Algorithms for Decision Support

Online Algorithms (3/3)

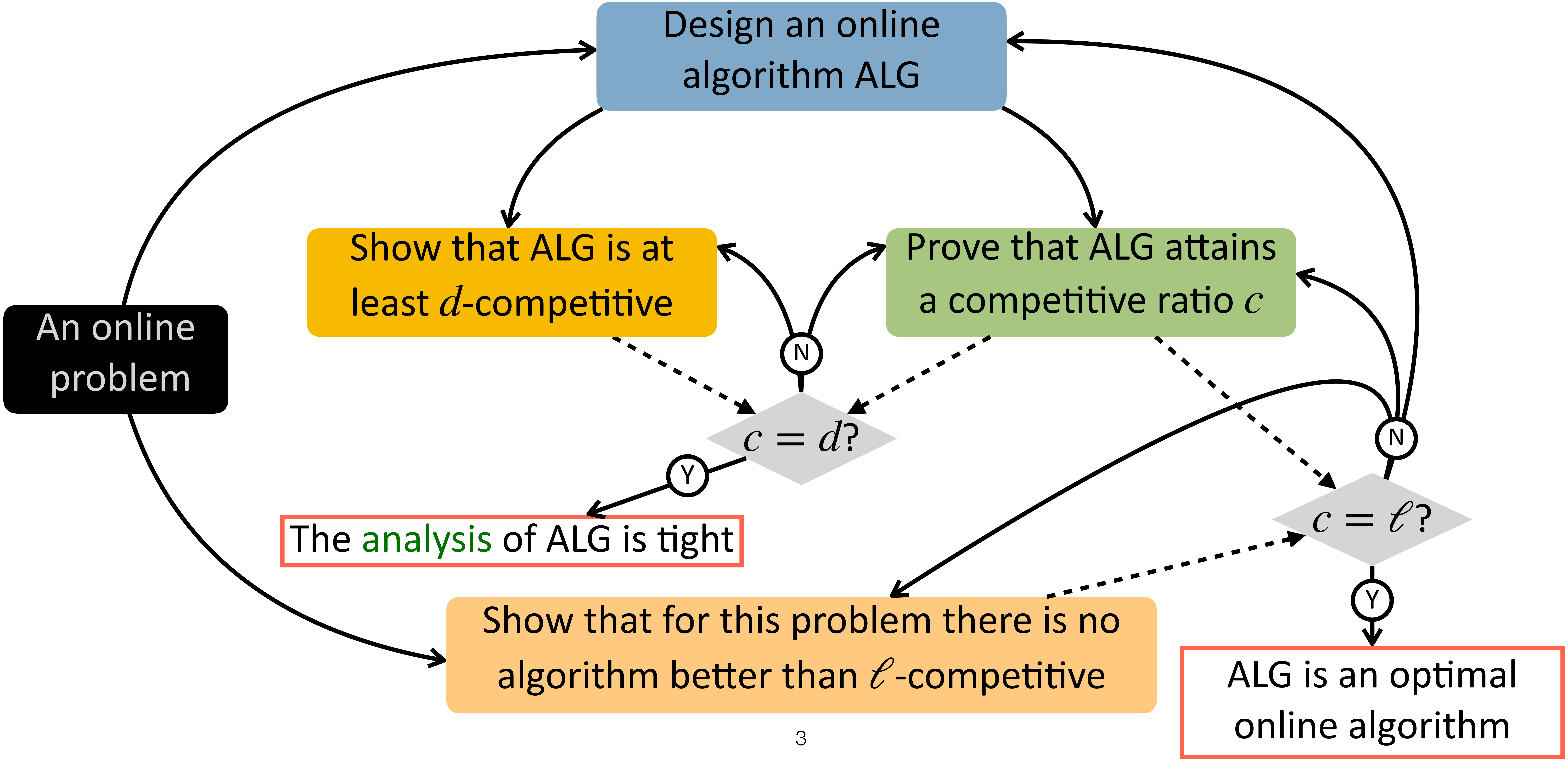
Bin Packing and Paging

Outline

- **Bin Packing** problem
    - Assume that we know the **ALG** cost

- **Paging** problem
    - We know very little about the **ALG** or the **OPT**

Research cycle of online algorithms



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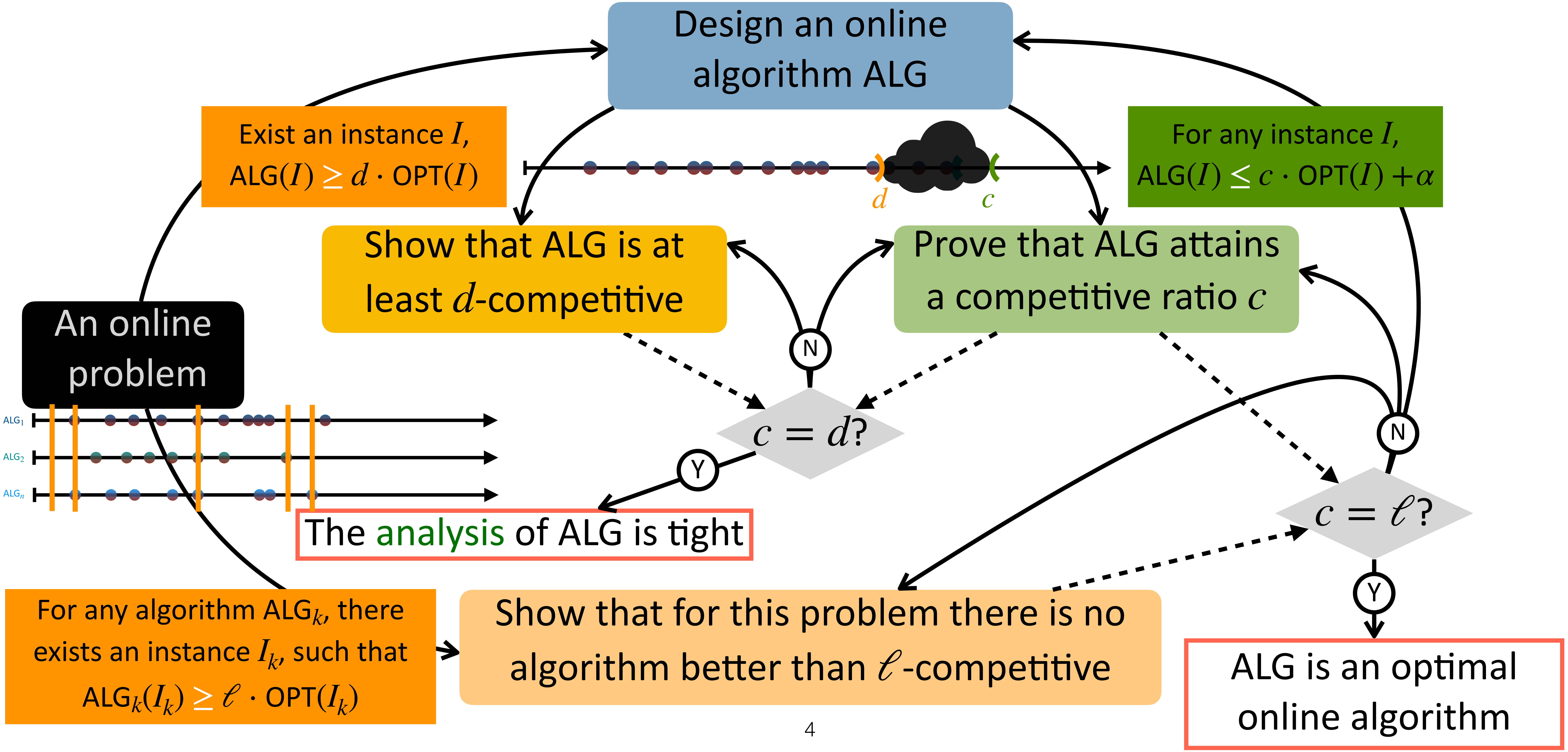

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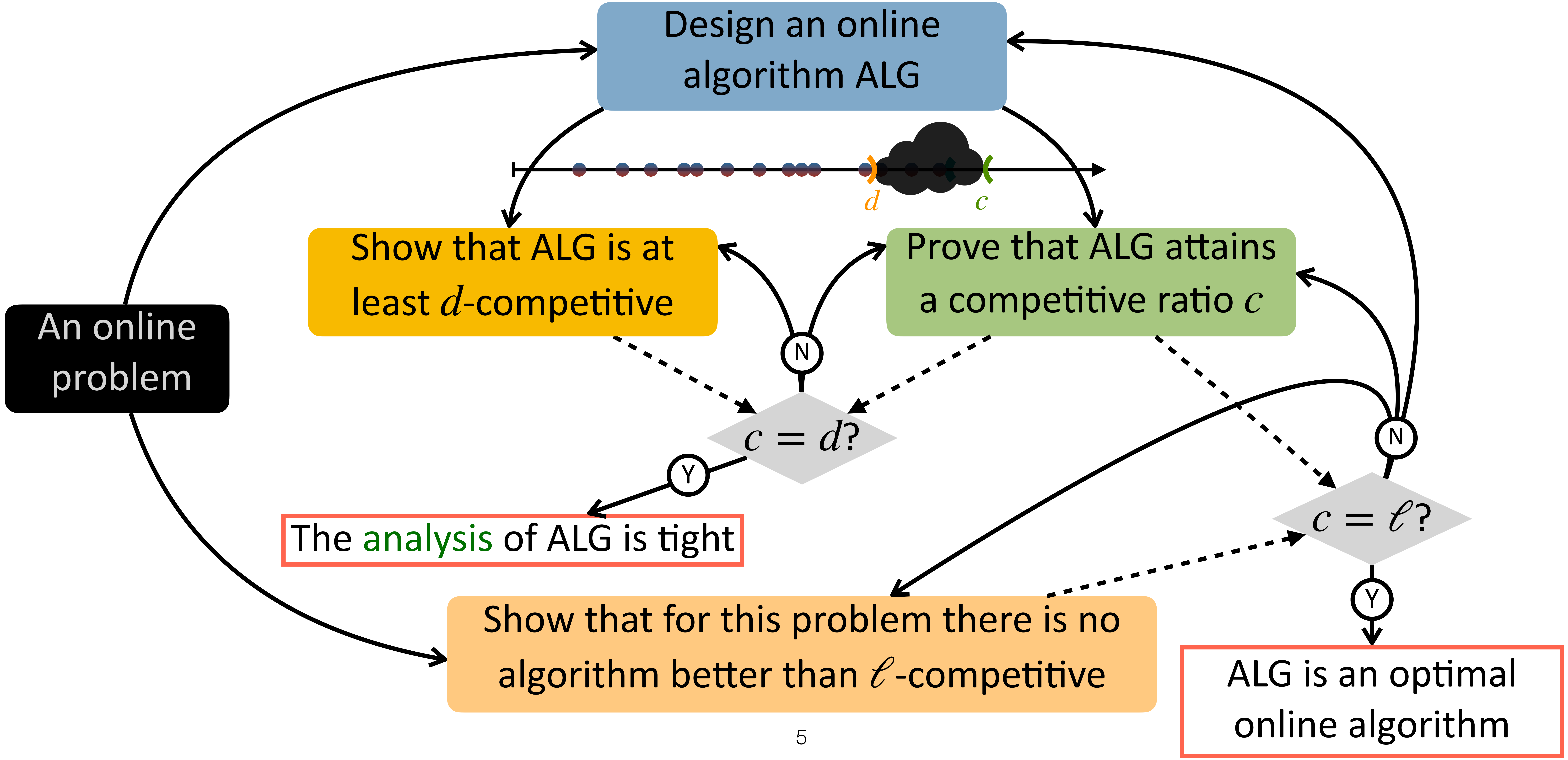

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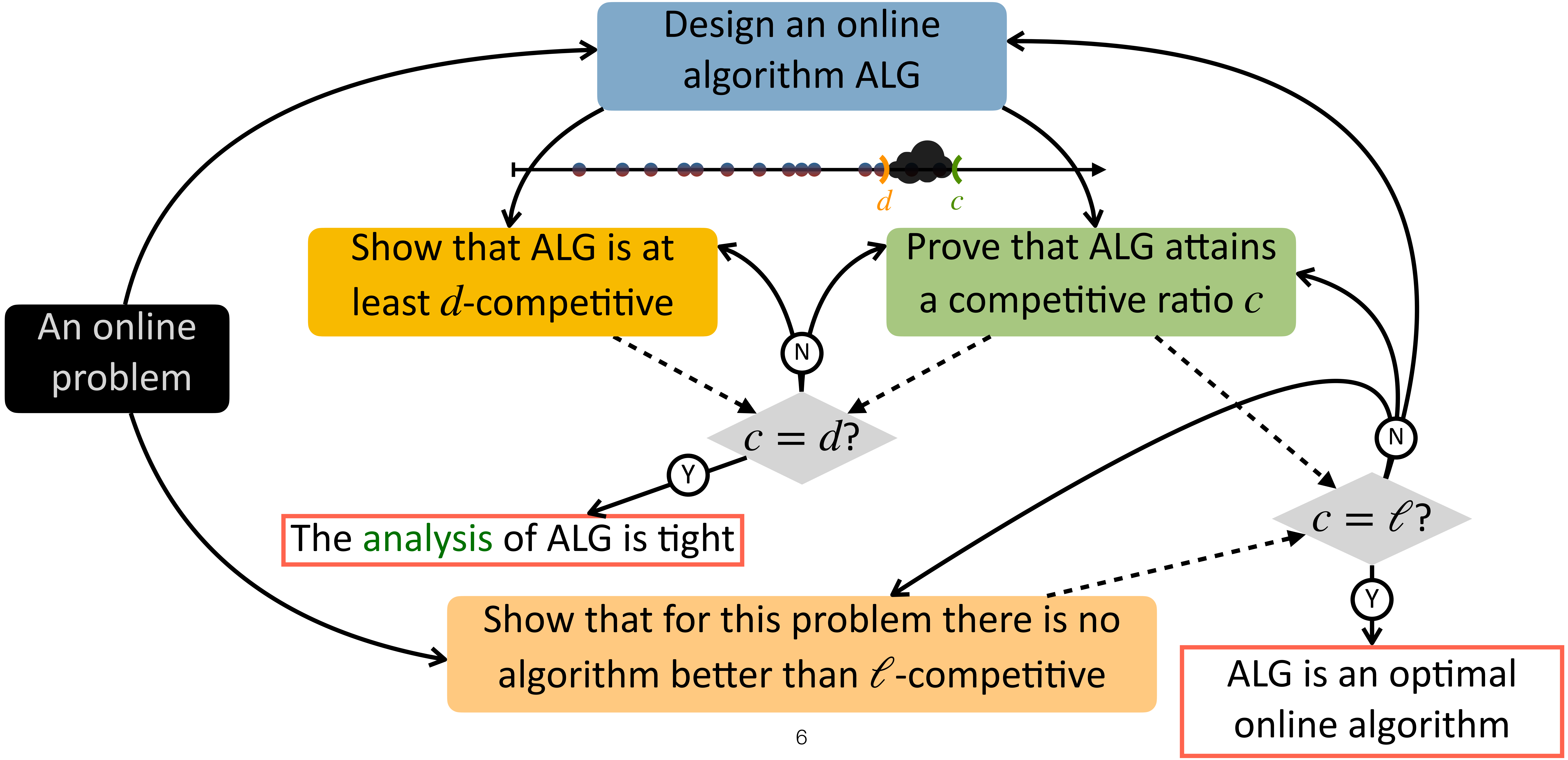
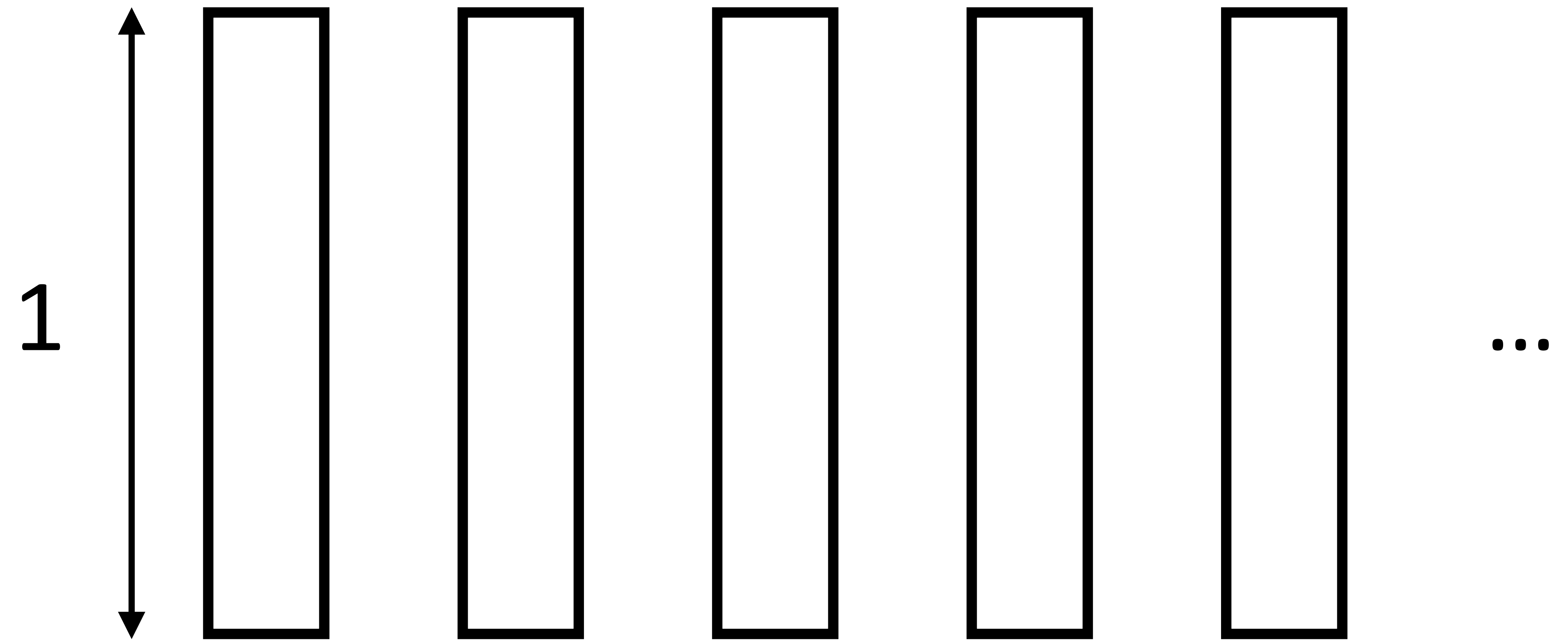


Online Bin Packing Problem

- There are infinite number of **capacity-1** ***bins***



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- There are infinite number of capacity-1 *bins*
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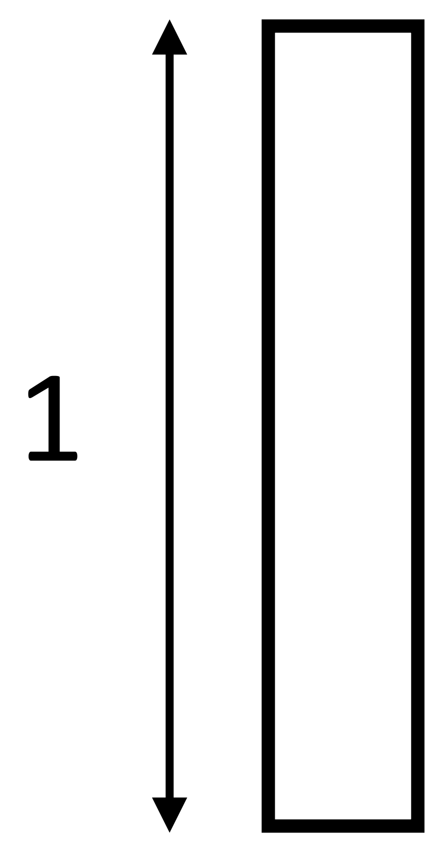

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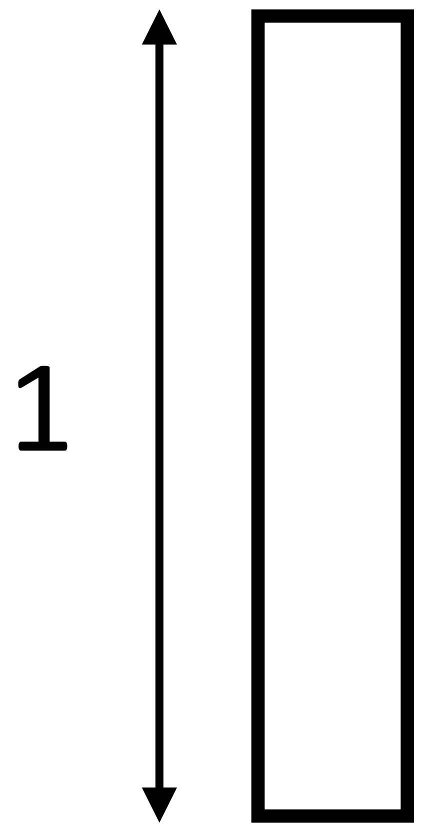

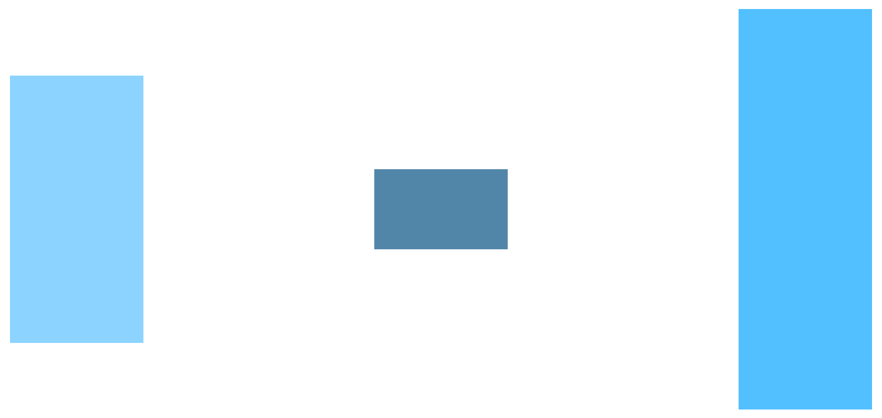

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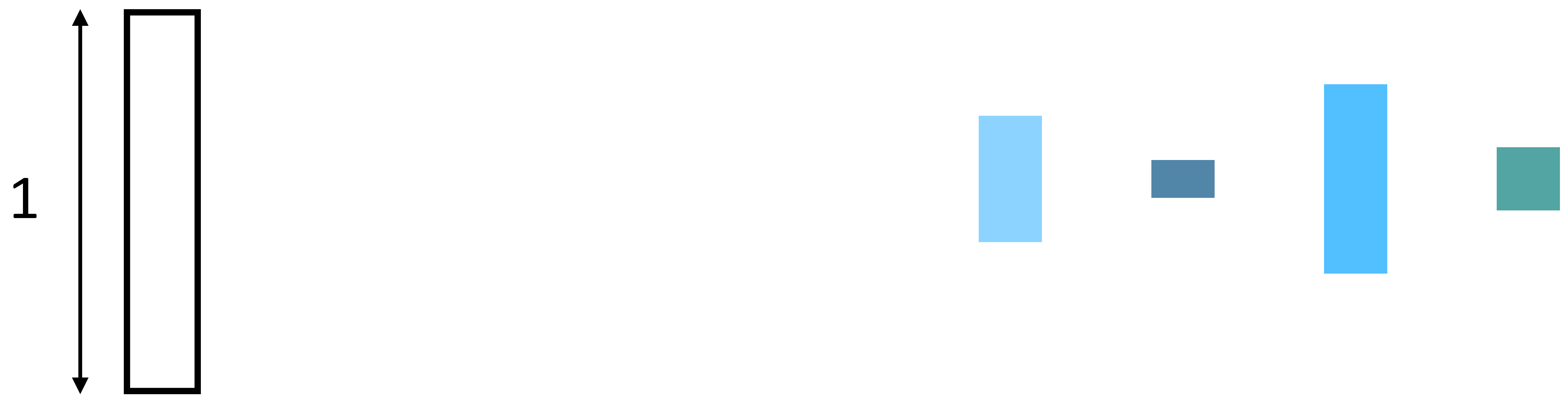

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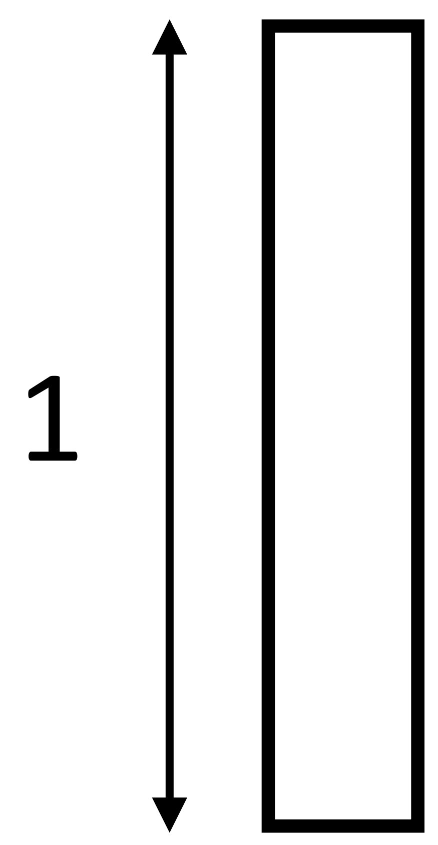

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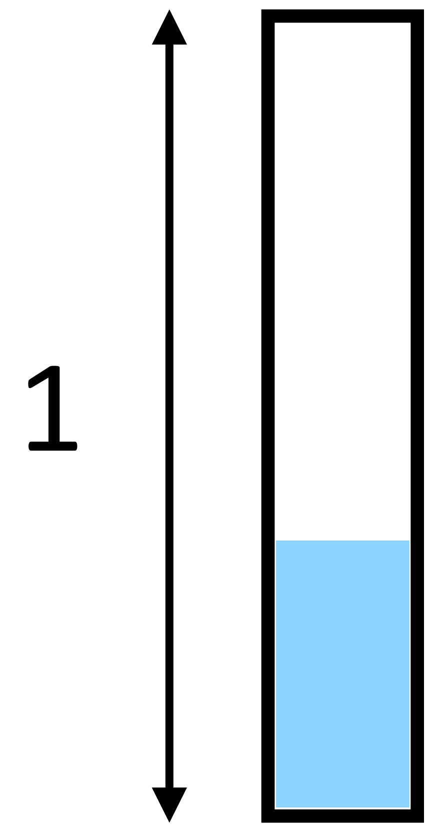

