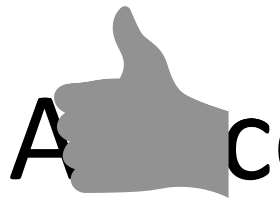
Algorithms for Decision Support

Online Algorithms (2/3)

Online Bidding

Outline

- Recap from the last lecture
- **Online bidding**
    - A  competitive algorithm
- A general technique for designing online algorithms
- “Best” online algorithms

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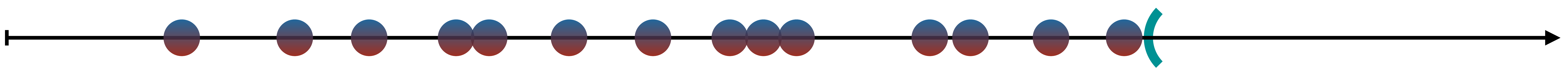
Recap

- Online problems, instances, and online algorithms

- Measurement of online algorithms: **competitive ratio**

No "ball" (which is $\frac{\text{ALG}(I)}{\text{OPT}(I)}$ for some instance I) is on the right of it



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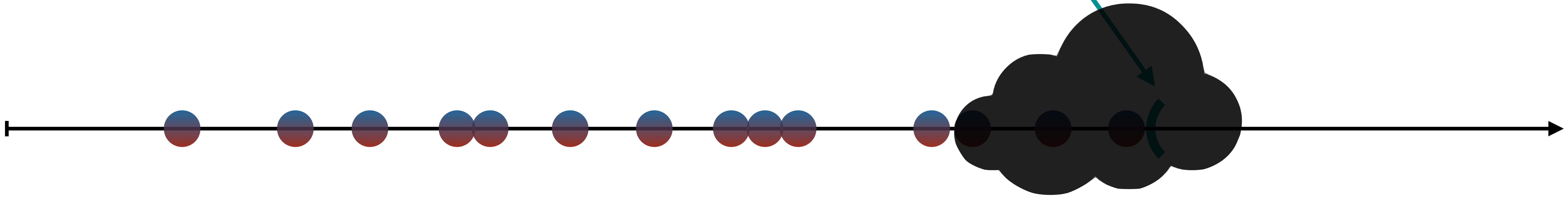
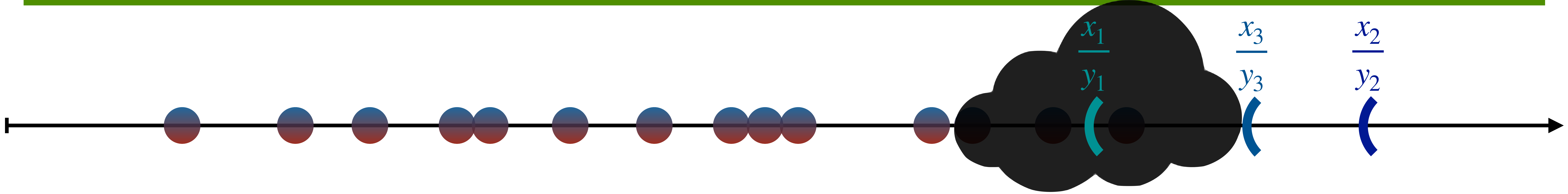


Recap

Method 1: To analyze the competitive ratio of an algorithm, one can argue that: For any instance I, ALG$(I) \leq x$ and OPT$(I) \geq y$. Therefore,

$$\frac{\text{ALG}(I)}{\text{OPT}(I)} \leq \frac{x}{y}$$



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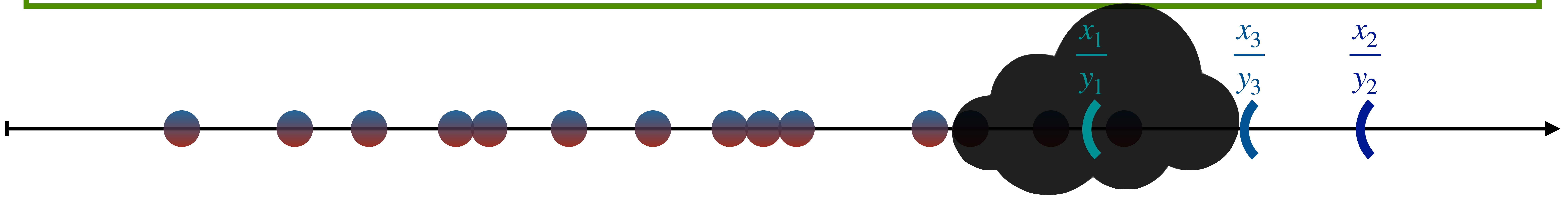
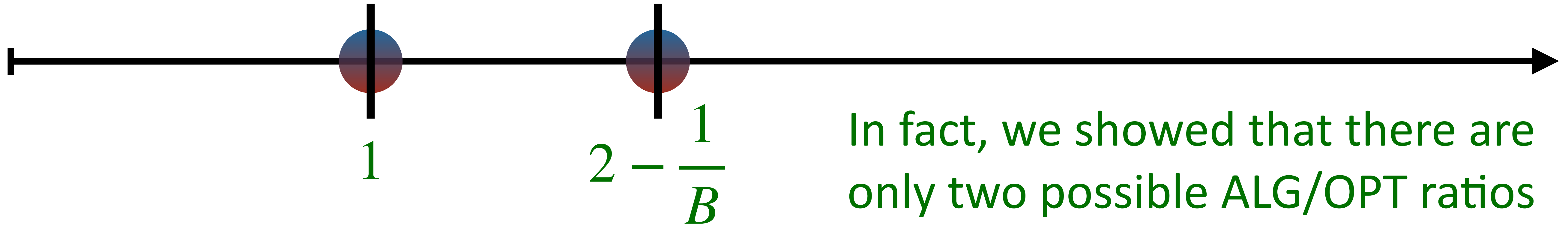


For the Ski-Rental problem, we showed that the "Buy on the B-th day" strategy is $(2 - \frac{1}{B})$-competitive



In fact, we showed that there are only two possible ALG/OPT ratios

This Lecture

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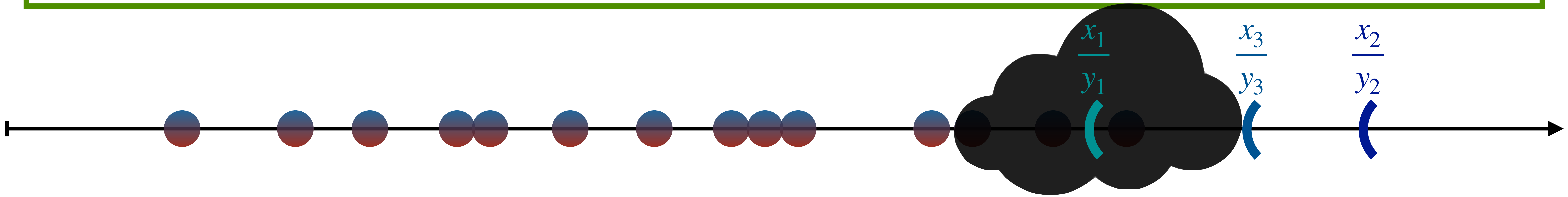
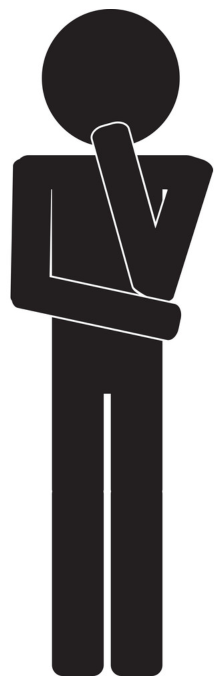


This lecture: how to (upper-) bound the competitive ratio in a more uncertain case?

- **Method 2: Assume we *know* the optimal solution cost, the algorithm cannot cost too much (hopefully…)**

Online Bidding

- The online algorithm (player)



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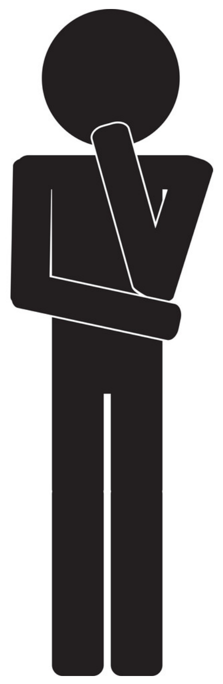

player

seller

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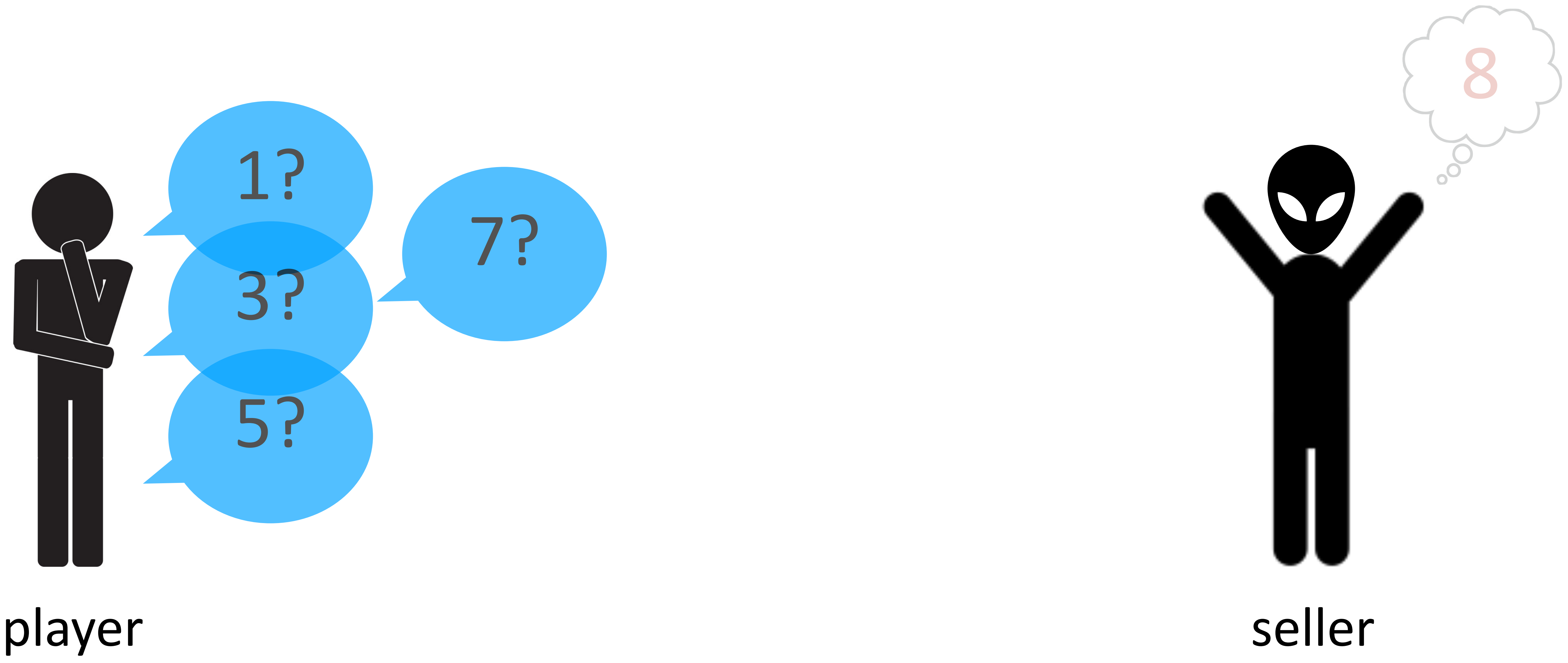

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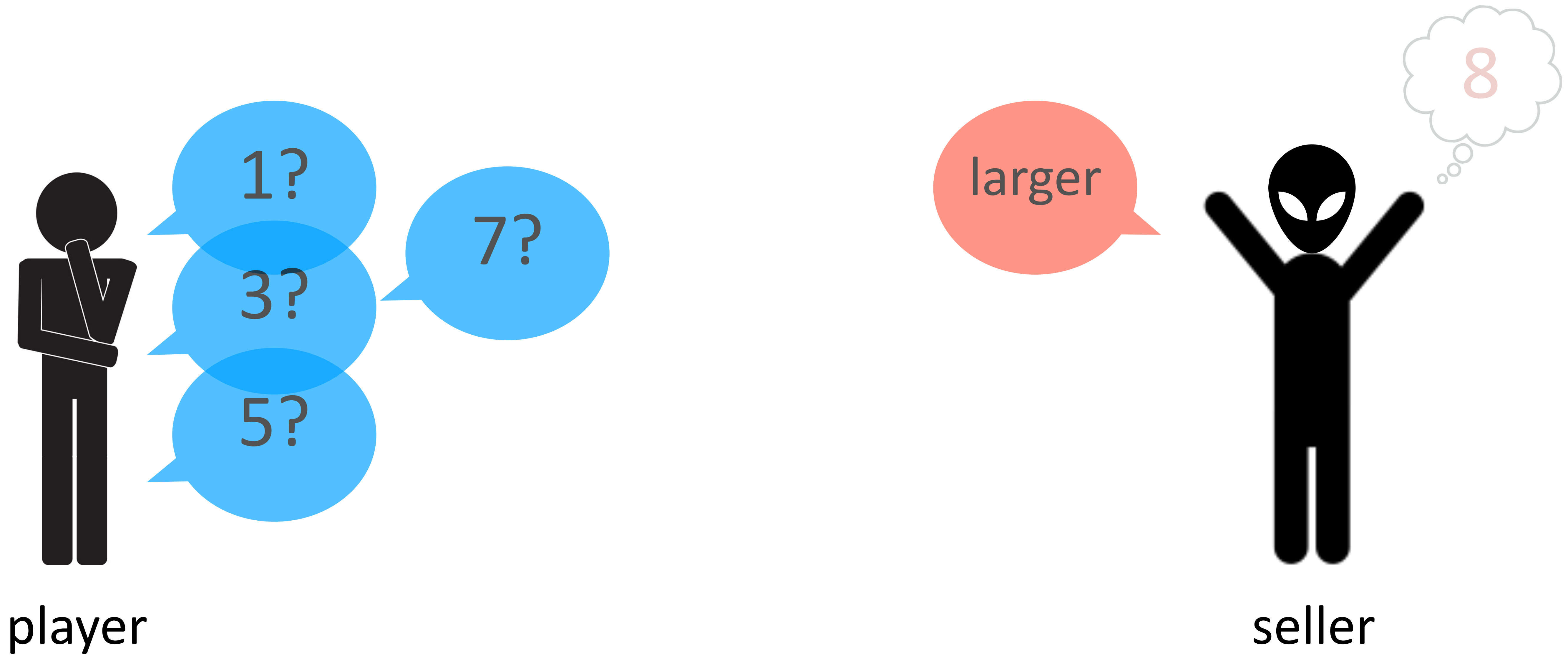

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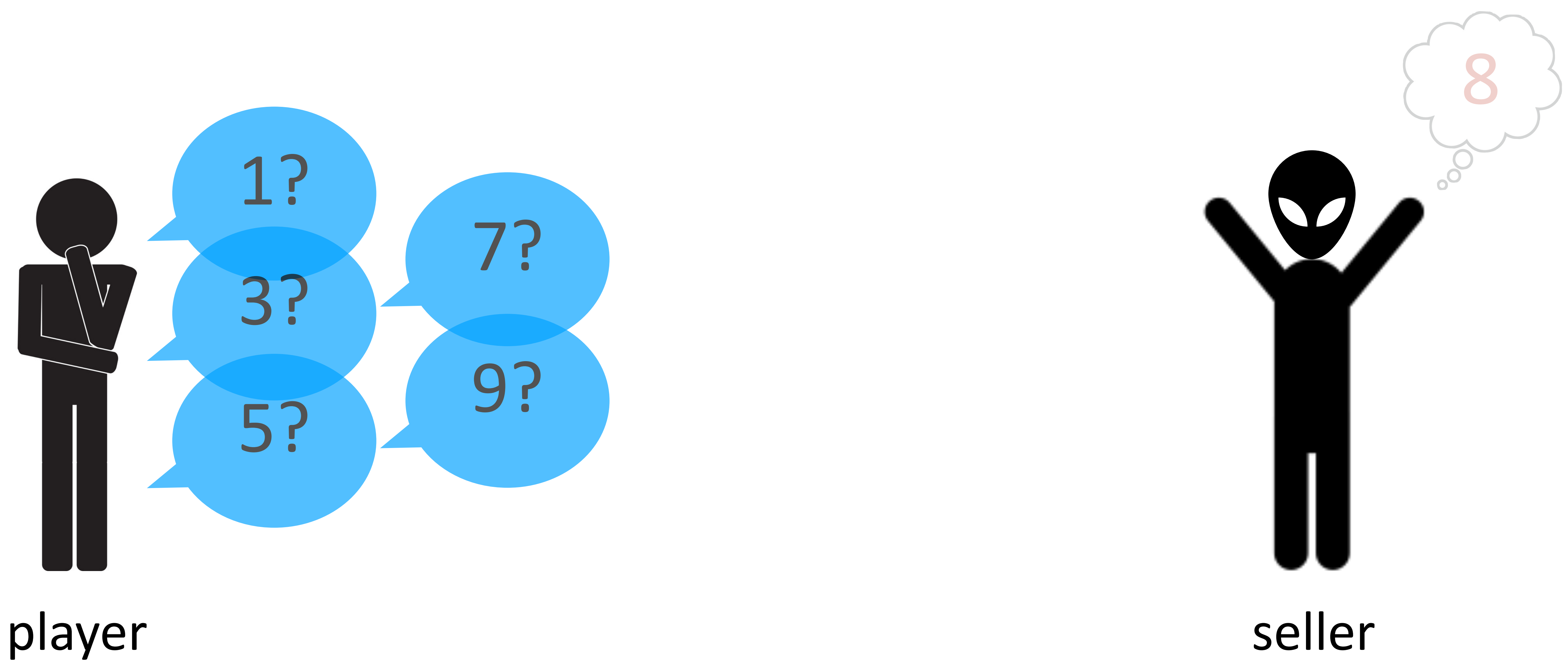

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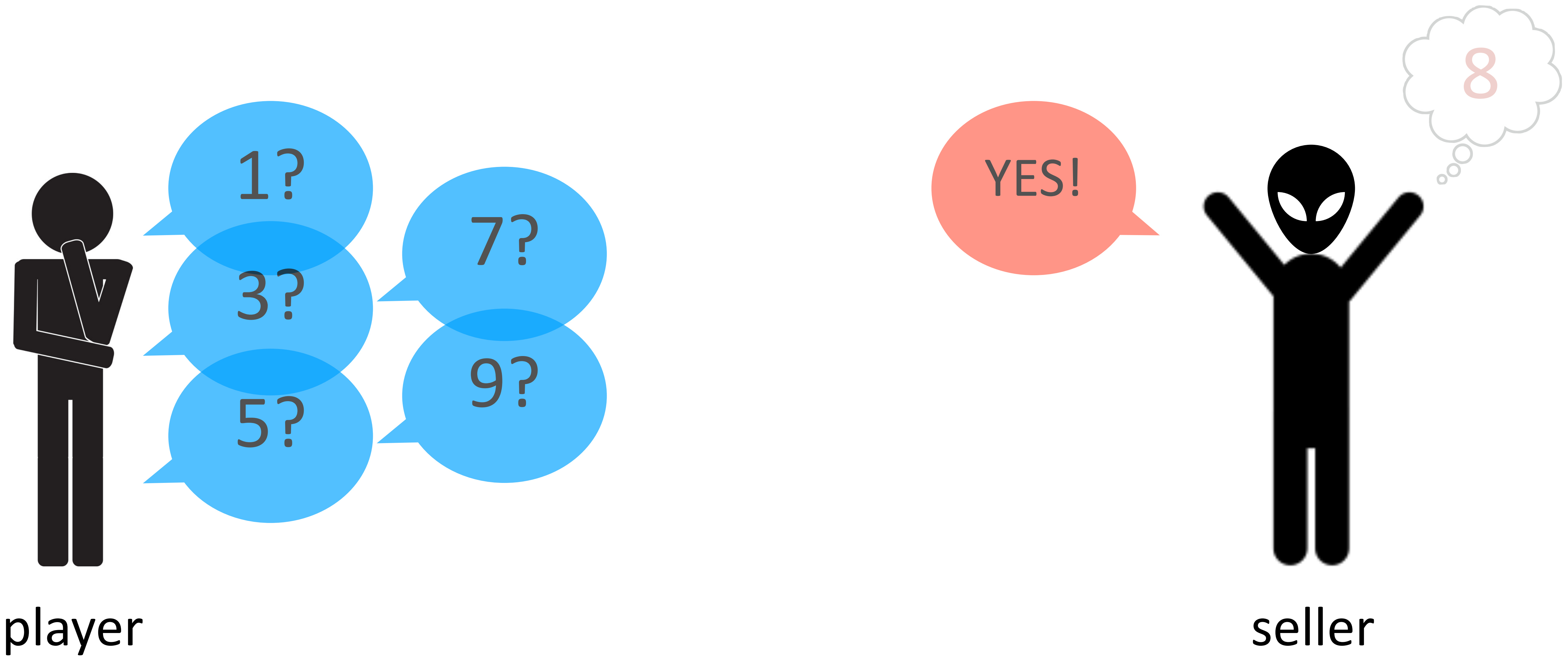

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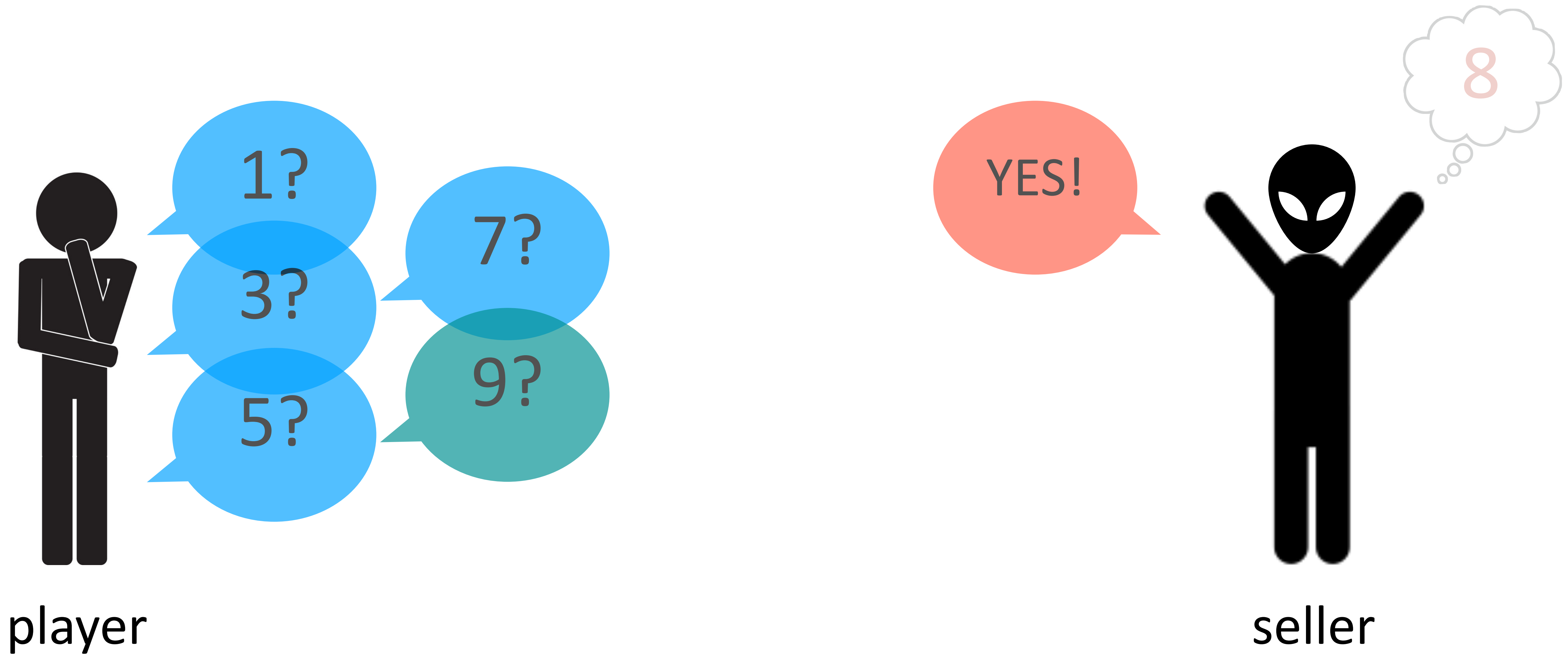
Total cost:
1+3+5+7+9 = 25



