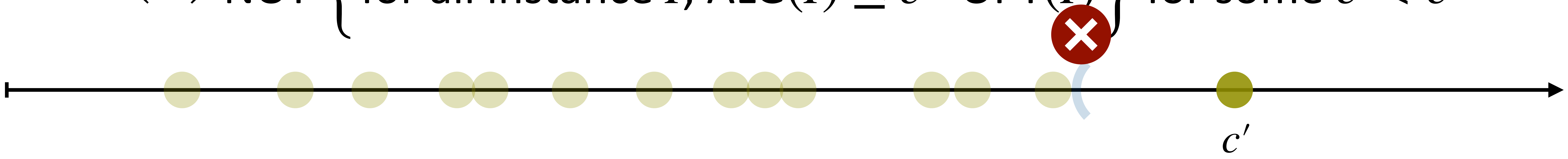

- An online algorithm ALG is c-**competitive** if for all instance I,

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Adversary

- By designing an adversarial input for an algorithm ALG, one can find the *lower bound* of the competitive ratio of ALG
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ALG$_B$: Buy the ski on the B-th skiing day

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ALG$_B$: Buy the ski on the B-th skiing day

- Which is/are the *strongest* adversary?

① $d = B - 1$

② $d = B$

③ $d = 2B$

What Happened

- The analysis of online algorithms can be seen as a game between the online algorithms and an adversary

- The adversary designs the next input according to the previous decisions of the online algorithm
  - The adversary tries its best to torture the online algorithm, punishes it for everything it does

- An adversary provides a *lower bound* of the competitive ratio

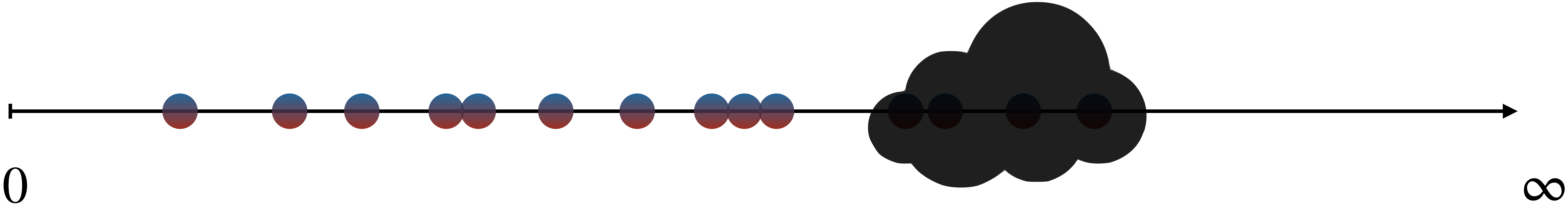
What is the goal for the project

- If you want to have some theoretical results for the online setting, you have to
  - Design an online algorithm
  - Prove that for any set of images and any set of unavailable time intervals, the cost (transmission completed time) of the online algorithm is at most c times of the optimal cost for some c

- You can start with finding the competitive ratio lower bound by designing an online algorithm and finding an adversary against it

Lower Bound and Tight Analysis

- In many cases, we don't know the optimal solution cost
  - There may be infinite instances
  - Even there are finite instances, we may not know the optimal solution behavior of all the instances

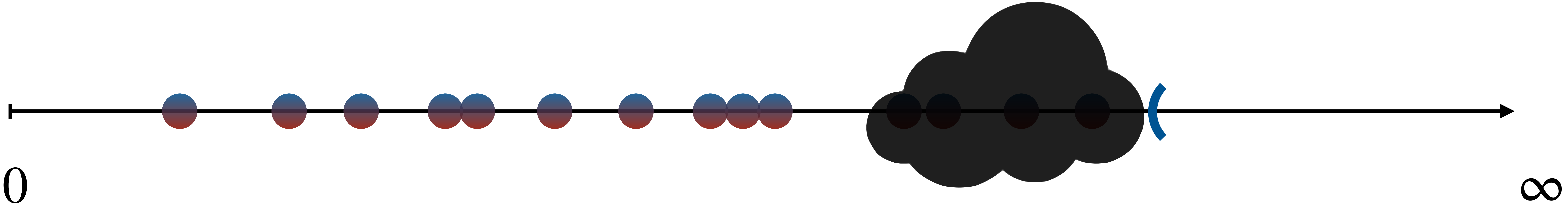


Lower Bound and Tight Analysis

- Sometimes we can only *upper-bound* the competitive ratio:

Argue that: For any instance I, ALG$(I) \leq x$ and OPT$(I) \geq y$. Therefore,

$$\frac{\text{ALG}(I)}{\text{OPT}(I)} \leq \frac{x}{y}$$



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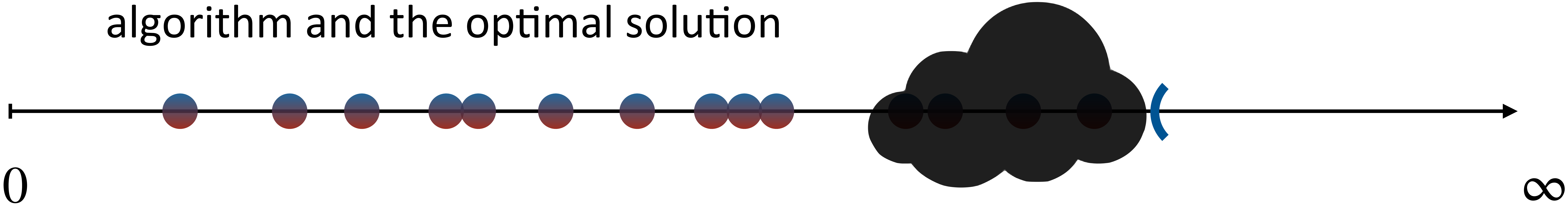