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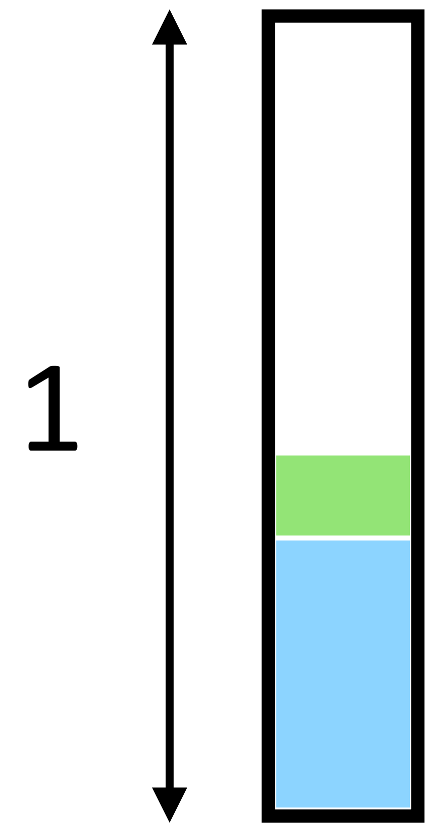

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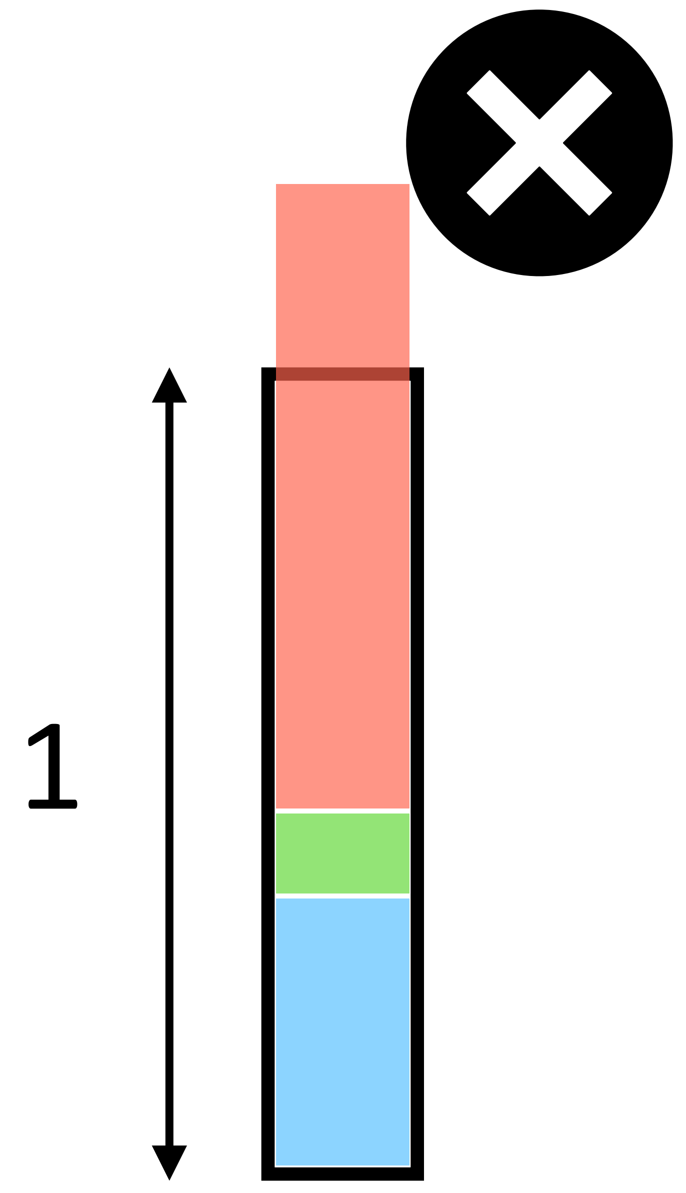

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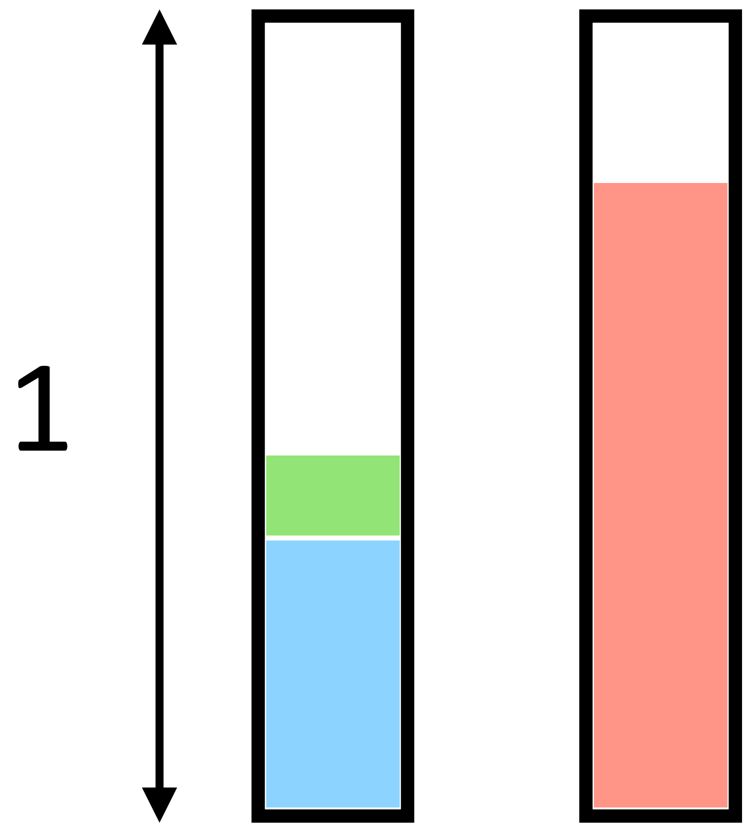

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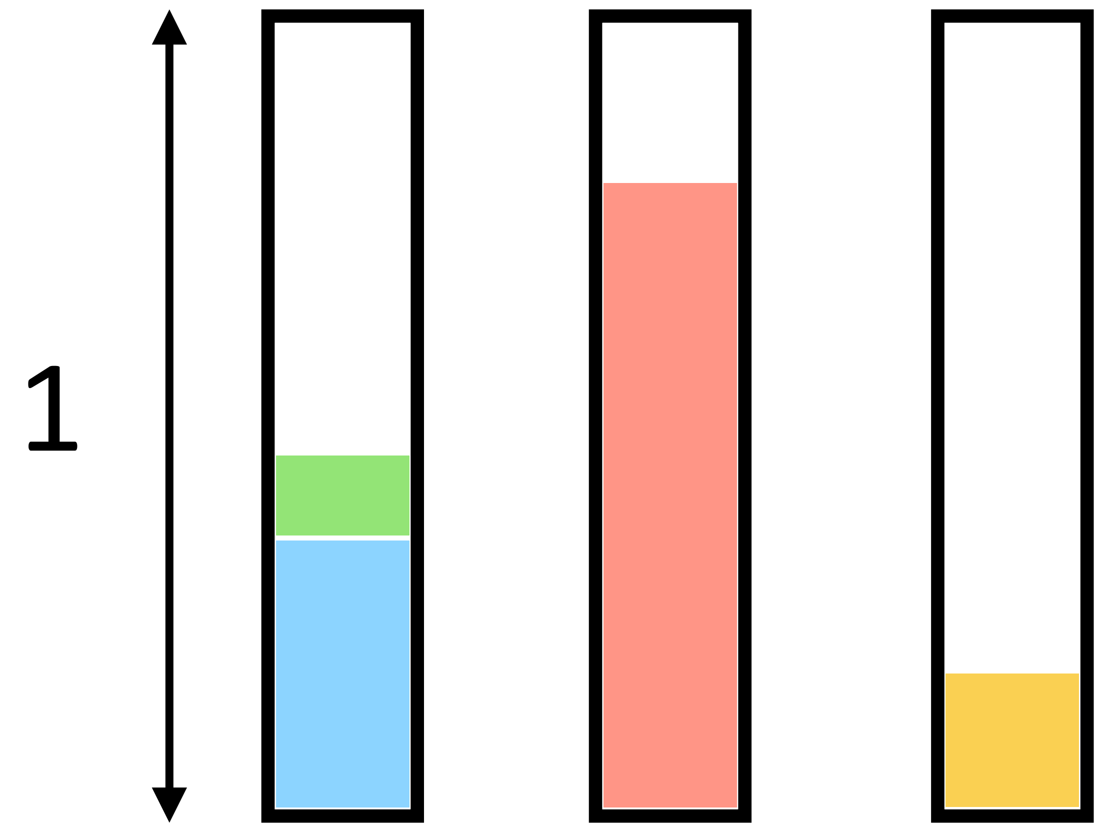

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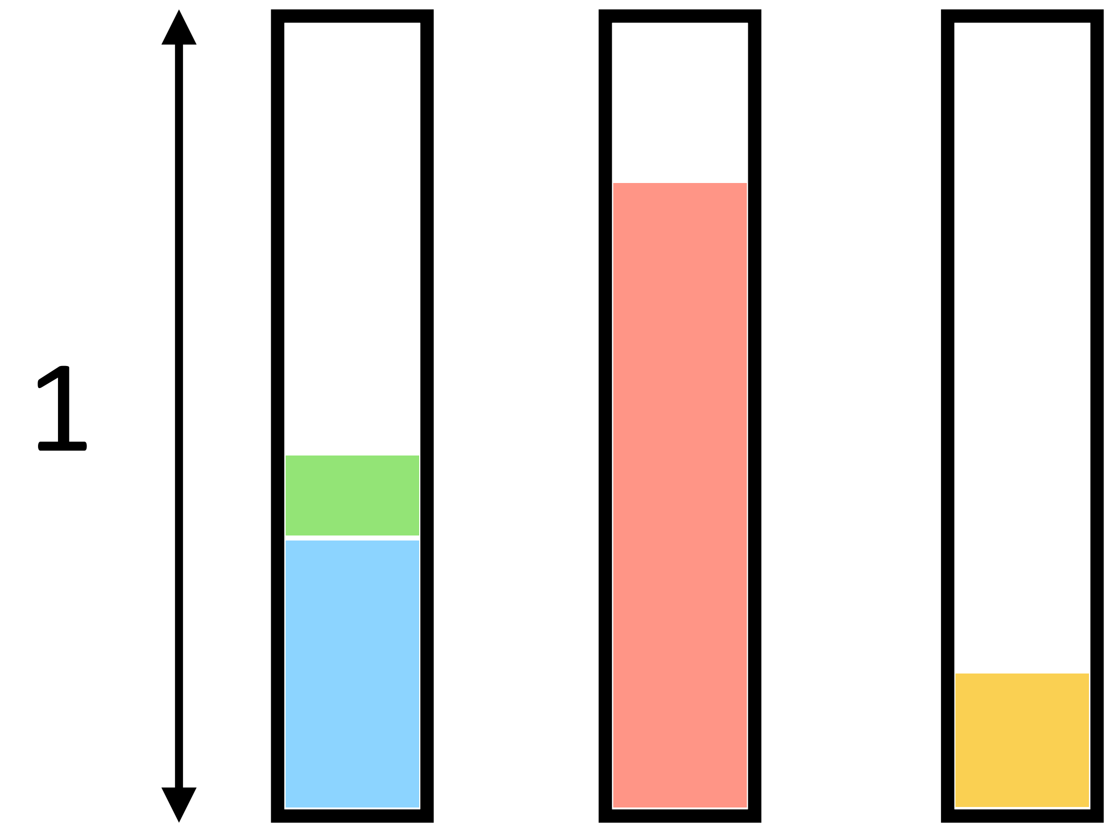
- The objective is to **put all items in a minimum number of bins**



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- **Bin Packing** problem
  - Assume that we know the **ALG** cost

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FirstFit Algorithm

FirstFit:

Once an item arrives:

- Scan through all (opened) bins and put it into the first bin that can accommodate it
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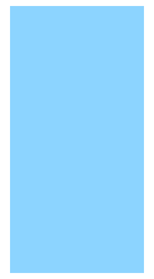

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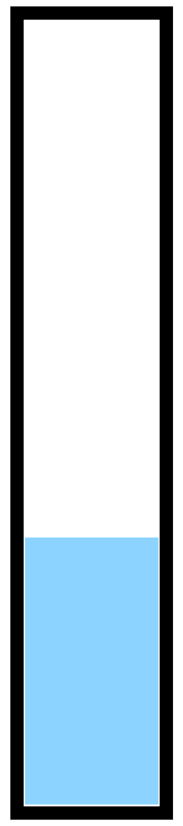

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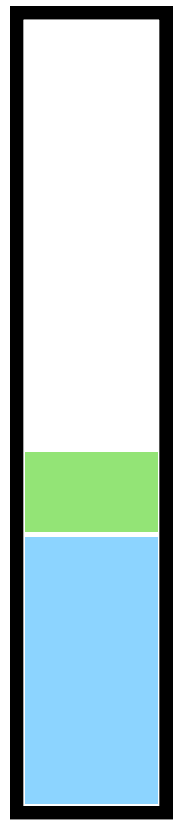

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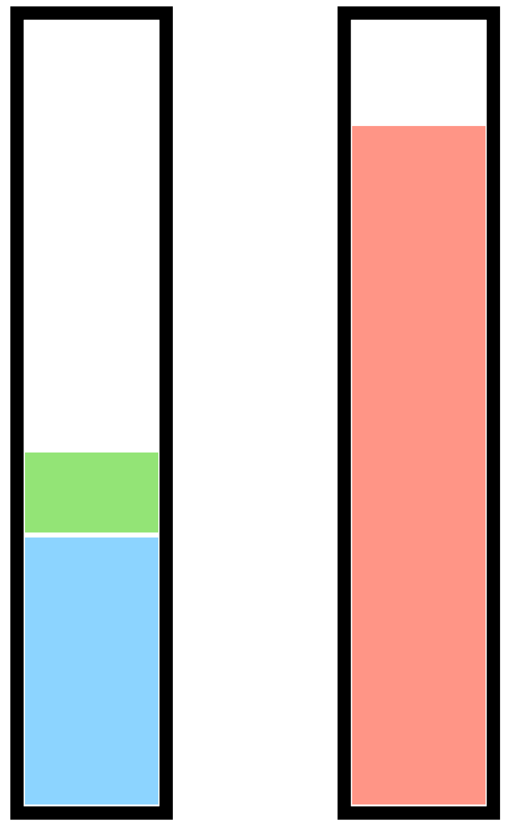

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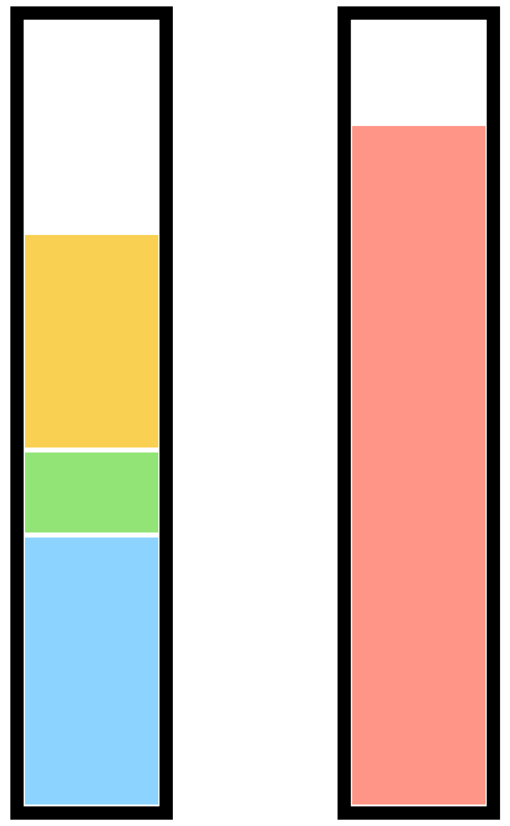

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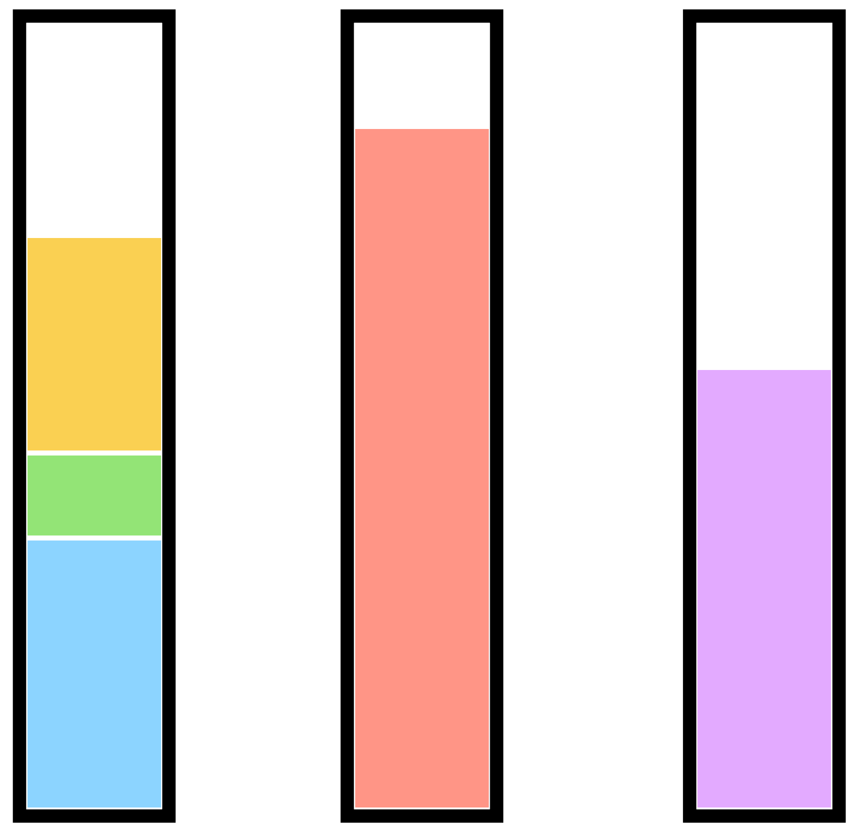

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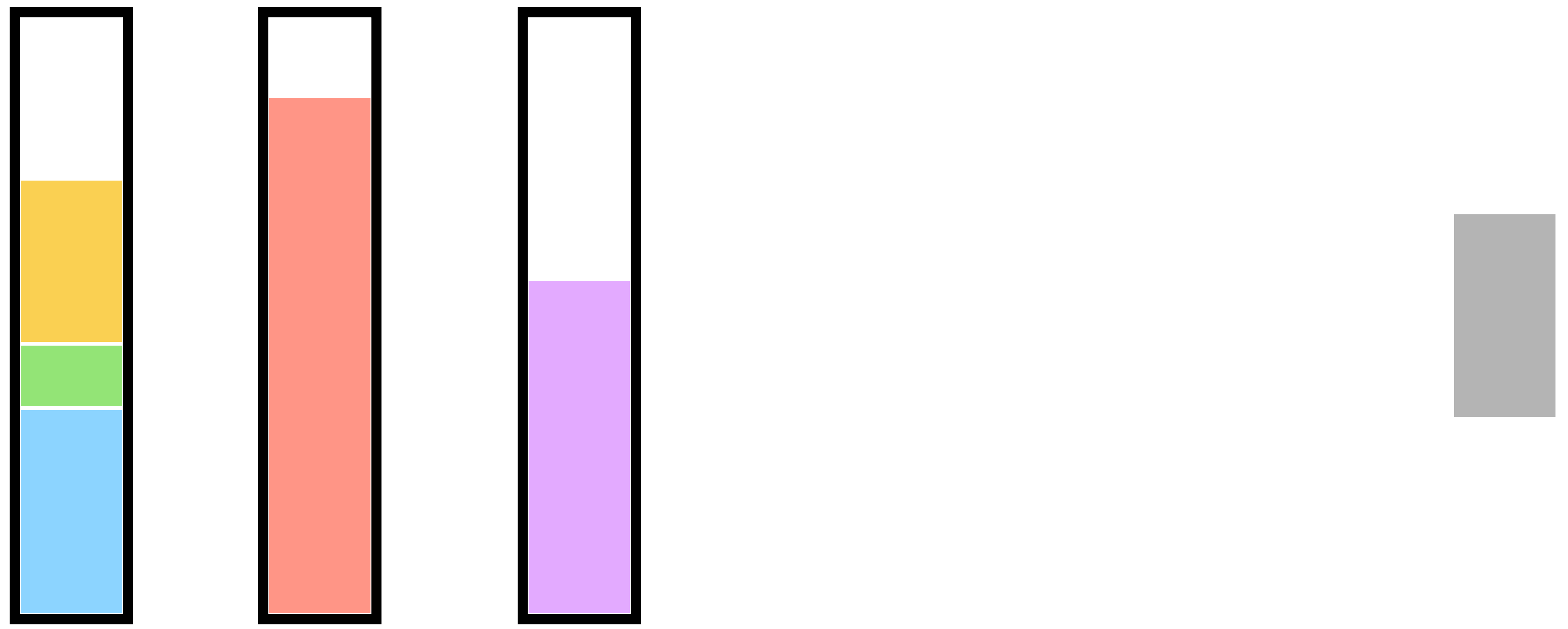

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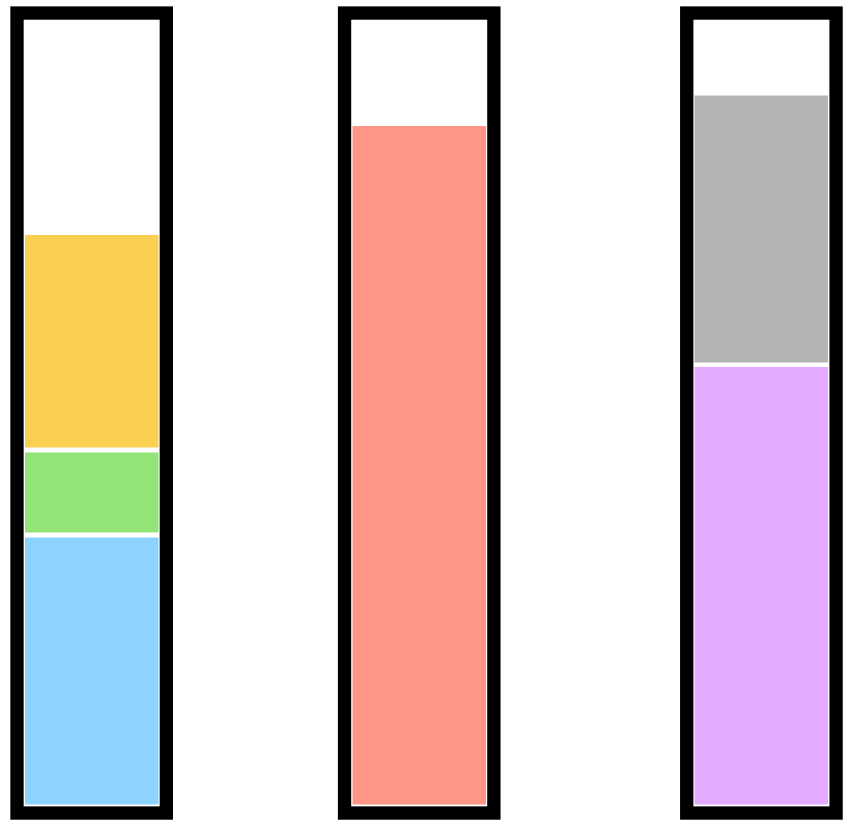

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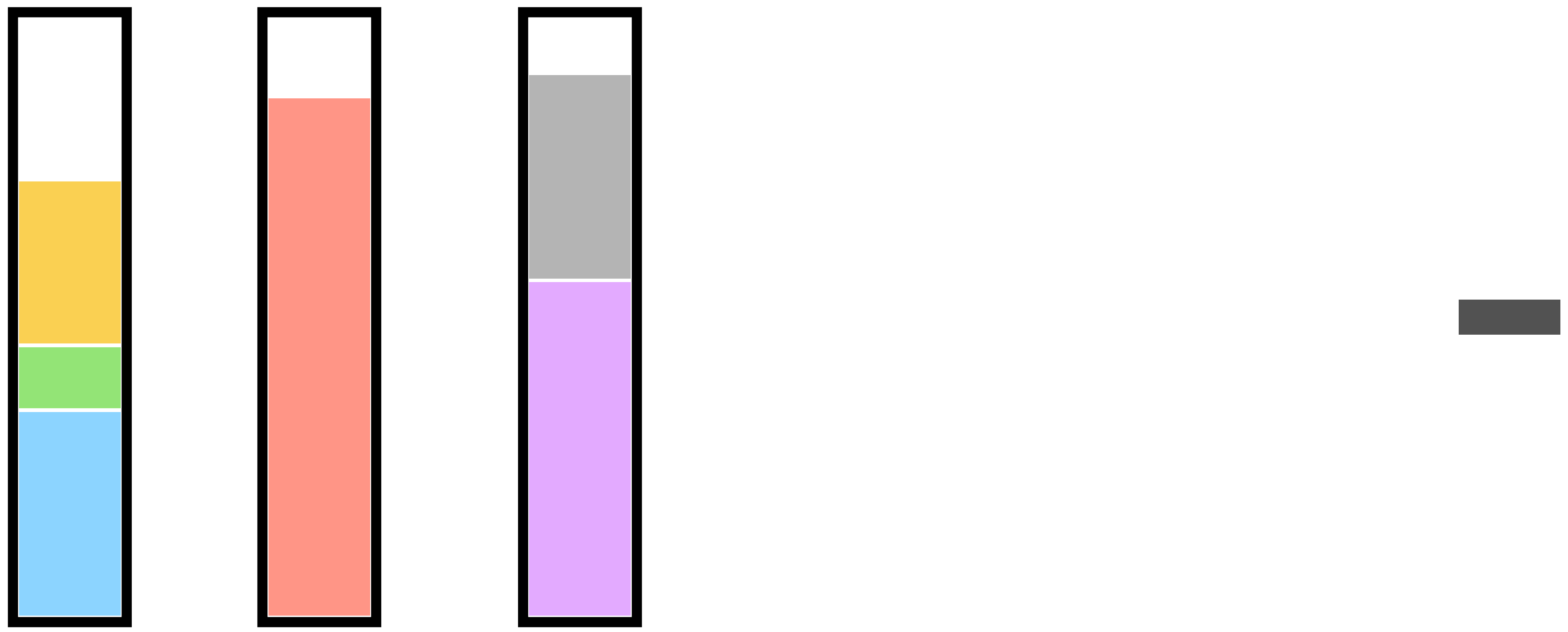