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Total cost:
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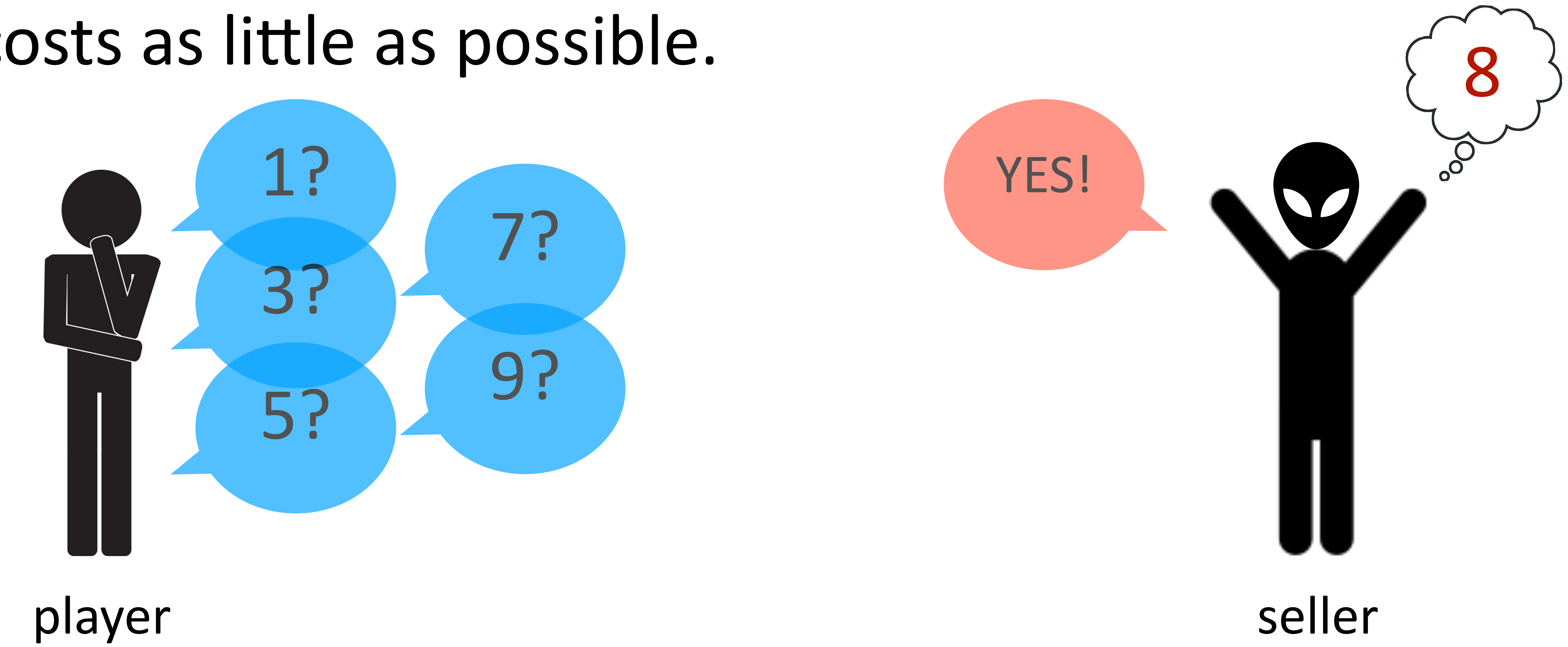
Optimal cost:
8



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- Example:
  $t = 8$
  $B_1 = \{1,2,3,4,5,6,7,8,9,10,\cdots\}$
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  $t = 8$
  $B_1 = \{1,2,3,4,5,6,7,8,9,10,\cdots\}$ Cost of $B_1 = 1 + 2 + \cdots + 8 = 36$
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$$B_1(8) = 4.5 \cdot \text{OPT}(8)$$

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$B_2(8) = 3.125 \cdot \text{OPT}(8)$

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  - Uncertain factor: we don't know what t is
  - The adversary knows the player's strategy and can choose a worst-case target t

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- **Online bidding**
    - A 4-competitive algorithm
- A general technique for designing online algorithms: **doubling**
- Different types of adversaries

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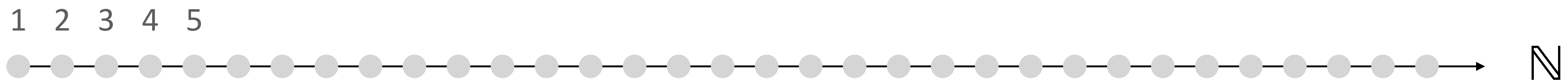

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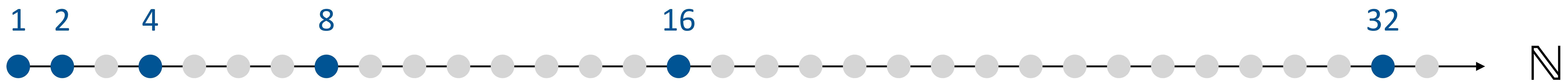

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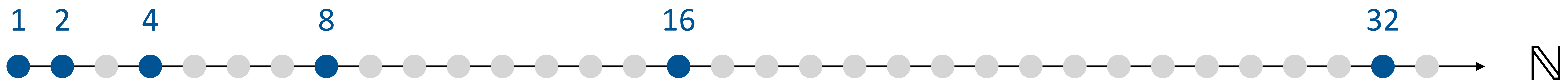
- How bad is **Double**?



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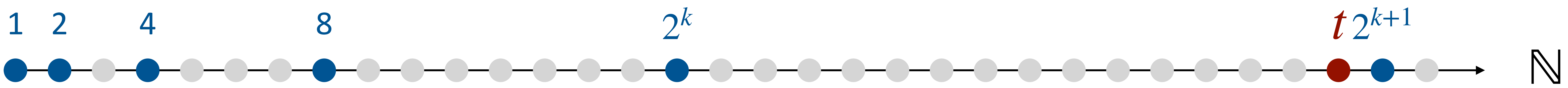
- How bad is **Double**?

Consider the adversary where $t = 2^k + x$ for some $k \geq 1$ and $x \in [1, 2^k - 1]$.

OPT $= 2^k + x$, and **Double** $= 1 + 2 + 4 + \cdots + 2^k + 2^{k+1} = 2 \cdot 2^{k+1}$.



$$\frac{\text{ALG}(t = 2^k + 1)}{\text{OPT}(t = 2^k + 1)} = \frac{2 \cdot 2^{k+1}}{2^k + x} = \frac{4 \cdot (2^k + x) - 4x}{2^k + x} = 4 - \frac{4x}{2^k + x} \geq 4 - \frac{4}{2^k + x}$$

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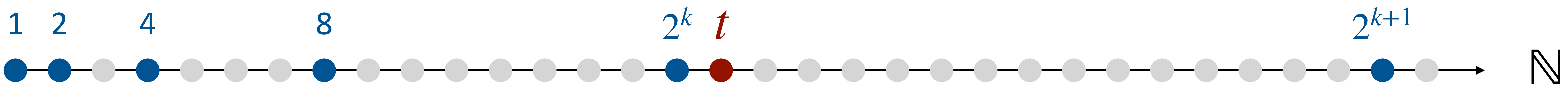

$$\frac{\text{ALG}(t = 2^k + 1)}{\text{OPT}(t = 2^k + 1)} = \frac{2 \cdot 2^{k+1}}{2^k + 1} = \frac{4 \cdot (2^k + 1) - 4}{2^k + 1} = 4 - \frac{4}{2^k + 1}$$

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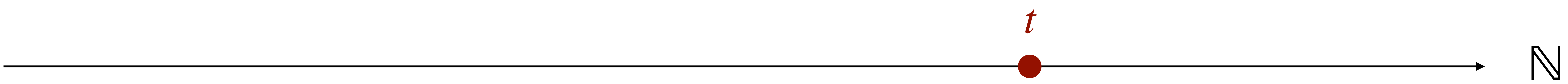

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Assume that the target is t. It must be between $2^k + 1$ to 2^{k+1} for some $k \geq 0$.

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$\text{OPT} = t$

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In this case, Double hits the target at bid 2^{k+1} and pays $1 + 2 + 4 + \cdots + 2^k + 2^{k+1}$.

$$\frac{\text{Double}(t)}{\text{OPT}(t)} = \frac{1 + 2 + 4 + \cdots + 2^{k+1}}{t}$$

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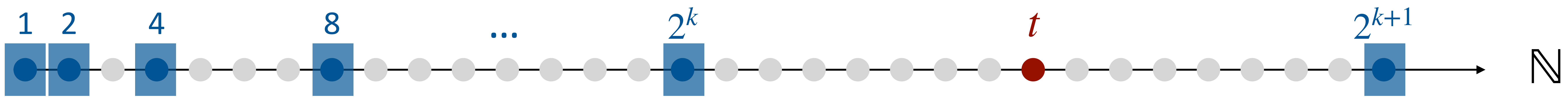

$$\frac{\text{Double}(t)}{\text{OPT}(t)} = \frac{1 + 2 + 4 + \cdots + 2^{k+1}}{t} < \frac{2 \cdot 2^k}{t}$$

2 is a magic number:
$1 + 2^0 + 2^1 + 2^2 + \cdots 2^i = 2 \cdot 2^i$

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$$\frac{\text{Double}(t)}{\text{OPT}(t)} = \frac{1 + 2 + 4 + \cdots + 2^{k+1}}{t} < \frac{2 \cdot 2^k}{t} \leq \frac{2^{k+2}}{2^k + 1}$$

$t \in (2^k, 2^{k+1}]$

$t \geq 2^k + 1$

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$$\frac{\text{Double}(t)}{\text{OPT}(t)} = \frac{1 + 2 + 4 + \cdots + 2^{k+1}}{t} < \frac{2 \cdot 2^k}{t} \leq \frac{2^{k+2}}{2^k + 1} < \frac{2^{k+2}}{2^k} = 4$$

What Happened

- For the analysis of the **Double** algorithm for the online bidding problem, we assume that the optimal solution is t, and then show that **the algorithm cost cannot be too much than** t by the behavior of the **Double** algorithm:
  - The (assumed) value of t tells us what the last bid is (2^{k+1})

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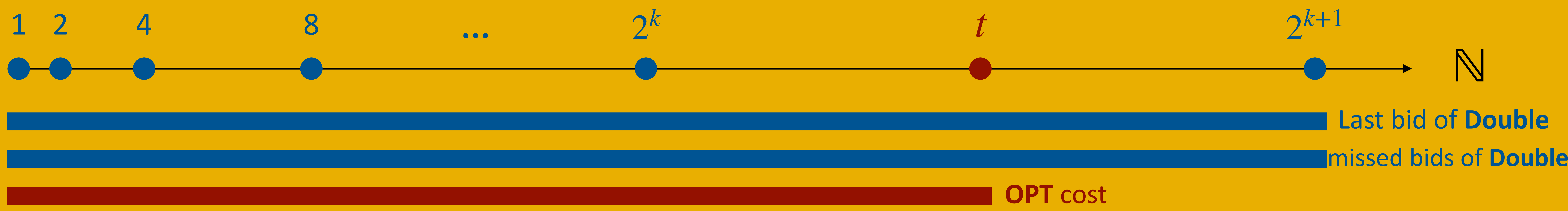

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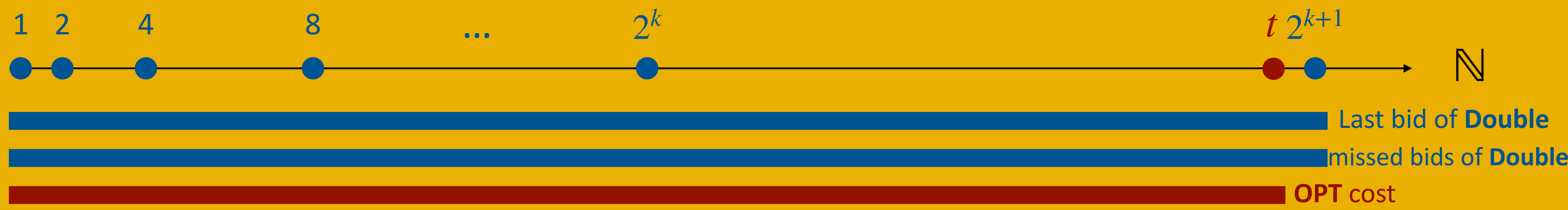

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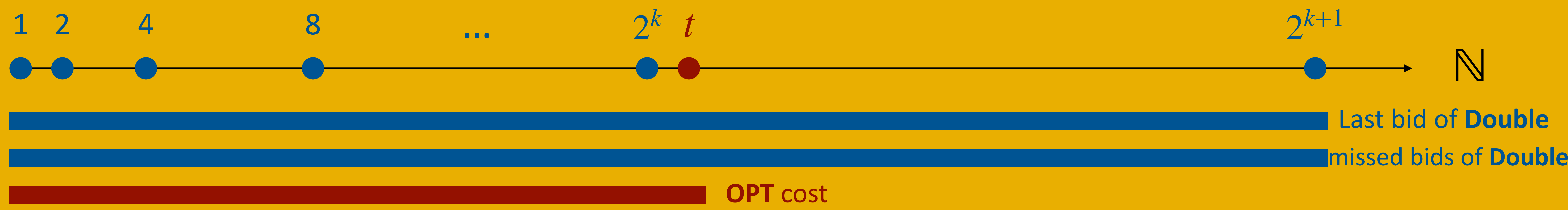

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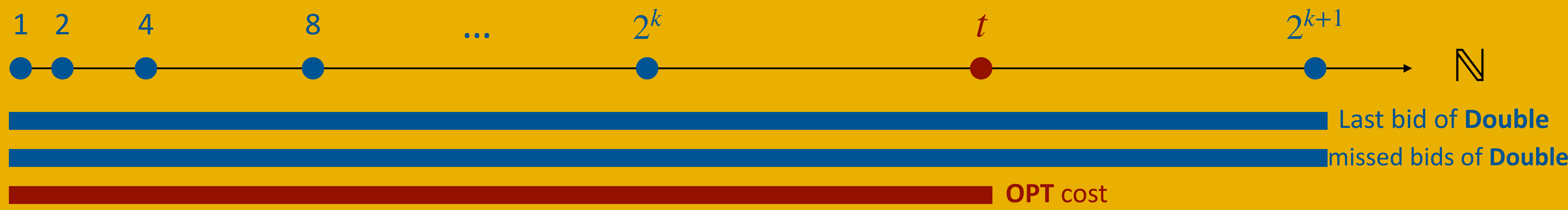

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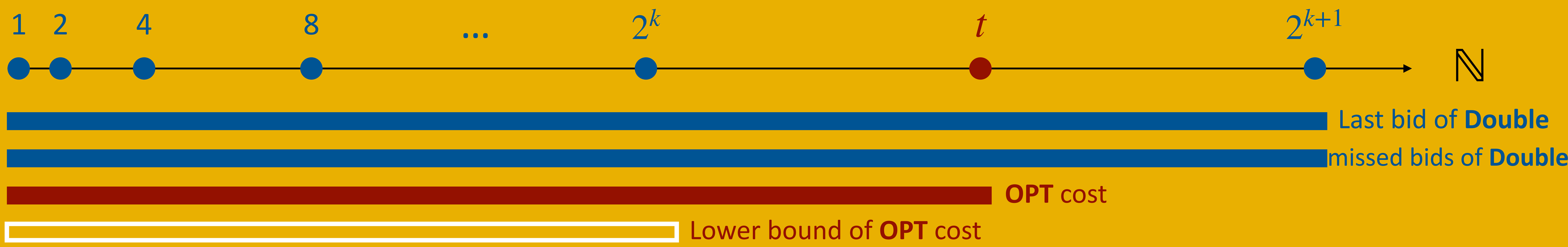

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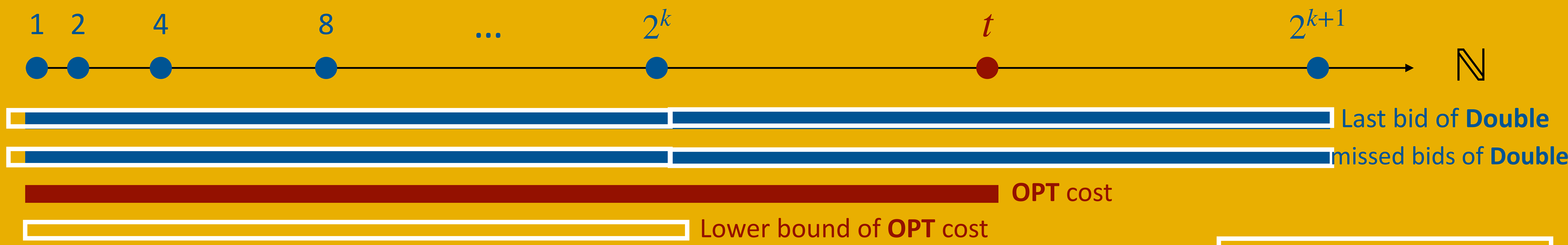

Double $< 4 \cdot$ **OPT**

Outline

- Recap from the last lecture
- **Online bidding**
  - A 4-competitive algorithm
- A general technique for designing online algorithms: **doubling**
- “Best” online algorithms

Doubling Technique

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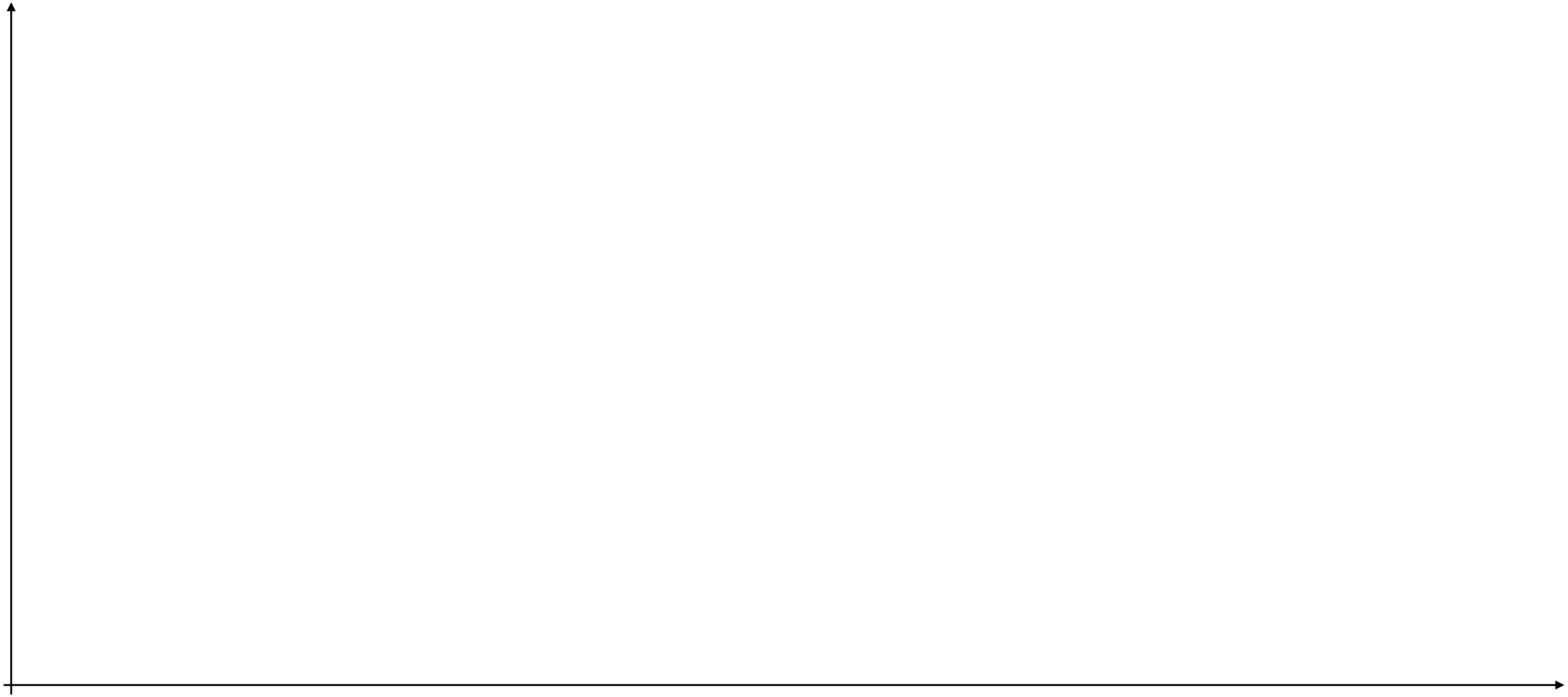

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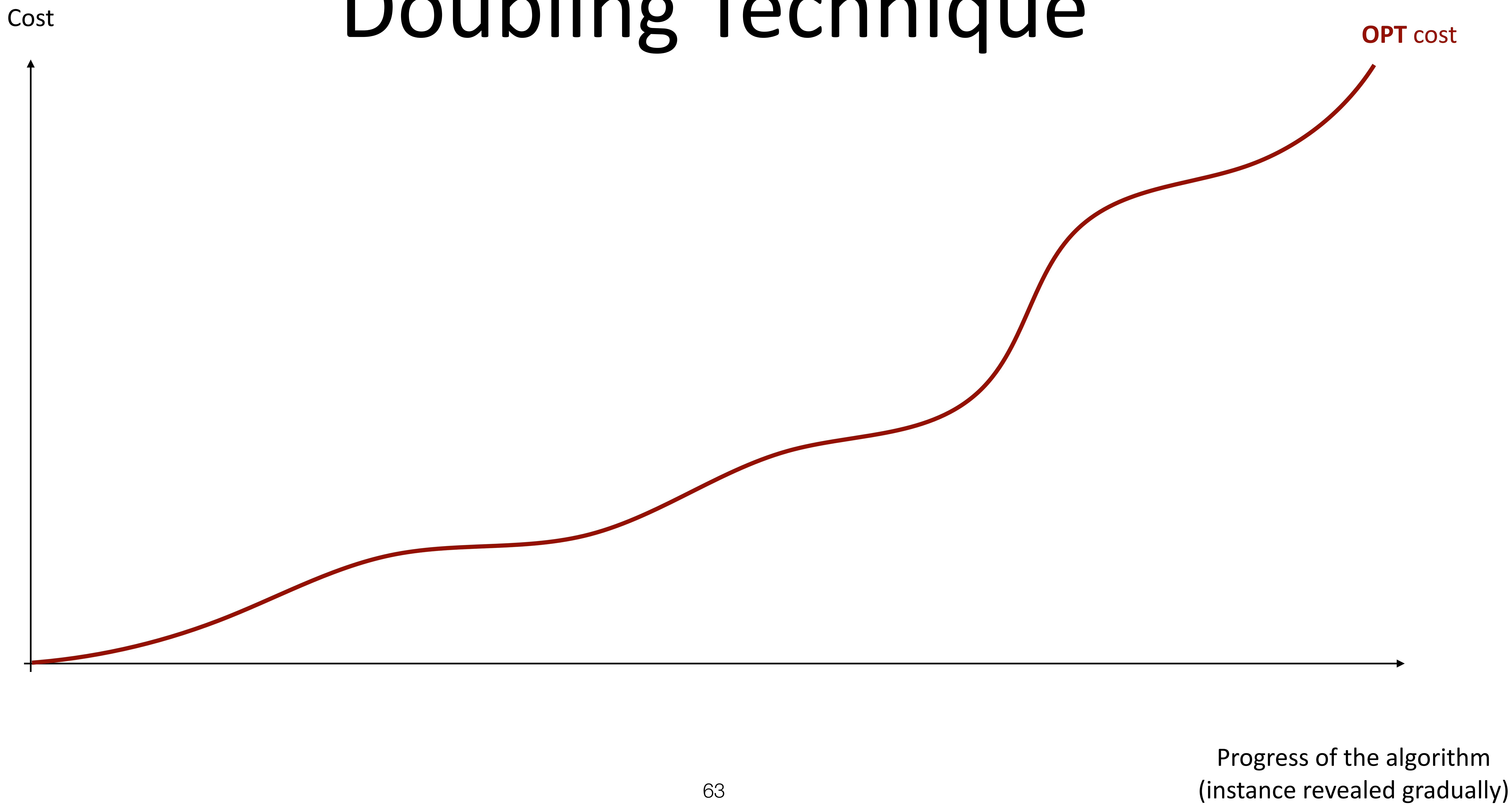

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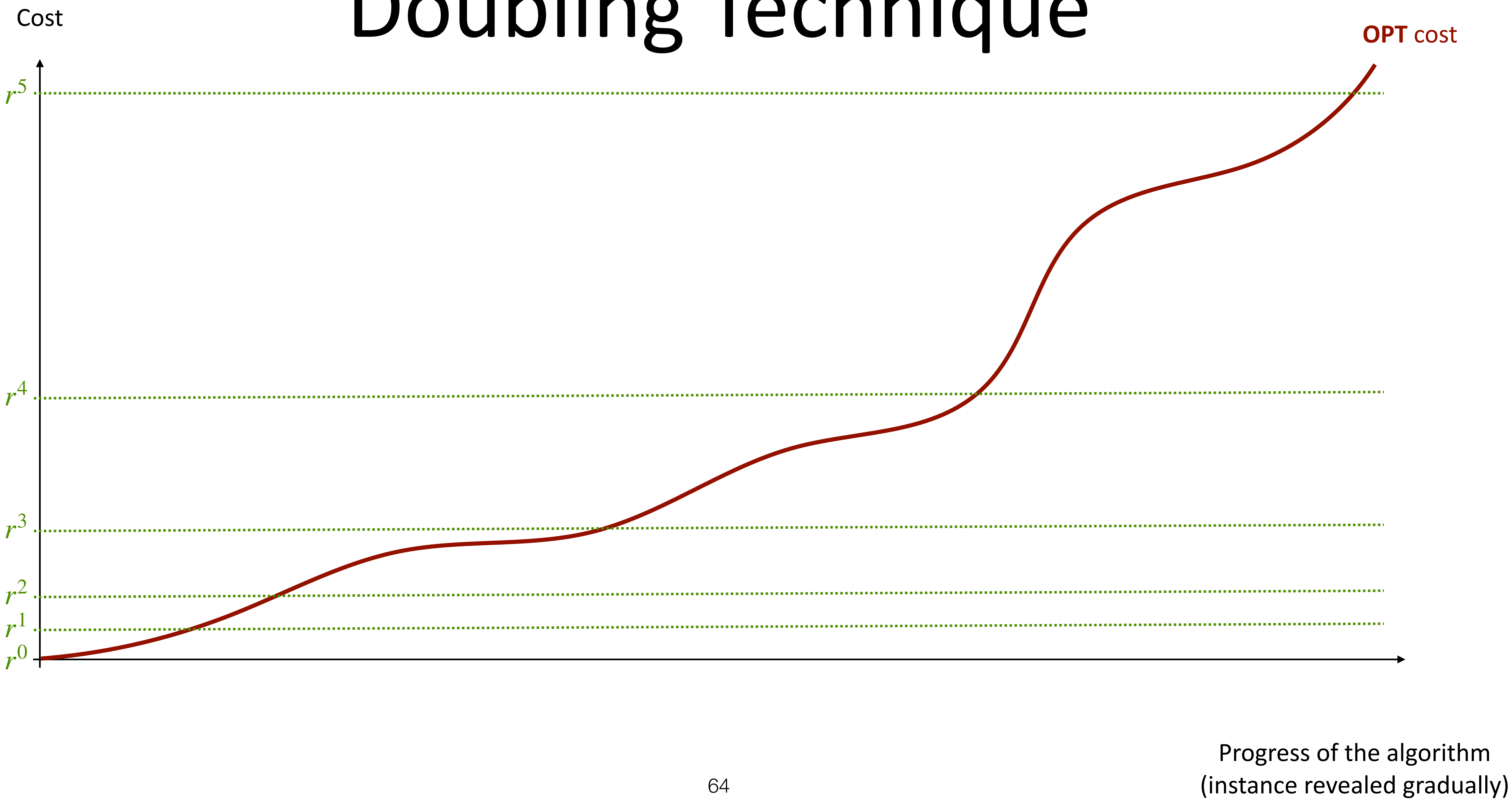