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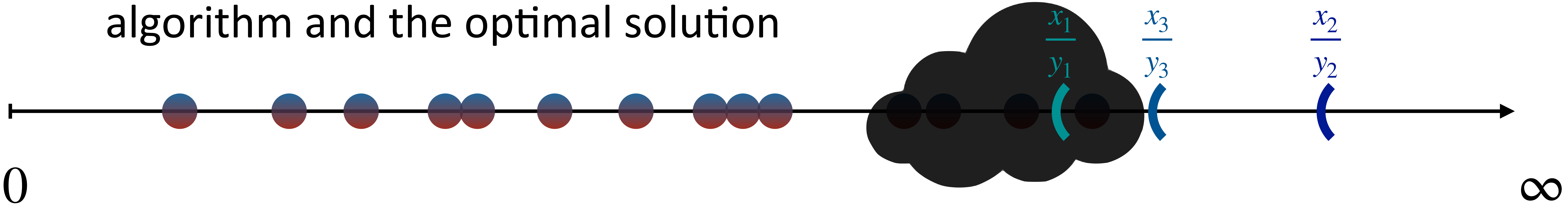
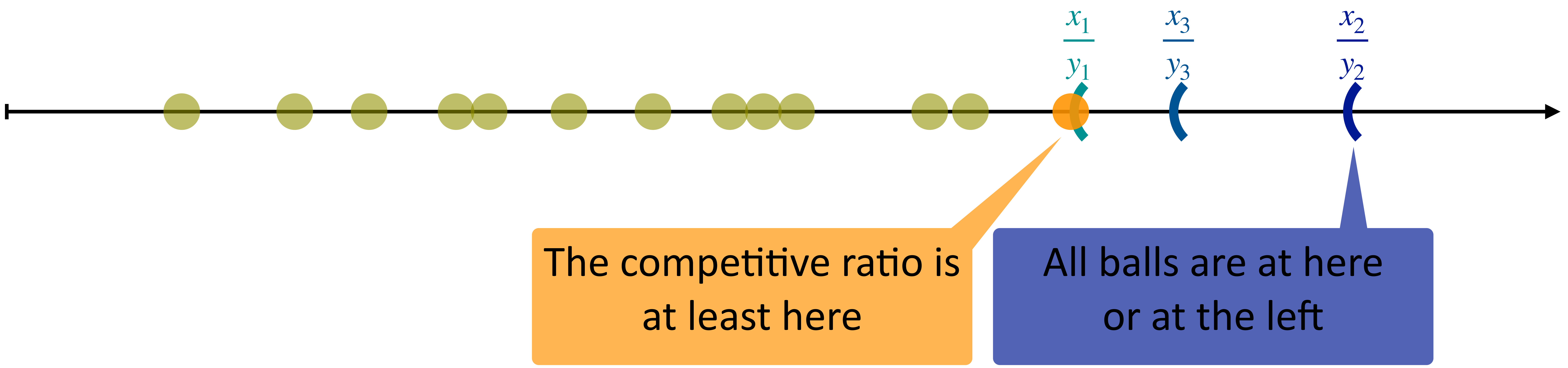


- Different x's and y's provide different upper bounds

Lower Bound and Tight Analysis

- An upper bound c is *tight* if

there exists an instance I^* such that $\dfrac{\mathrm{ALG}(I^*)}{\mathrm{OPT}(I^*)} = c$



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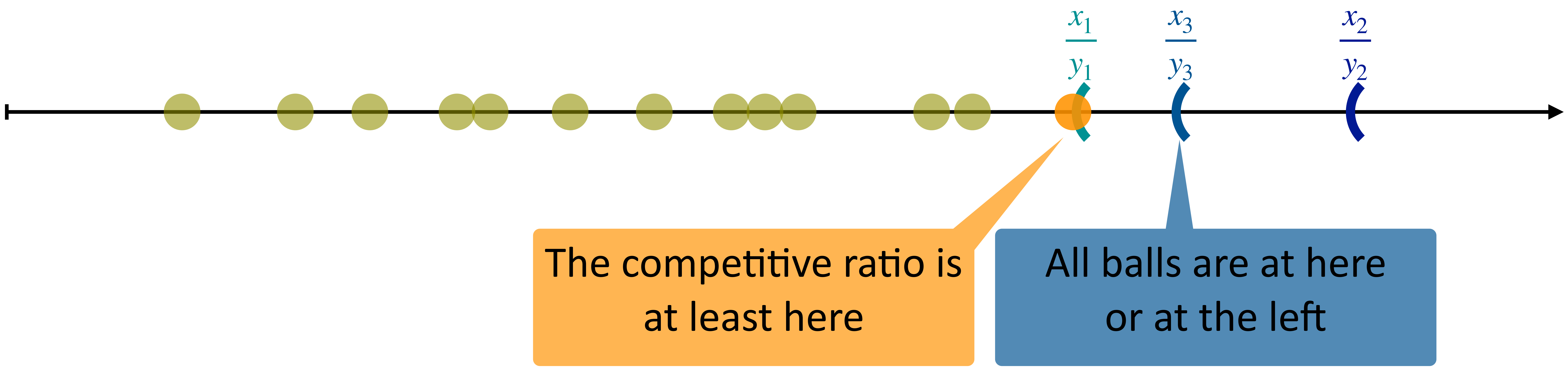