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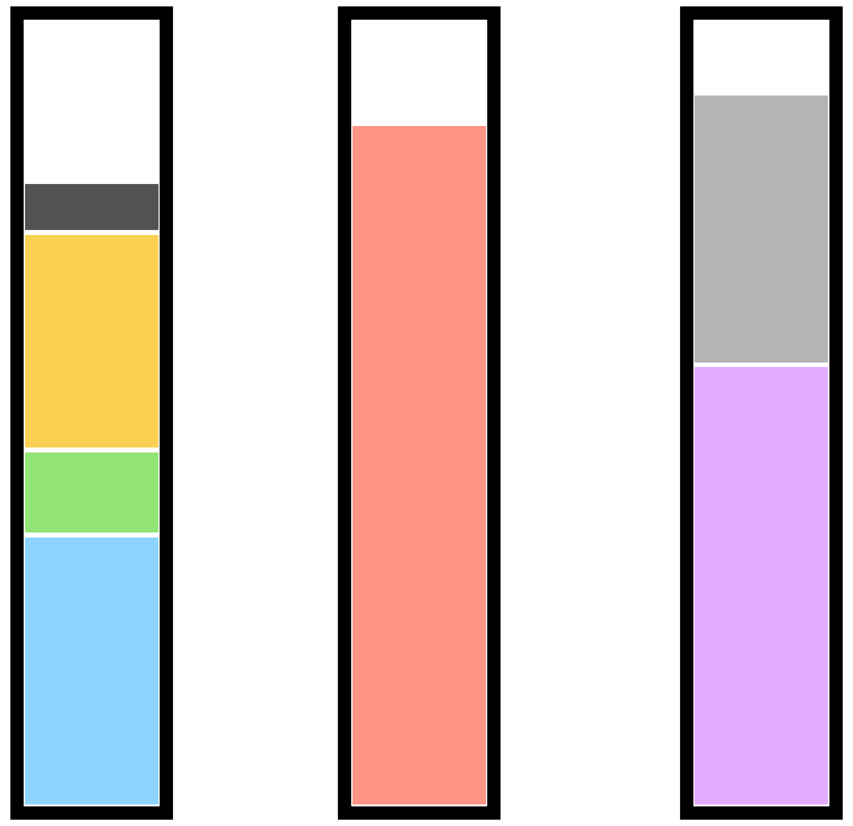

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$\text{FirstFit} \leq 2 \cdot \text{OPT} + 1$

<Proof idea>

- Observation: There is at most one bin at least *half empty*

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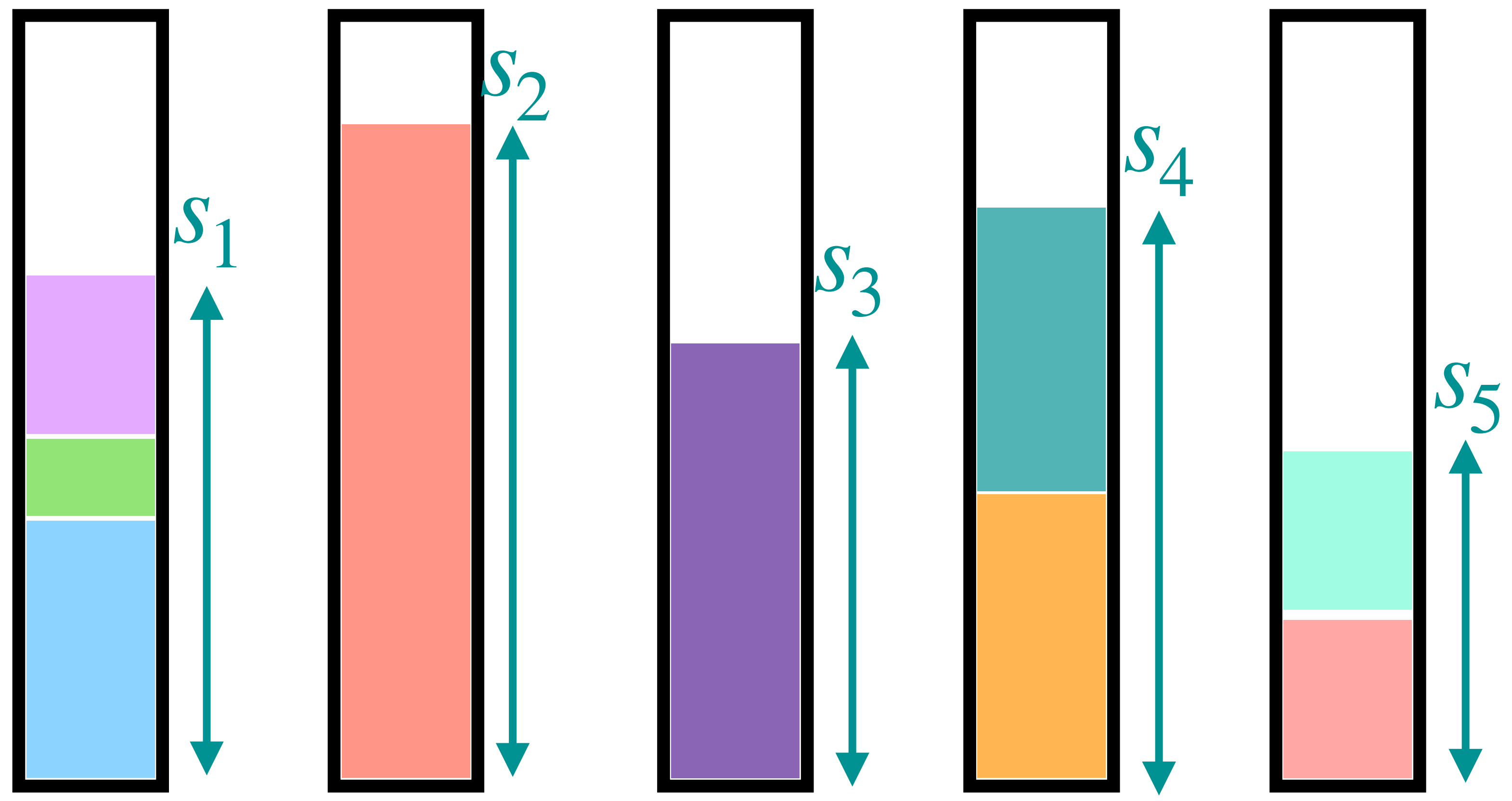

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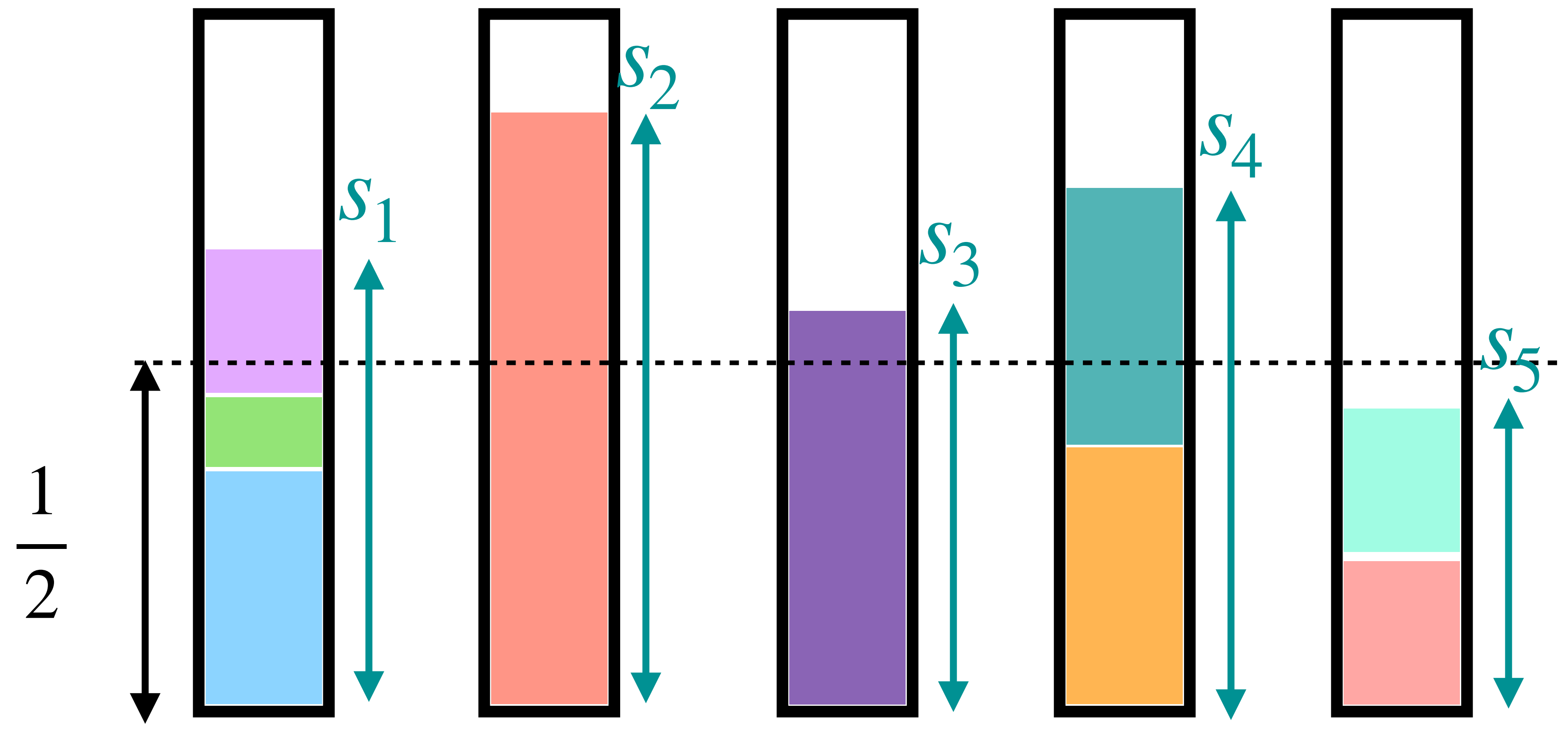

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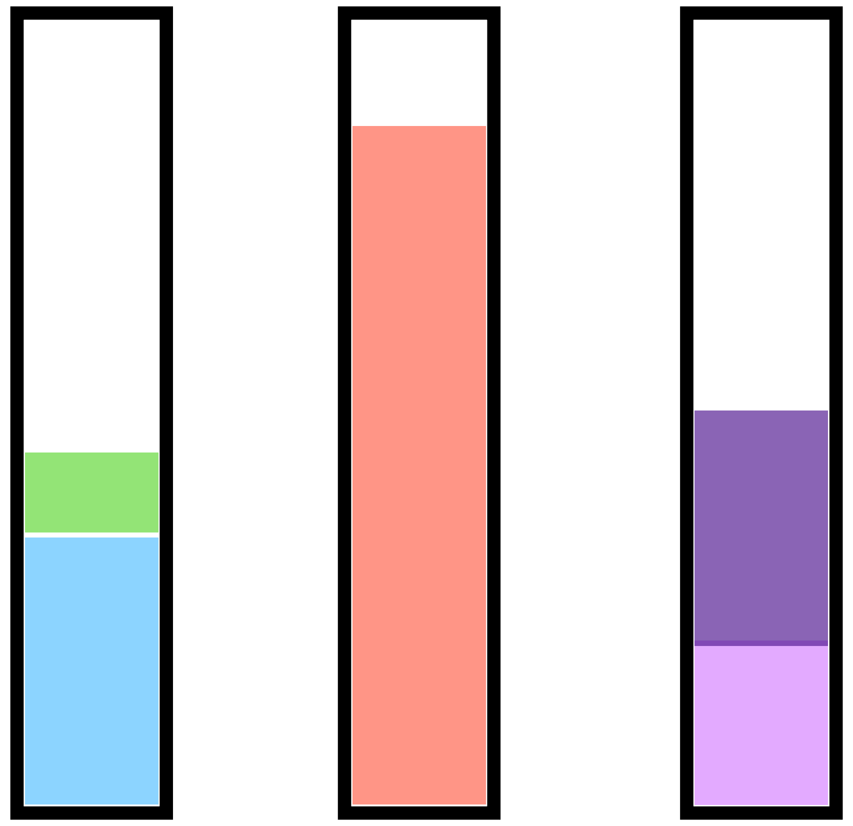

Prove by contradiction: Assume that there are two bins with size less than $1/2$

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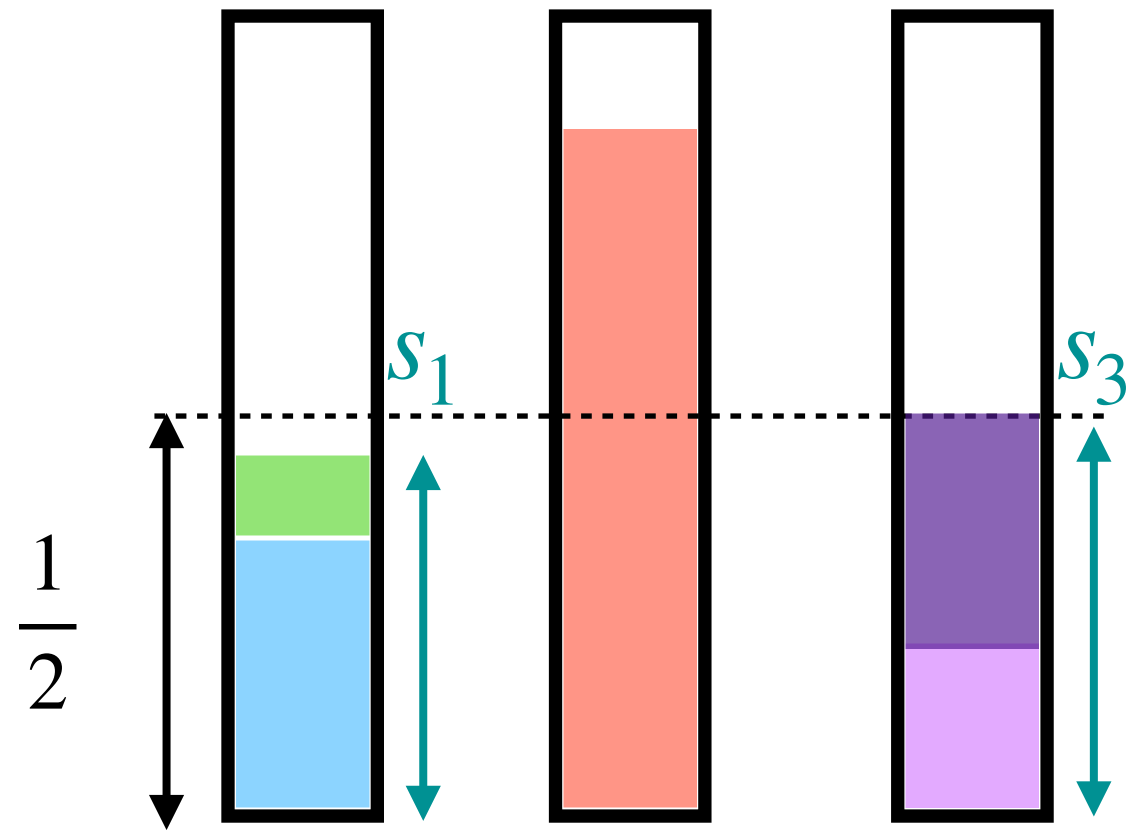

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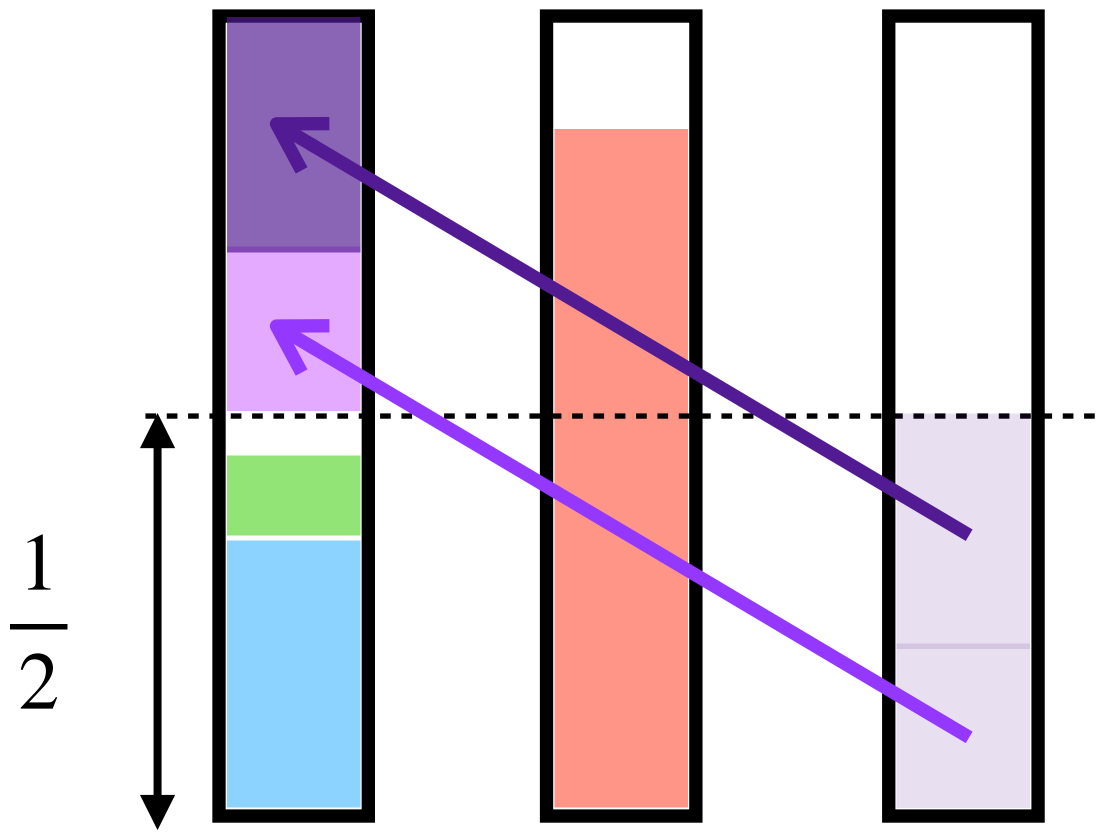

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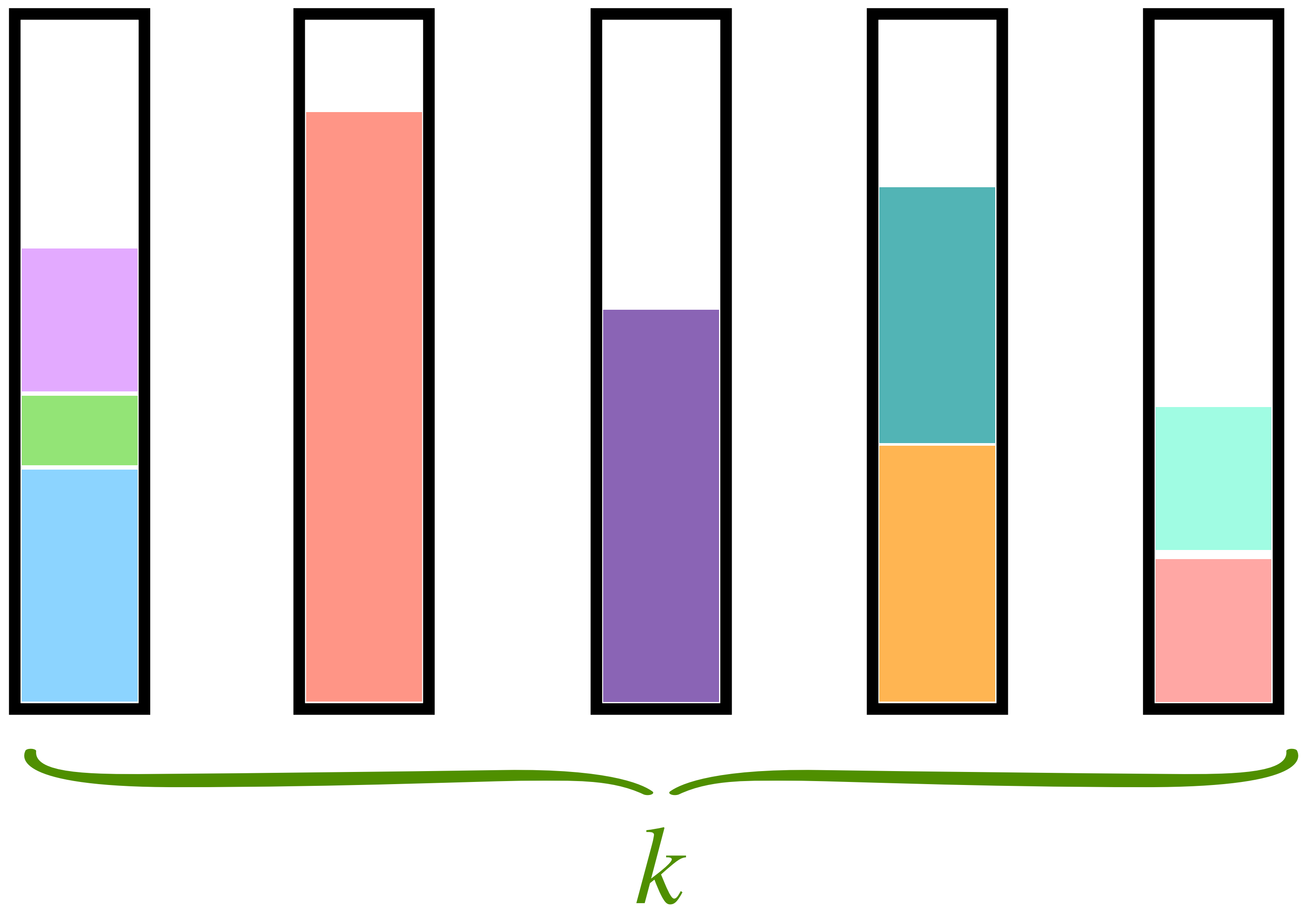

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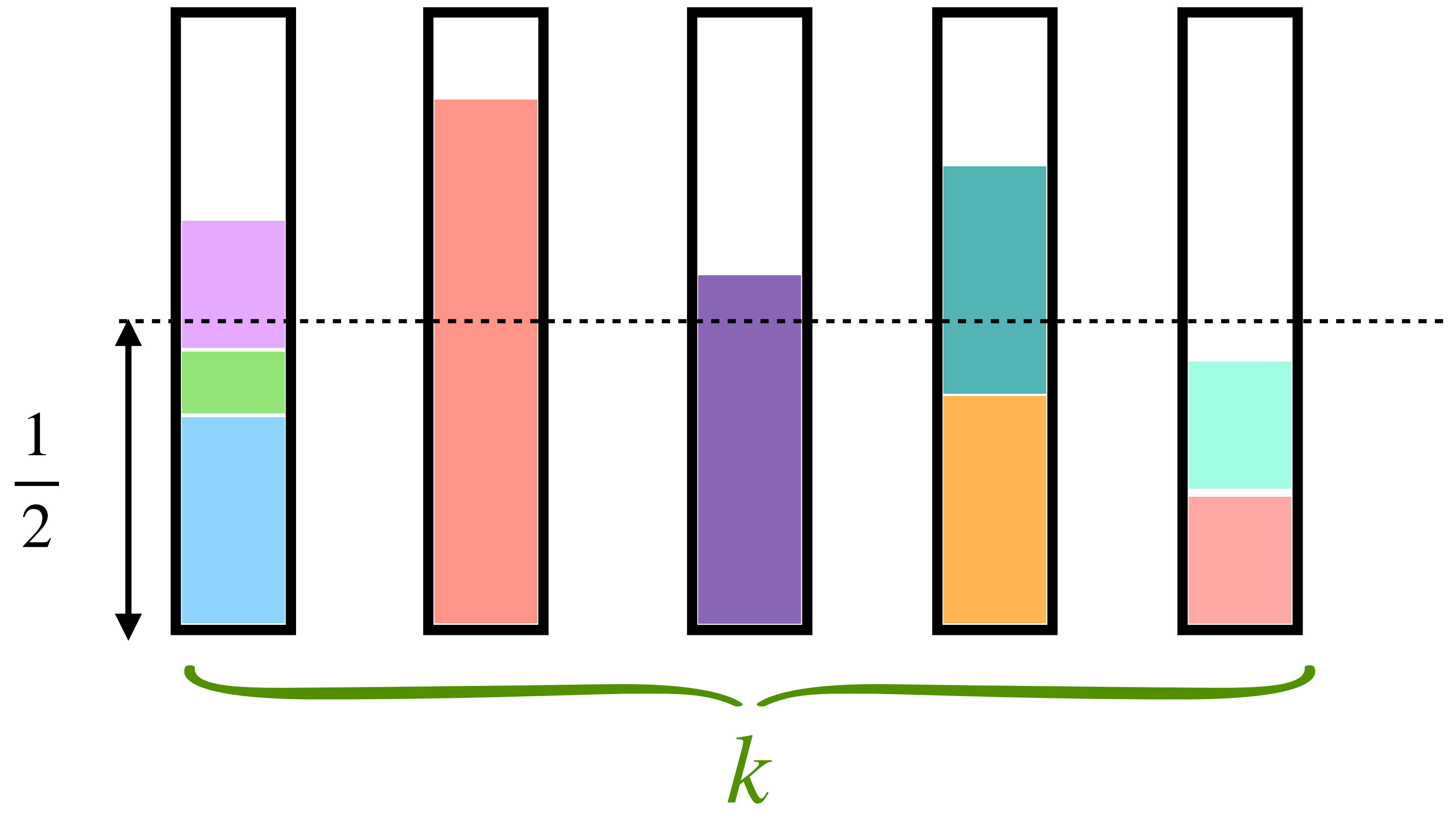

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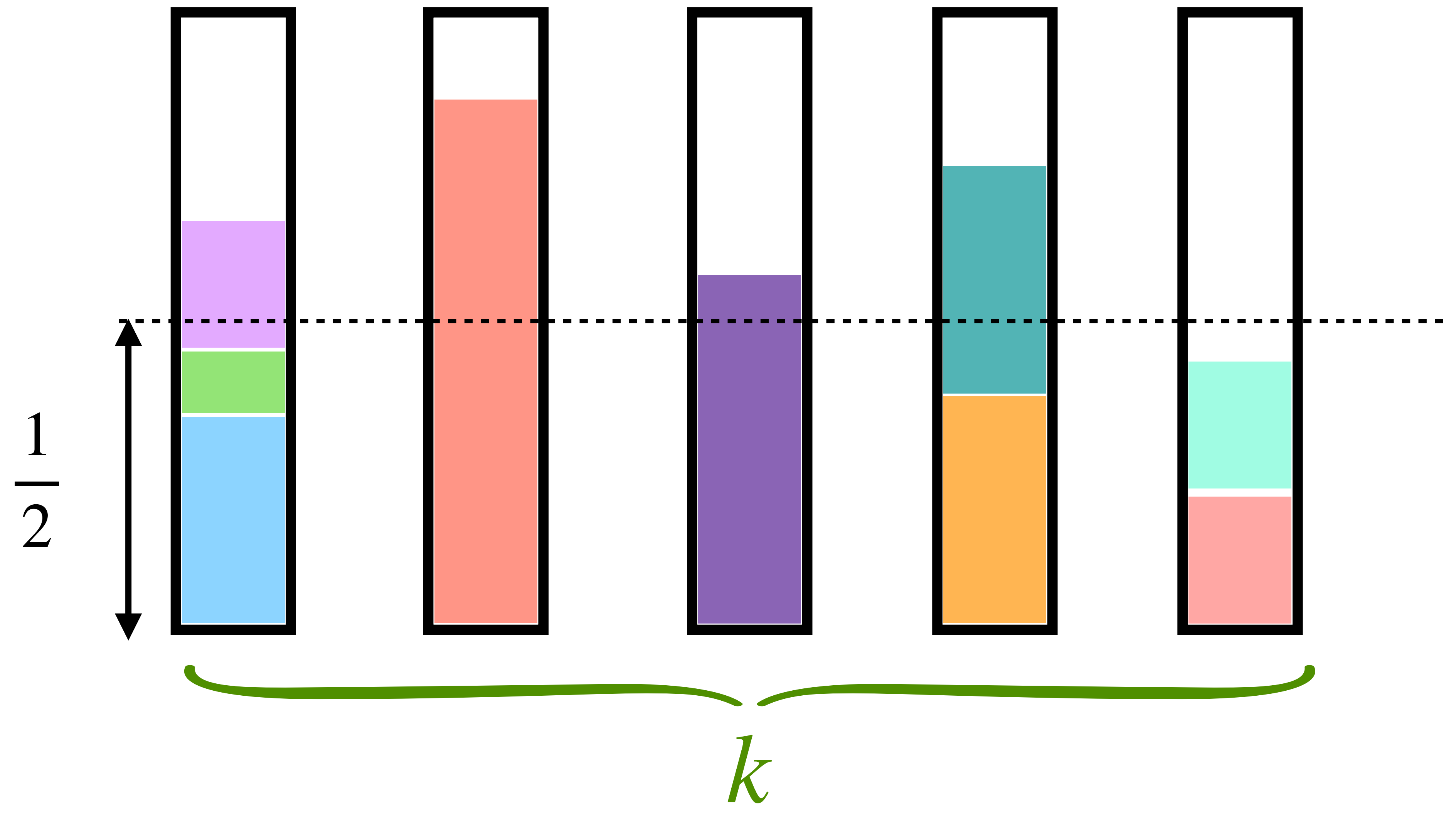