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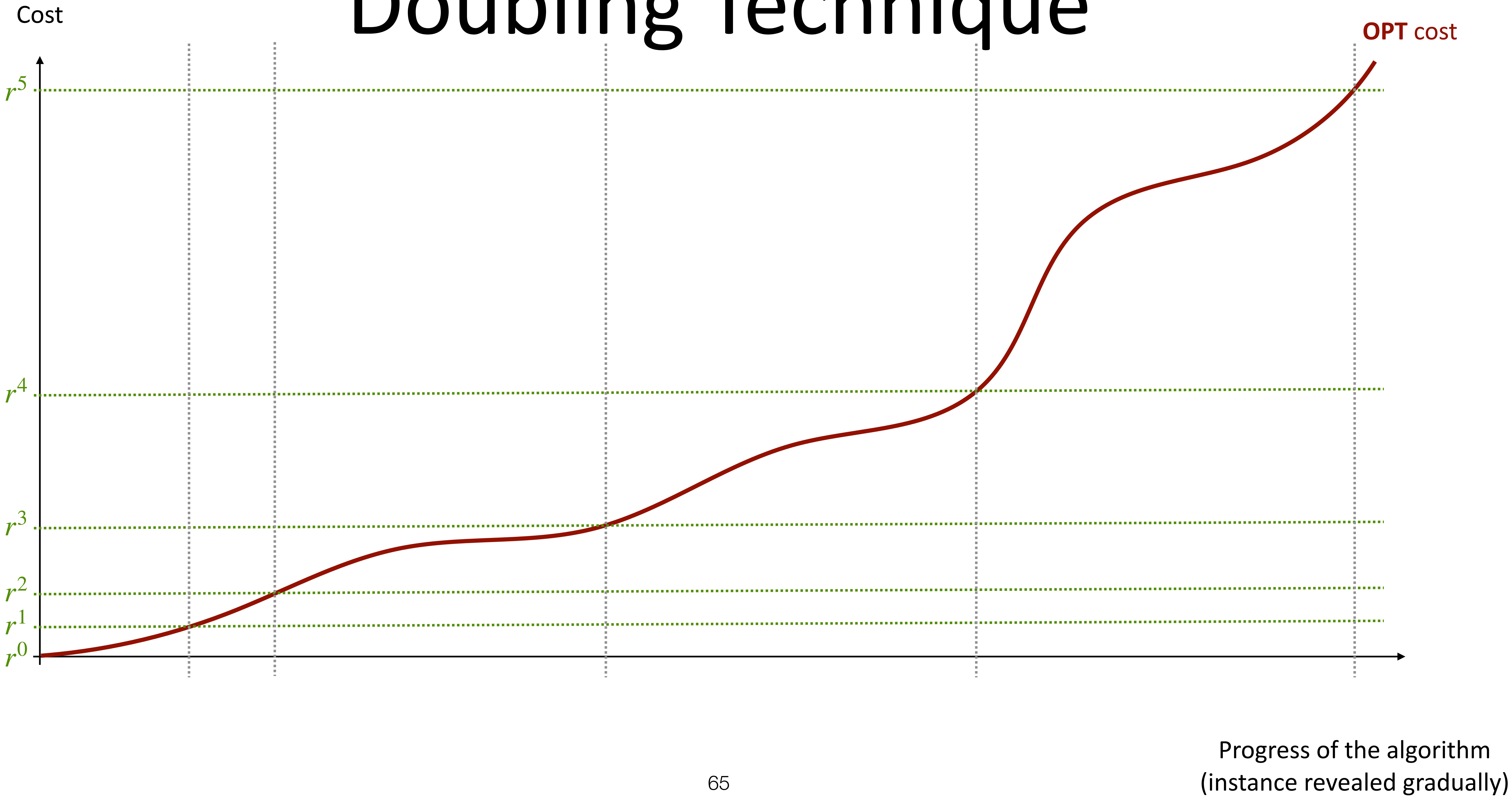

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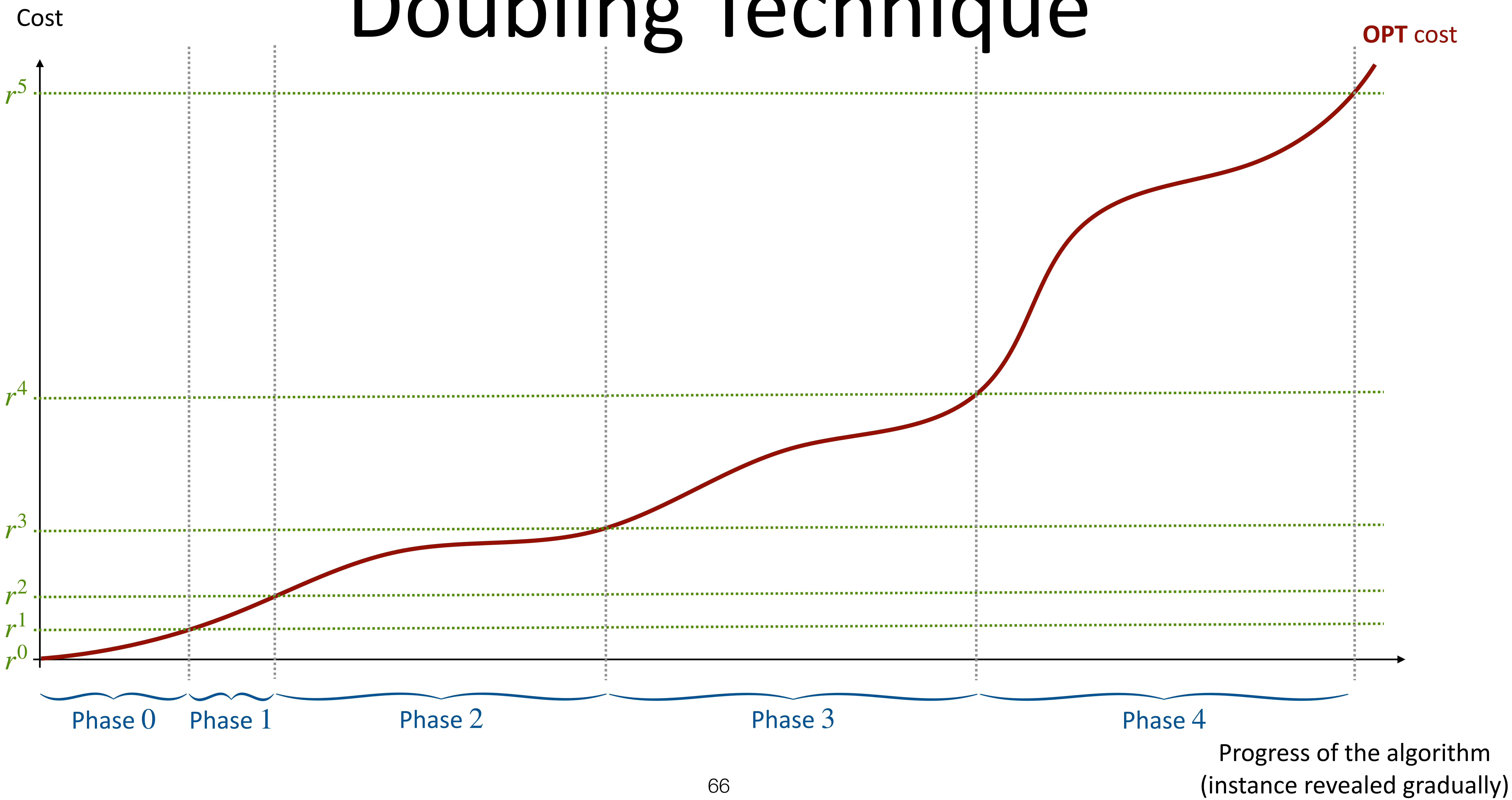

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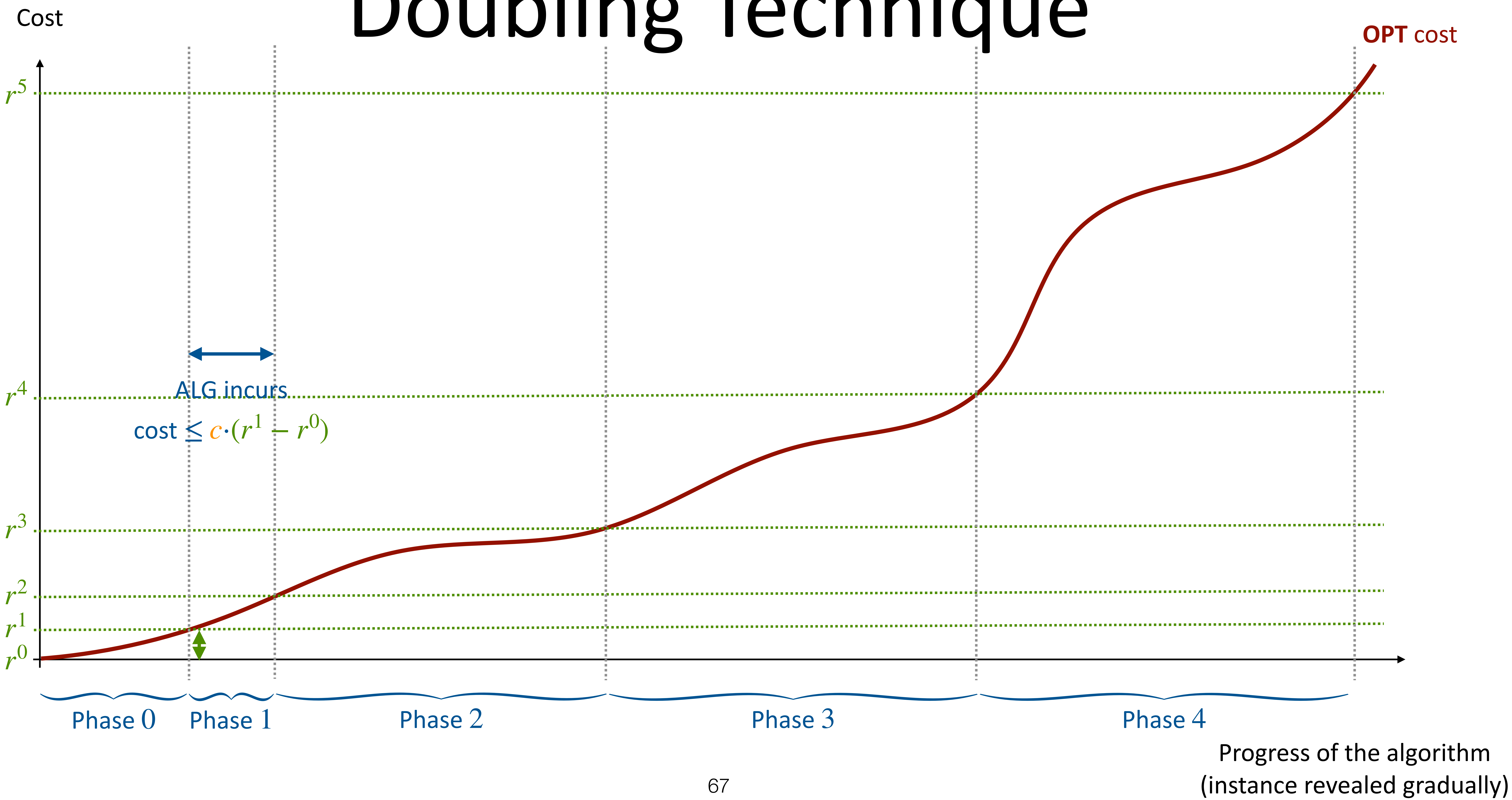

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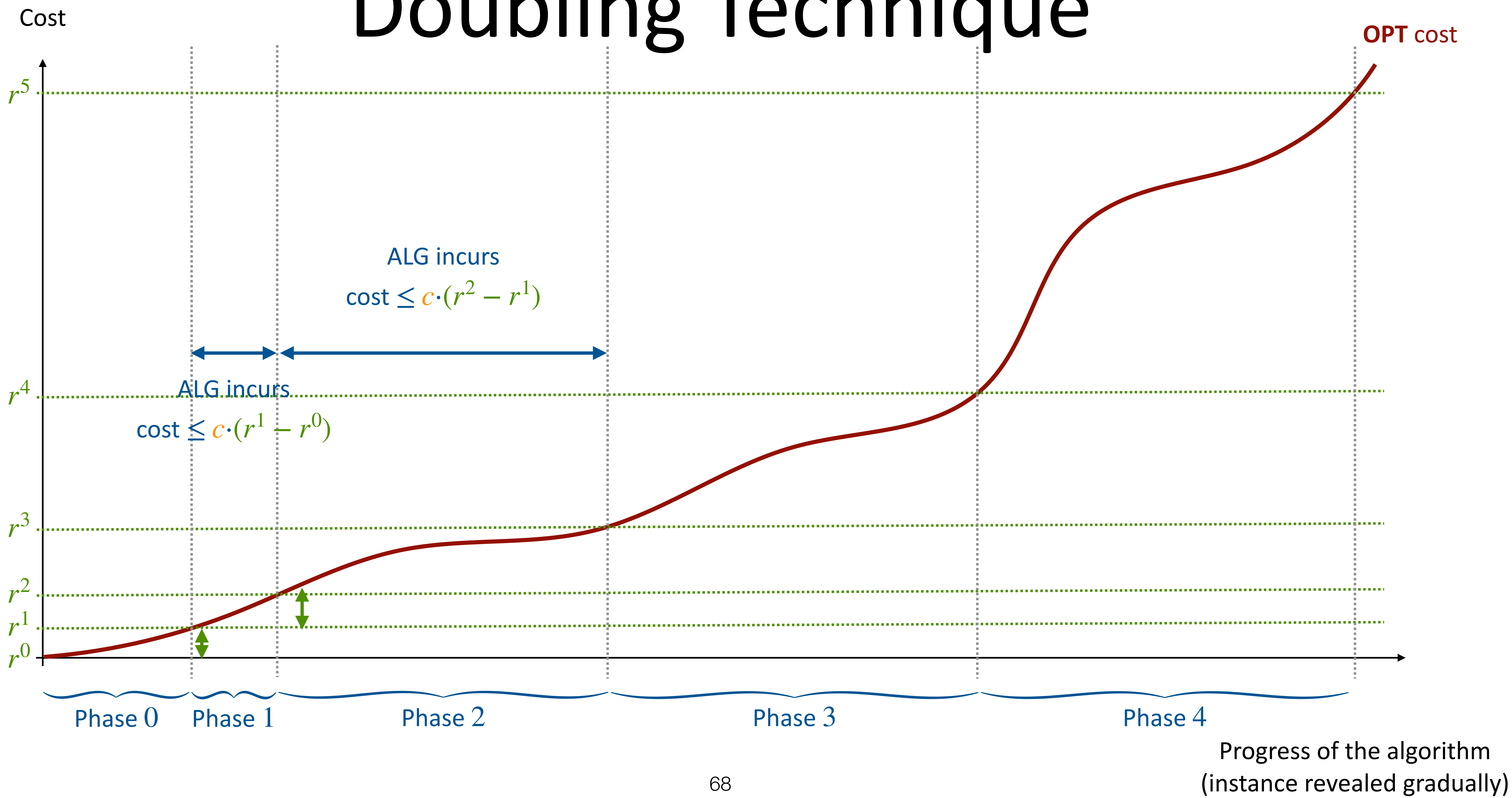

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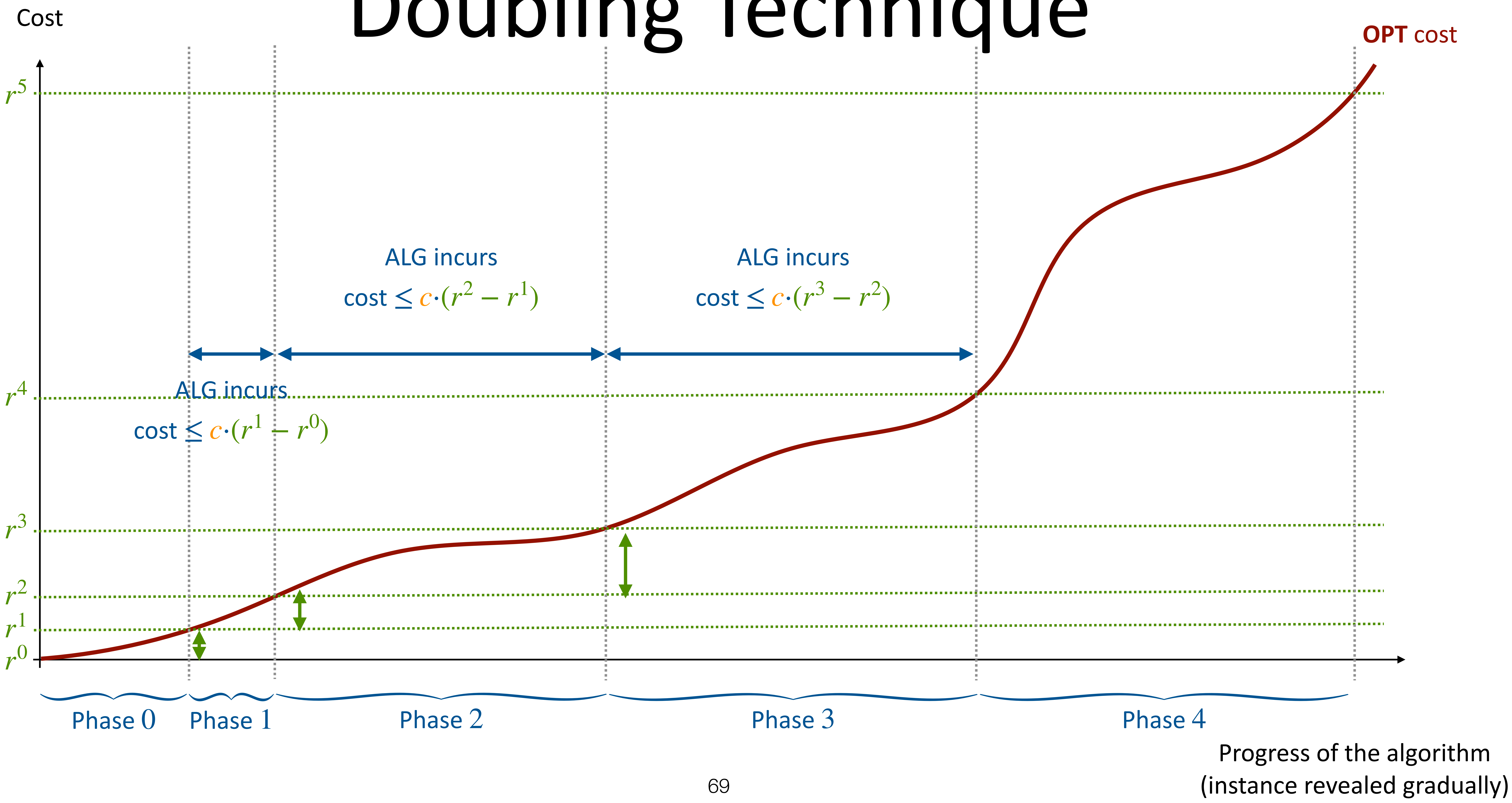

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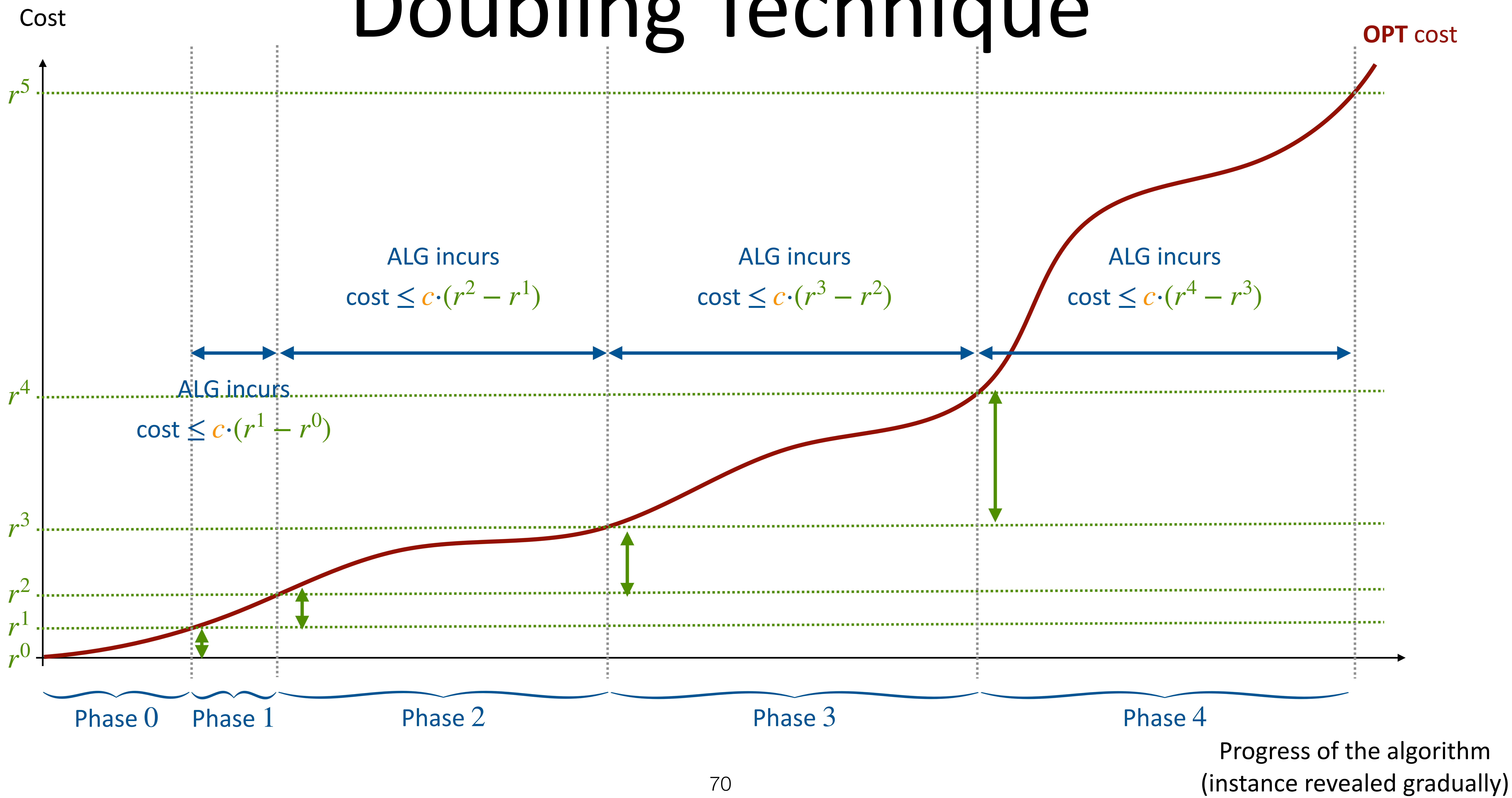