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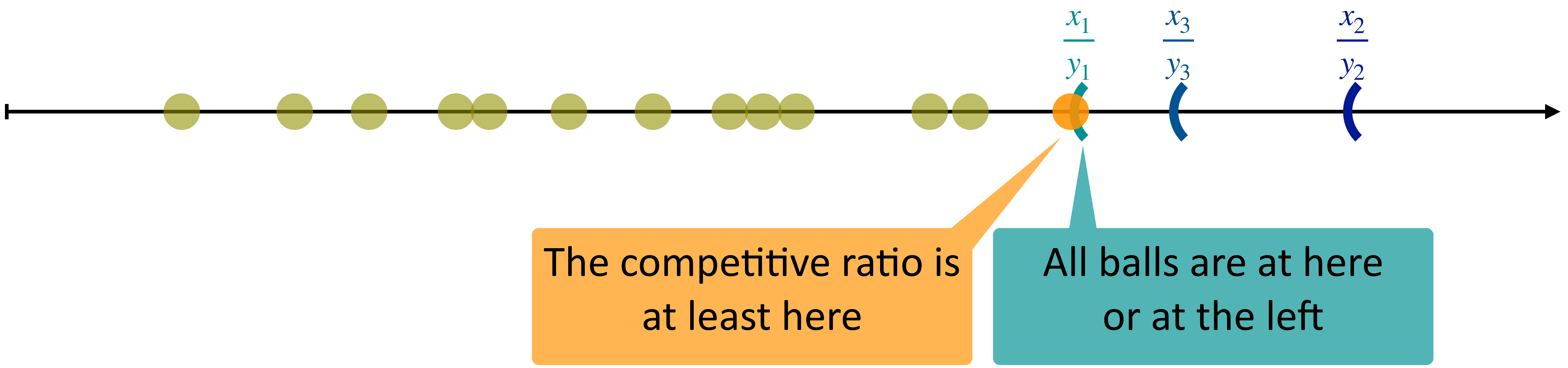

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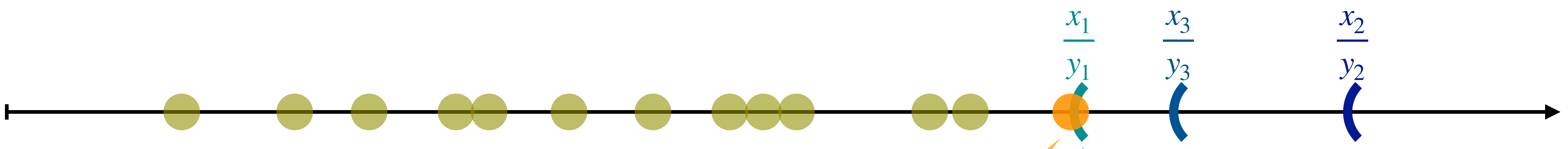

The competitive ratio is at least here

All balls are at here or at the left

The upper bound hits a "real ball" and cannot be pushed left further

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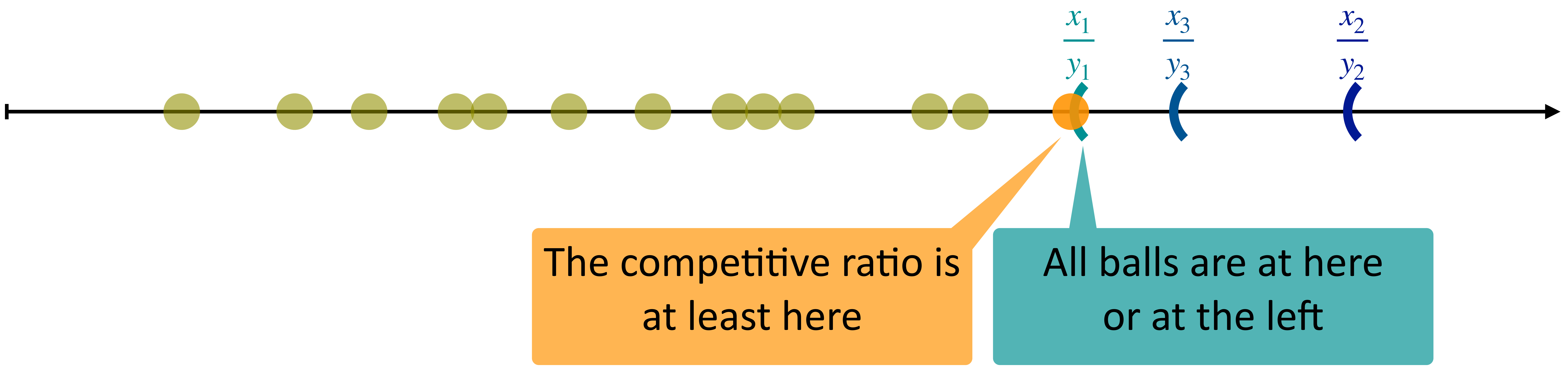