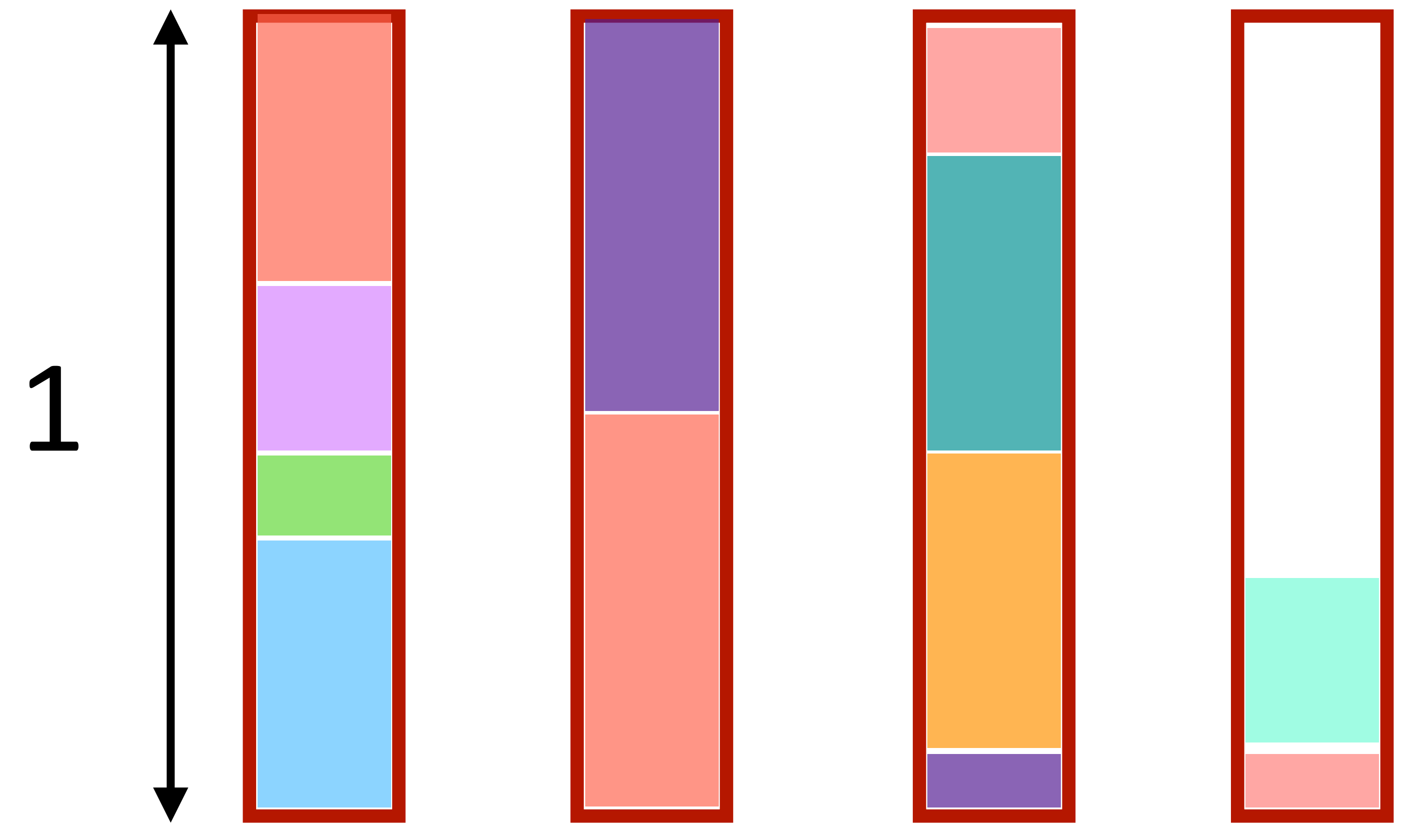

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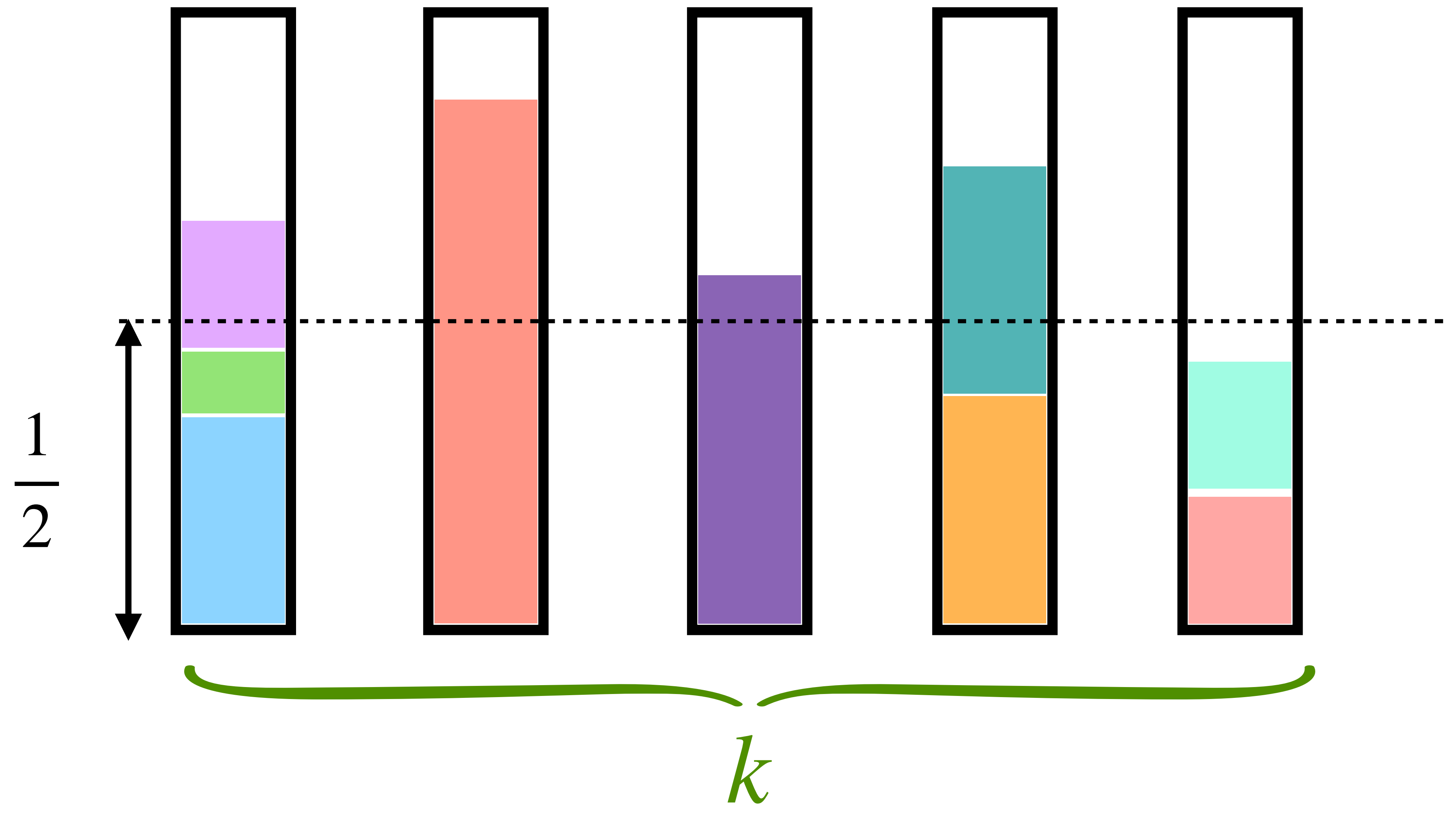

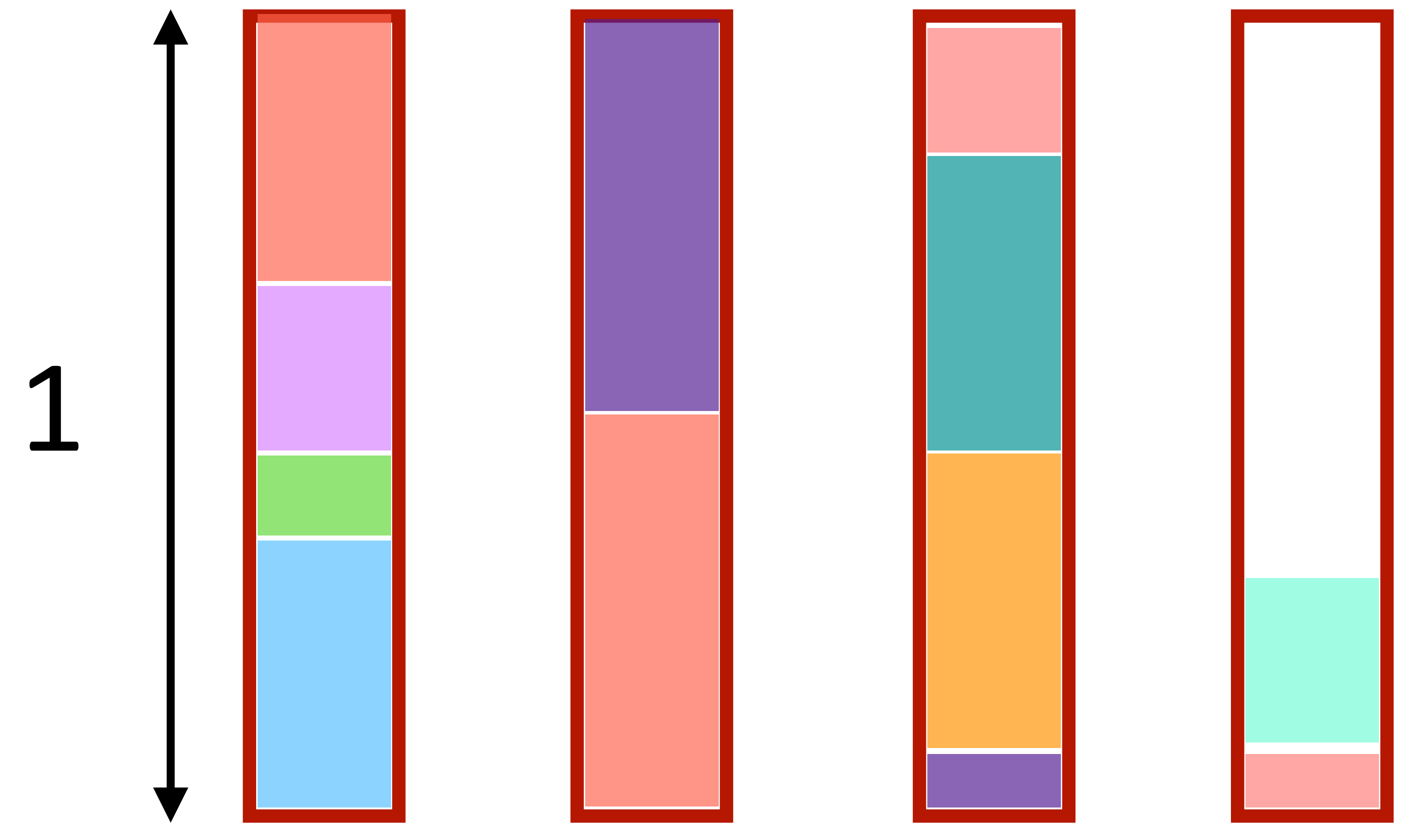

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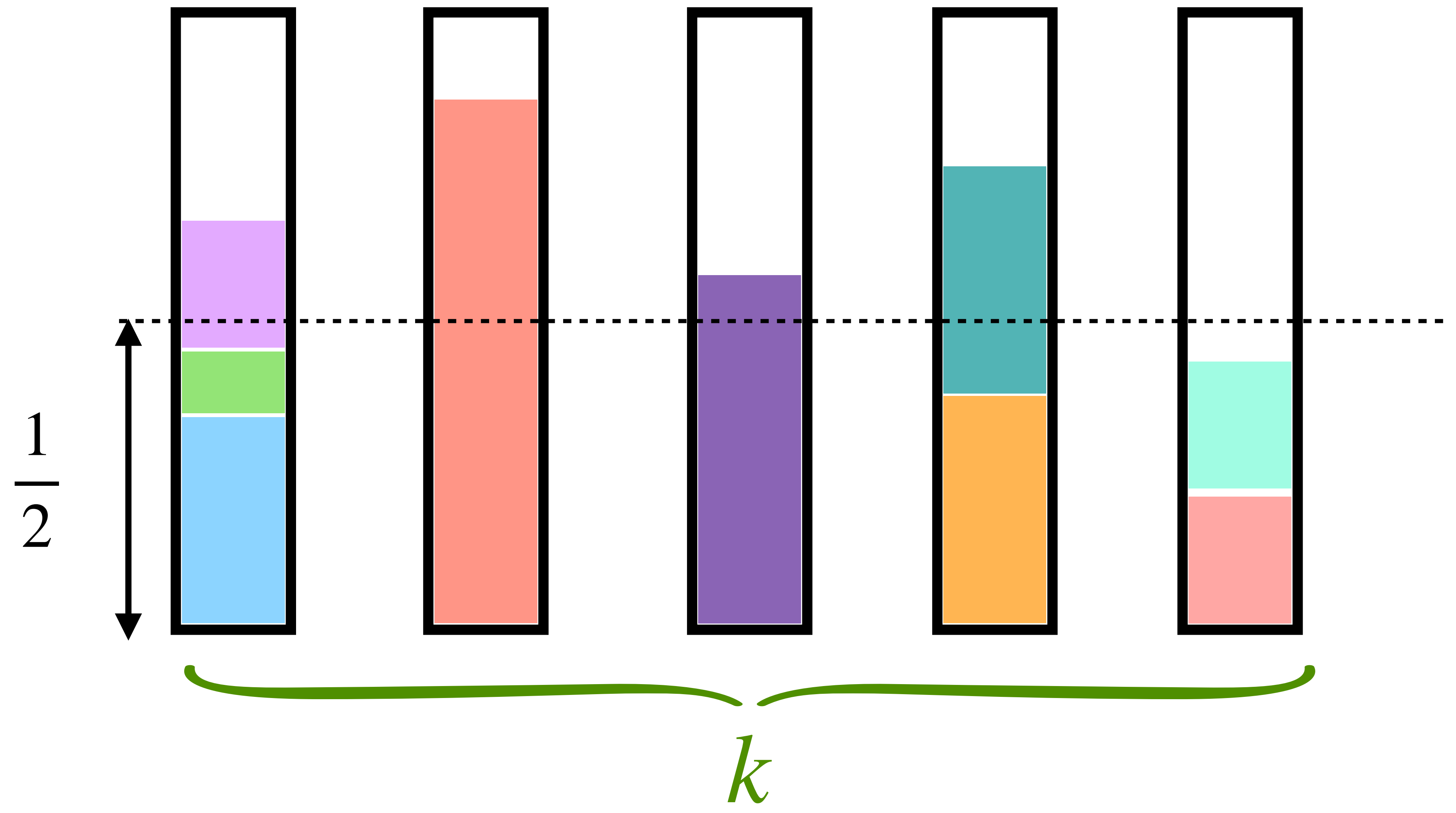

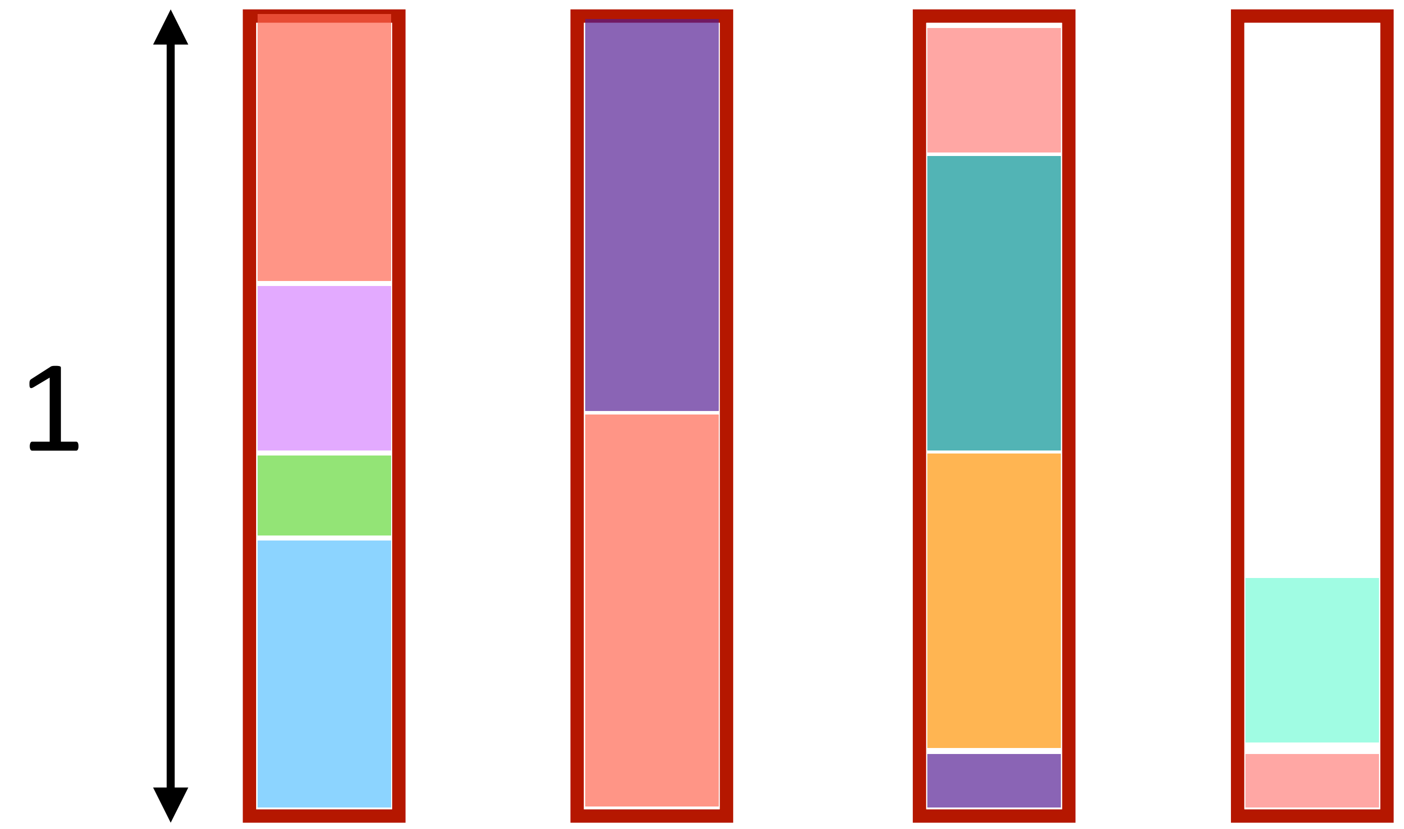

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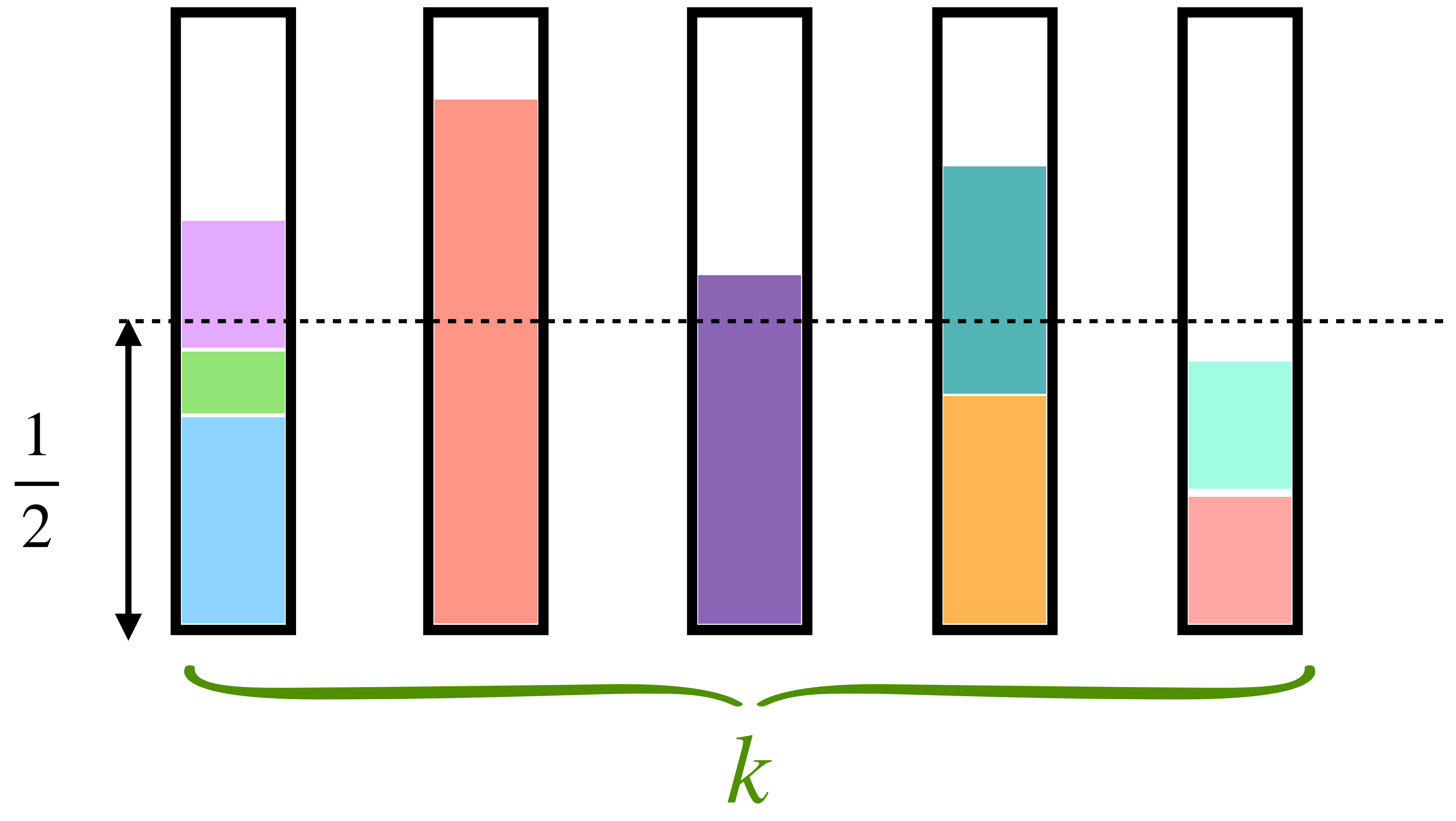

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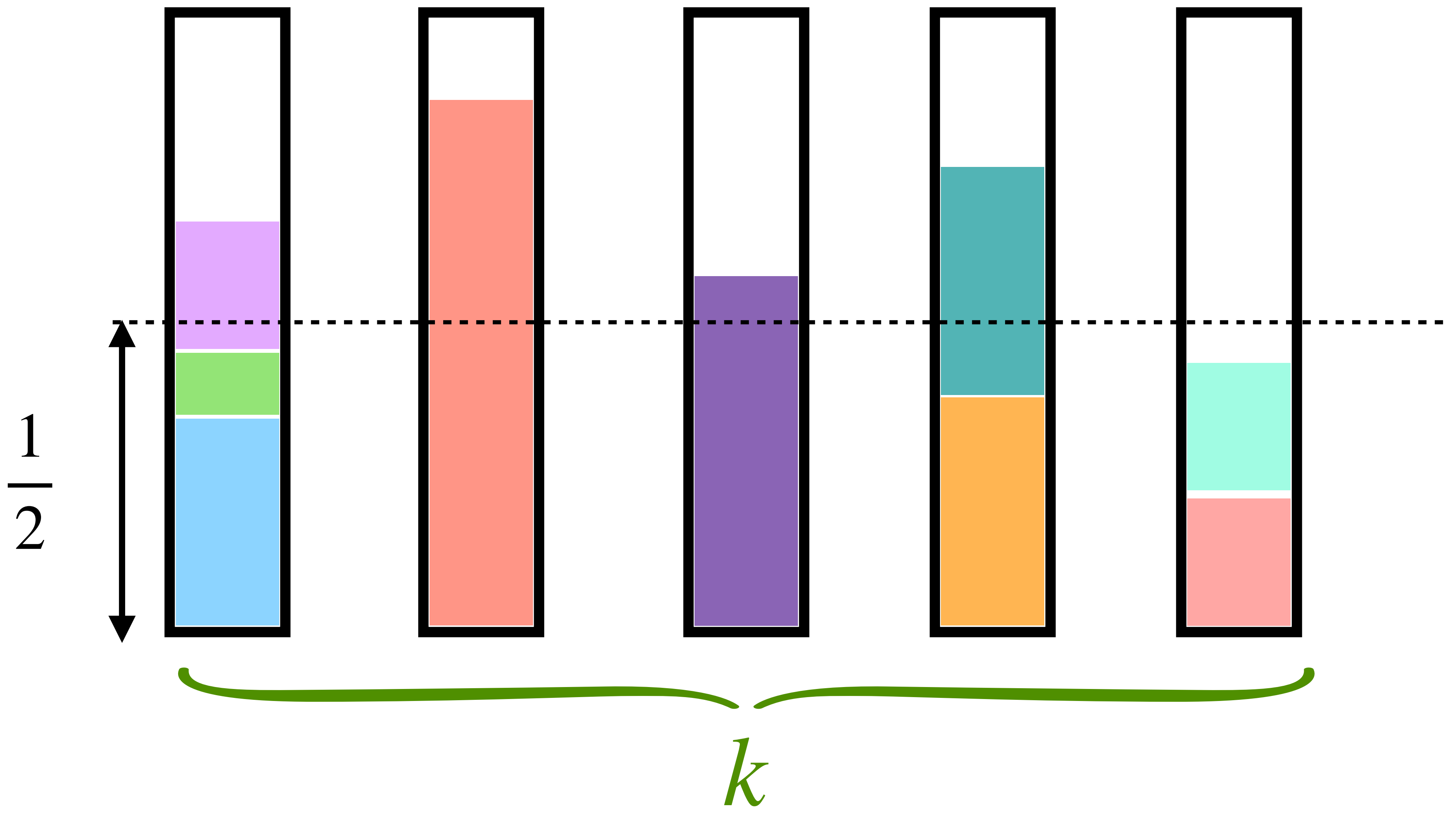