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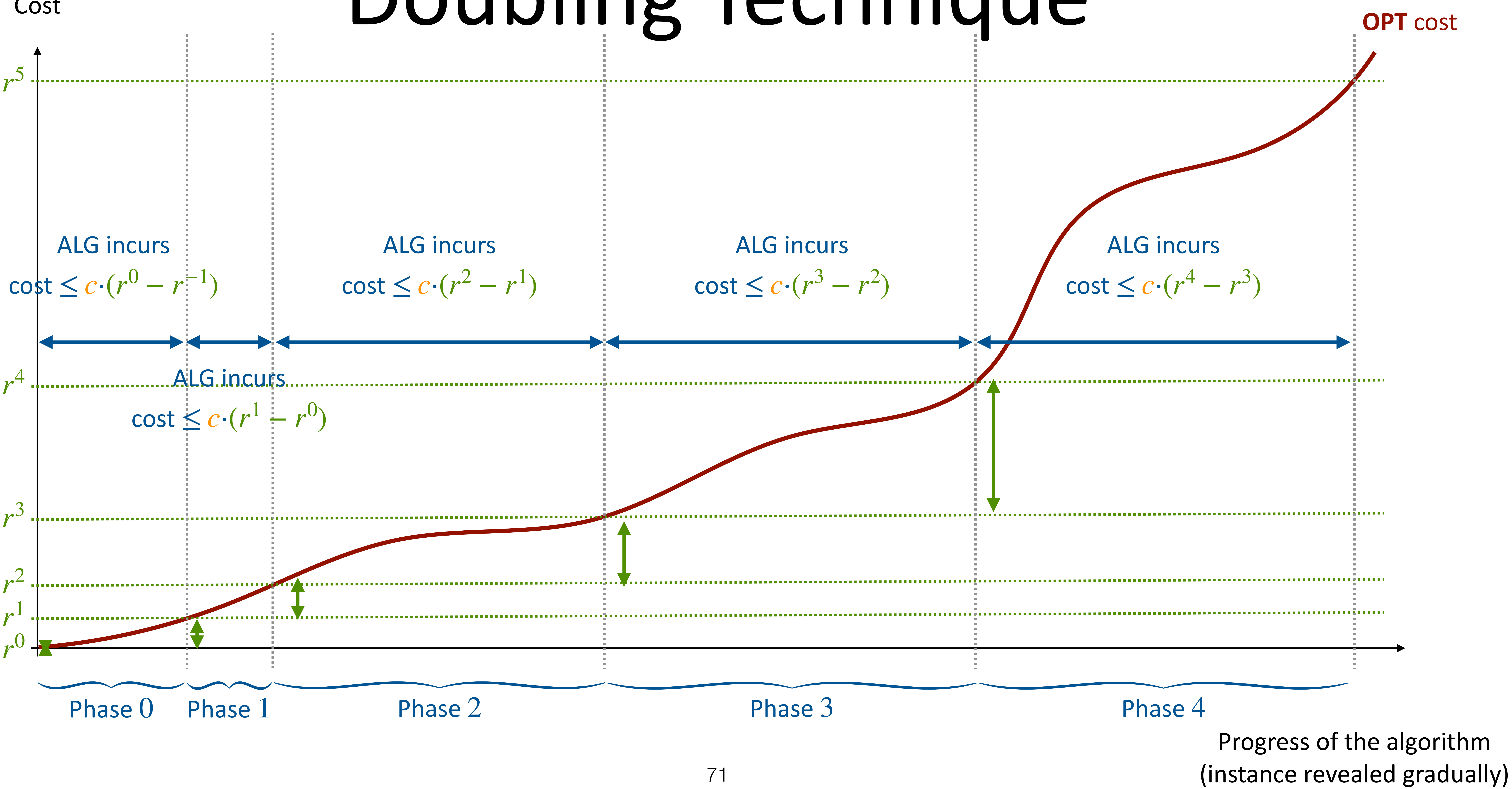

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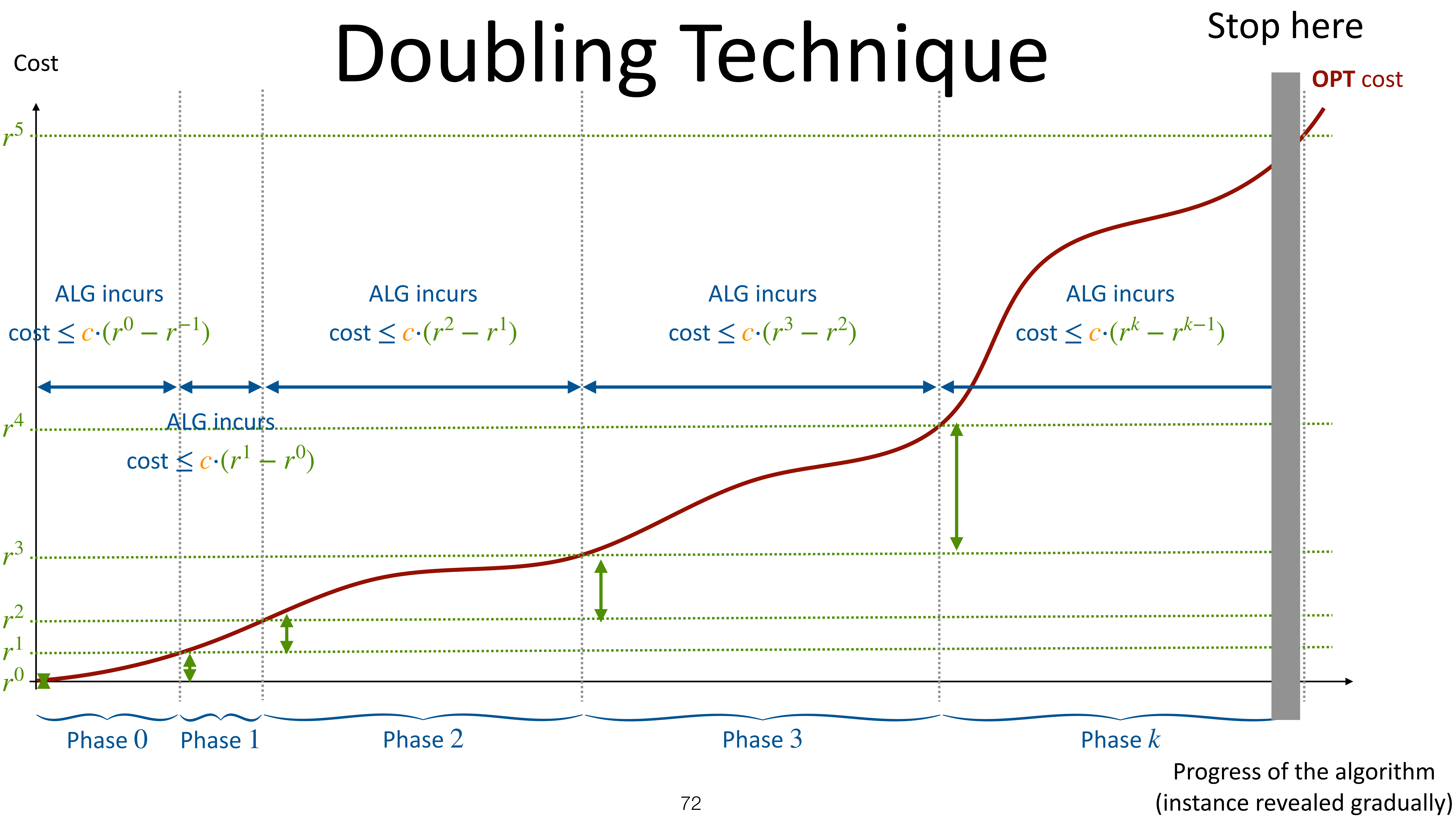

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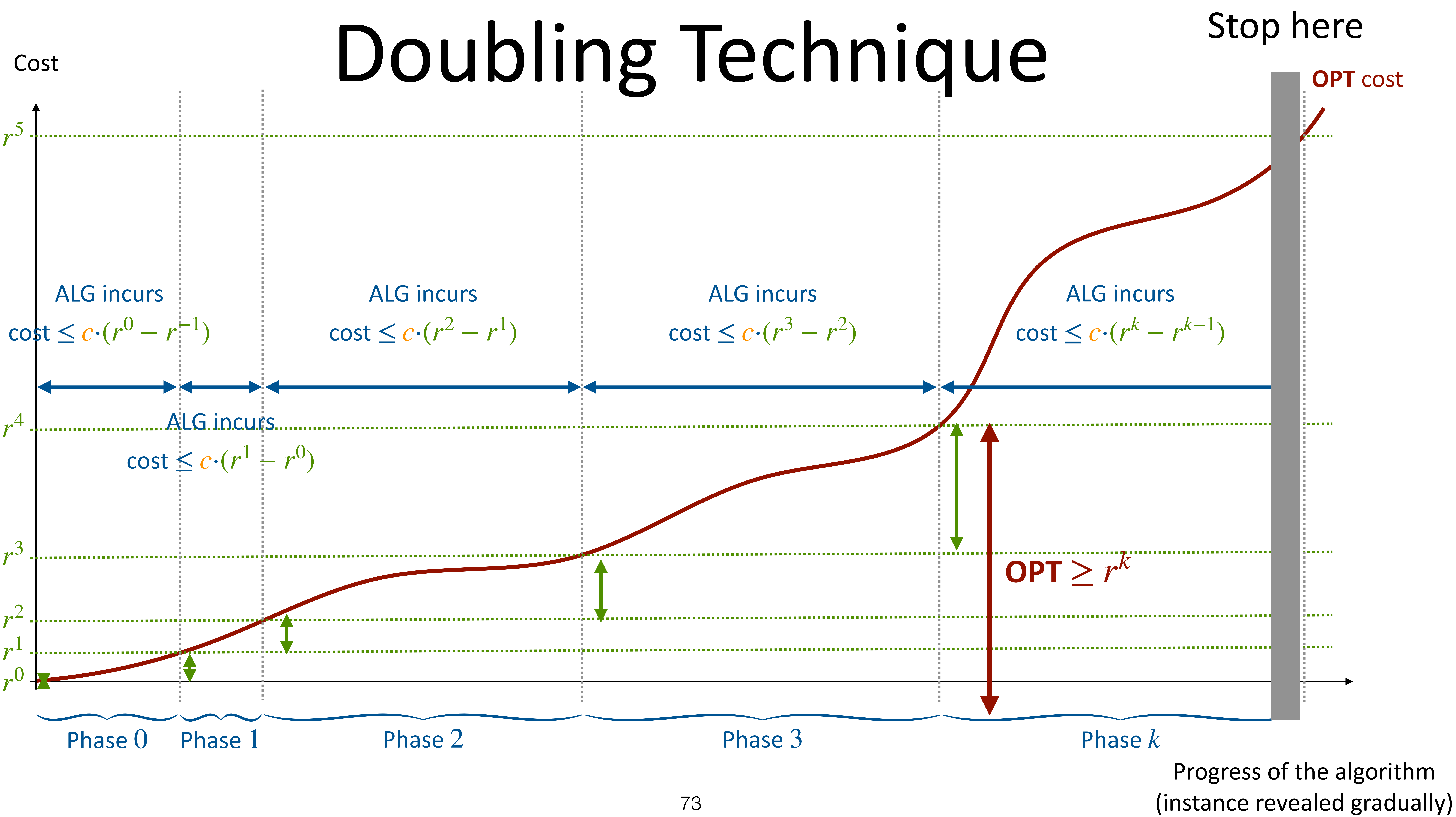

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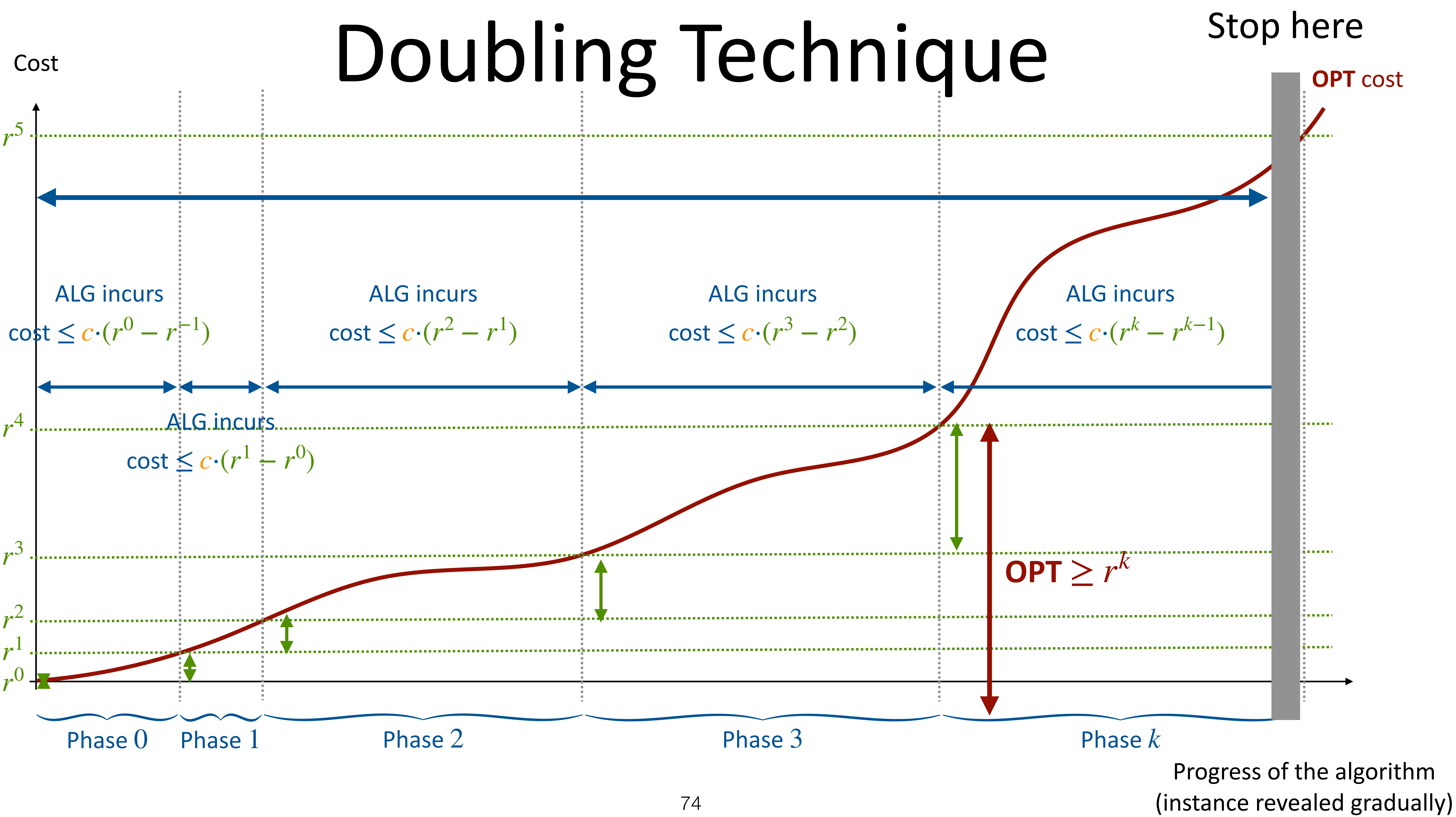

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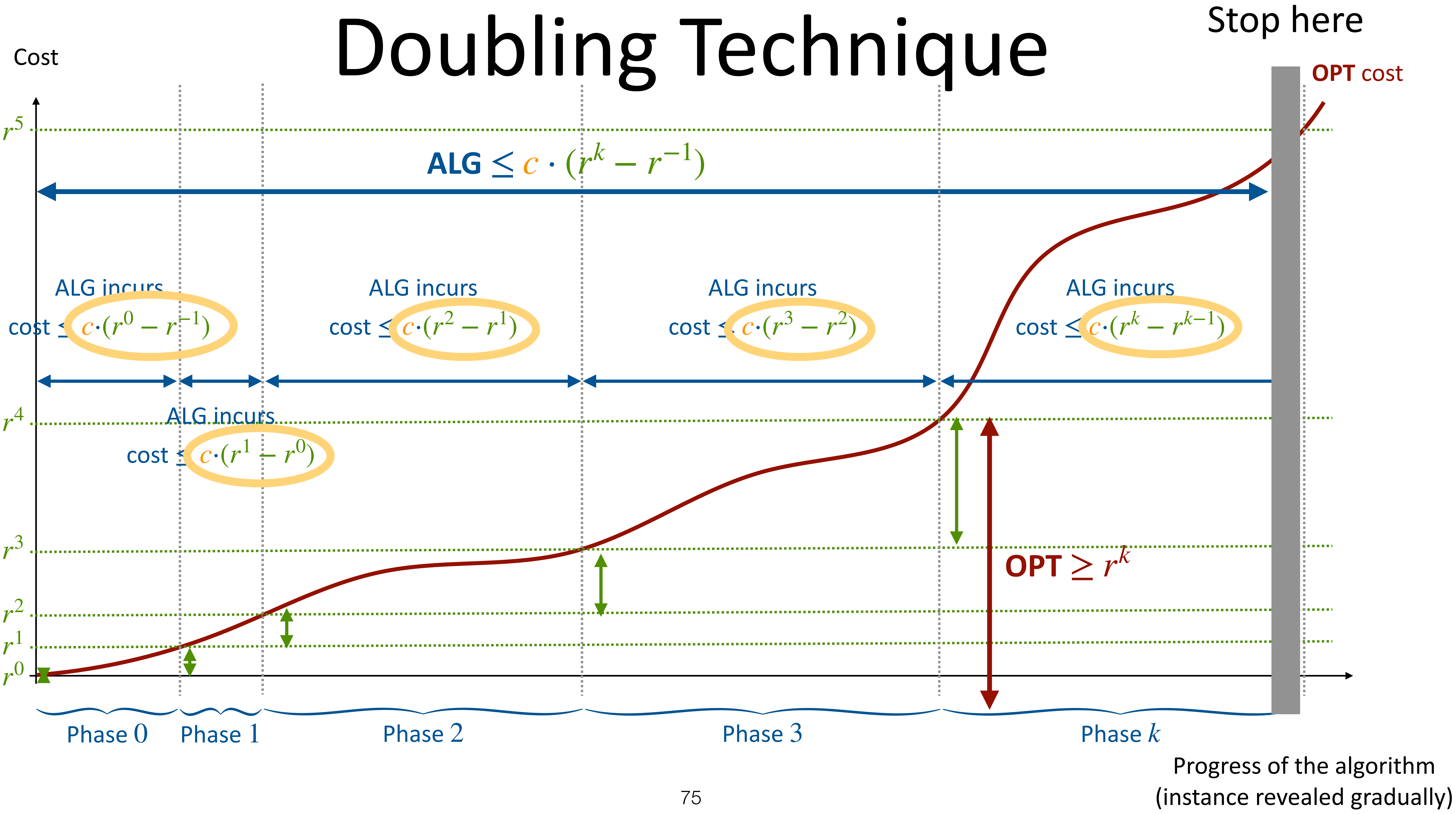

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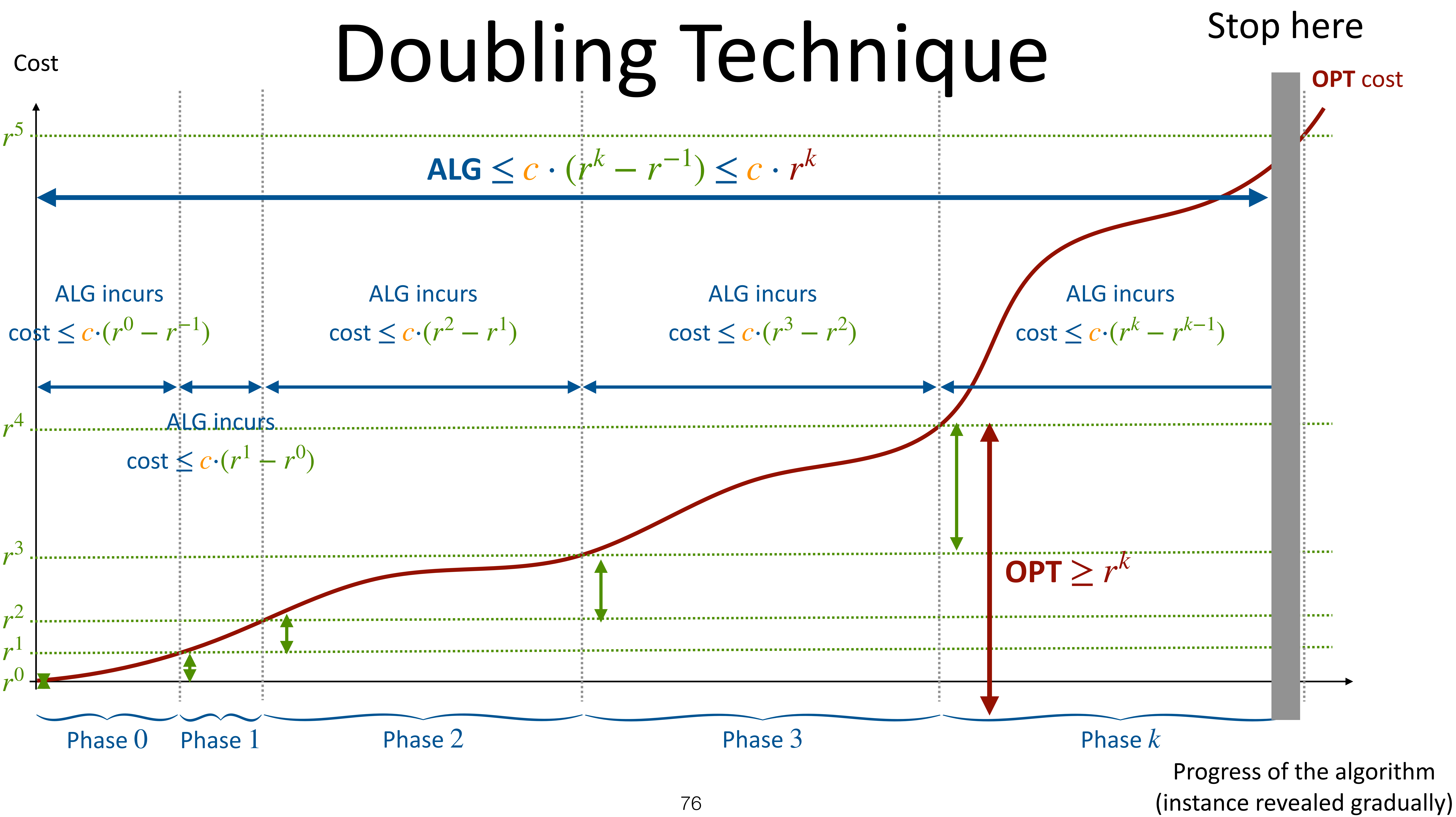

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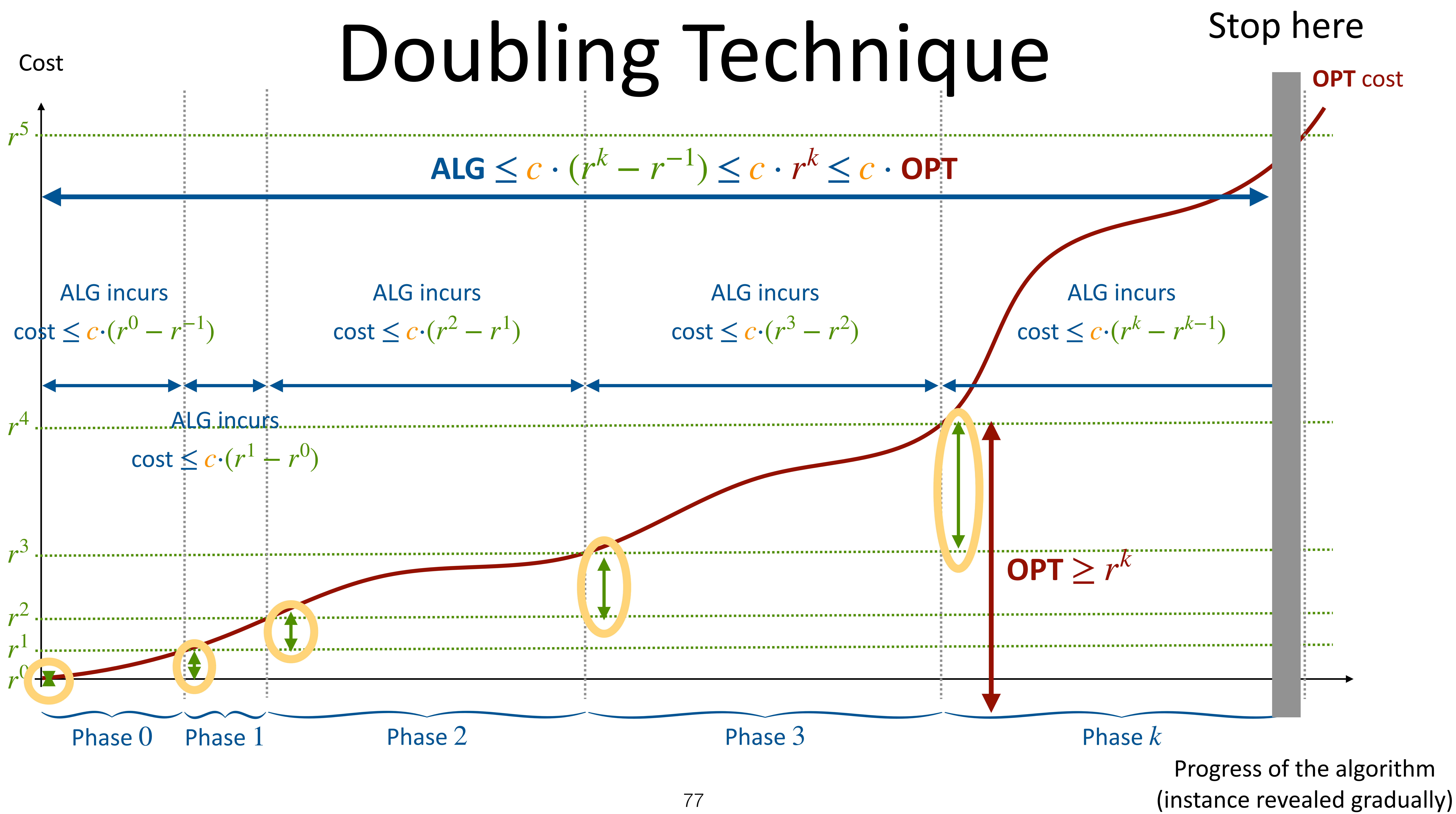

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Your main effort when using doubling!

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  - Using geometrically increasing estimation on the optimal solution, we can produce fragments of the algorithm's solution.
  - The term "doubling" is a bit misleading; sometimes we also use factors other than 2.