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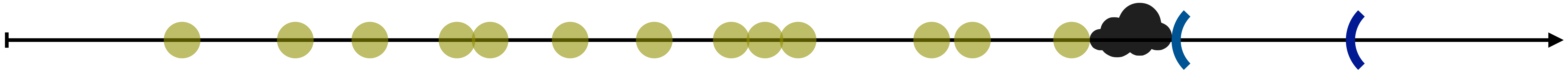

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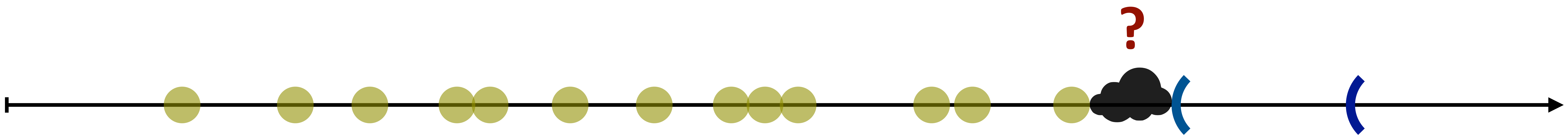

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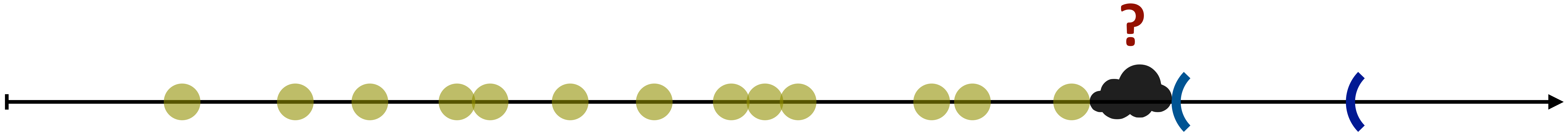

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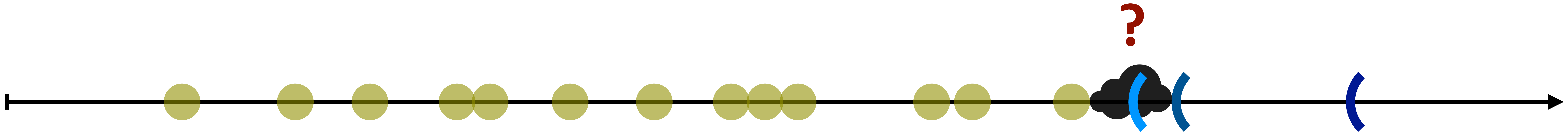

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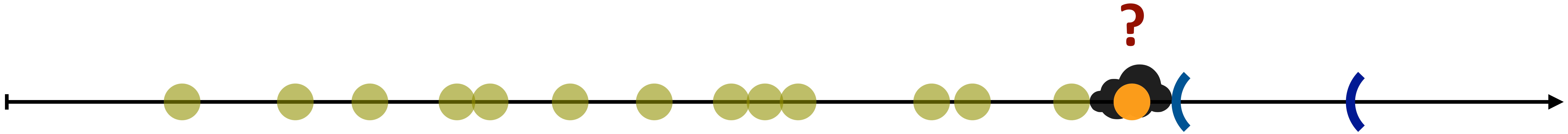