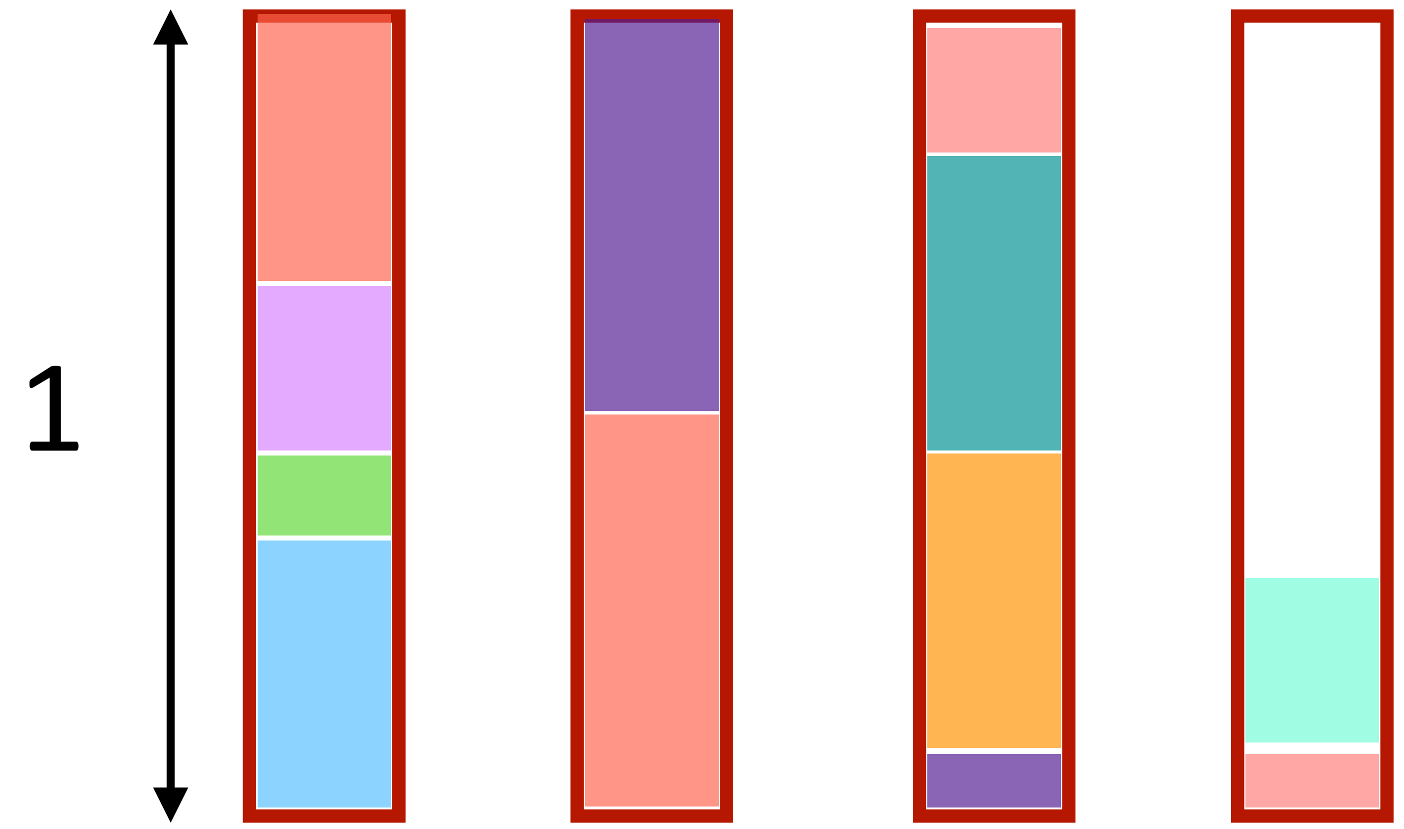

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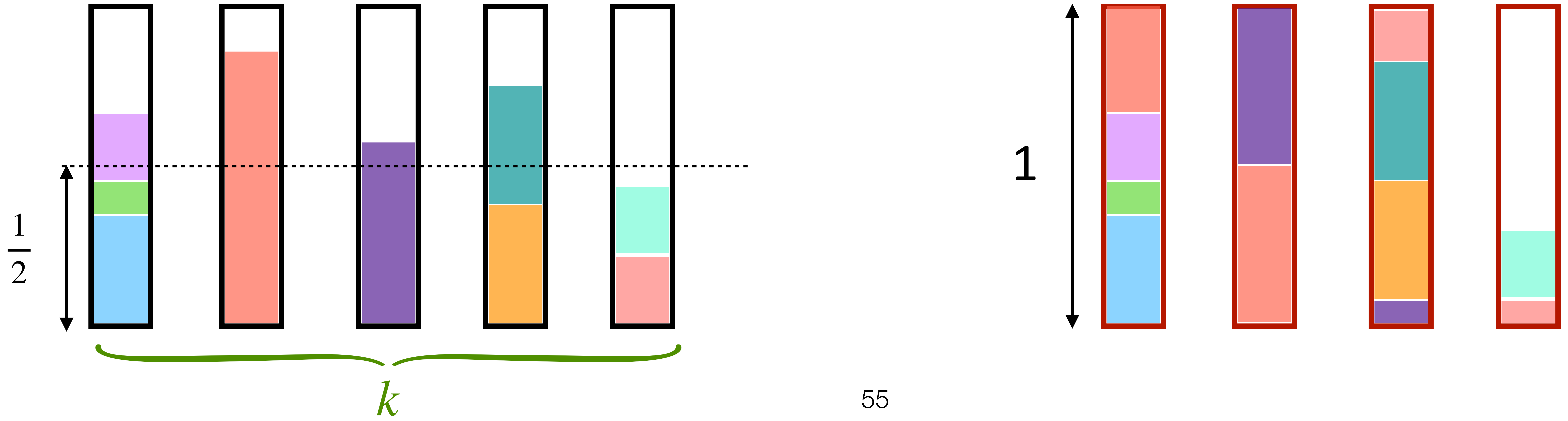
FirstFit $\leq 2 \cdot \text{OPT} + 1$

<Proof idea>

- Observation: There is at most one bin at least *half empty* (that is, let s_j be the total size of the items in bin B_j. There is at most one bin B_j such that $s_j \leq 1/2$)
  - Let k be the number of bins opened by **FirstFit**. Total size of all items $\geq (k-1) \cdot \frac{1}{2}$
- OPT $\geq$ (total size of all items)/1 $\geq (k-1)/2 \iff$ **FirstFit** $= k \leq 2 \cdot$ OPT $+1$



$\square$

What Happened

- To analyze the competitive ratio of FirstFit, we assume that it takes k bins

- By the property of FirstFit, at most one bin is half-empty
  - The total size of all items is bounded by below, so is the number of bins optimal solution needs

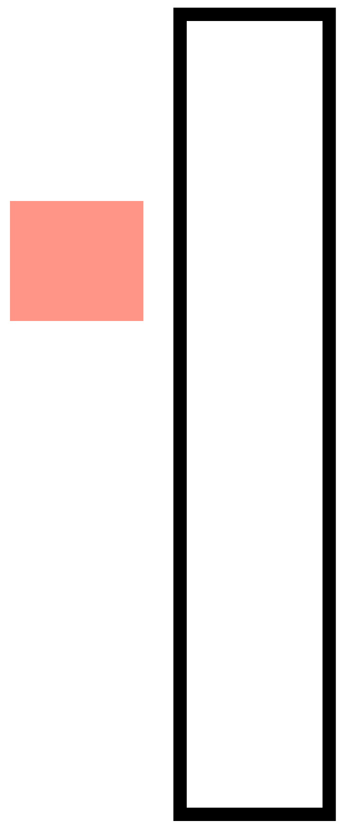
Outline

- **Bin Packing** problem
  - Assume that we know the **ALG** cost

- **Paging** problem
  - We know very little about the **ALG** or the **OPT**

FirstFit is at least 1.667-competitive

- Consider a sequence of requests
  - 6k items, each with size $\frac{1}{6} - 2\epsilon$



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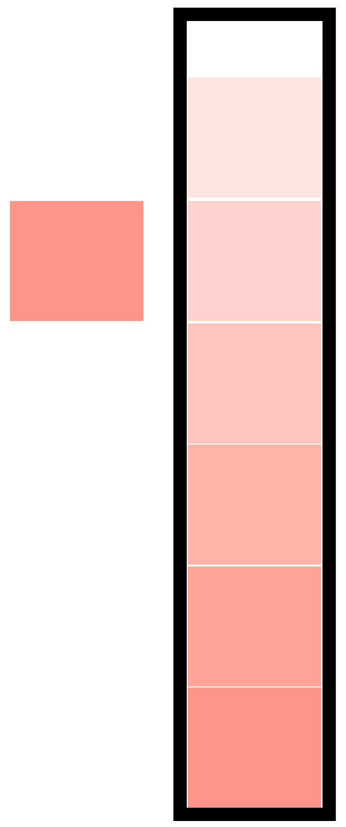

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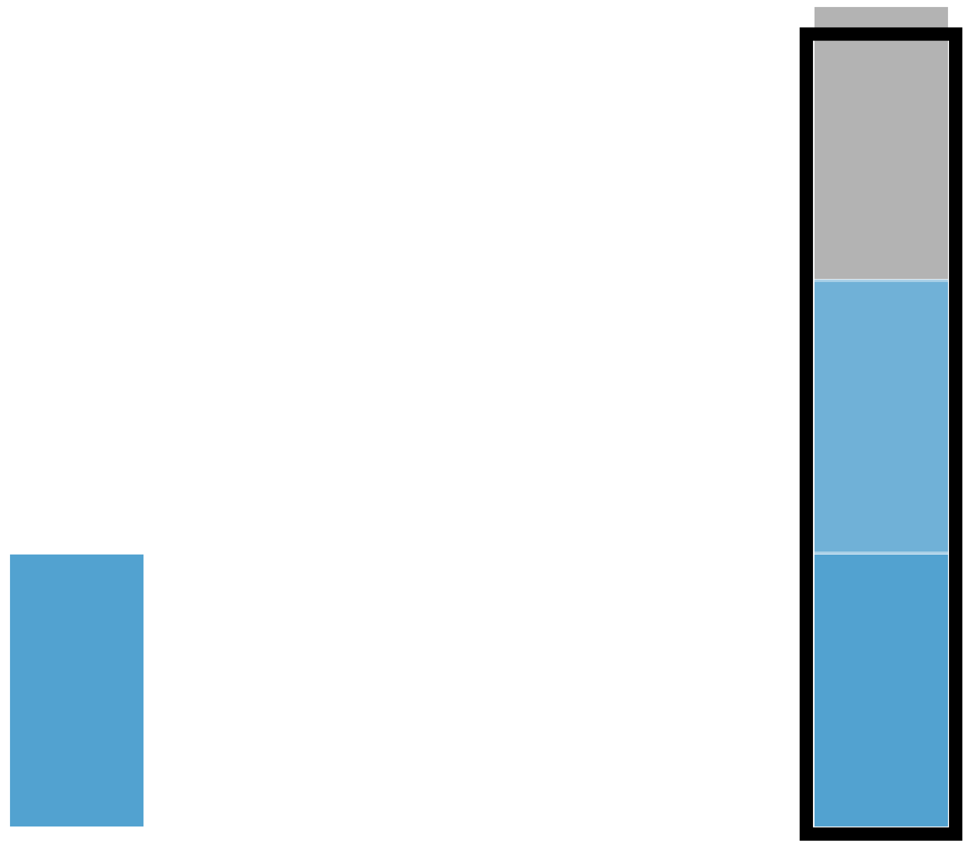

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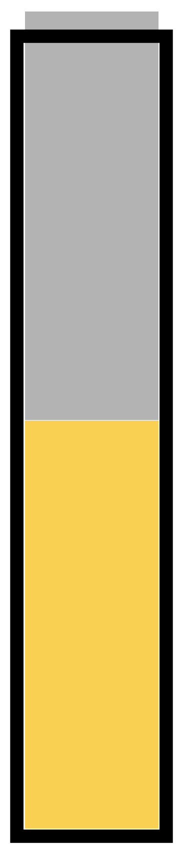

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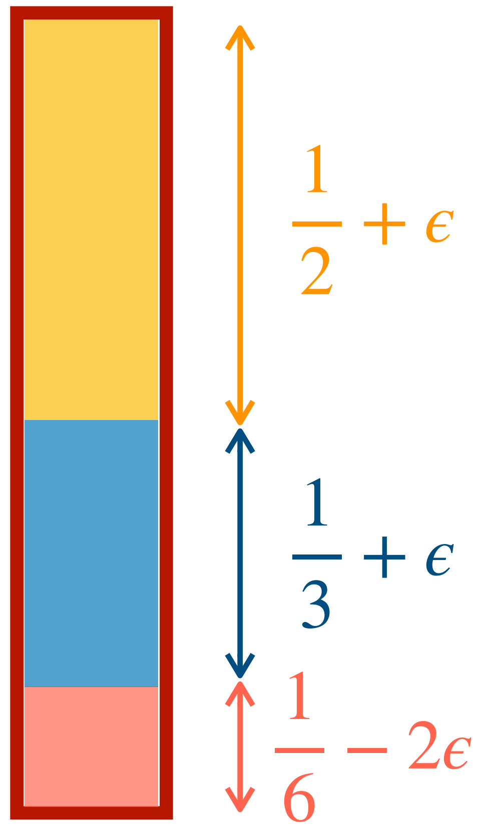

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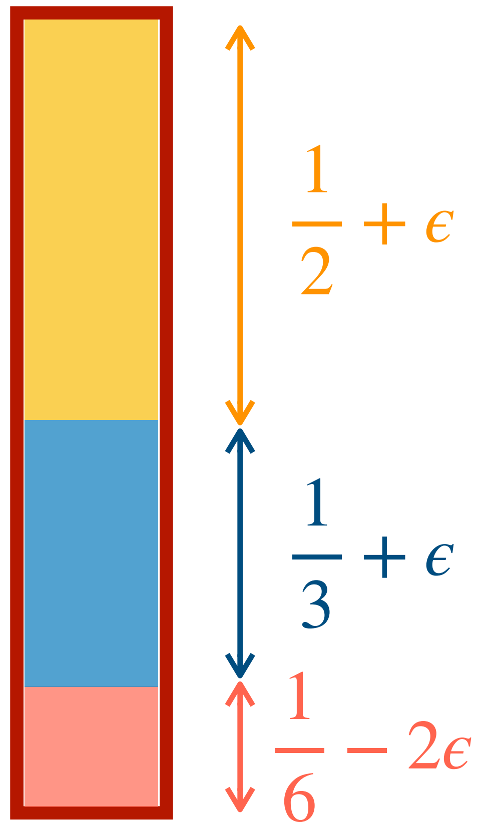

$\times$ 6k

OPT = 6k

FirstFit is at least 1.667-competitive

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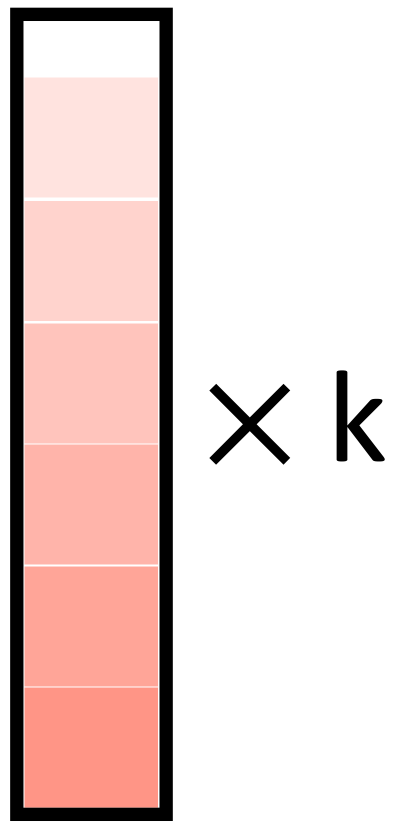

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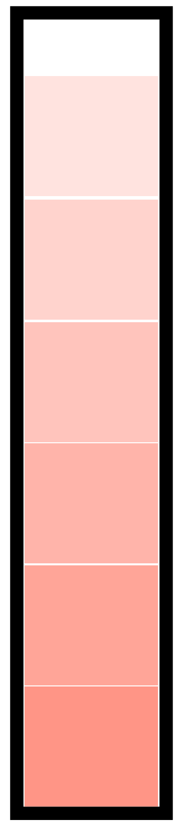
$\times$ k

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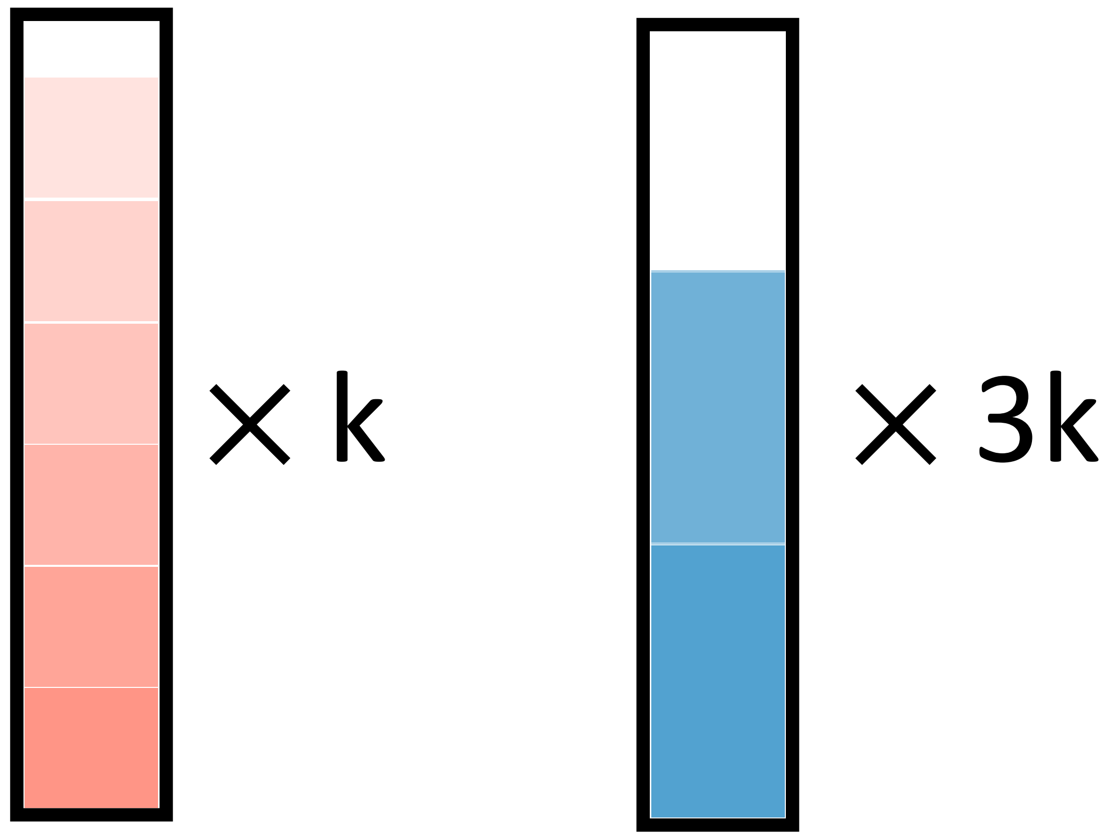

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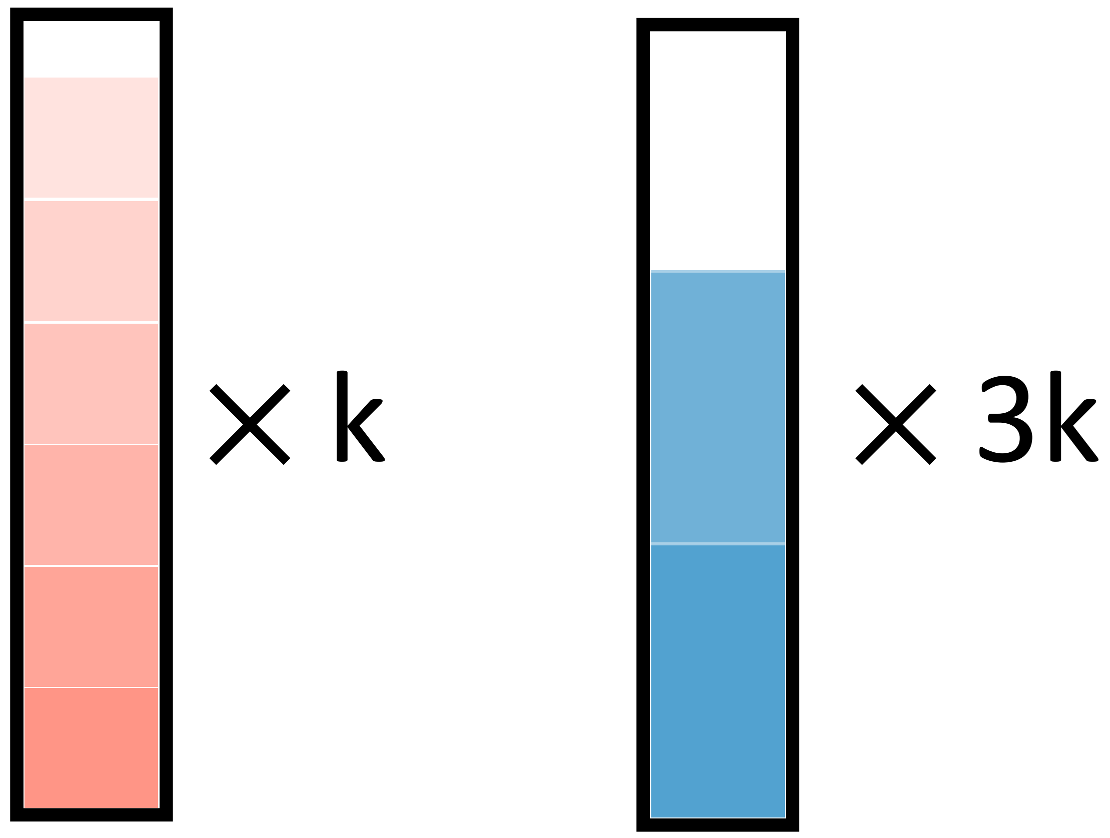

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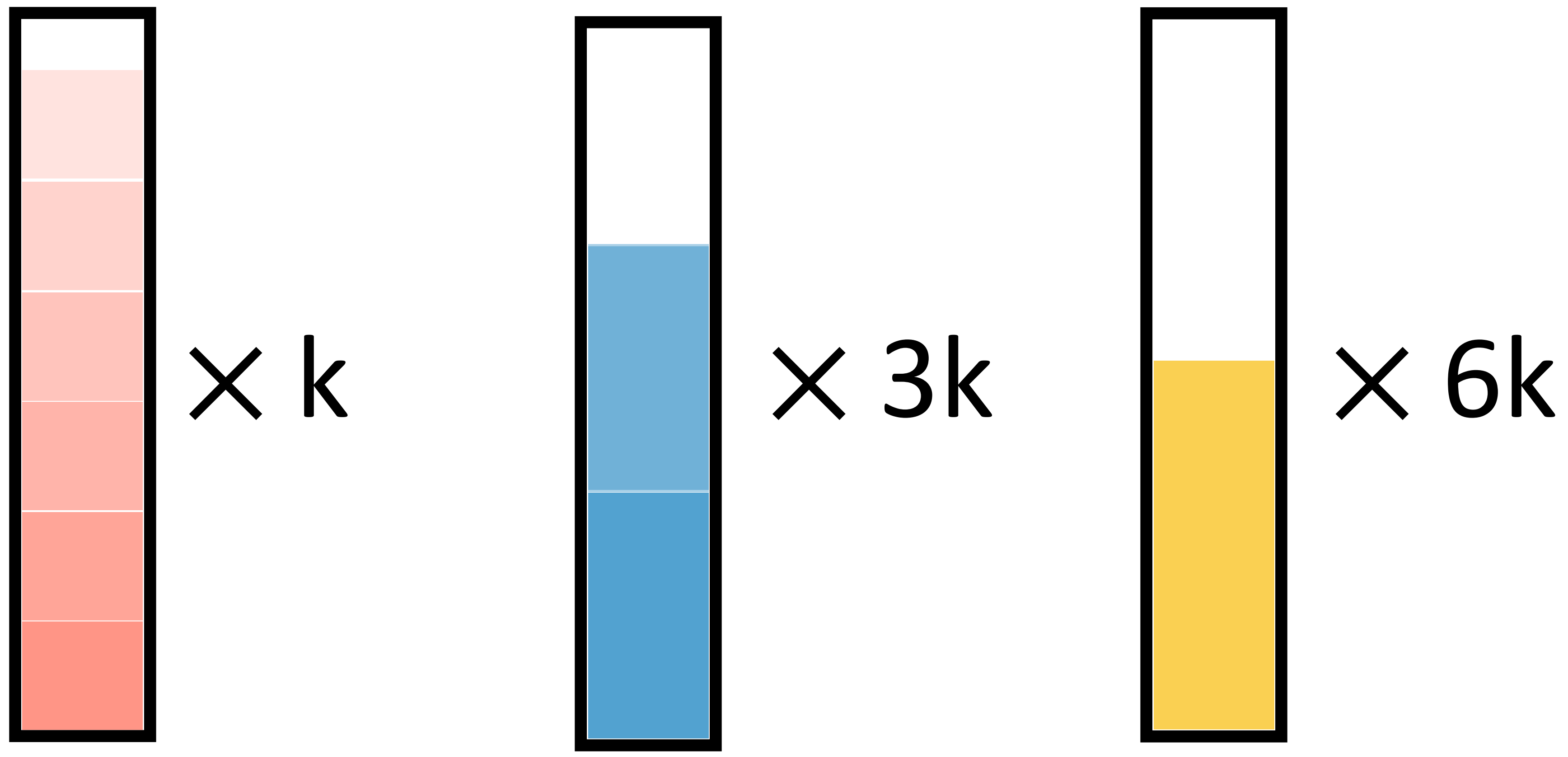