Doubling Technique — Bidding

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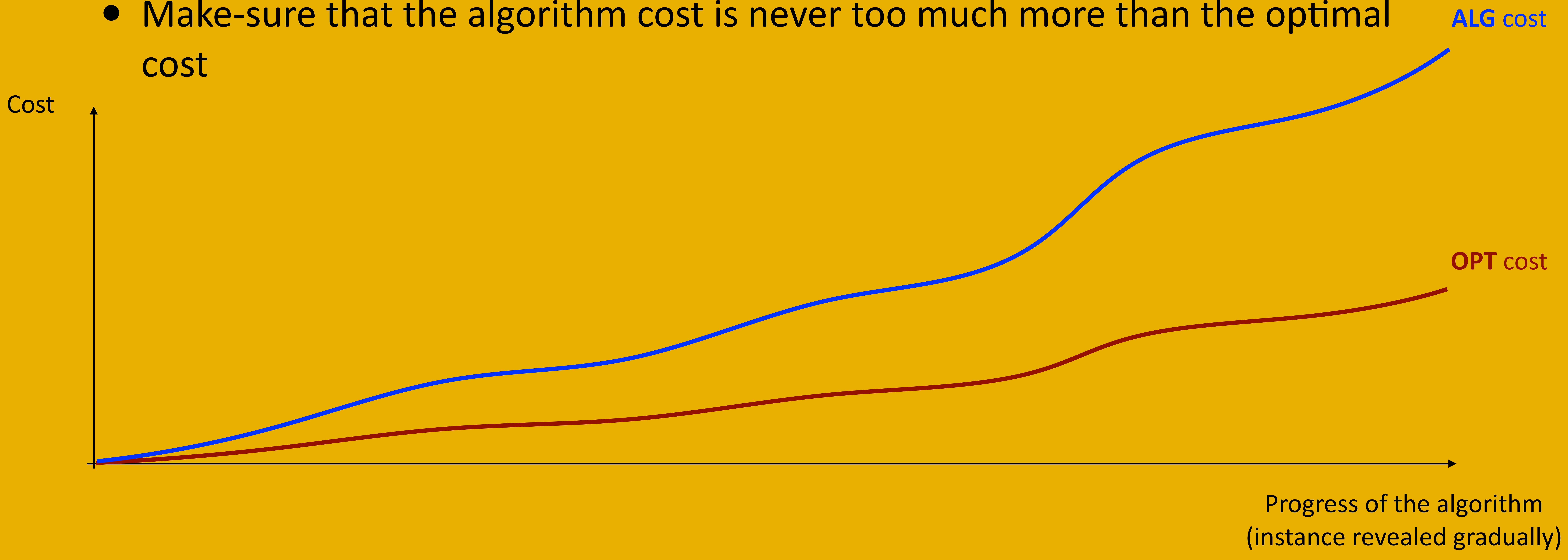
  - If **DOUBLE** never creates (total) cost that exceeds $c \cdot (r^i - r^{i-1})$ in every phase i, **DOUBLE** is c-competitive
    - **The DOUBLE algorithm incurs cost 2^{i+1} in phase i, and $2^{i+1} \leq 4 \cdot (2^i - 2^{i-1})$**

What Happened

- The idea of **doubling technique** for designing online algorithms:
  - Keep track of the optimal solution throughout the process
  - Make-sure that the algorithm cost is never too much more than the optimal cost



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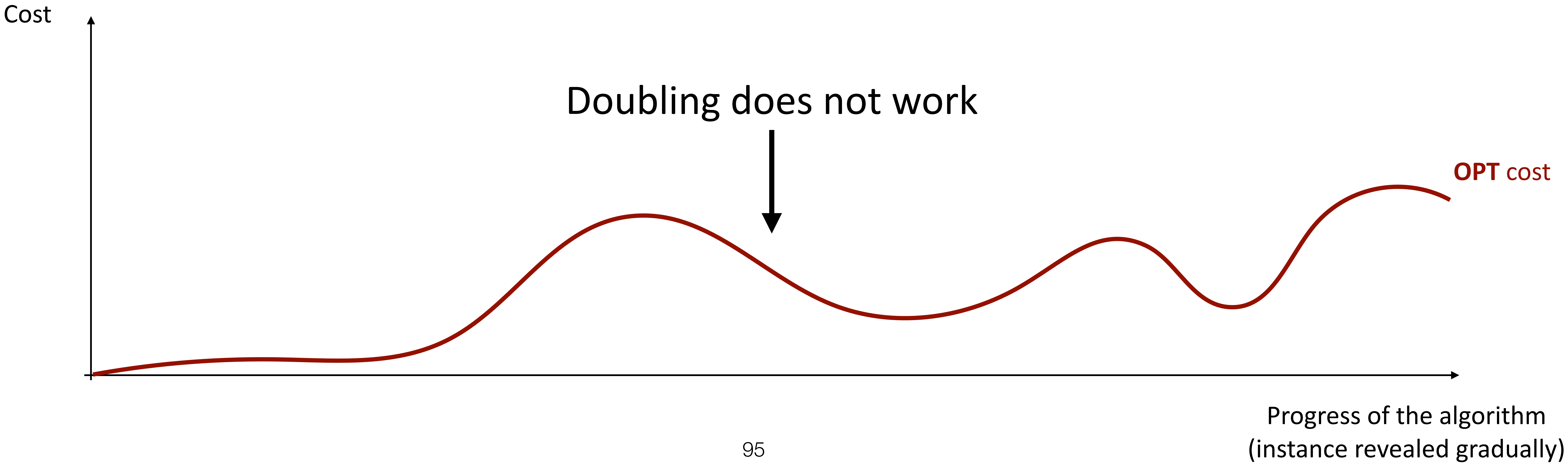

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Dominating set:
A subset D of vertices such that
Every vertex which is not in D has at least one neighbor in D

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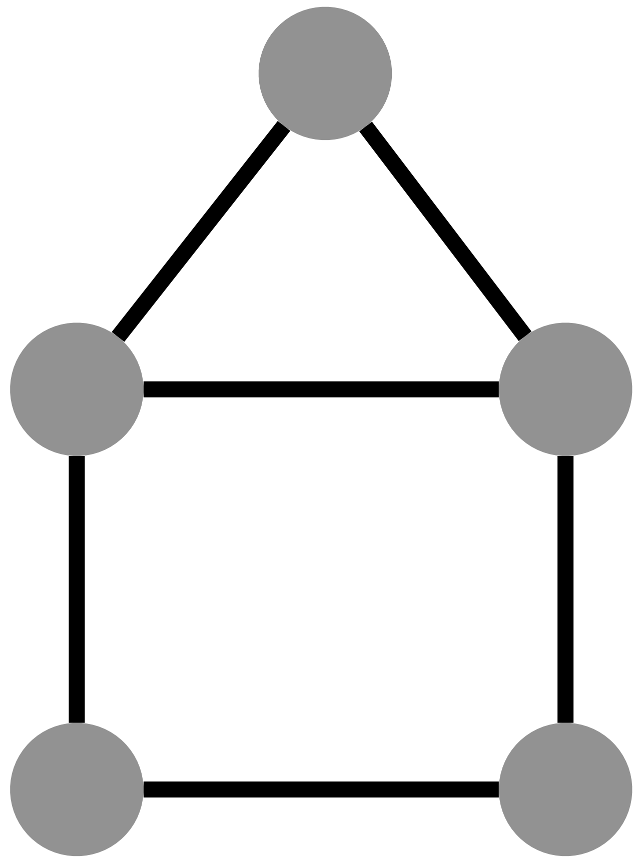

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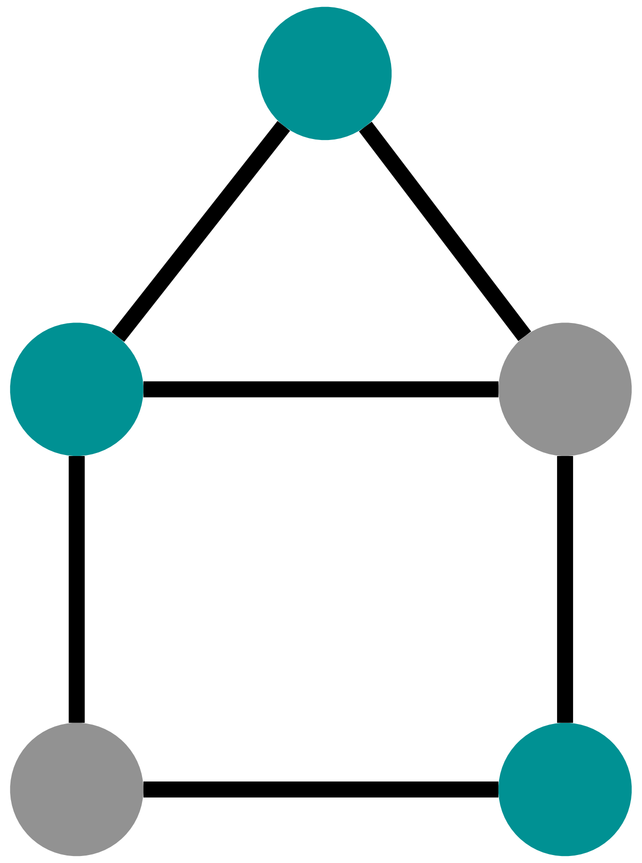

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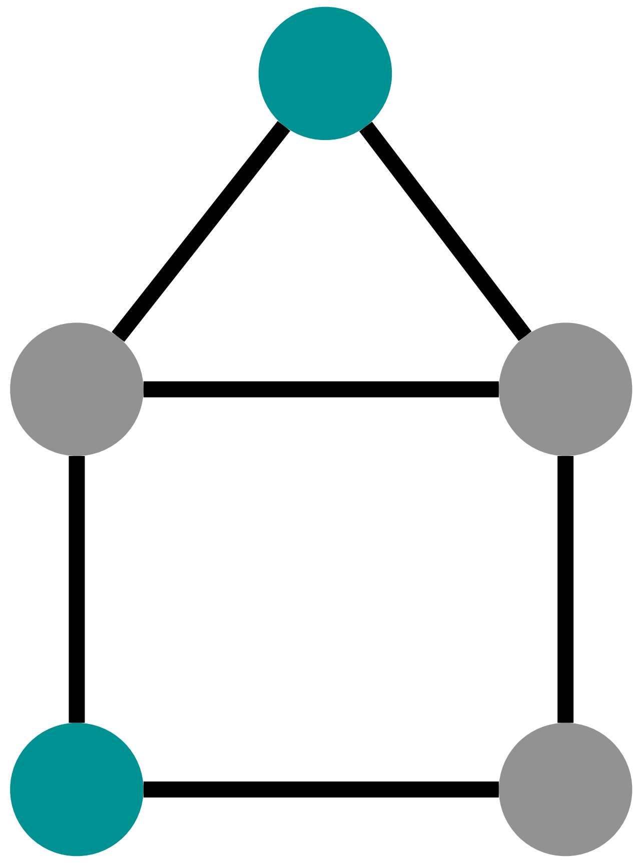

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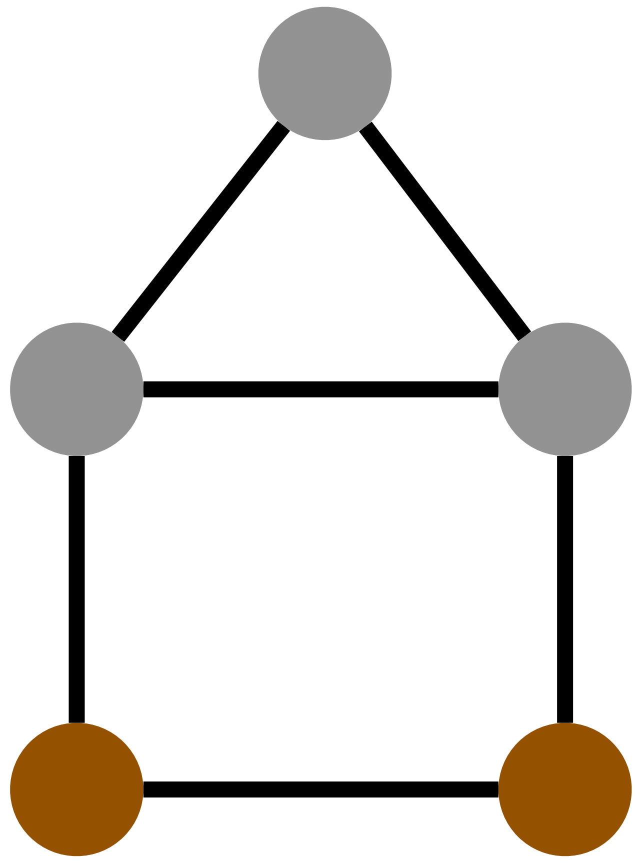

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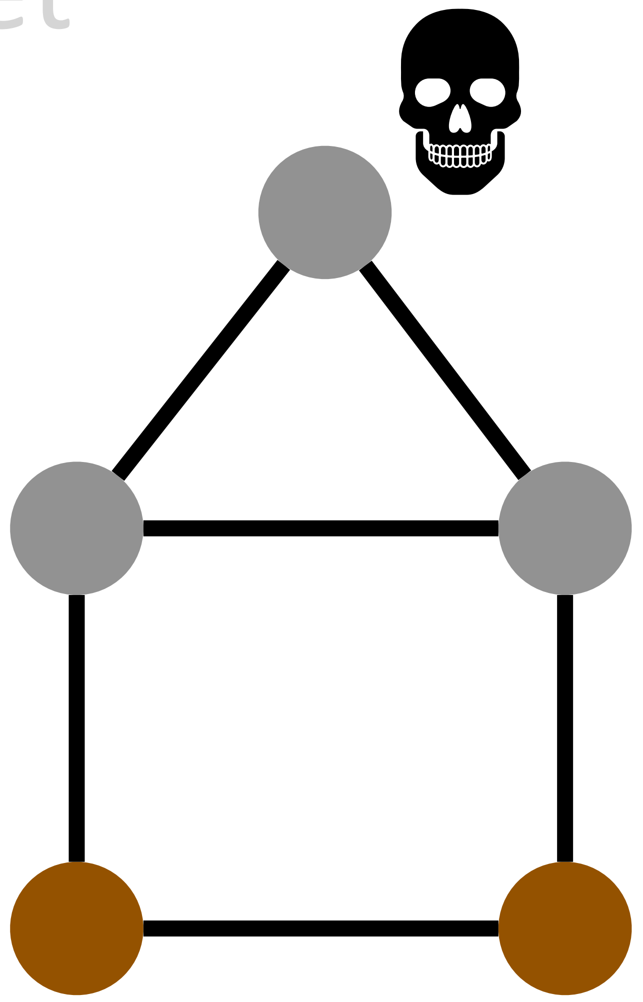

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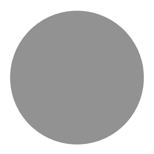

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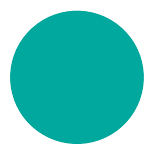

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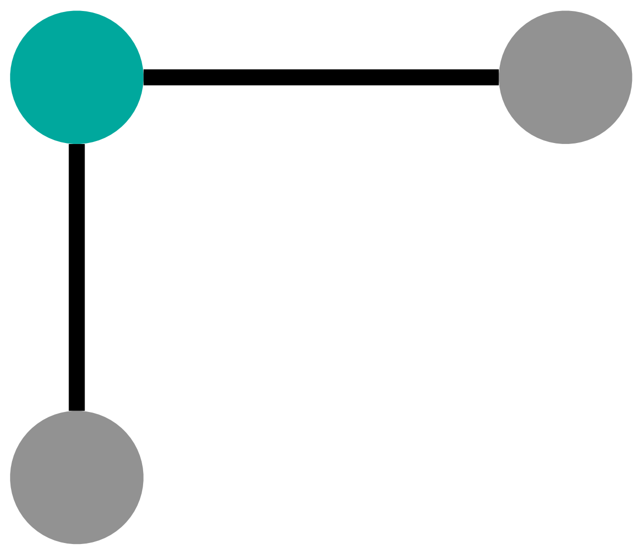

Dominating set:

A subset D of vertices such that

Every vertex which is not in D has at least one neighbor in D

Problems that Doubling Does not Help

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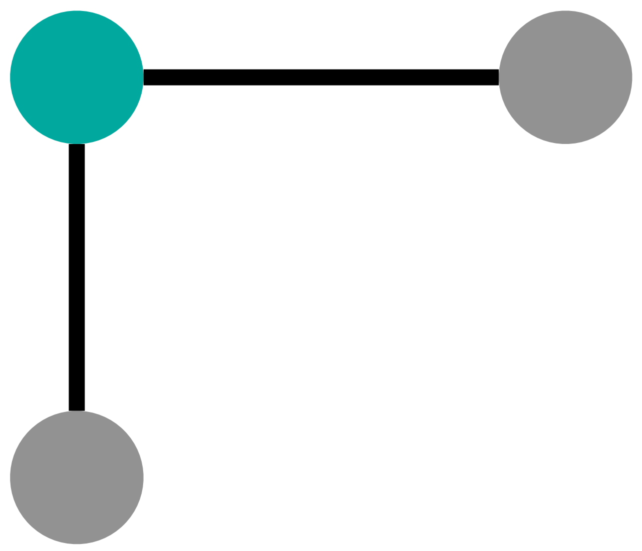

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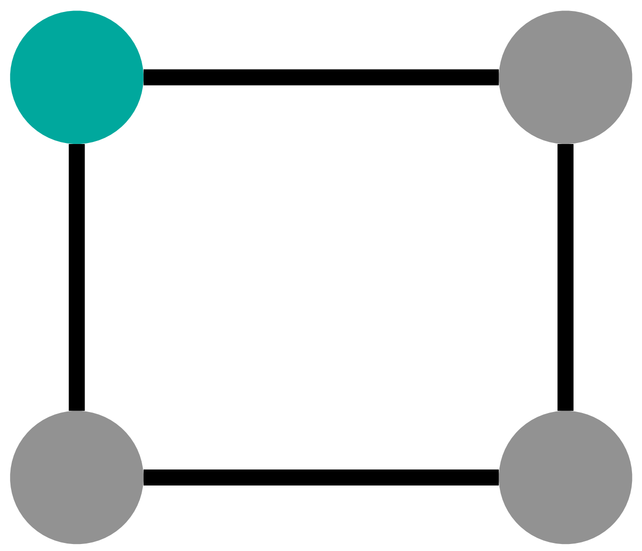

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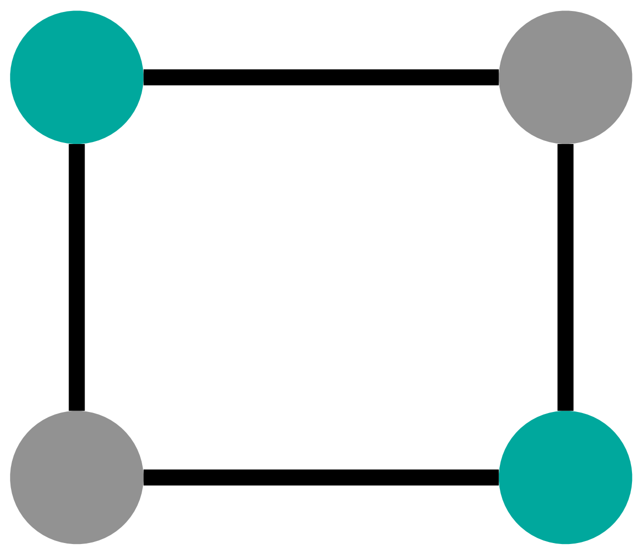

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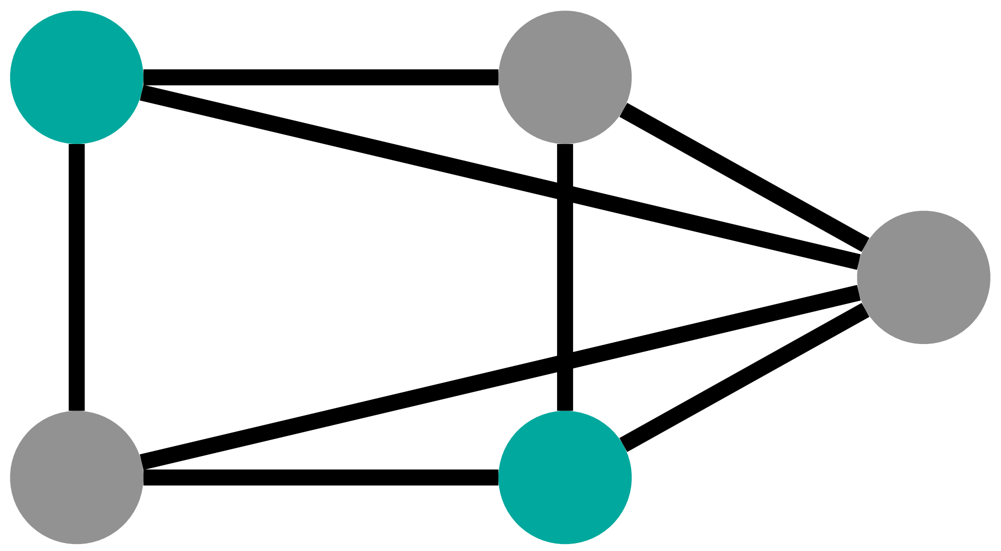

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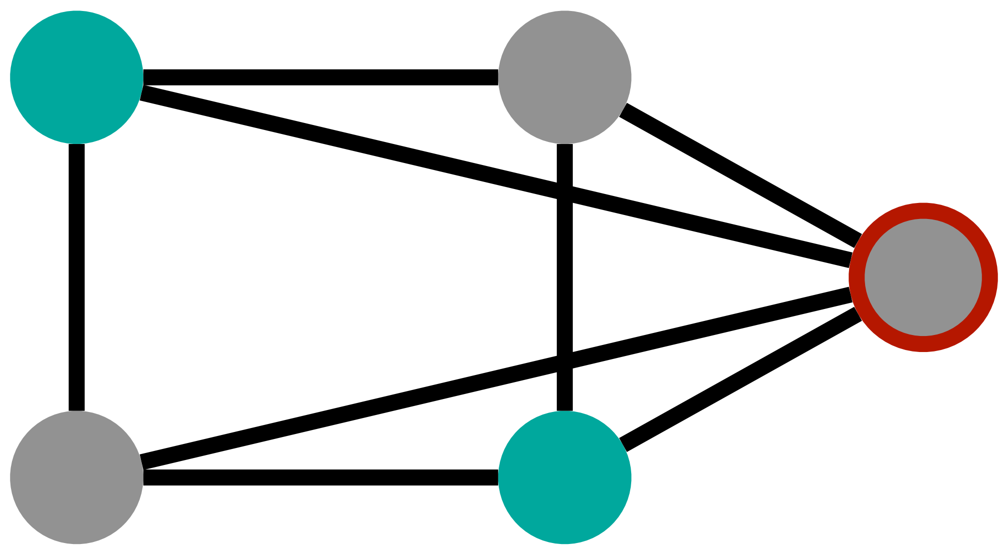

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What Happened

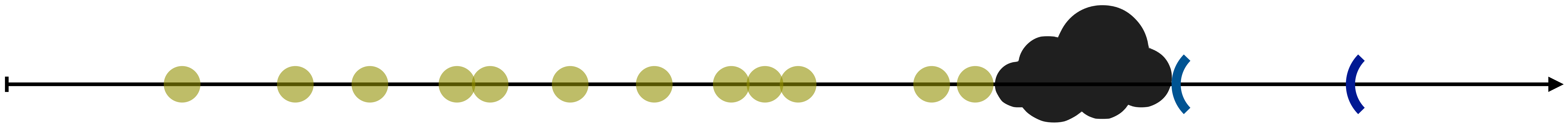
- Doubling helps only when the optimal cost is non-decreasing when more inputs are revealed

Outline

- Recap from the last lecture
- **Online bidding**
  - A 4-competitive algorithm
- A general technique for designing online algorithms: **doubling**
- "Best" online algorithms

Problem Competitive Ratio Lower Bound

- Recall that for any algorithm, we can prove that its competitive ratio has a lower bound (by designing an adversarial input against it)



Problem Competitive Ratio Lower Bound

- Recall that for any algorithm, we can prove that its competitive ratio has a lower bound (by designing an adversarial input against it)

- By designing adversarial instances, one can prove that for a problem, there is a performance lower bound L for all online algorithm. That is, **any (deterministic) online algorithm is at least L-competitive**.