Loop of Online Algorithms Design

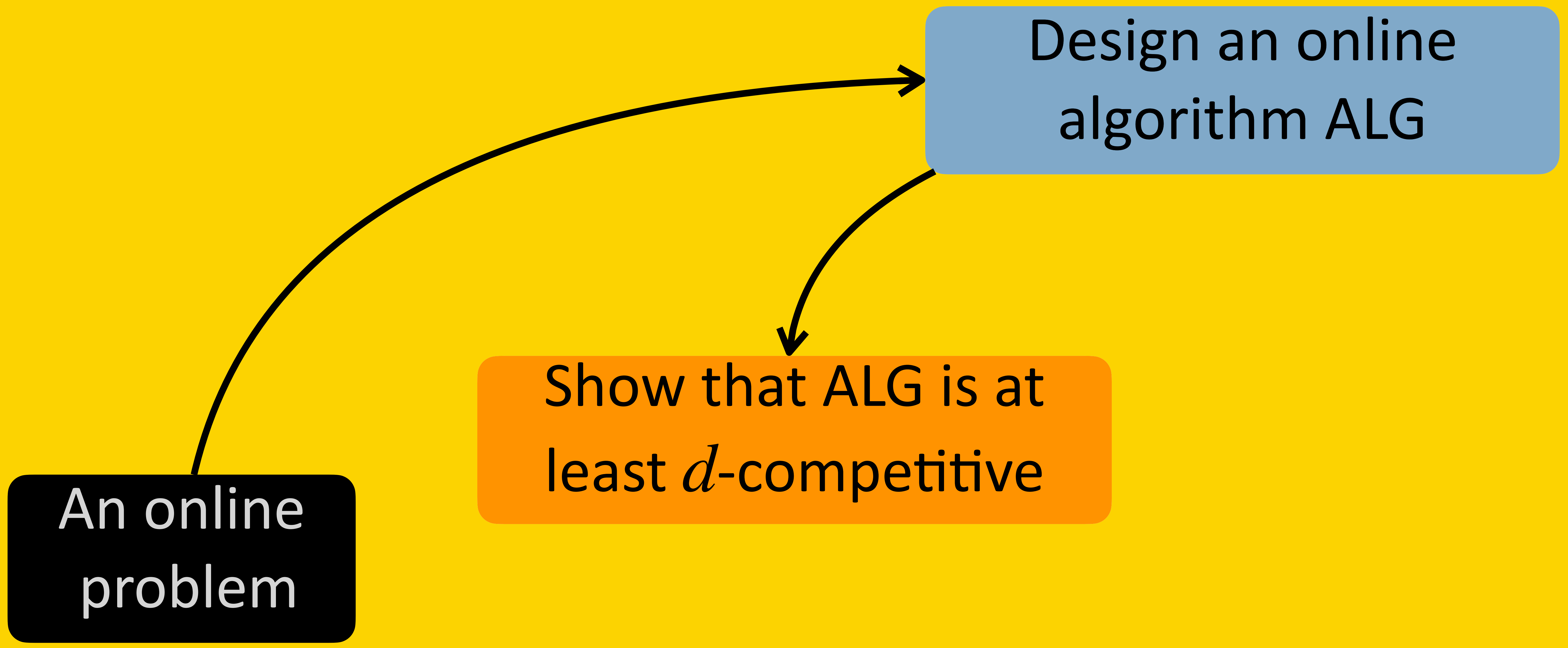
An online
problem

Loop of Online Algorithms Design

Design an online algorithm ALG

An online problem

Loop of Online Algorithms Design



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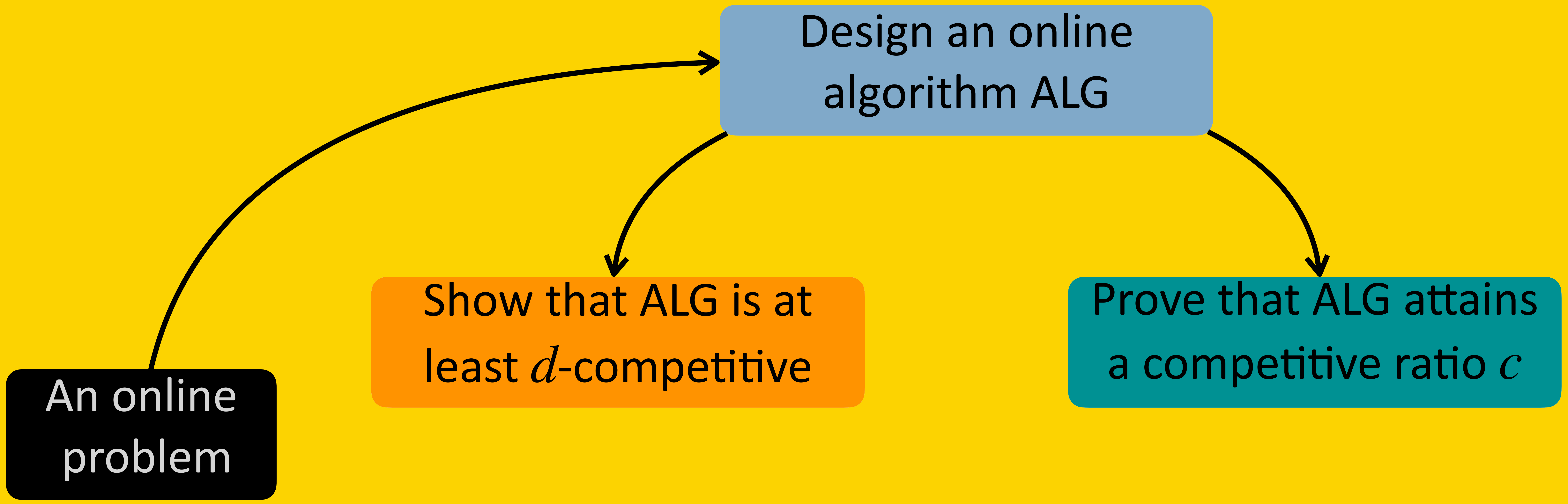