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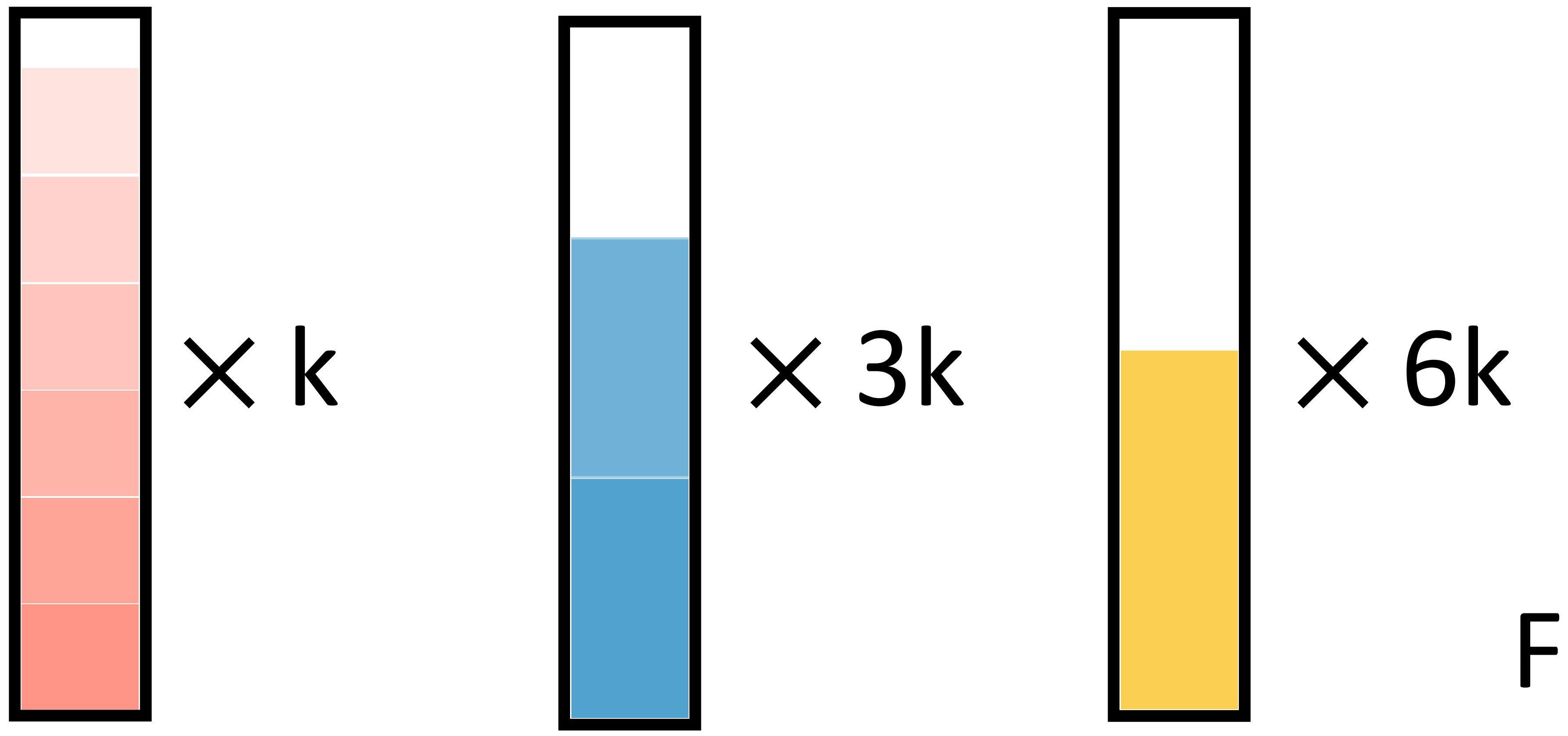

FirstFit = 10k

FirstFit is at least 1.667-competitive

- Consider a sequence of requests
  - 6k items, each with size $\frac{1}{6} - 2\epsilon$
  - 6k items, each with size $\frac{1}{3} + \epsilon$
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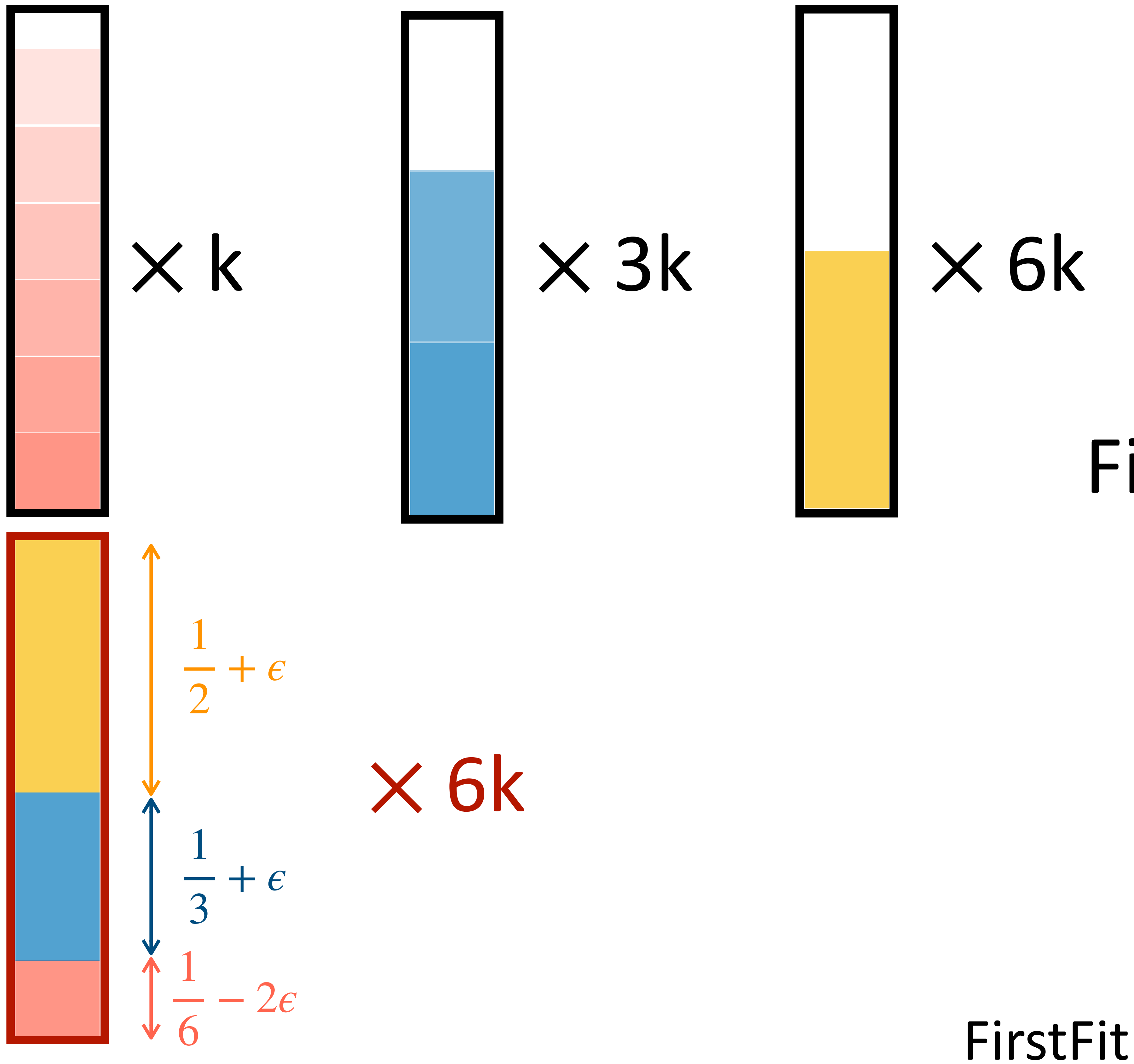

FirstFit = 10k

OPT = 6k

$$\frac{\text{FirstFit}}{\text{OPT}} = \frac{10}{6} = 1.667$$

Outline

- **Bin Packing** problem
  - Assume that we know the **ALG** cost

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Any deterministic online algorithm is at least 1.333-competitive

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<Proof idea>

Prove by contradiction: design an instance such that any algorithm ALG that is (4/3-ϵ)-competitive for the first half of the instance, it cannot be (4/3-ϵ)-competitive for the whole instance.

Any deterministic online algorithm is at least 1.333-competitive

<Proof idea>
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$$\underbrace{\frac{1}{2}-\epsilon, \frac{1}{2}-\epsilon, \cdots, \frac{1}{2}-\epsilon}_{m}, \underbrace{\frac{1}{2}+\epsilon, \frac{1}{2}+\epsilon, \cdots, \frac{1}{2}+\epsilon}_{m}$$

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<Proof idea> Assume ALG is (4/3-ϵ)-competitive

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OPT(I) = $\dfrac{m}{2}$

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OPT(I) = $\frac{m}{2}$

$\text{ALG}(I) < \frac{4}{3} \cdot \frac{m}{2}$

Any deterministic online algorithm is at least 1.333-competitive

<Proof idea> Assume ALG is (4/3-ϵ)-competitive

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OPT(I) = $\frac{m}{2}$

ALG(I) $< \frac{4}{3} \cdot \frac{m}{2}$ = $\frac{2}{3} \cdot m$

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<Proof idea> Assume ALG is (4/3-ϵ)-competitive

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$$\mathrm{OPT}(I) = \frac{m}{2}$$

$$\mathrm{ALG}(I) < \frac{4}{3} \cdot \frac{m}{2} = \frac{2}{3} \cdot m$$

$$= a_1 + a_2$$

a_1: #bins with 1 item in ALG(I)

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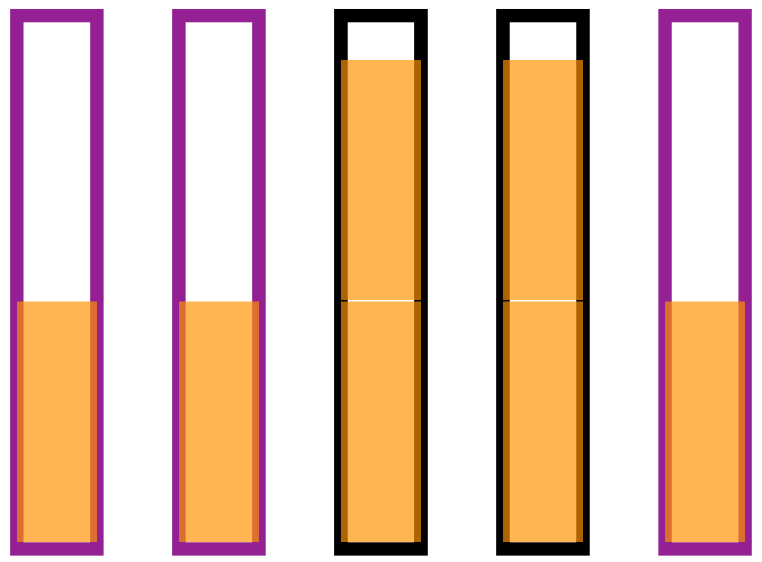

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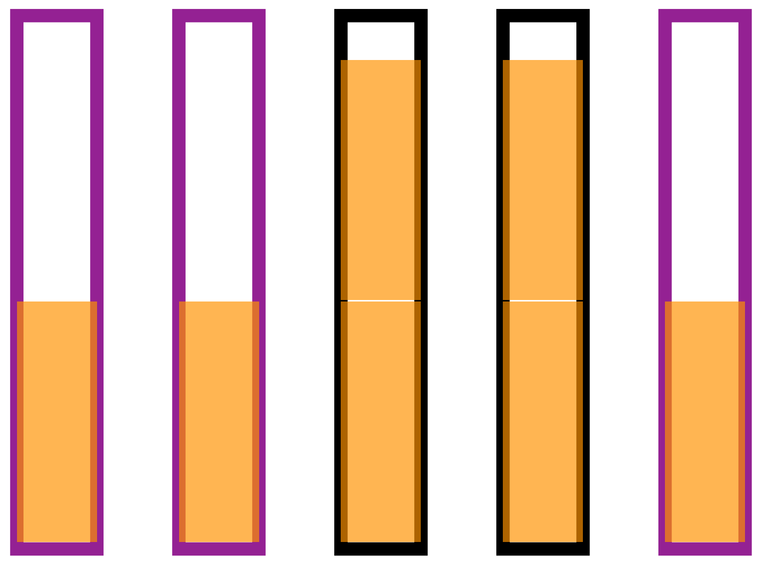
$m = a_1 + 2a_2$

There are m items



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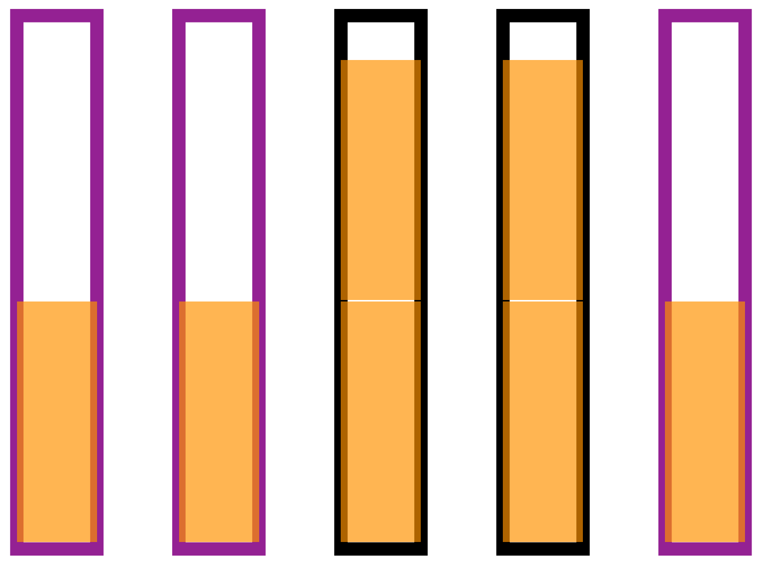

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<Proof idea> Assume ALG is $(4/3-\epsilon)$-competitive

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$$\mathrm{OPT}(I) = \frac{m}{2}$$

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OPT(I) = $\frac{m}{2}$

OPT(I+I) = m

$$\boxed{\text{ALG}(I) < \frac{4}{3} \cdot \frac{m}{2}} = \frac{2}{3} \cdot m$$

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ALG(I+I) = $a_1 + a_2 + x$

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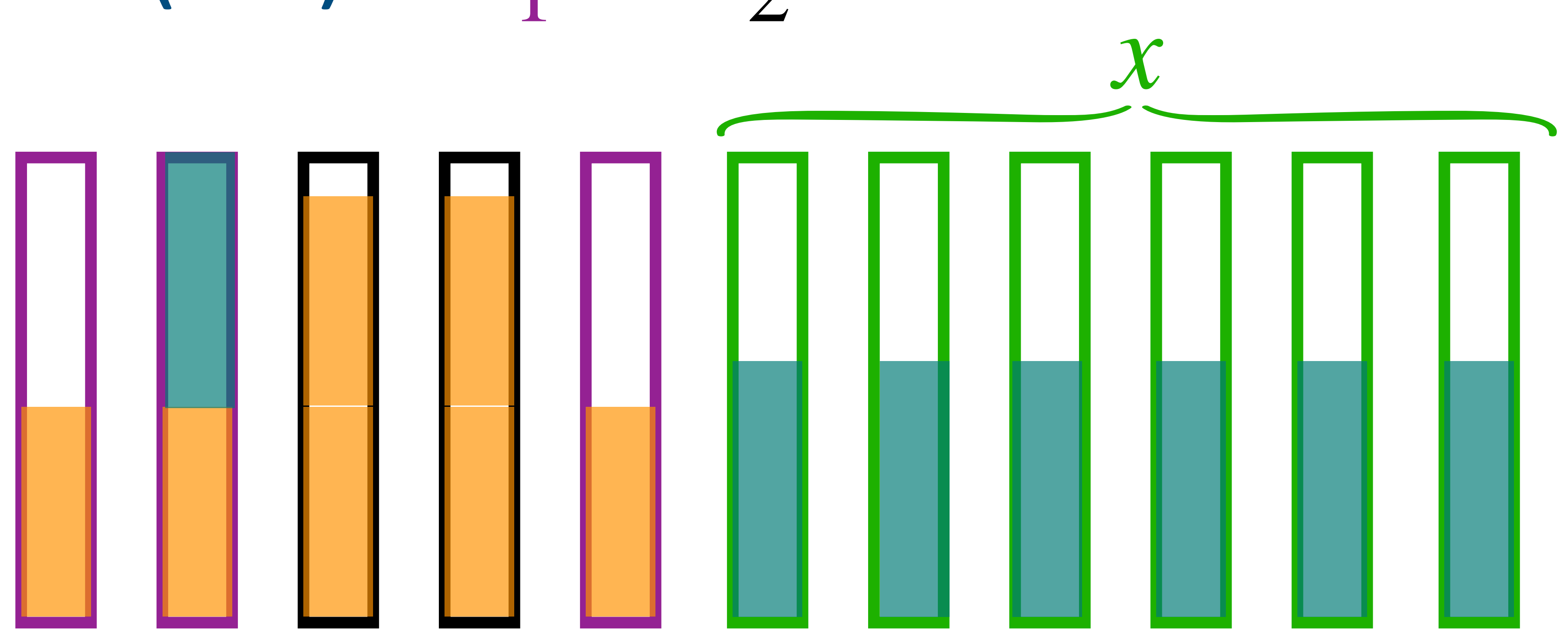

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OPT(I) = $\dfrac{m}{2}$

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$$\text{ALG}(I) < \frac{4}{3} \cdot \frac{m}{2} = \frac{2}{3} \cdot m$$

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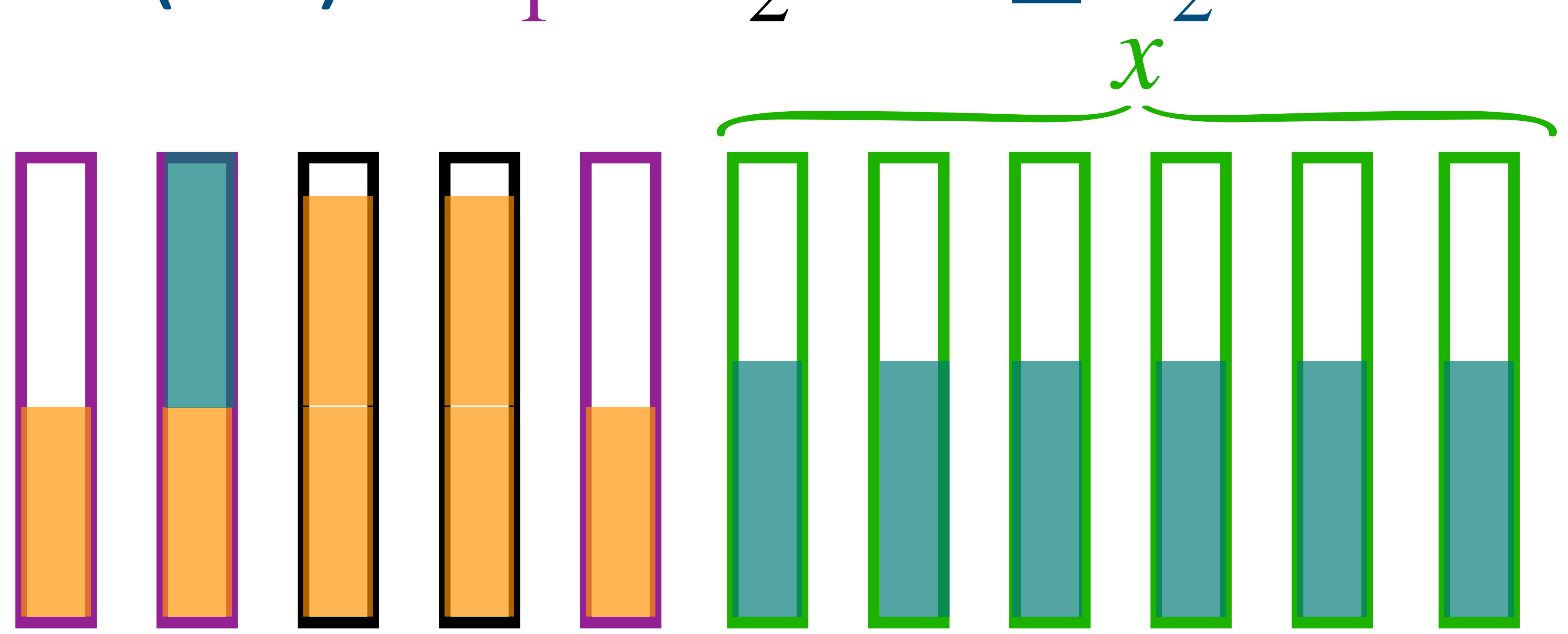

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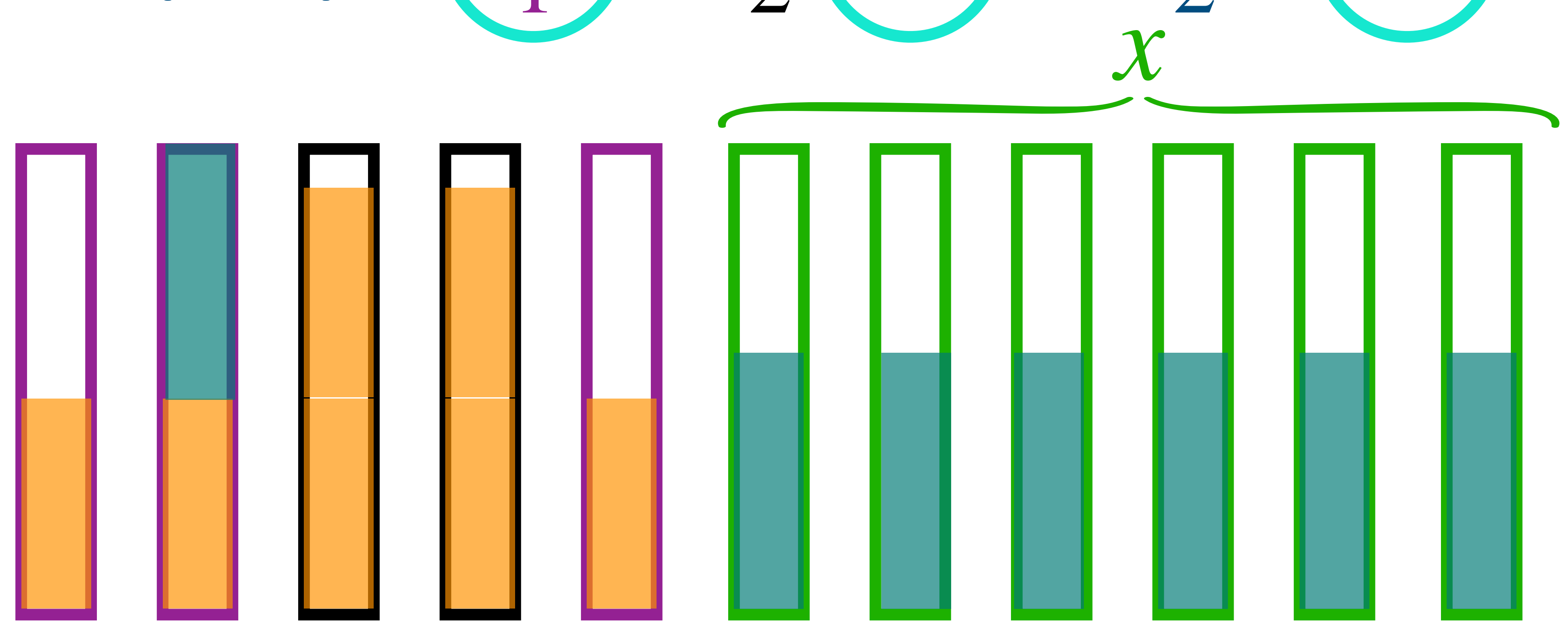

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OPT(I) = $\frac{m}{2}$

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ALG(I) $< \frac{4}{3} \cdot \frac{m}{2} = \frac{2}{3} \cdot m$

$= a_1 + a_2 = m - a_2$

ALG(I+I) = $a_1 + a_2 + x \geq a_2 + m$

ALG(I+I) $< \frac{4}{3} \cdot$ OPT(I+I)

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$$\text{OPT}(I) = \frac{m}{2}$$

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a_1: #bins with 1 item in ALG(I)

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$m = a_1 + 2a_2$

$$\text{OPT}(I+I) = m$$

$$\text{ALG}(I+I) = a_1 + a_2 + x \geq a_2 + m$$

$$\text{ALG}(I+I) < \frac{4}{3} \cdot \text{OPT}(I+I) = \frac{4}{3} \cdot m$$

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$$\mathrm{OPT}(I) = \frac{m}{2}$$

$$\mathrm{OPT}(I+I) = m$$

$$\mathrm{ALG}(I) < \frac{4}{3} \cdot \frac{m}{2} = \frac{2}{3} \cdot m$$

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$$\mathrm{ALG}(I+I) = a_1 + a_2 + x \geq a_2 + m$$

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a_1: #bins with 1 item in ALG(I)

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$m = a_1 + 2a_2$

Any deterministic online algorithm is at least 1.333-competitive

<Proof idea> Assume ALG is $(4/3-\epsilon)$-competitive

Prove by contradiction: design an instance such that any algorithm ALG that is $(4/3-\epsilon)$-competitive for the first half of the instance, it cannot be $(4/3-\epsilon)$-competitive for the whole instance. Consider the adversarial input:

$$\underbrace{\frac{1}{2}-\epsilon, \frac{1}{2}-\epsilon, \cdots, \frac{1}{2}-\epsilon}_{m}, \underbrace{\frac{1}{2}+\epsilon, \frac{1}{2}+\epsilon, \cdots, \frac{1}{2}+\epsilon}_{m}$$

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Contradiction!

$$a_2 < \frac{m}{3} \Longleftrightarrow \boxed{\text{ALG}(I) = m - a_2 > \frac{2}{3} \cdot m}$$