Problem Competitive Ratio Lower Bound

- Formally, we prove that for any online algorithm ALG_i, there exists an instance I_i such that

$$\frac{\mathsf{ALG}_i(I_i)}{\mathsf{OPT}(I_i)} \geq L$$

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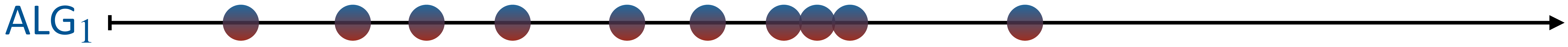

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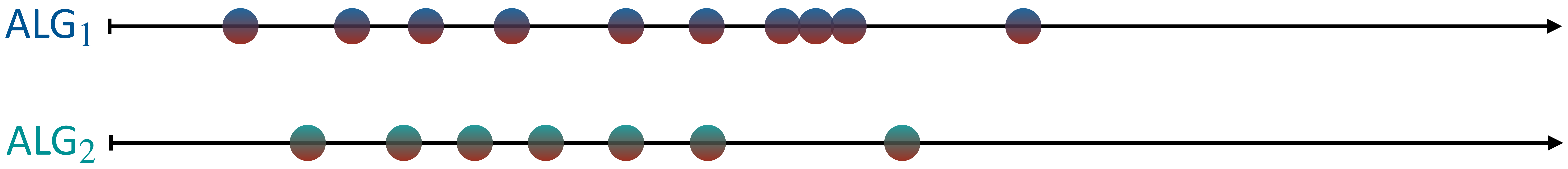

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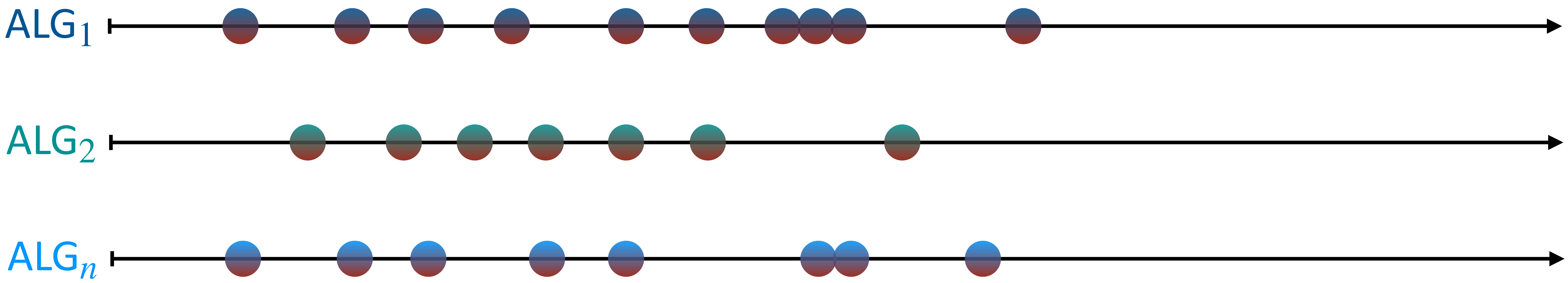

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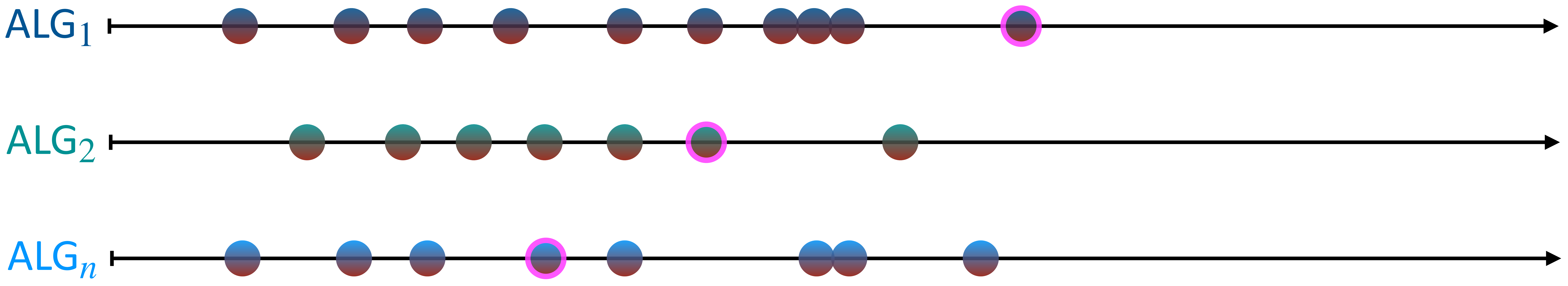

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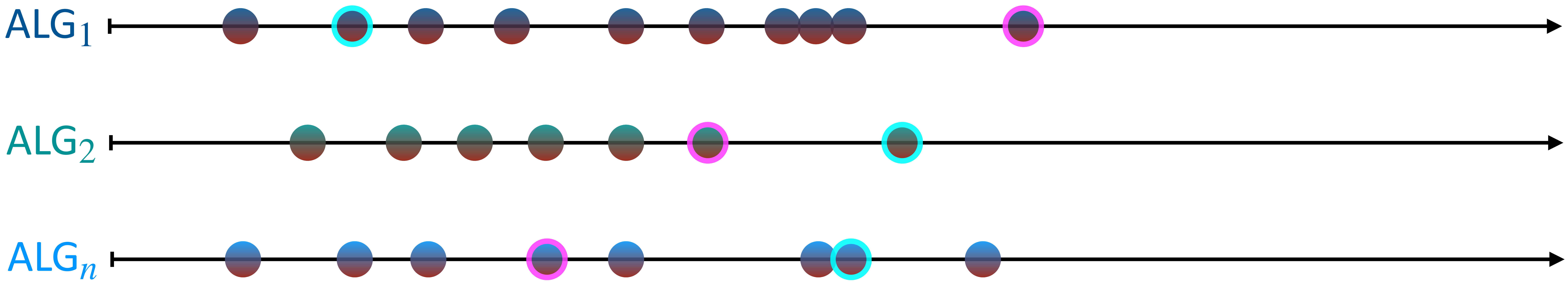

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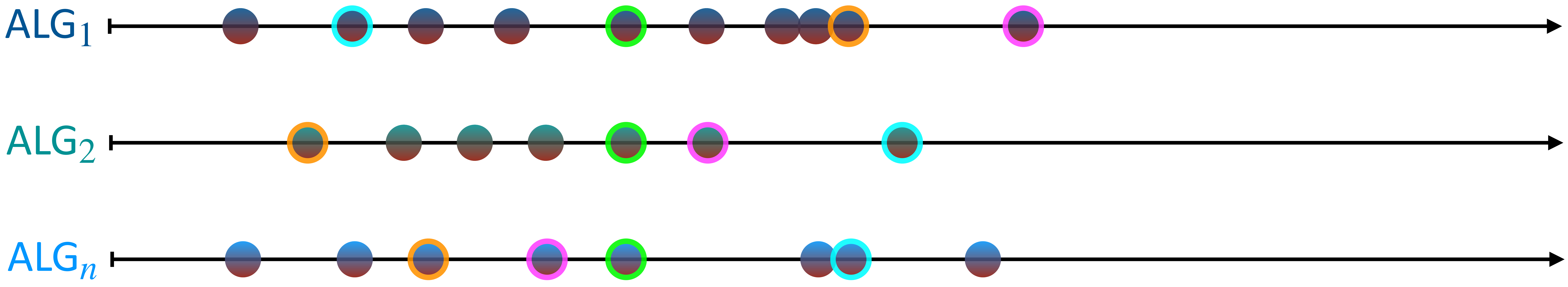

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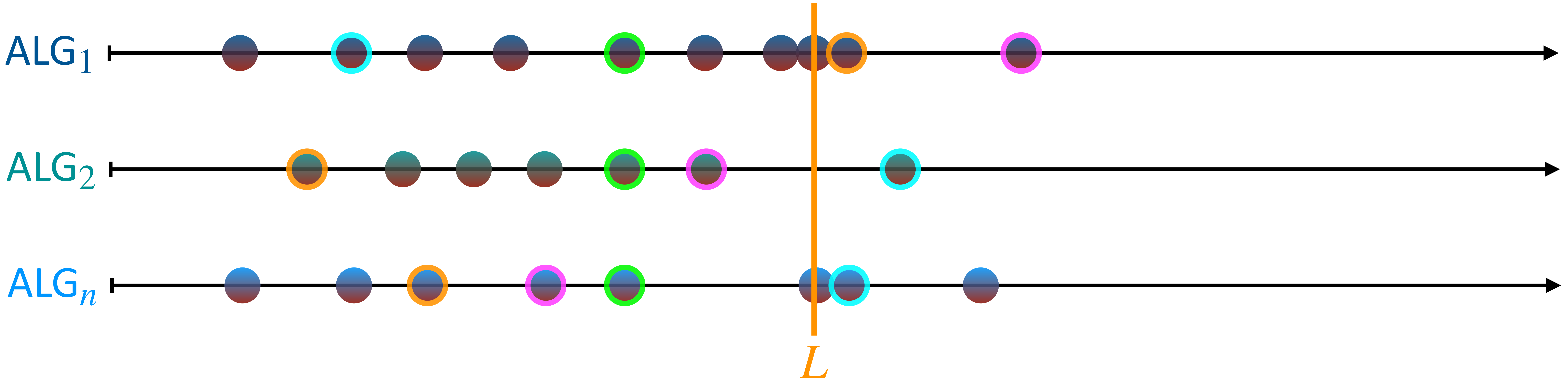

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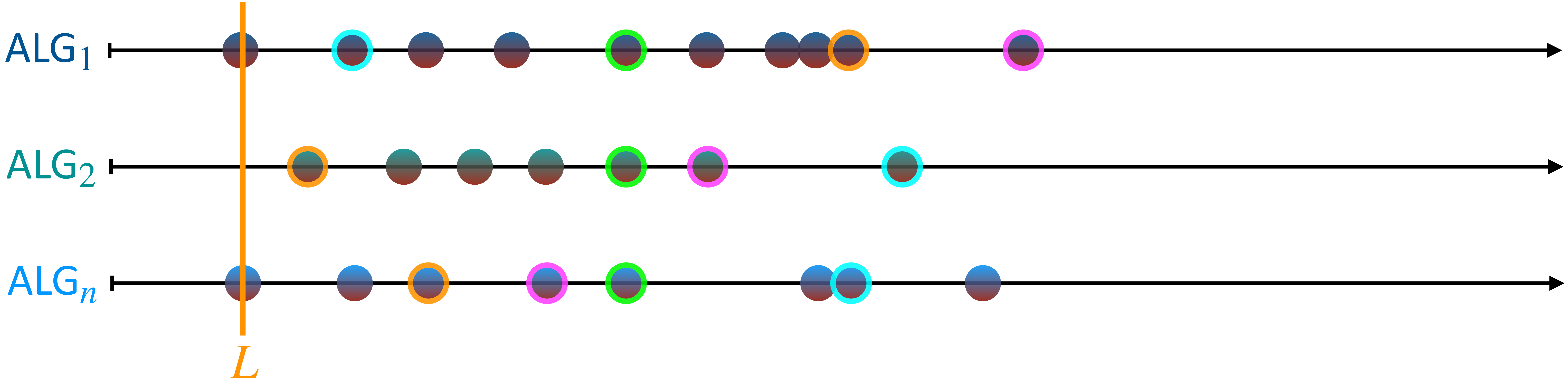

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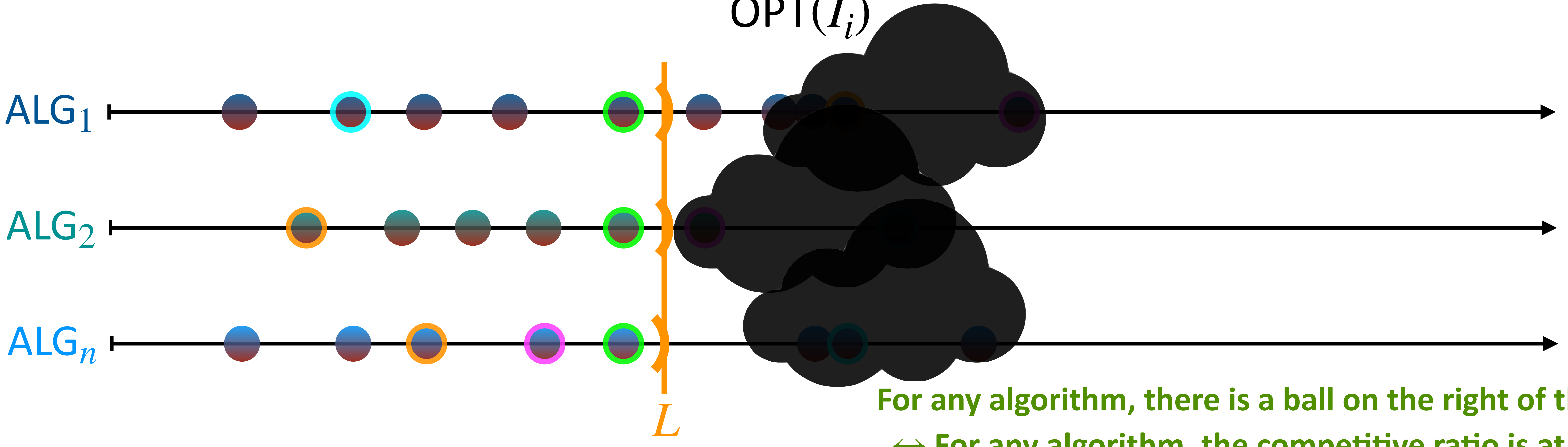
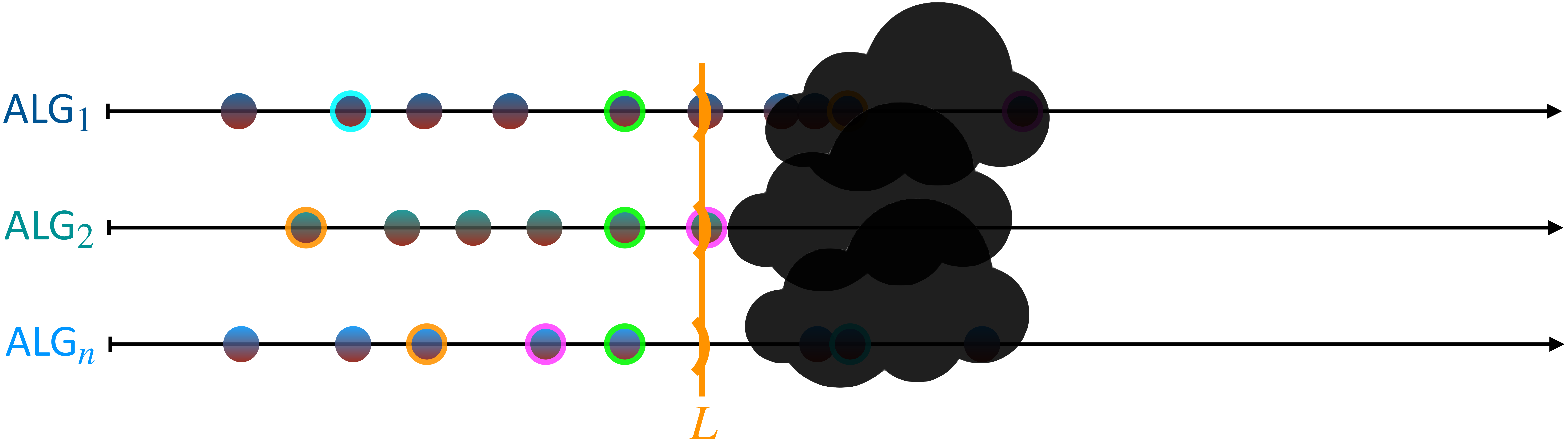


For any algorithm, there is a ball on the right of the bar L

$\leftrightarrow$ For any algorithm, the competitive ratio is at least L

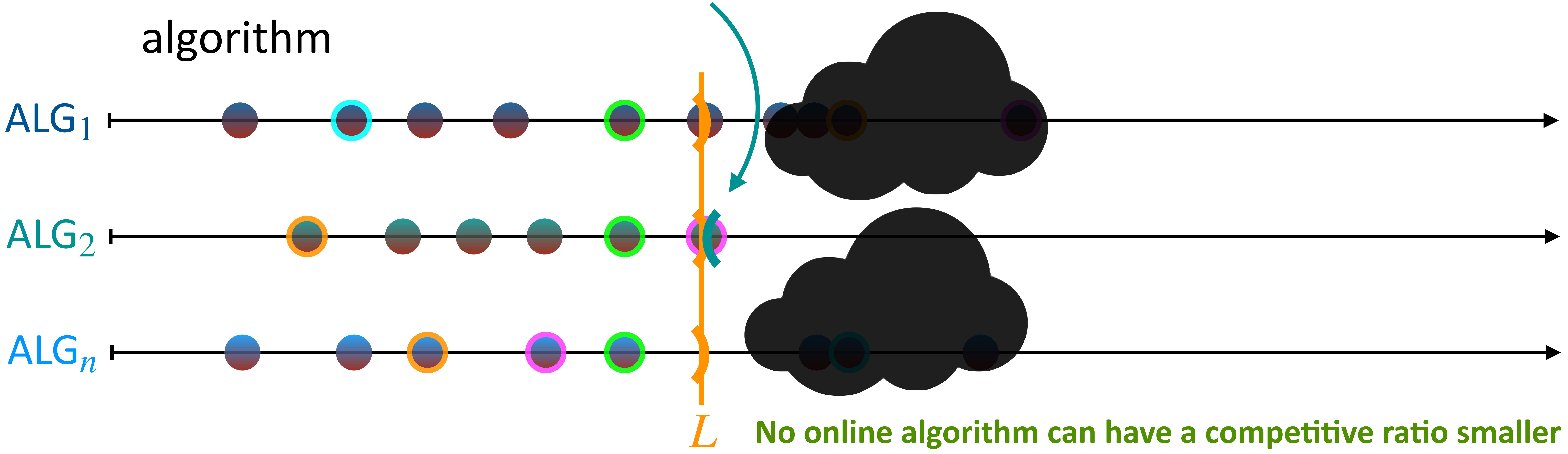
Problem Competitive Ratio Lower Bound

- For any algorithm, there is a ball at or on the right of the bar L
  $\leftrightarrow$ For any algorithm, the competitive ratio is at least L



Problem Competitive Ratio Lower Bound

- For any algorithm, there is a ball at or on the right of the bar L
  $\leftrightarrow$ For any algorithm, the competitive ratio is at least L
- If there is an algorithm that is L-competitive, it is the best online algorithm

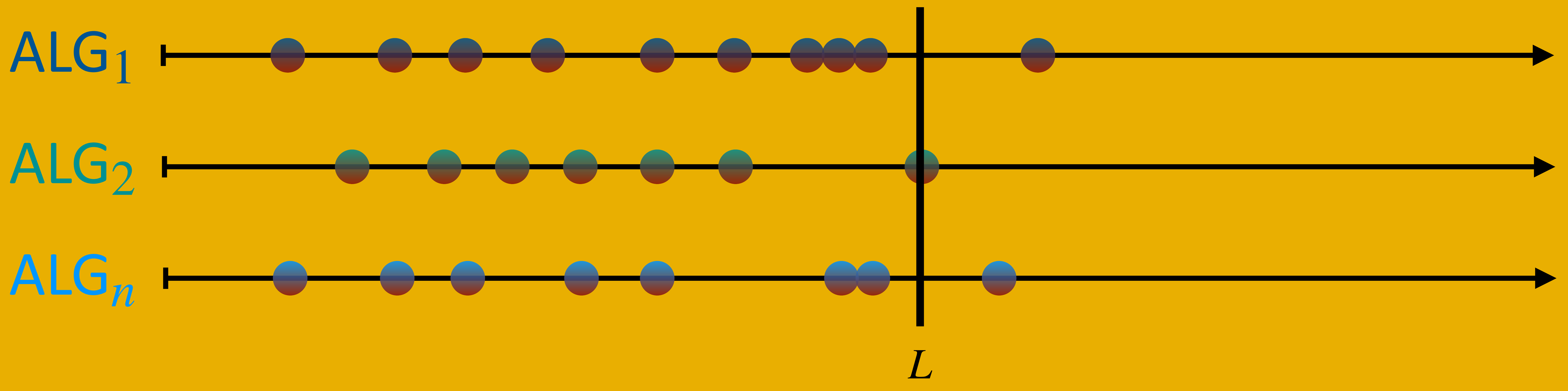


No online algorithm can have a competitive ratio smaller than L

What Happened

- If you find a way to design (a series of) instances such that for any online algorithm, the ratio between its cost and the optimal cost is at least L, you show that no online algorithm can be better than L-competitive

- In this case, if you have an online algorithm which is at most L -competitive, it is the best (optimal) online algorithm for this problem



Problem Competitive Ratio Lower Bound

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<Proof Idea>

Any online algorithm must buy the ski on some day.

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Any online algorithm must buy the ski on some day.

Assume that algorithm ALG$_k$ buys the ski on the k-th skiing day, we design the adversarial input I_k that there are exactly k skiing days.

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<Proof Idea>

Any online algorithm must buy the ski on some day.
Assume that algorithm ALG$_k$ buys the ski on the k-th skiing day, we design the adversarial input I_k that there are exactly k skiing days.

As long as we can prove that $\frac{\text{ALG}_k(I_k)}{\text{OPT}(I_k)} \geq 2 - \frac{1}{B}$ for all k, the theorem is proven.

Problem Competitive Ratio Lower Bound

- Theorem: For the Buy-or-Rent problem, there is no deterministic online algorithm better than $(2 - \frac{1}{B})$-competitive.

<Proof> Consider ALG$_k$ and I_k. Since I_k is the instance with exactly k skiing days. The cost of algorithm ALG$_k$ on instance I_k is $(k-1)+B$, while the optimal cost is $\min\{B, k\}$.

- If $k \geq B$, the optimal cost is B and the ratio

$$\frac{\text{ALG}_k(I_k)}{\text{OPT}_k(I_k)} = \frac{(k-1)+B}{B} \geq \frac{(B-1)+B}{B} = 2 - \frac{1}{B}$$

Problem Competitive Ratio Lower Bound

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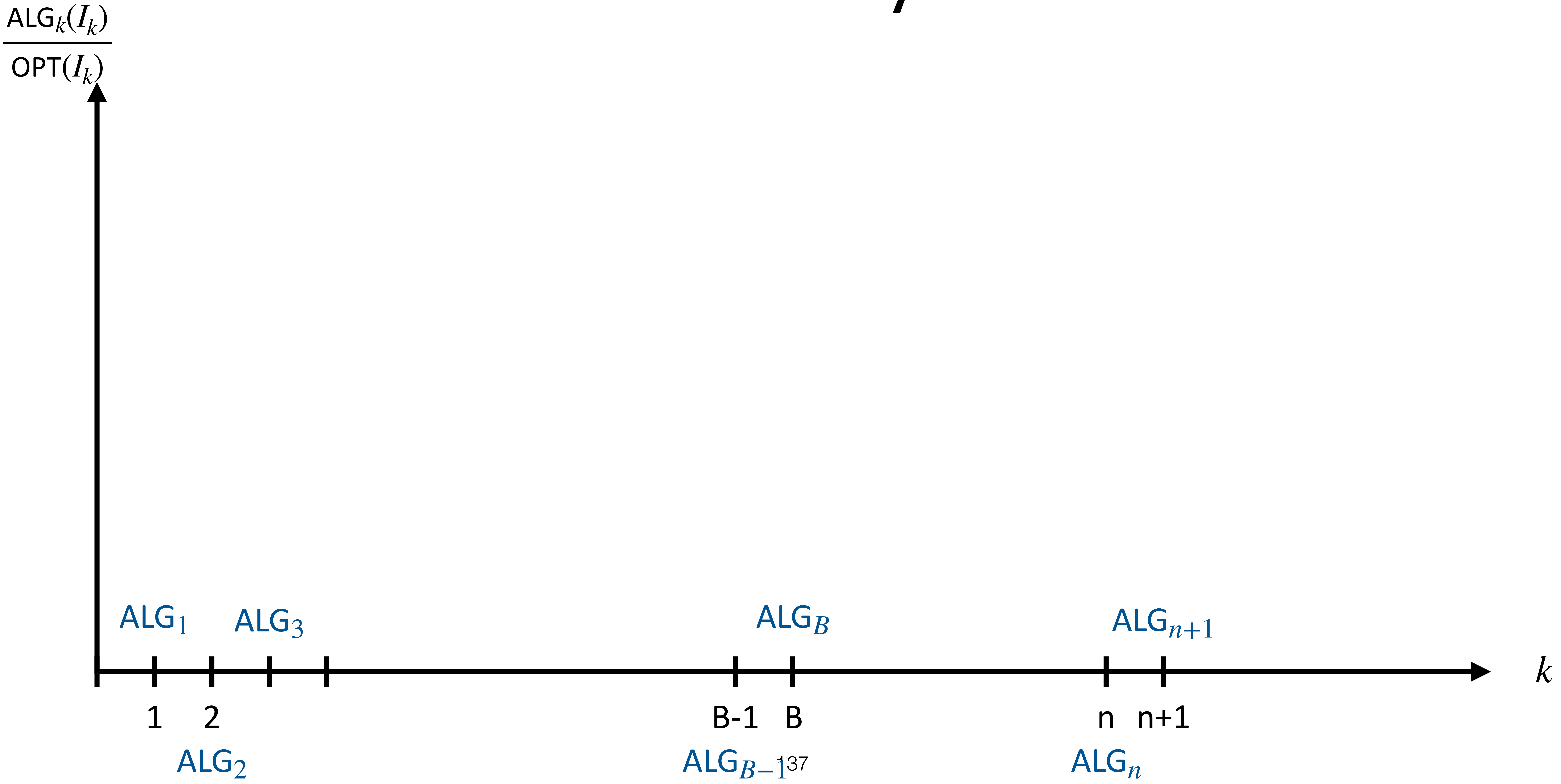
- If $k < B$, the ratio $\dfrac{\text{ALG}_k(I_k)}{\text{OPT}_k(I_k)} = \dfrac{(k-1)+B}{k}$. The ratio decreases as k increases.