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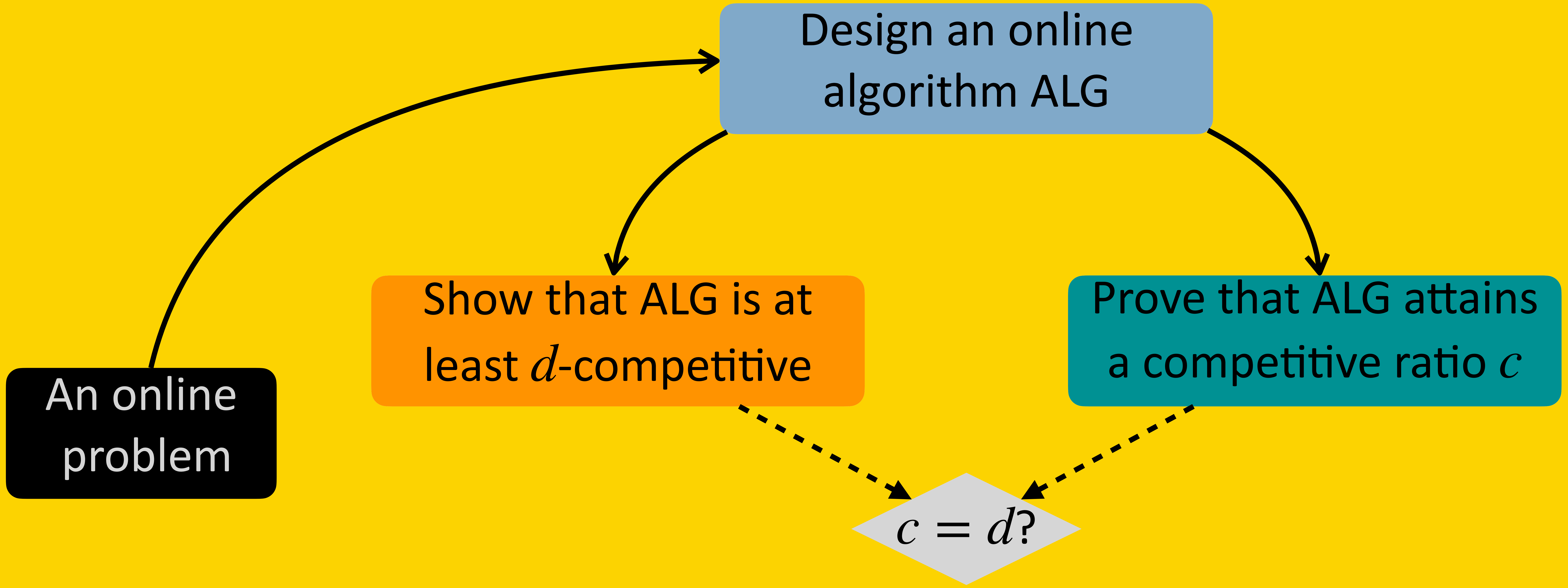

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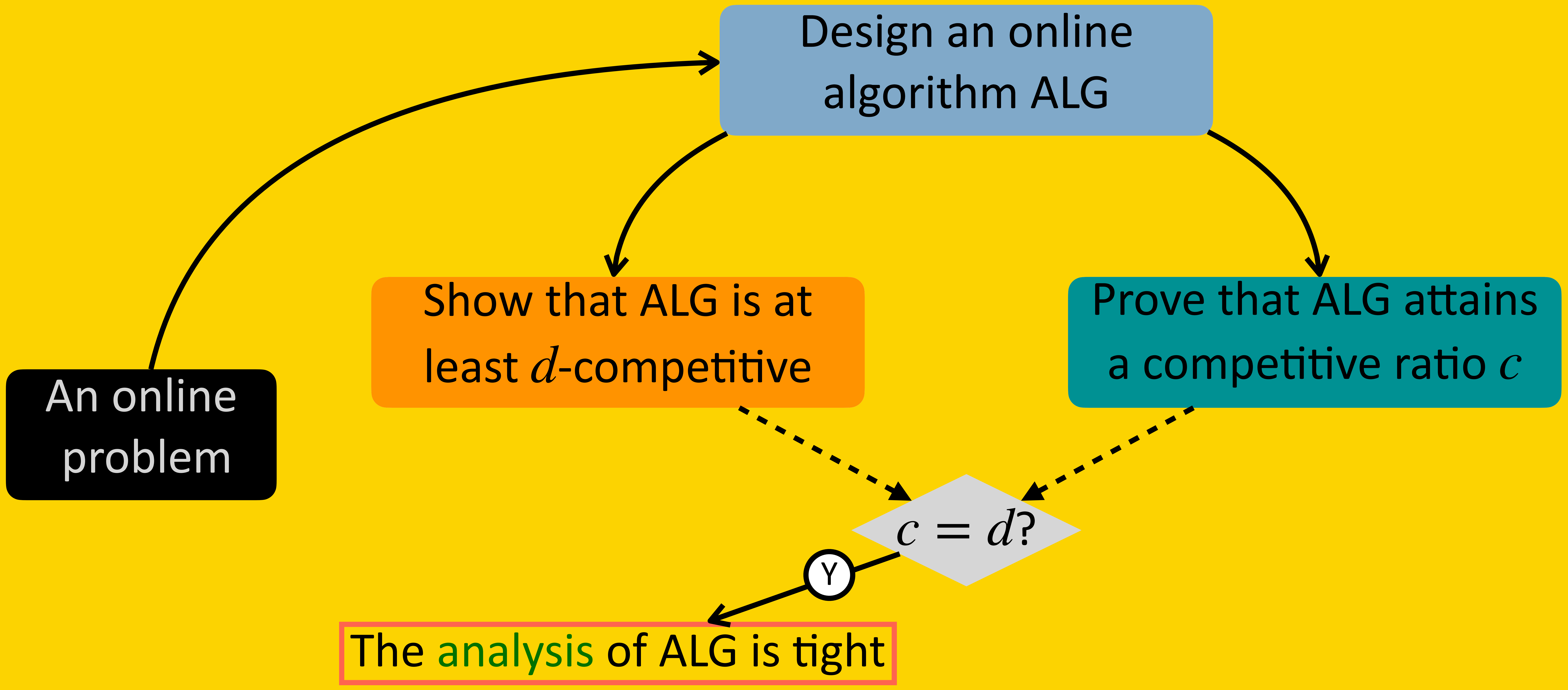