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Outline

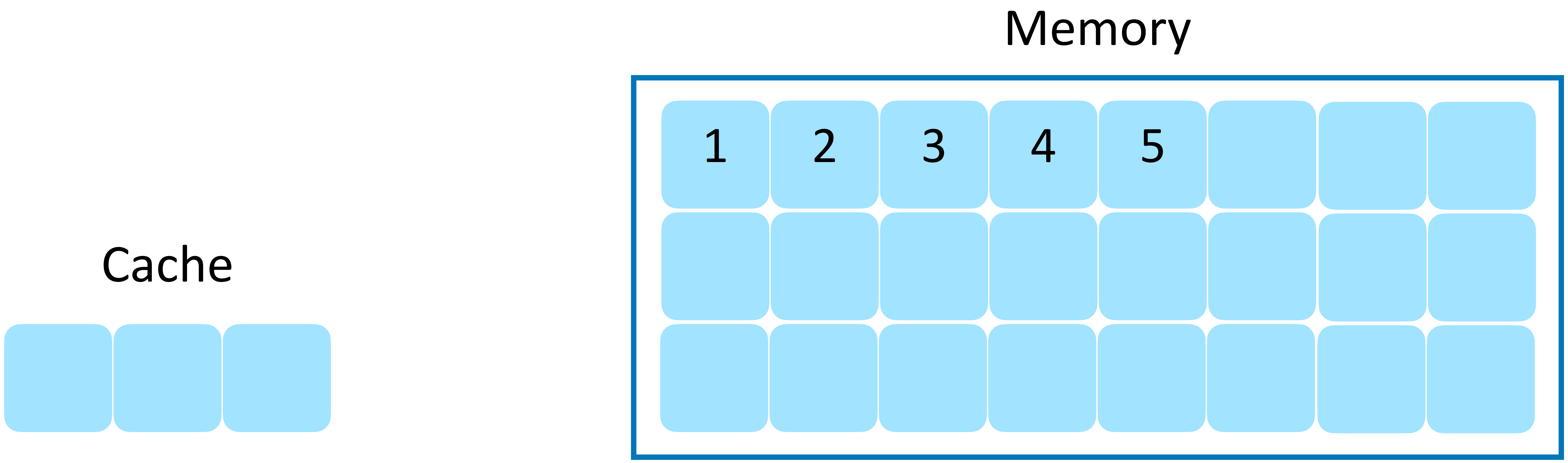
- **Bin Packing** problem
  - Assume that we know the **ALG** cost

- **Paging** problem
  - We know very little about the **ALG** or the **OPT**

Paging

Paging

- In computer systems, the memory system is two-level
  - Data (in blocks) are stored in the **memory**, which cannot accessed directly
    - A block of data is called a page
  - The **cache** can be accessed directly and is fast, but very small



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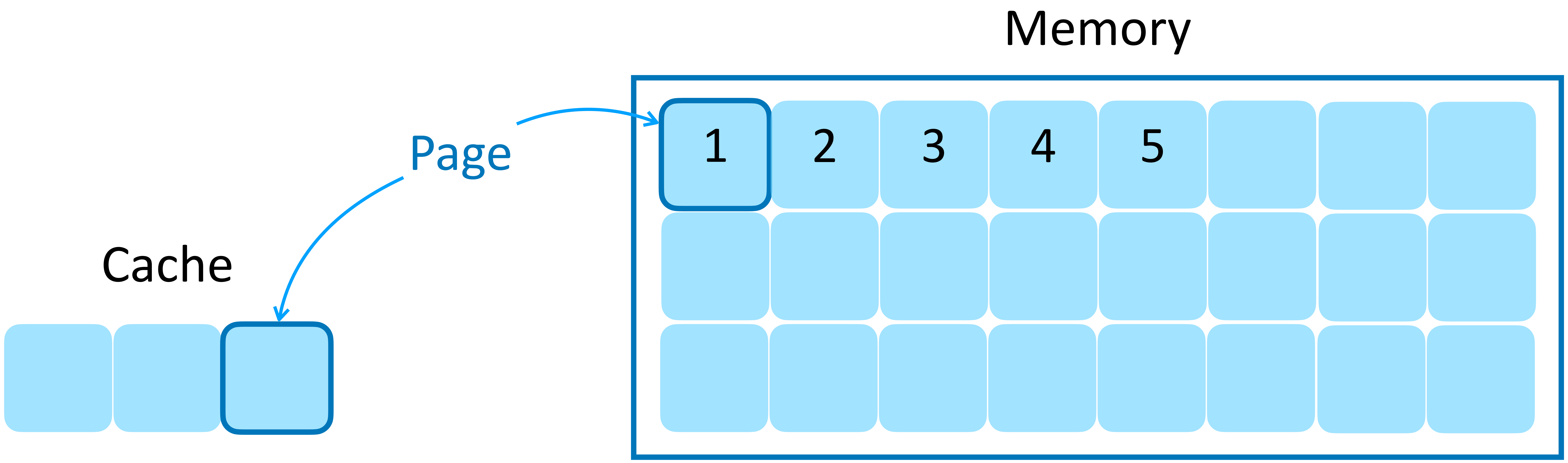

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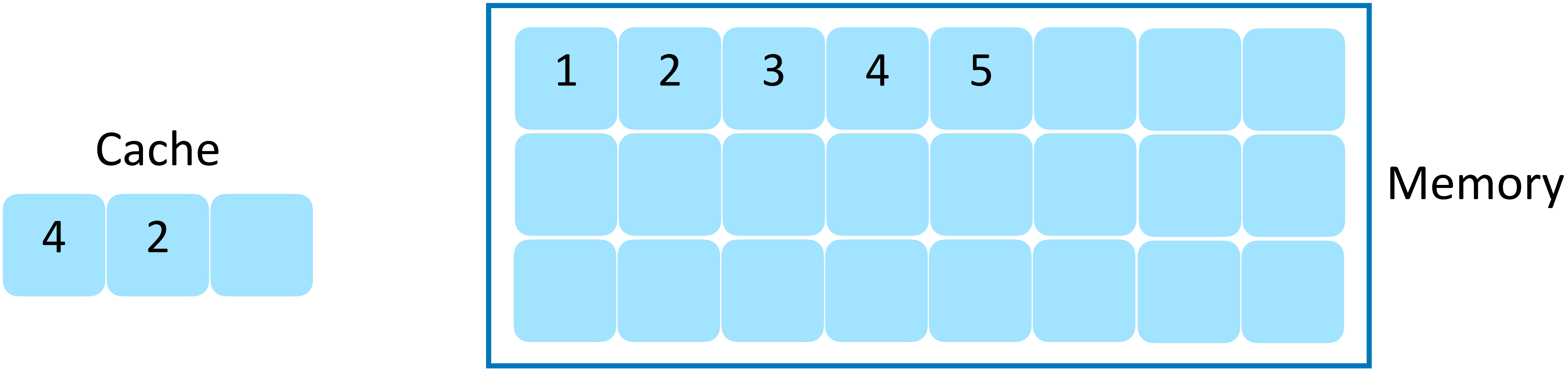

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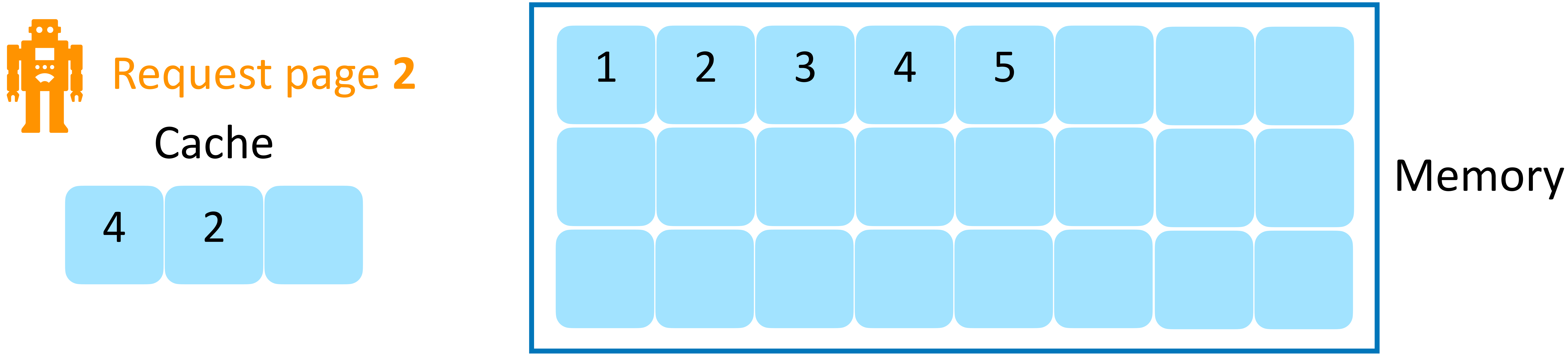

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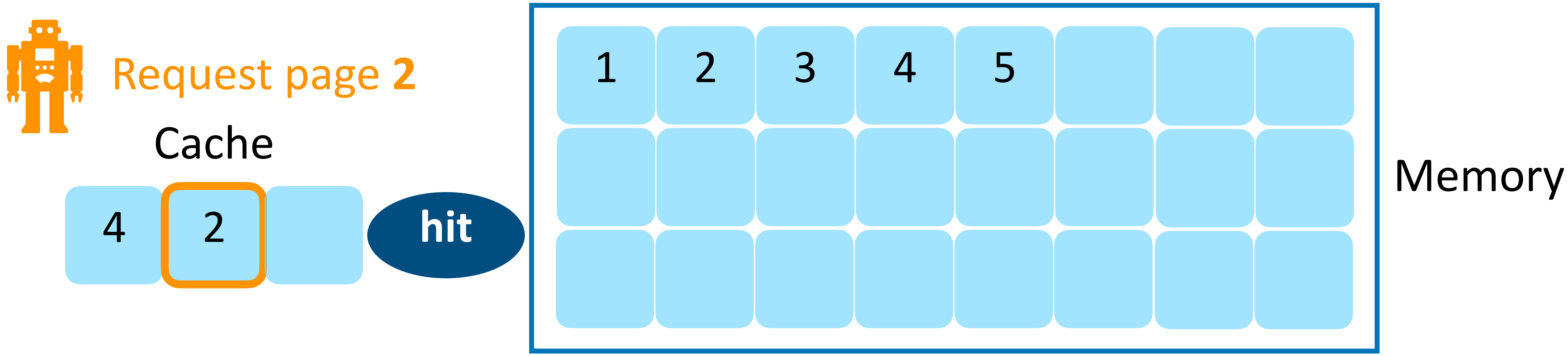

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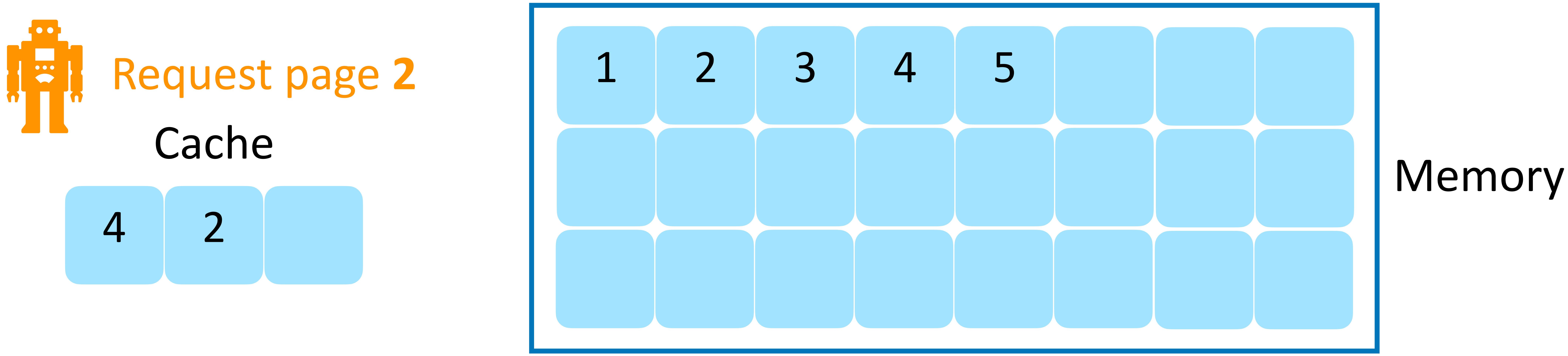

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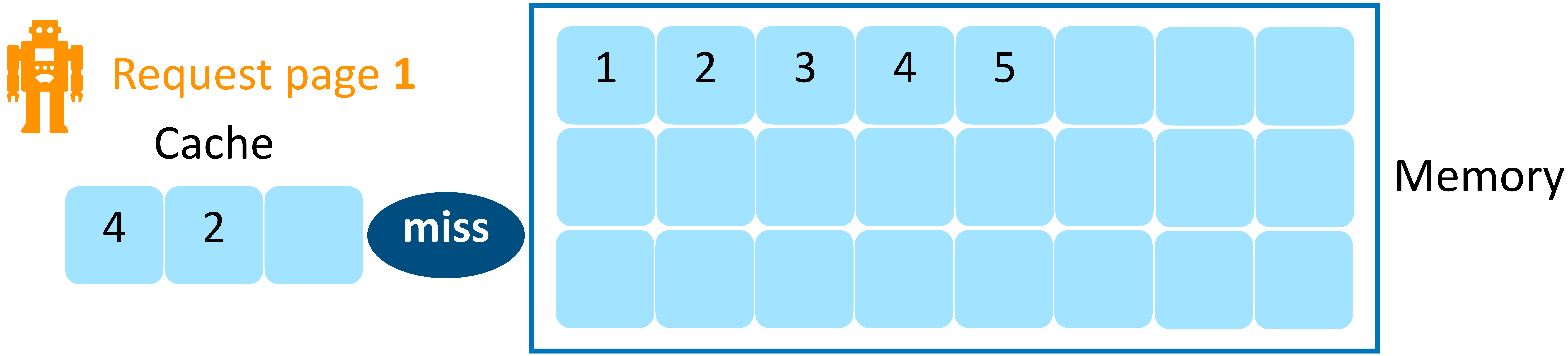

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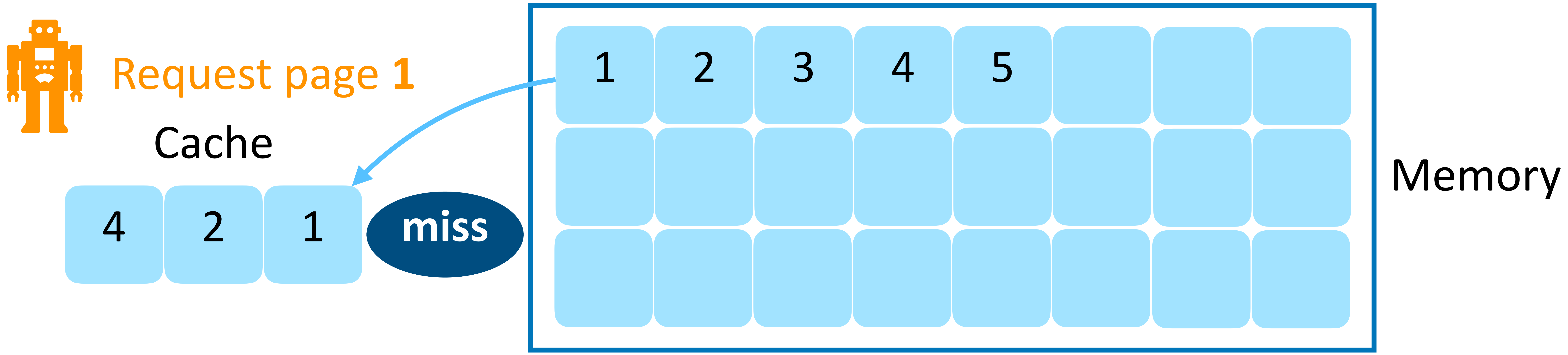

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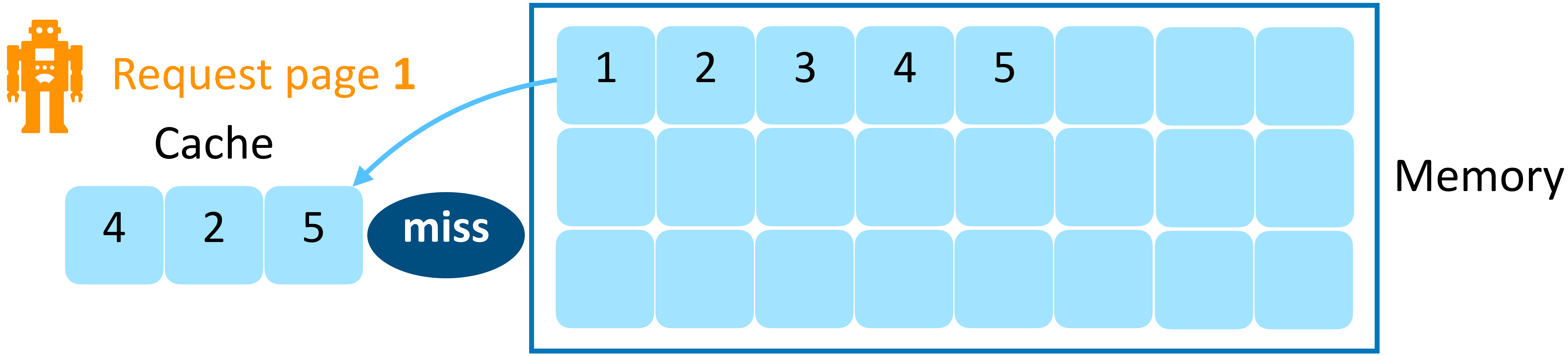

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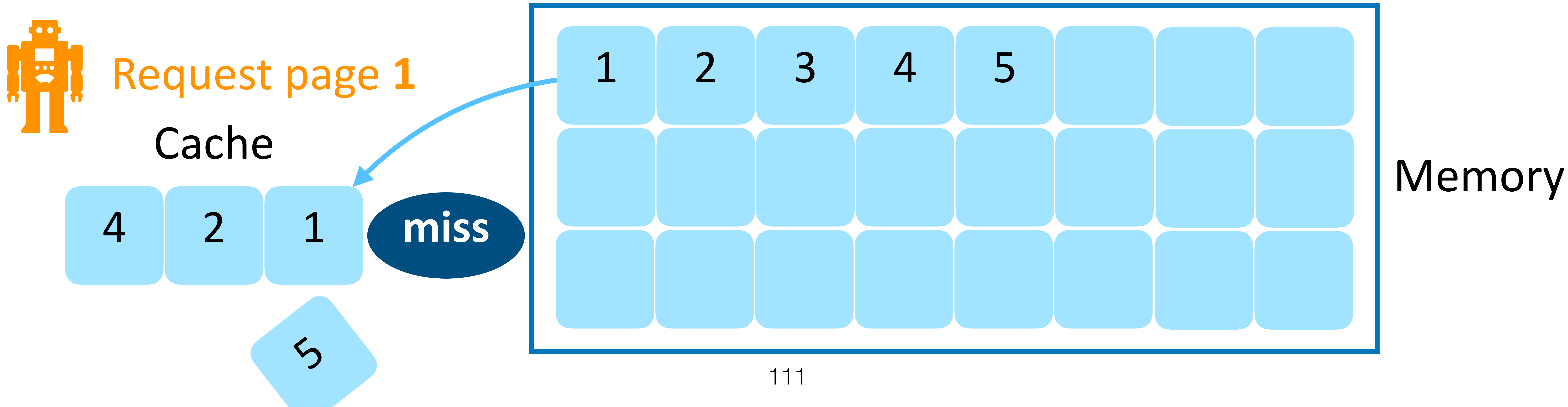

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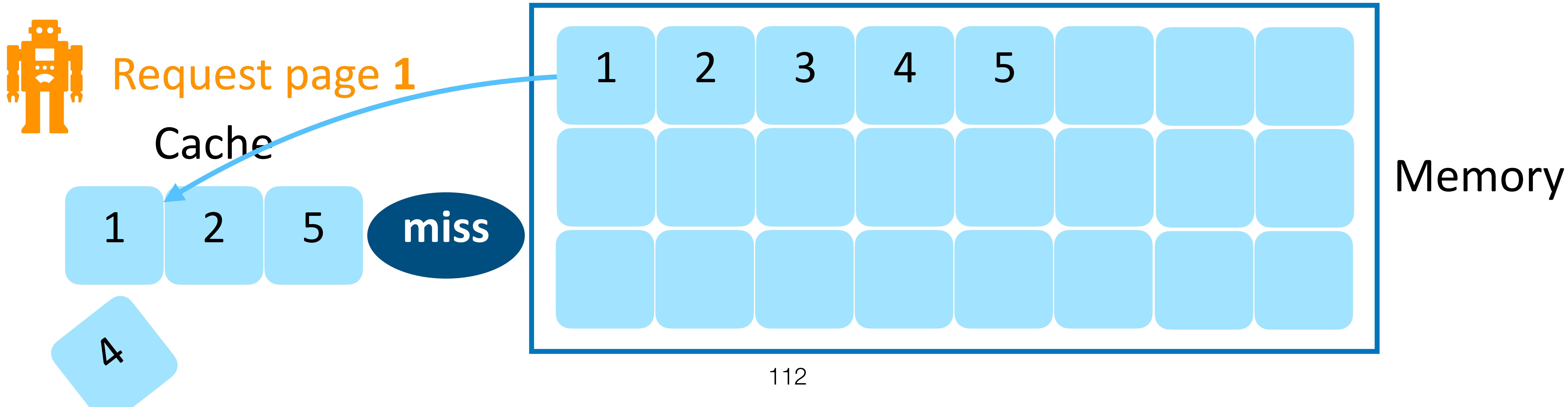

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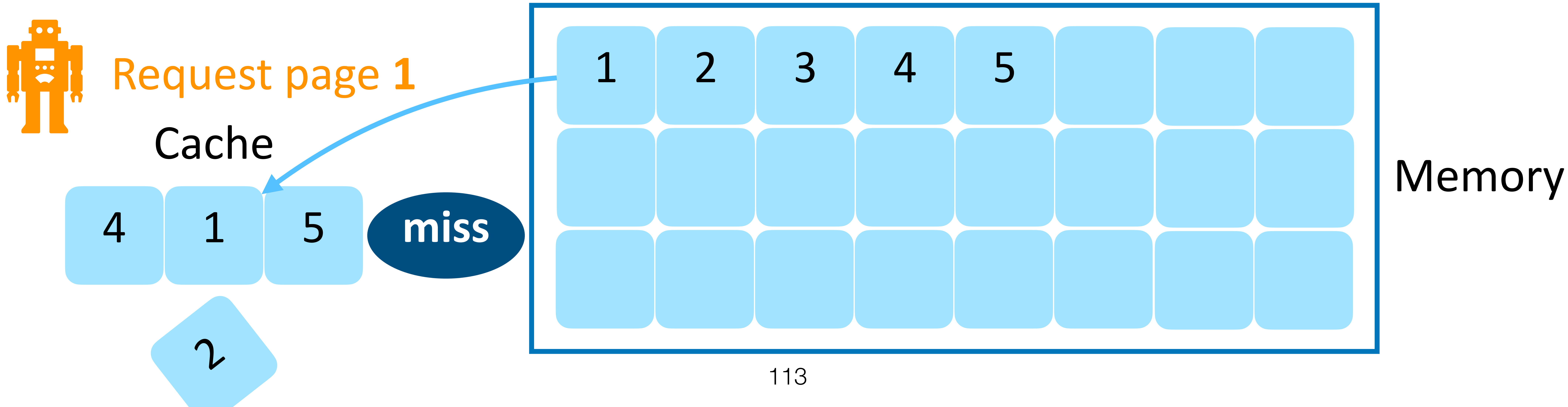
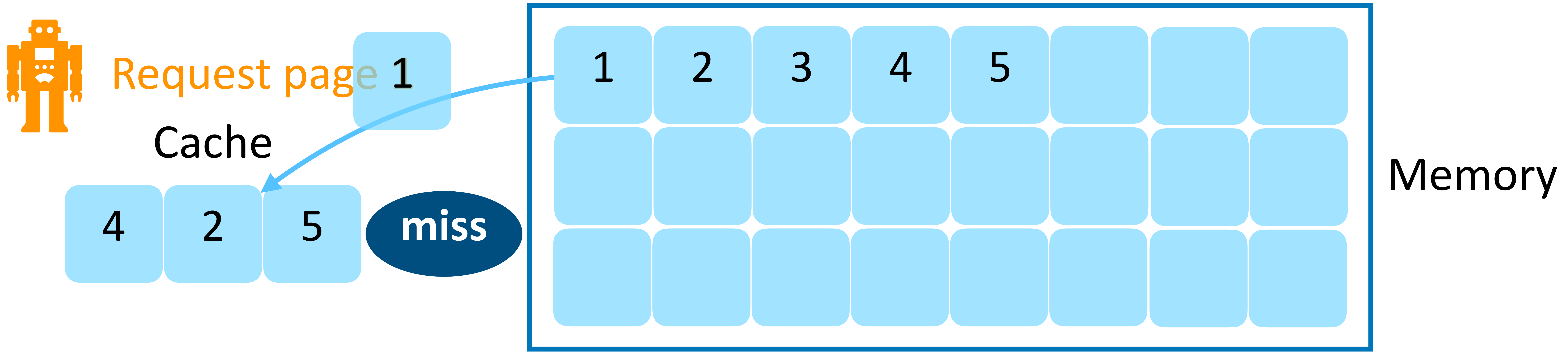


Paging

- Given a sequence of n requests of pages $r_1, r_2, \cdots, r_n$, revealed one by one
  - Set of pages $= \{1, 2, 3, \cdots, n\}$
- With a size-k cache, the algorithm has to serve all the requests with a minimum number of page faults
  - Choose which page to evict wisely



Paging Algorithms

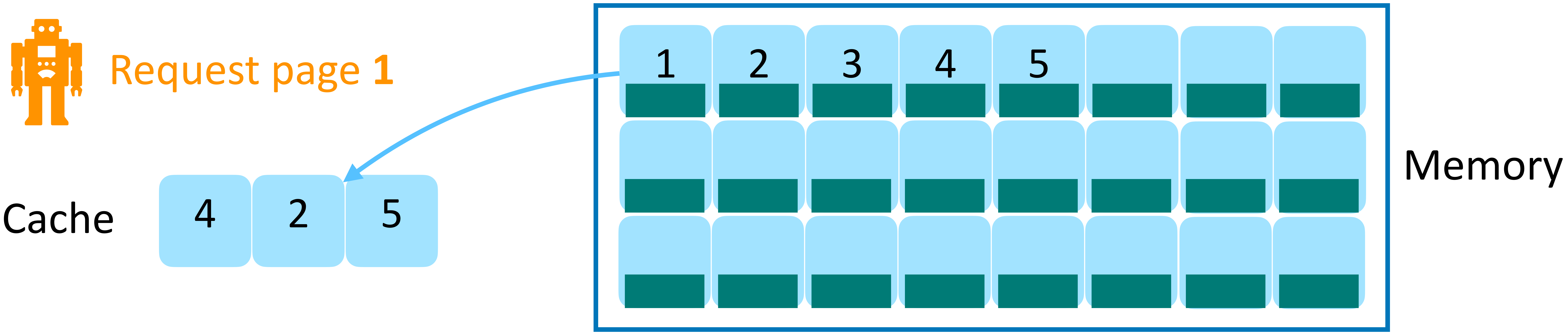
- LIFO (Last-In-First-Out)
- FIFO (First-In-First-Out)
- LFU (Least-Frequently-Used)
- LRU (Least-Recently-Used)
- CLOCK (CLOCK-replacement)
- LFD (Longest-Forward-Distance)

LFU (Least-Frequently-Used)

LFU (Least-Frequently-Used) algorithm:

Every page has a **counter** that keep the number of times it has been accessed

Once a page fault is incurred, evict the one with the lowest counter value (break tie arbitrarily)