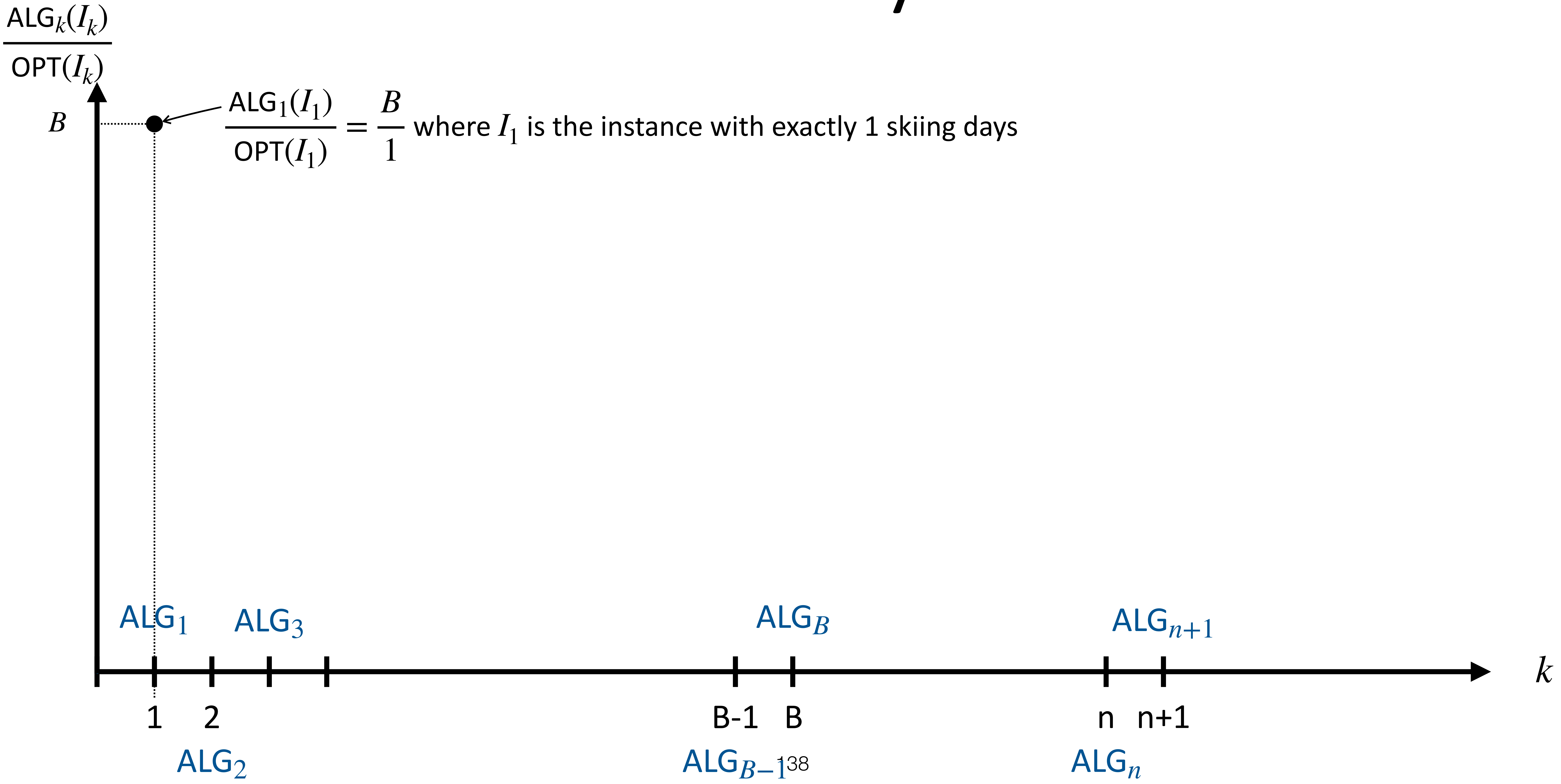
  Hence, the ratio is lower bounded by $\dfrac{(B-1)+B}{B}$ since $k < B$

Lower Bound for the Buy-or-Rent Problem



Lower Bound for the Buy-or-Rent Problem

$$\frac{\text{ALG}_1(I_1)}{\text{OPT}(I_1)} = \frac{B}{1}$$ where I_1 is the instance with exactly 1 skiing days



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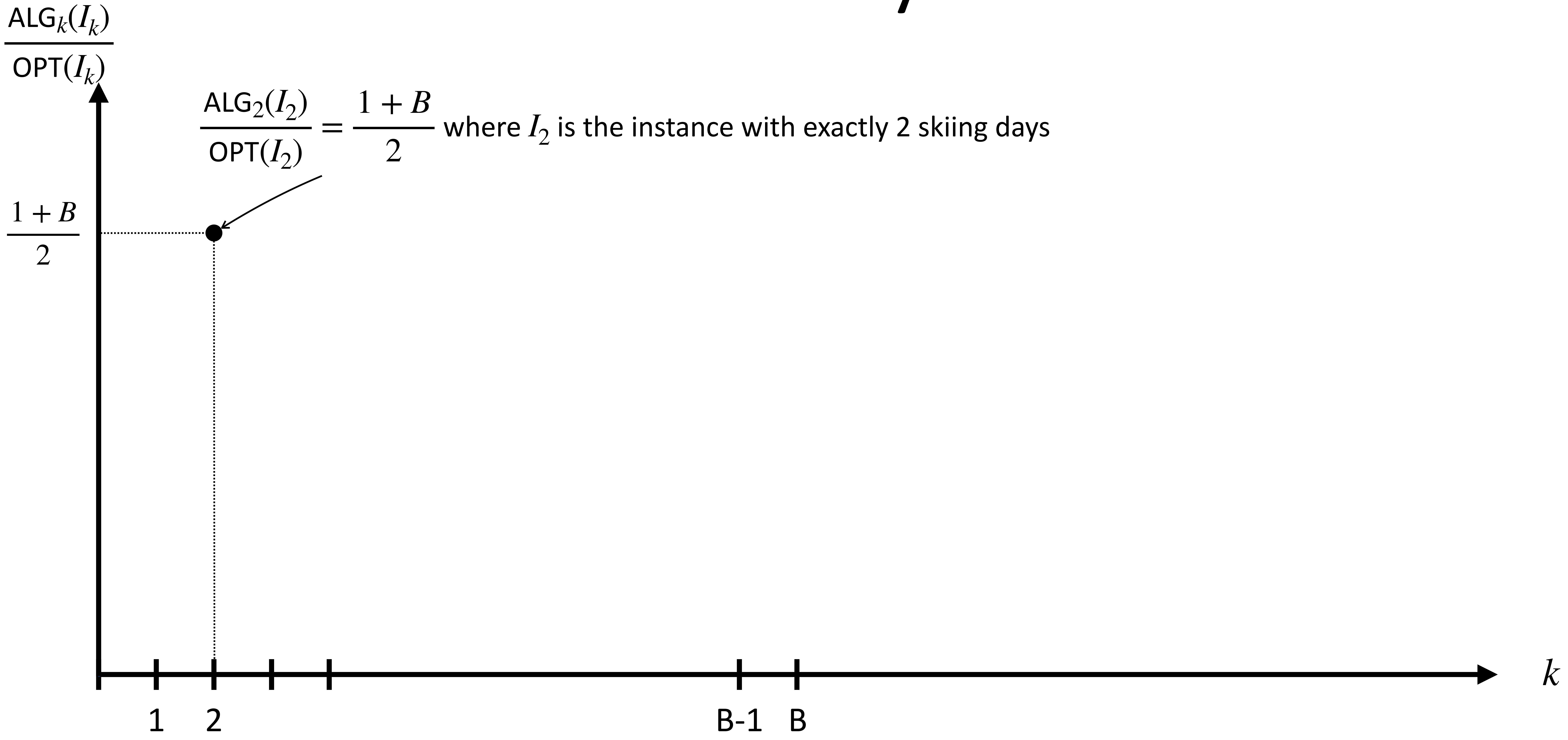

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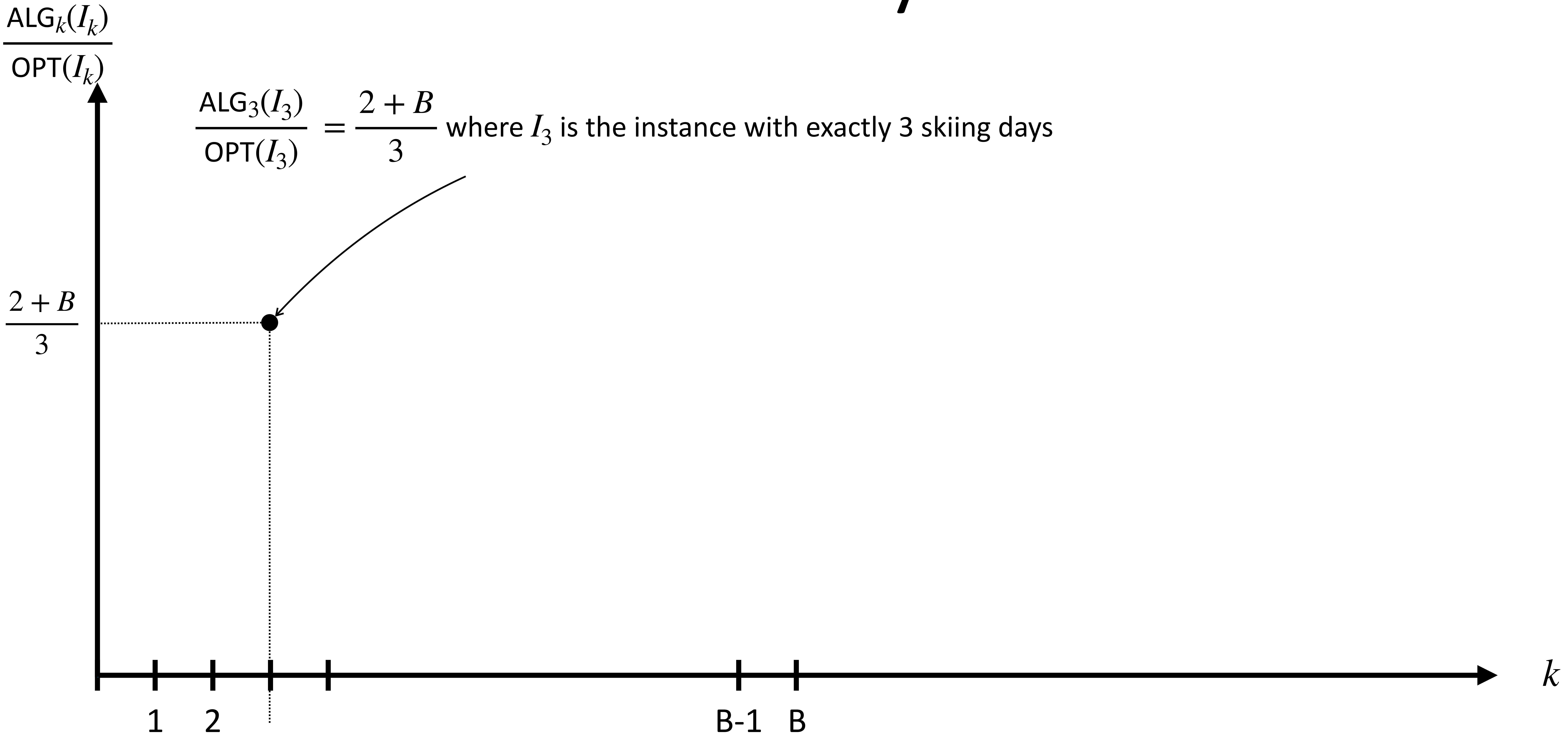
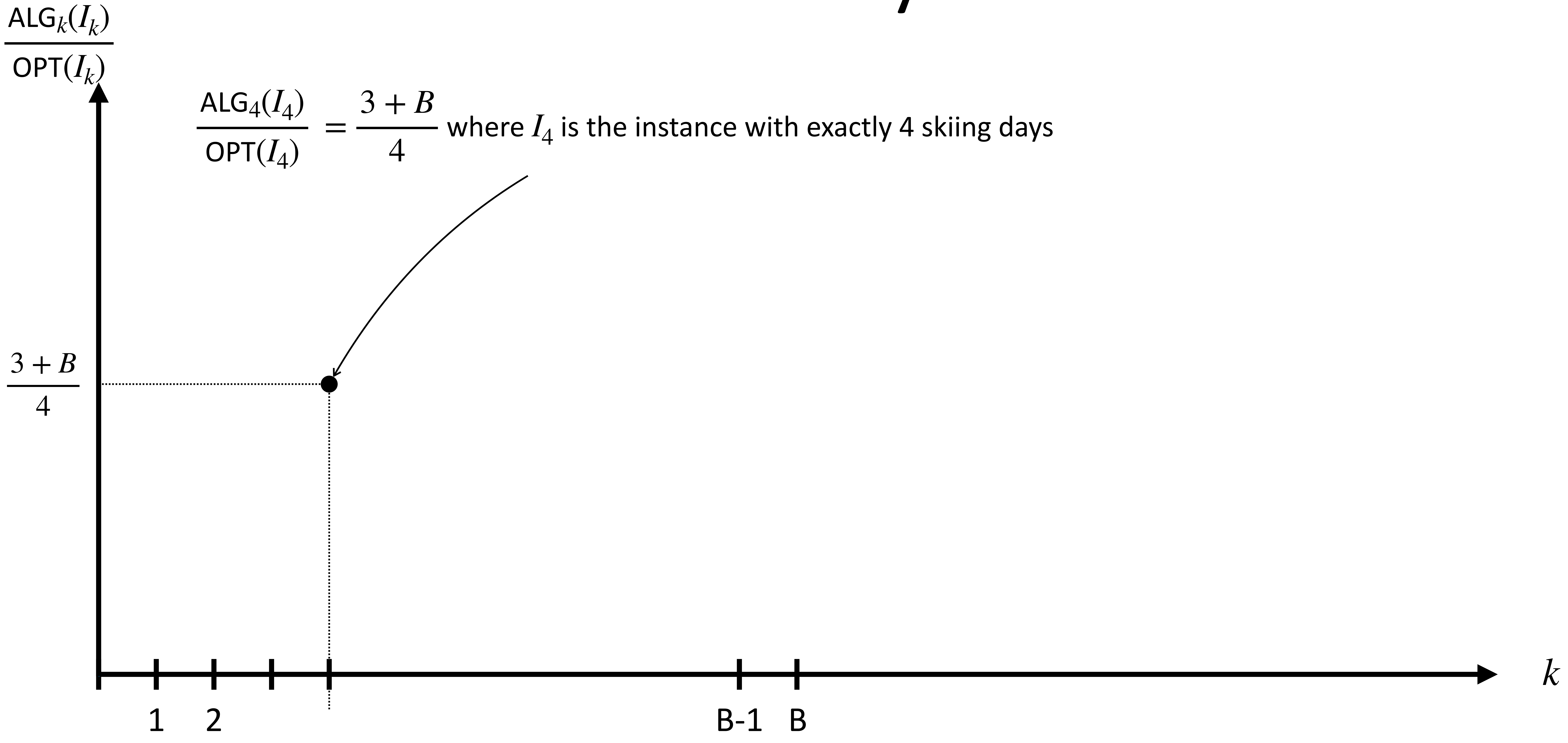


Lower Bound for the Buy-or-Rent Problem

$\frac{\text{ALG}_4(I_4)}{\text{OPT}(I_4)} = \frac{3+B}{4}$ where I_4 is the instance with exactly 4 skiing days



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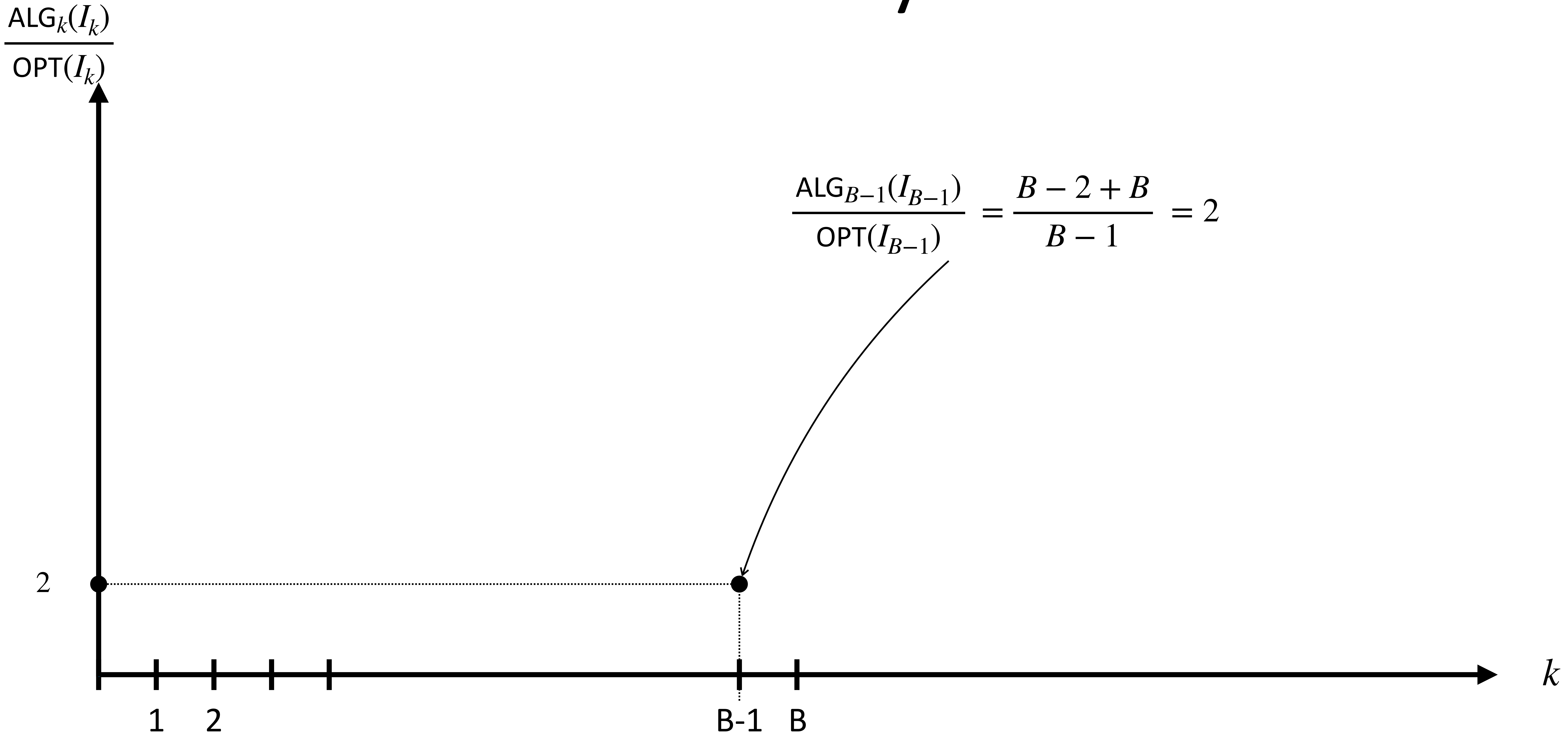

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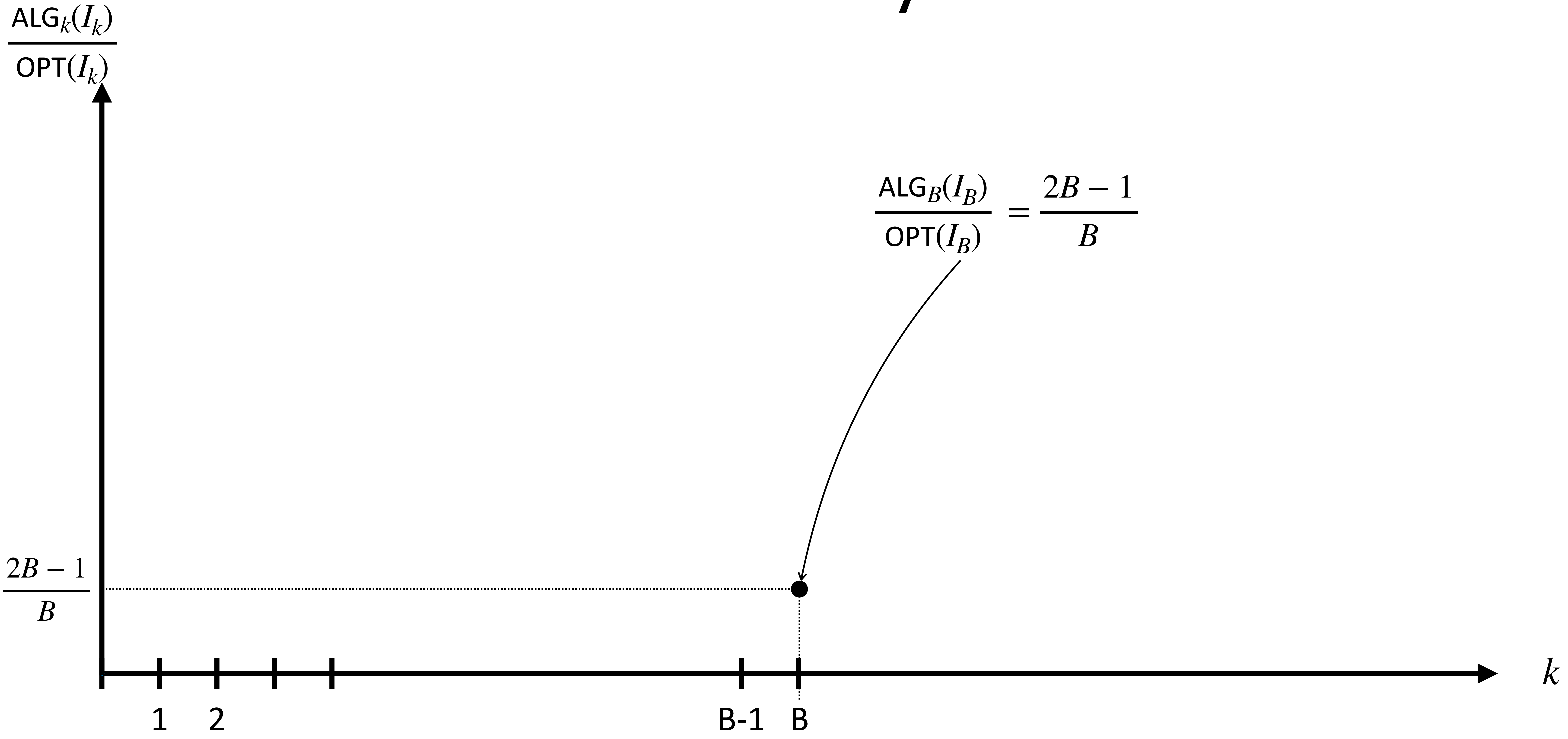

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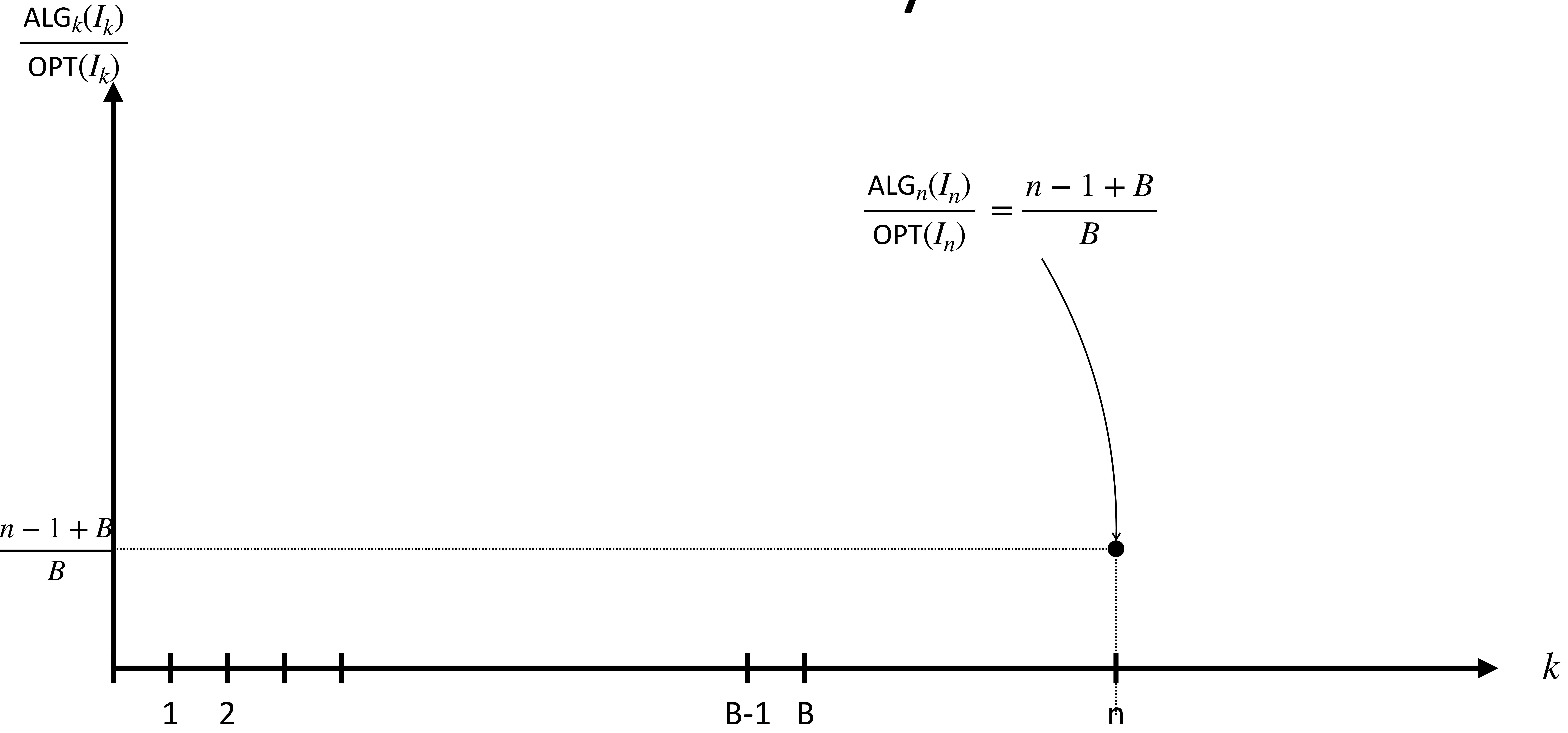