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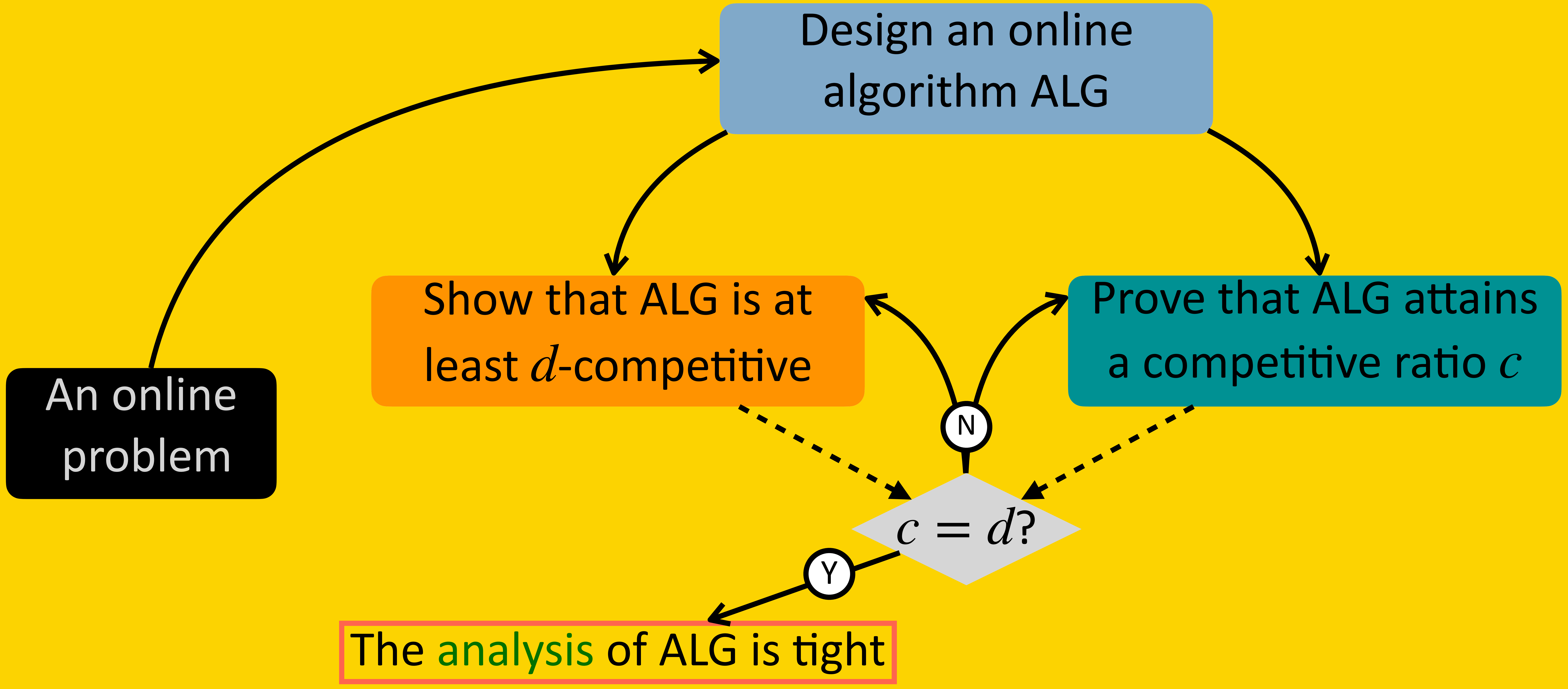

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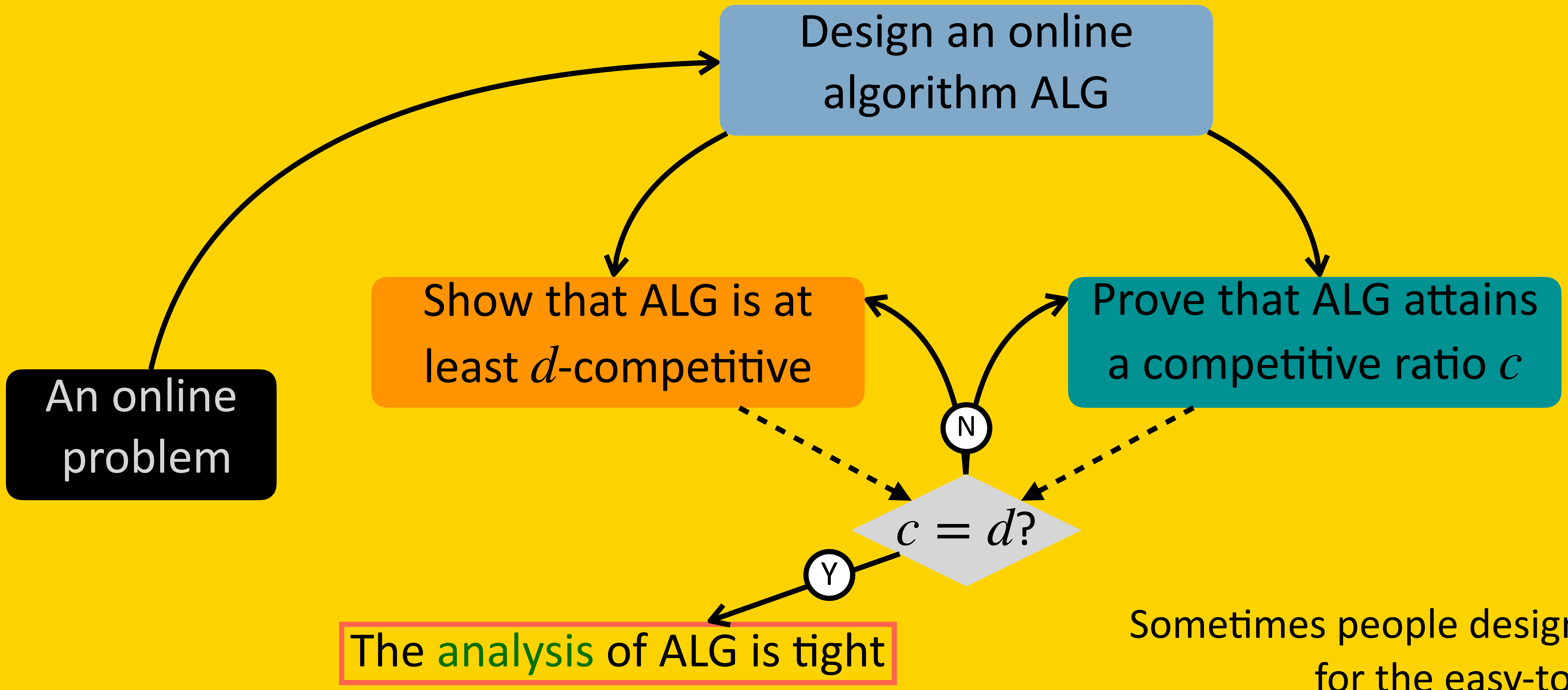