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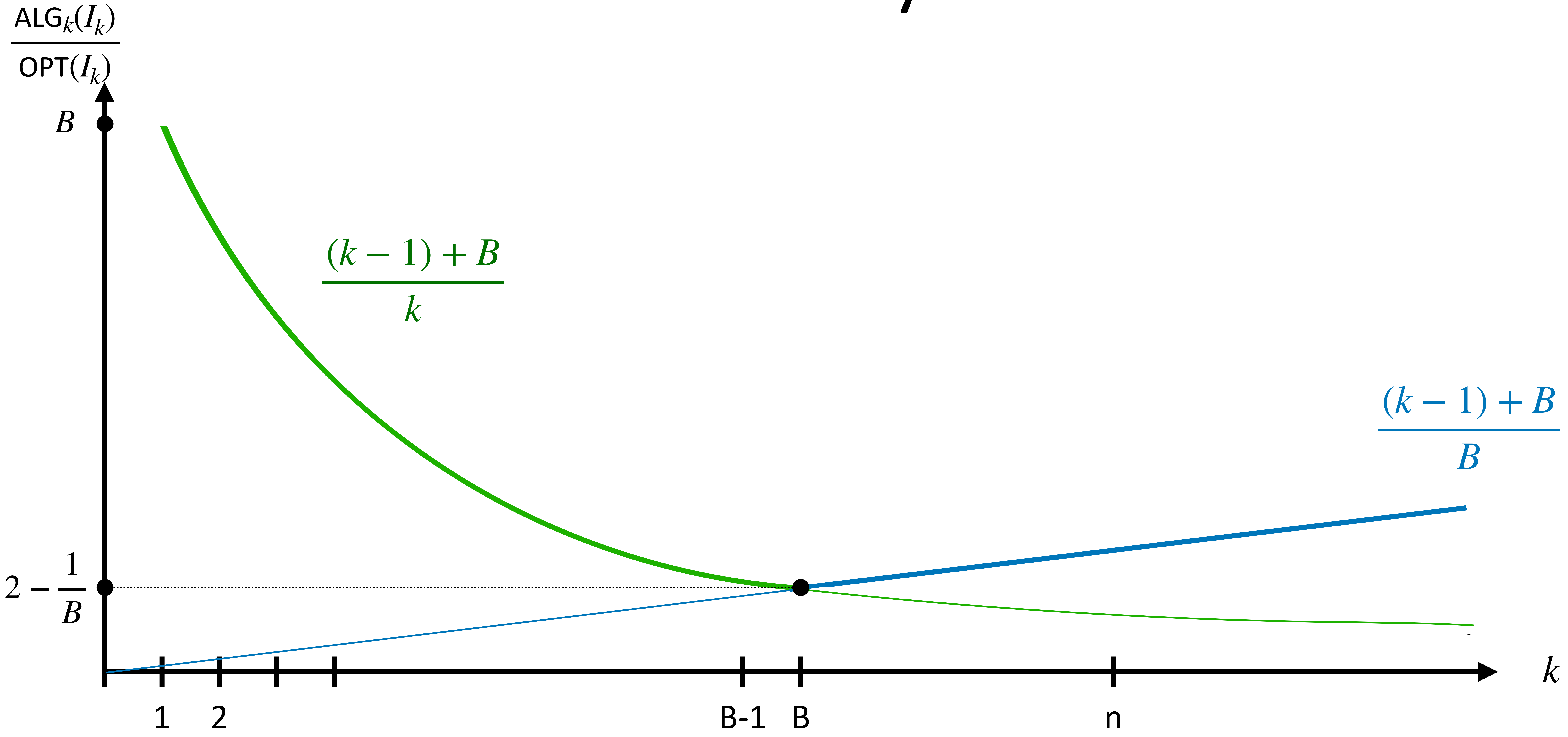


Optimal Online Algorithms

ALG: Buy the ski on the B-th skiing day

- Theorem: For the Buy-or-Rent problem, algorithm ALG is $(2 - \frac{1}{B})$-competitive.
- Theorem: For the Buy-or-Rent problem, there is no deterministic online algorithm better than $(2 - \frac{1}{B})$-competitive.
- Corollary: ALG is an optimal online algorithm
  - If an online algorithm attains the competitive ratio which matches the problem competitive ratio lower bound, the algorithm is an **optimal online algorithm**

Recap: Online Optimization

An online problem

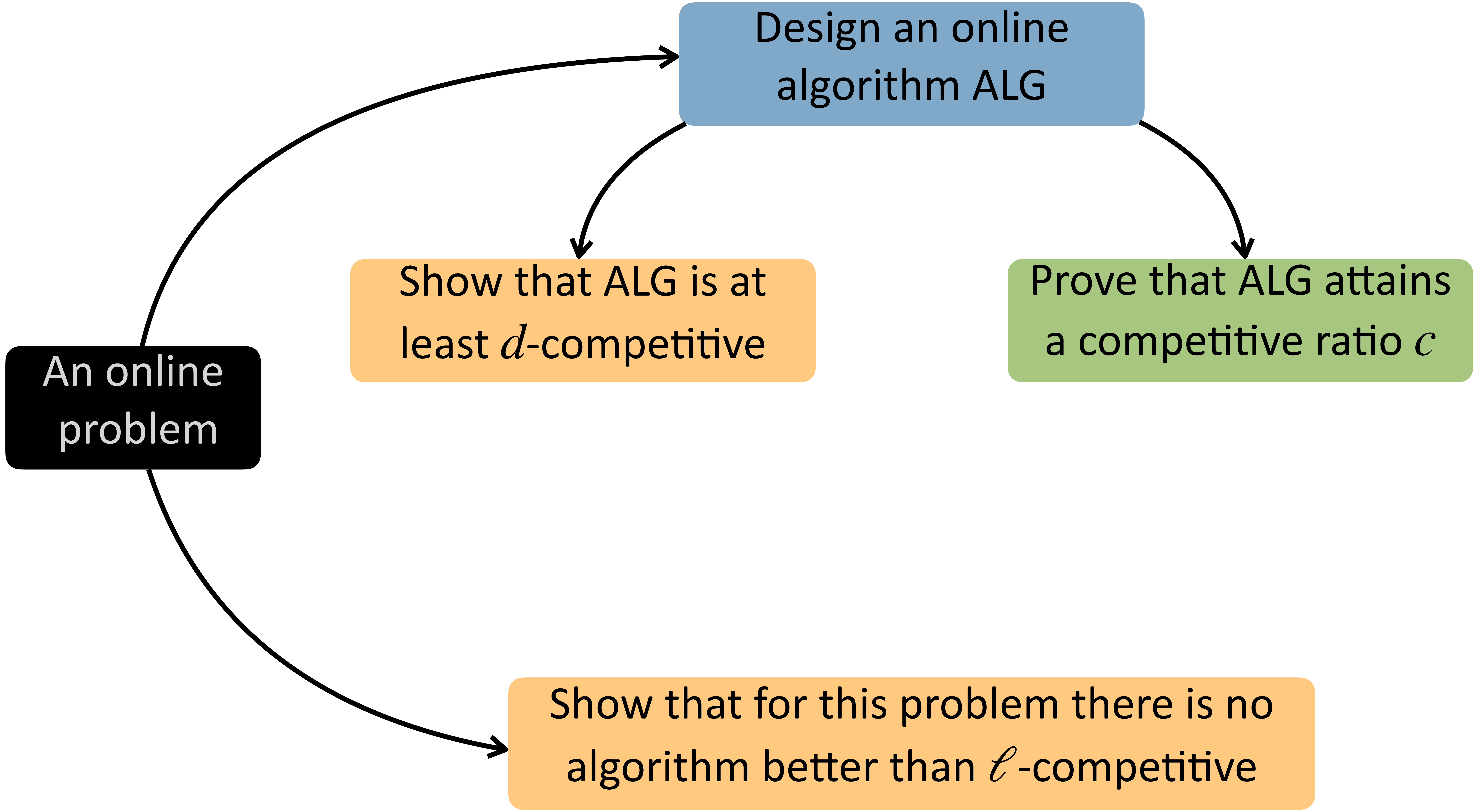
Recap: Online Optimization

Design an online algorithm ALG

An online problem

Show that for this problem there is no algorithm better than ℓ-competitive

Recap: Online Optimization



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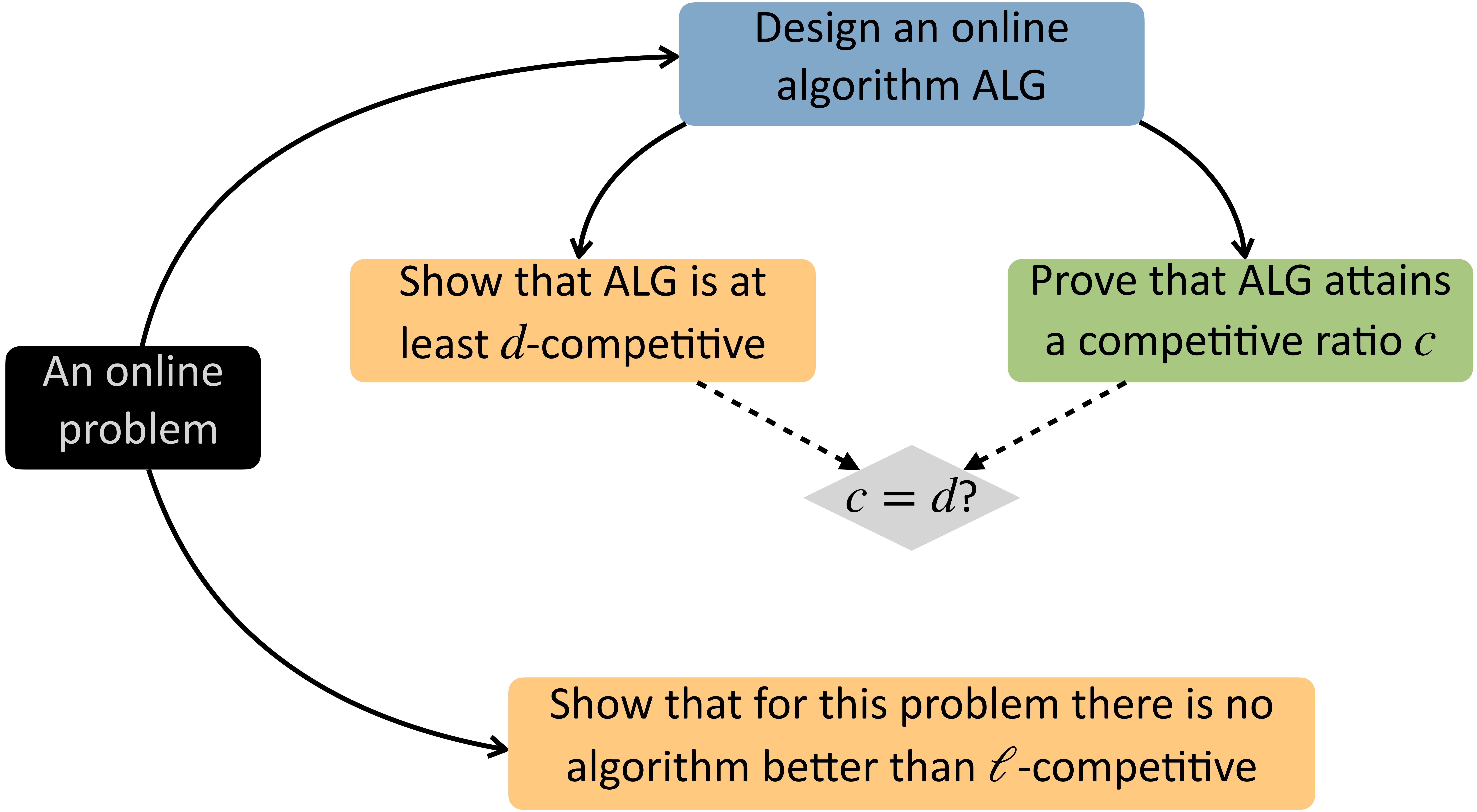

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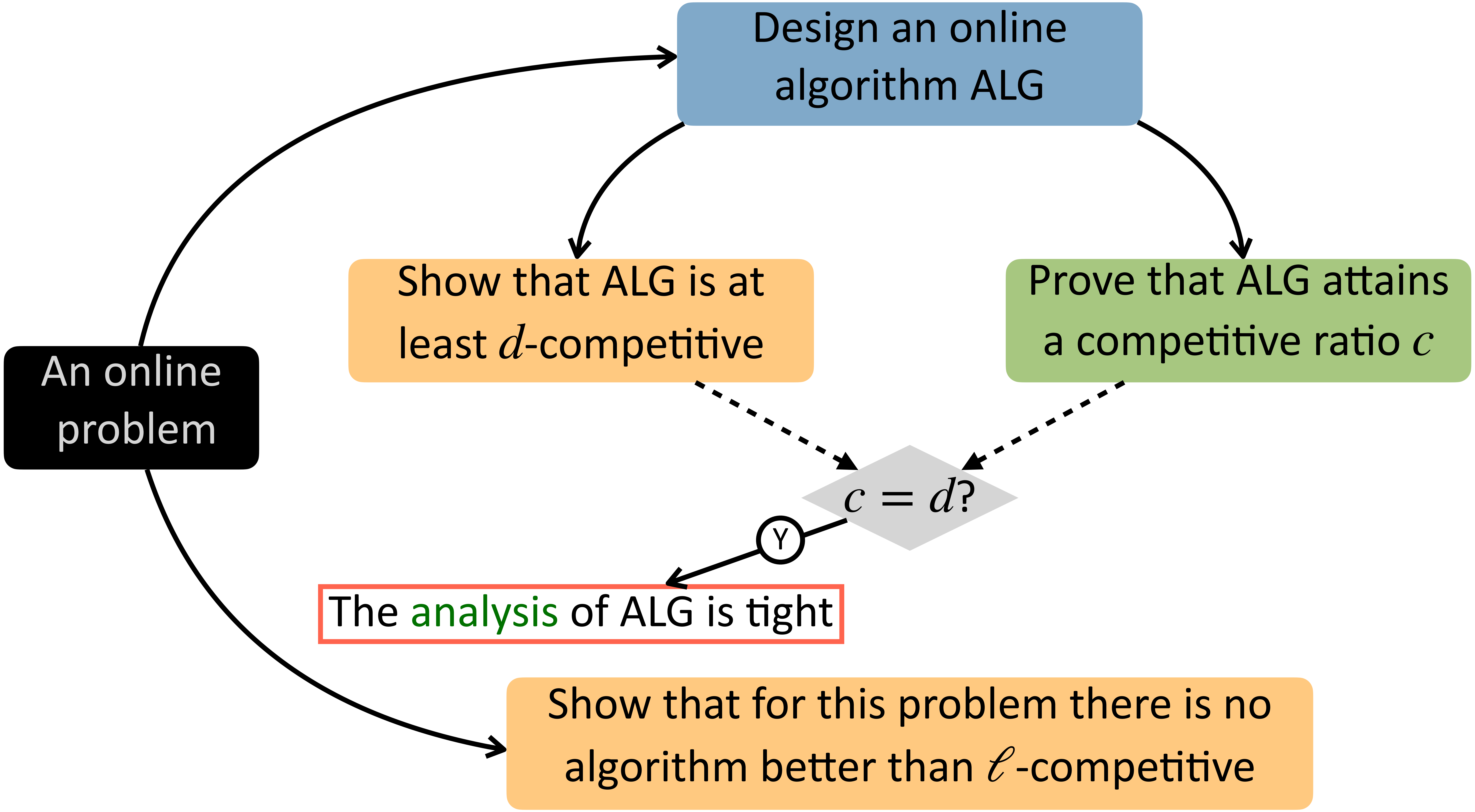

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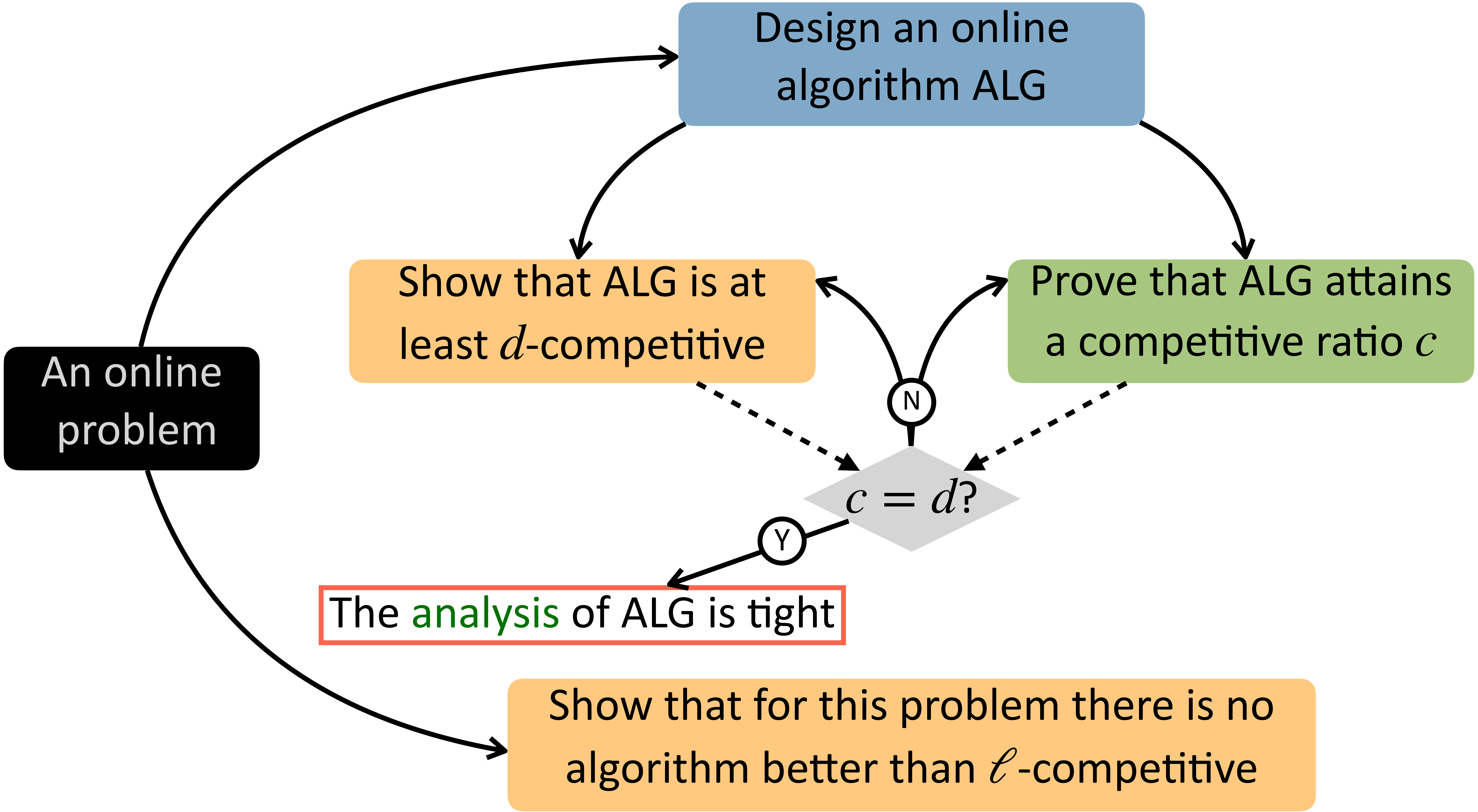

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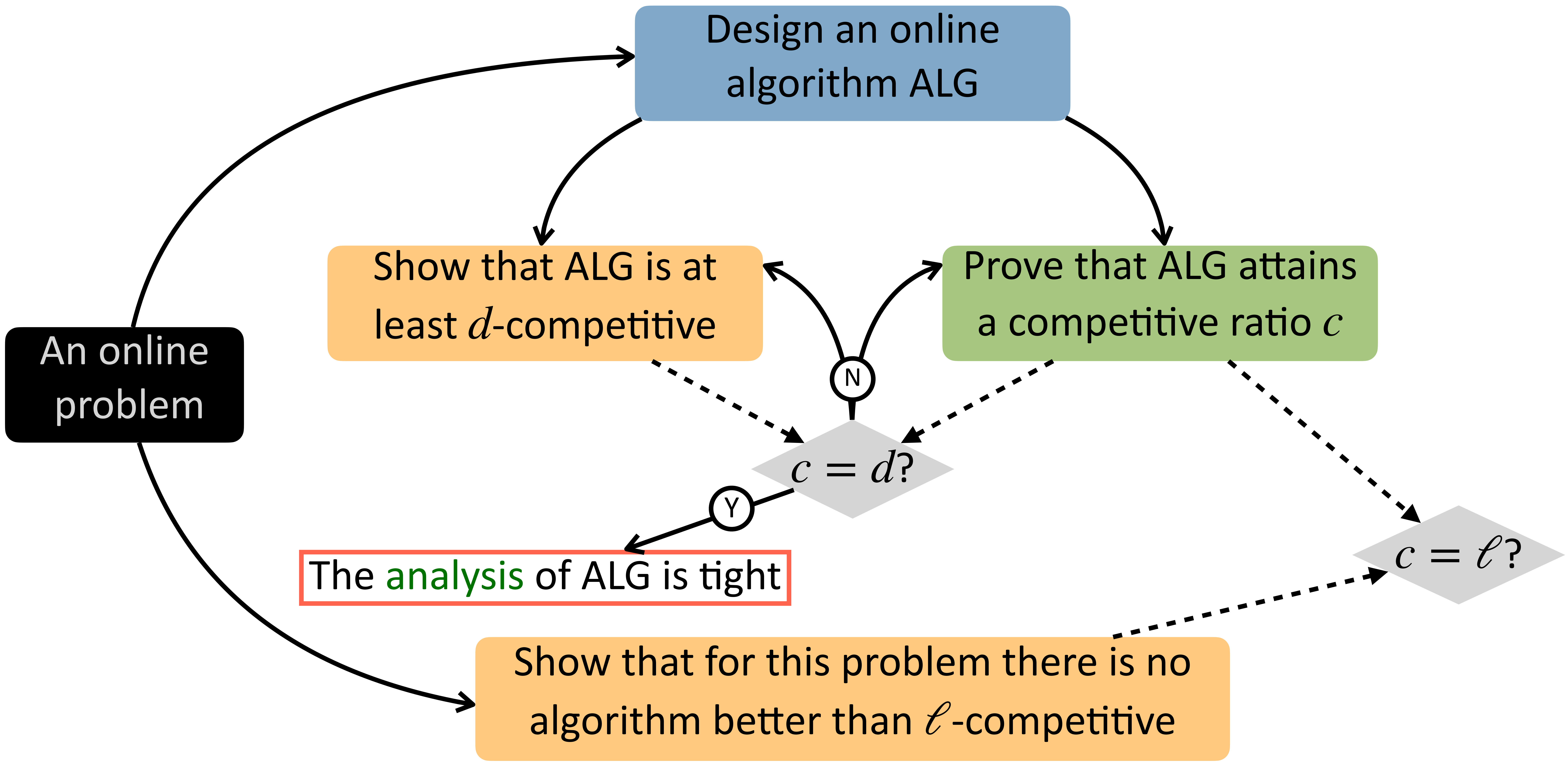

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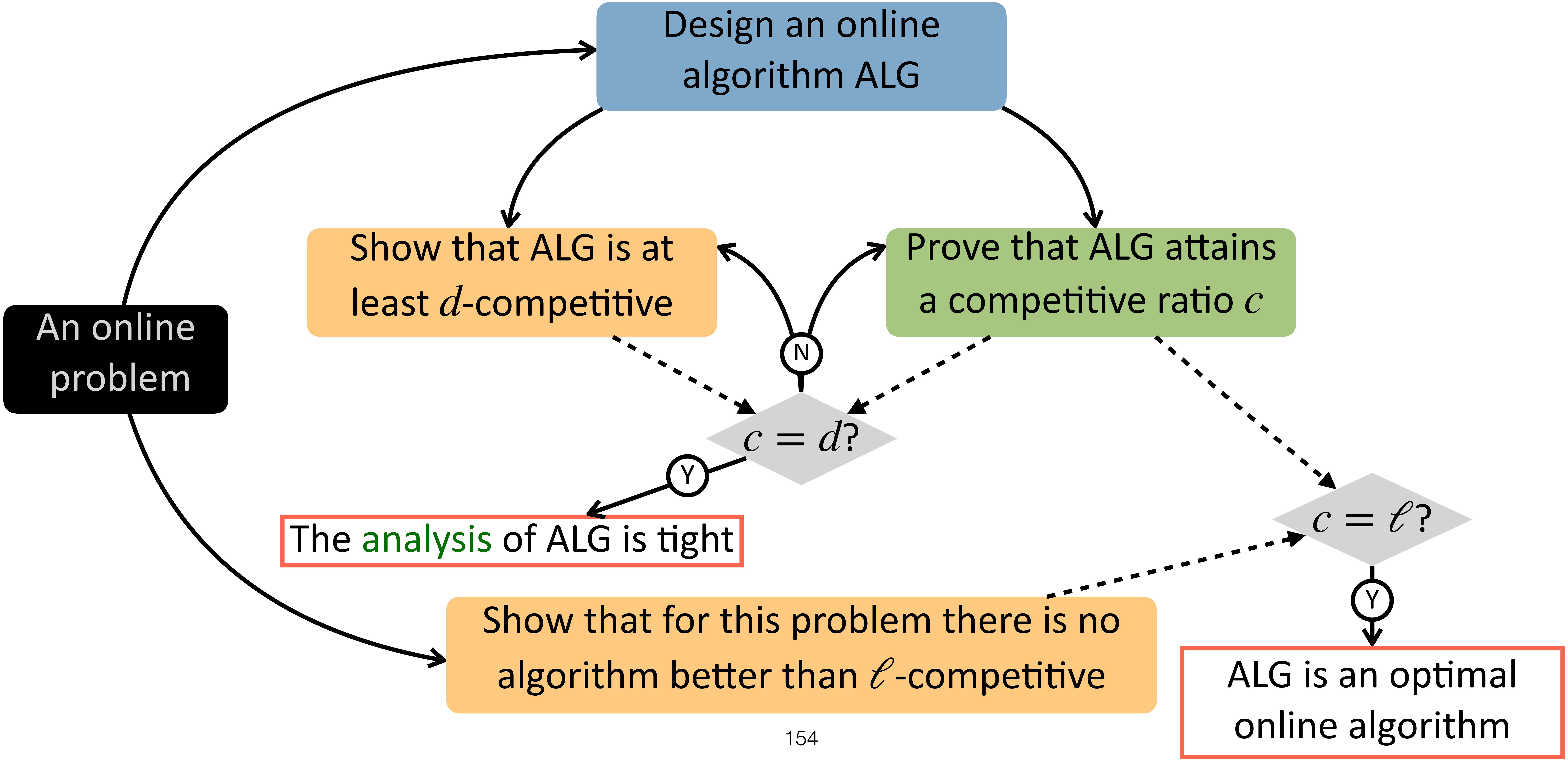

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