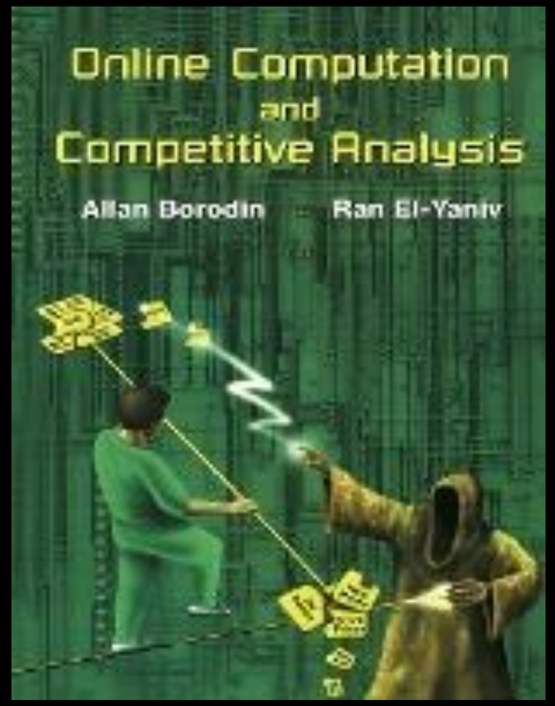
Sometimes people design online algorithms for the easy-to-analyze

Summary

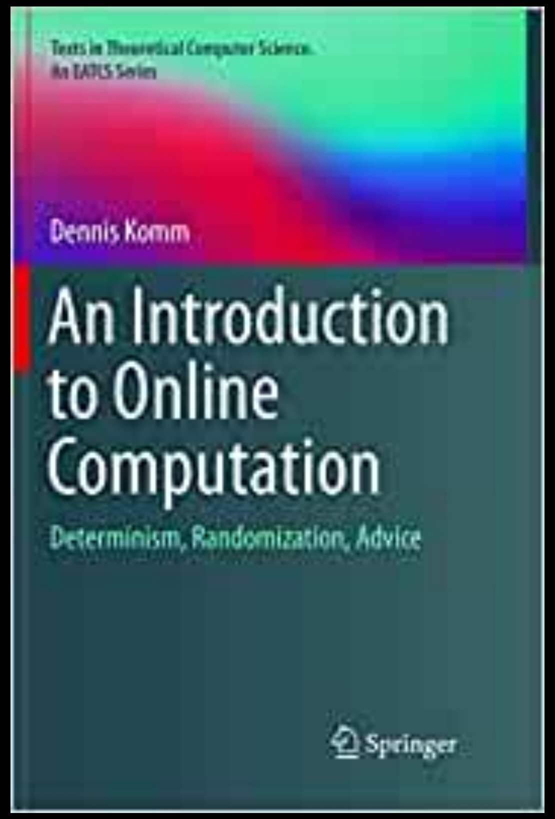
- Online optimization
- Measure the performance: **Competitive ratio**
  - How good is an online algorithm?
    - Show that the algorithm is (*at most*) c-competitive. (For all instance I, $\dfrac{\text{ALG}(I)}{\text{OPT}(I)} \leq c$)
  - How bad is an online algorithm?
    - Adversary game
    - Find an adversary for the algorithm and prove that it cannot be better than c-competitive
      - That is, it is *at least* c-competitive
  - **Tight analysis**: Find an adversary I' for the algorithm such that $\dfrac{\text{ALG}(I')}{\text{OPT}(I')}$ meets the lower bound.

Online Algorithms Books

- *Online Computation and Competitive Analysis Paperback English*
  by Allan Borodin and Ran El-Yaniv



- *An Introduction to Online Computation: Determinism, Randomization, Advice*
  by Dennis Komm



- *Online Algorithms: The State Of The Art*
  by Amos Fiat and Gerhard J. Woeginger