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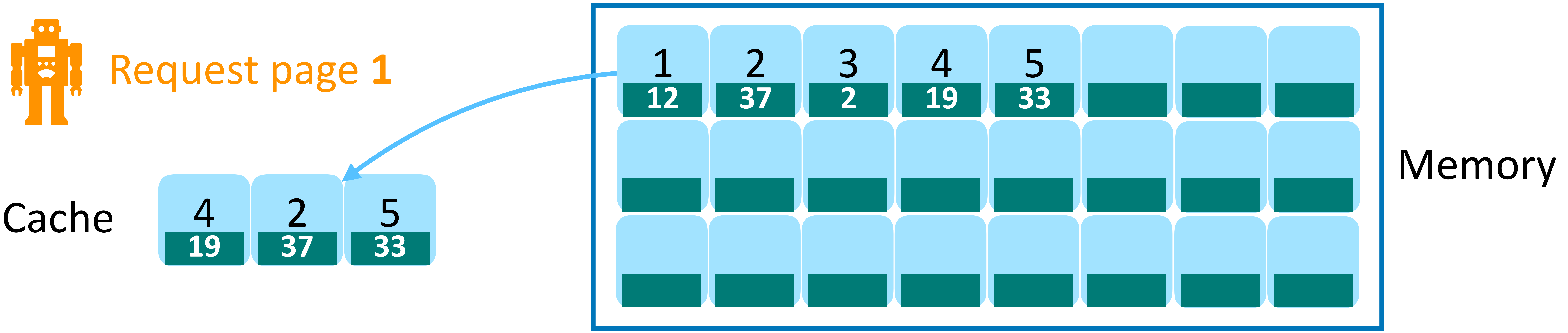

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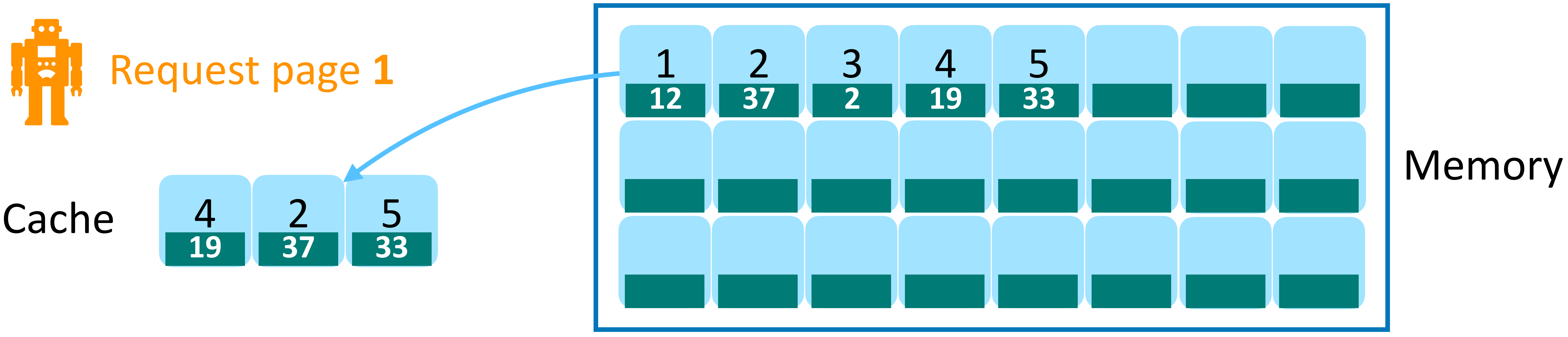

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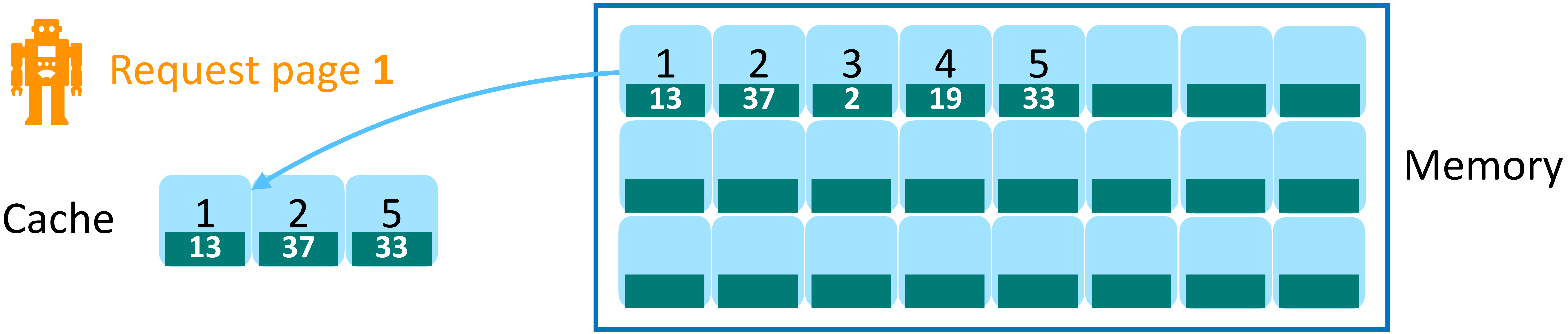

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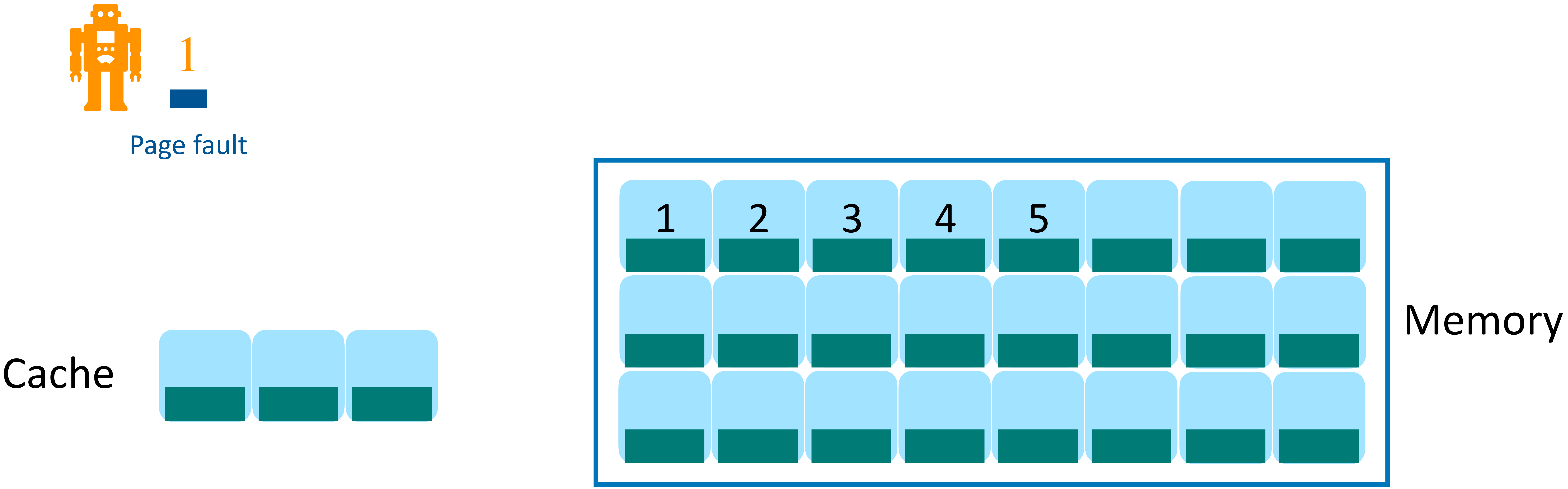

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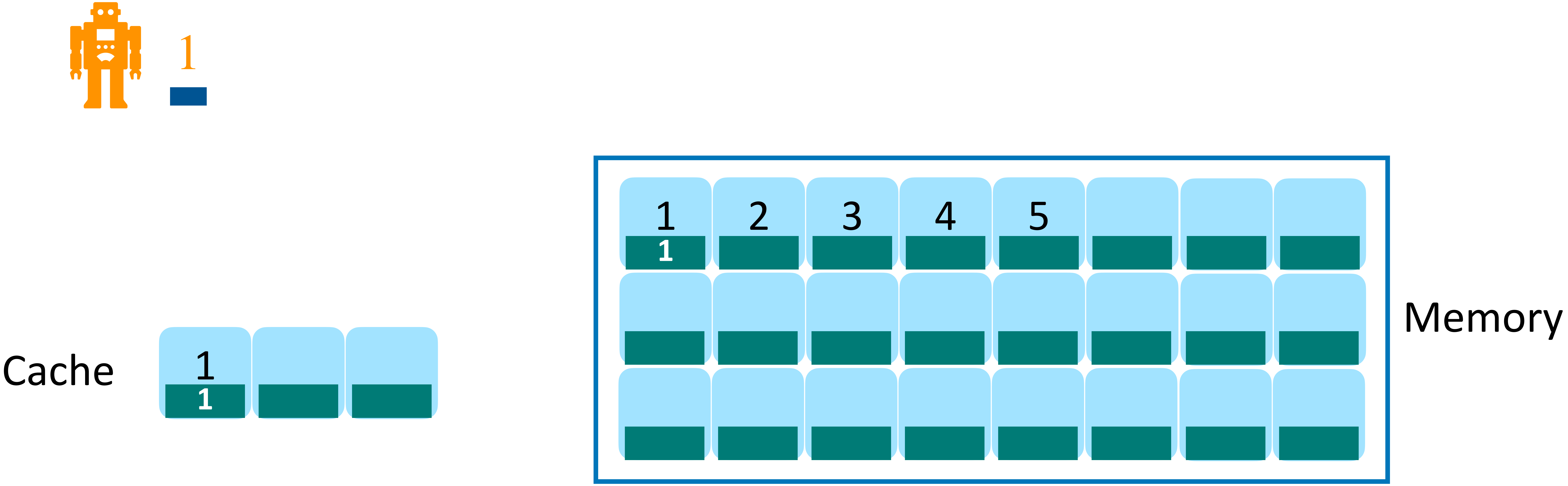

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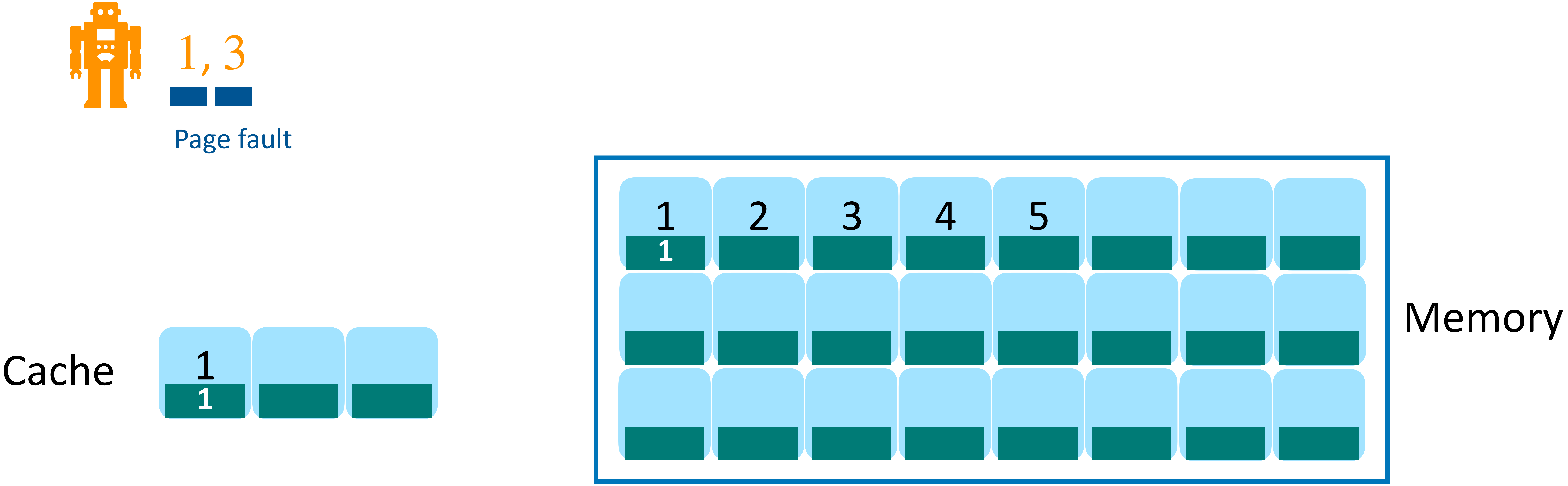

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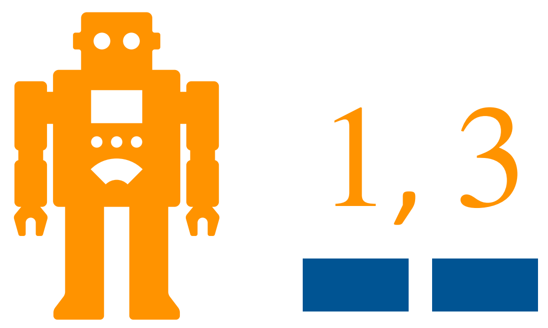
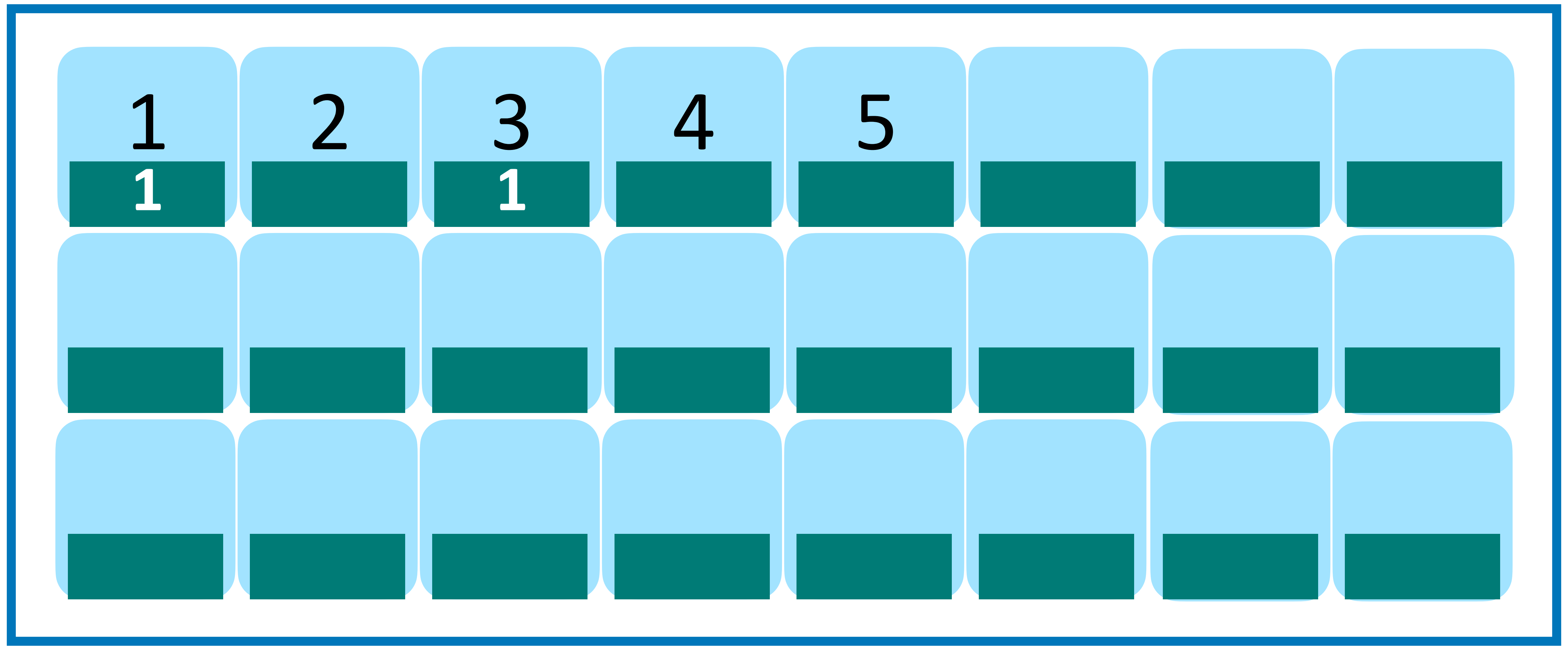


Cache



Memory

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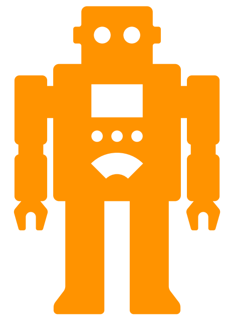
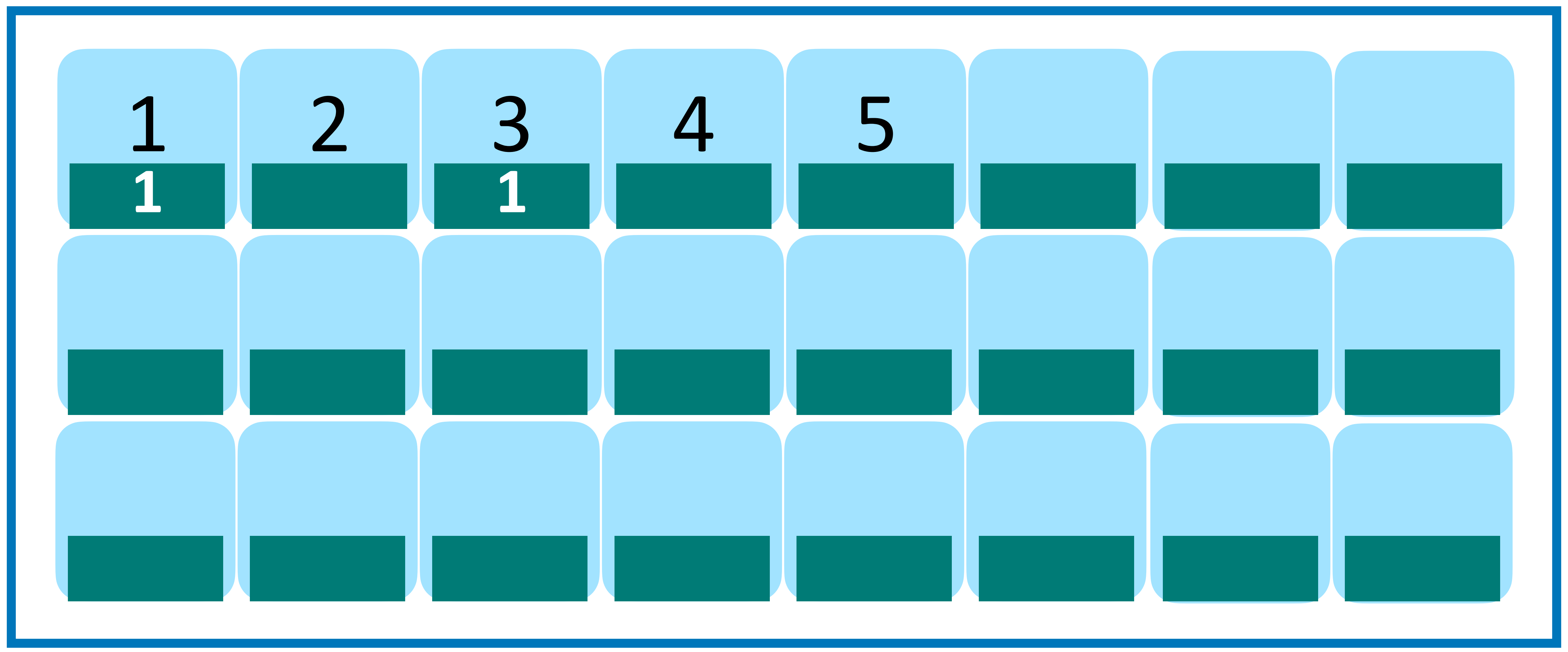


1, 3, 3

Cache



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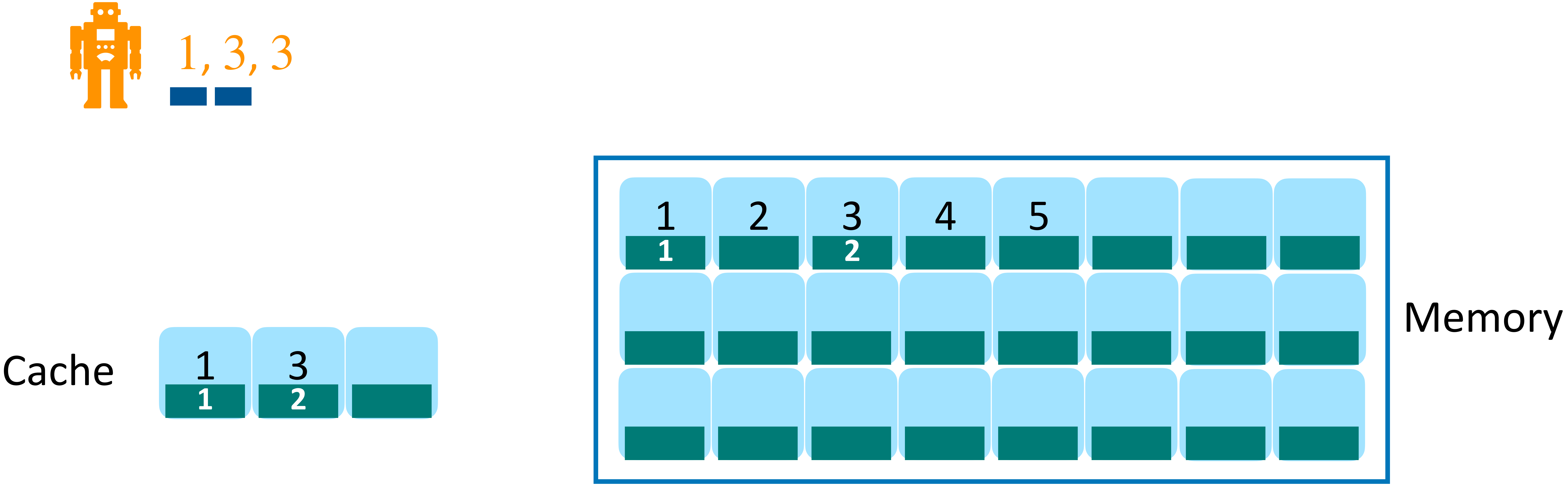

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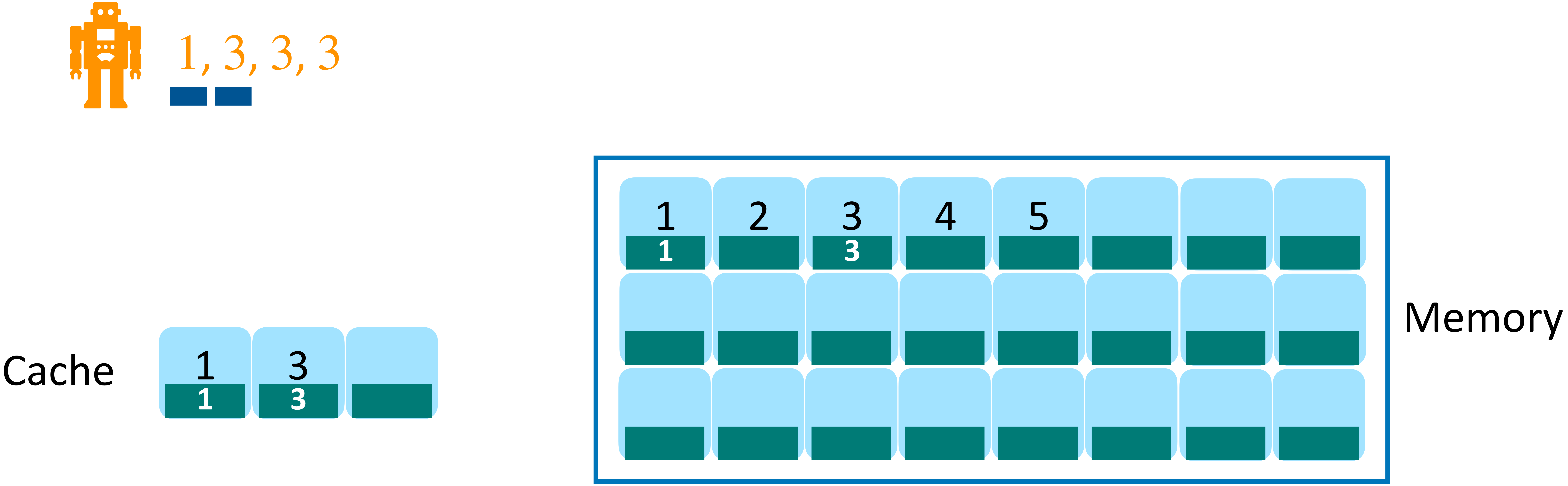

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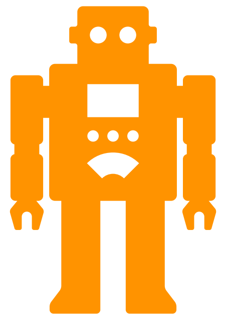
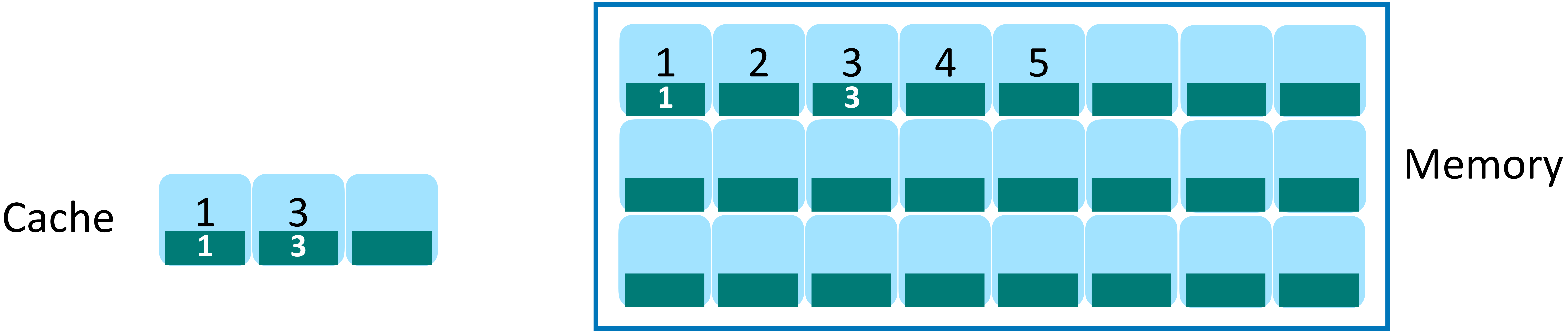


1, 3, 3, 3, 5



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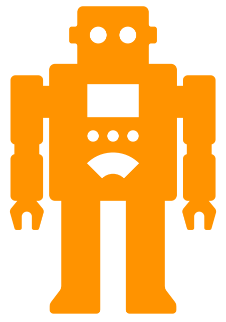

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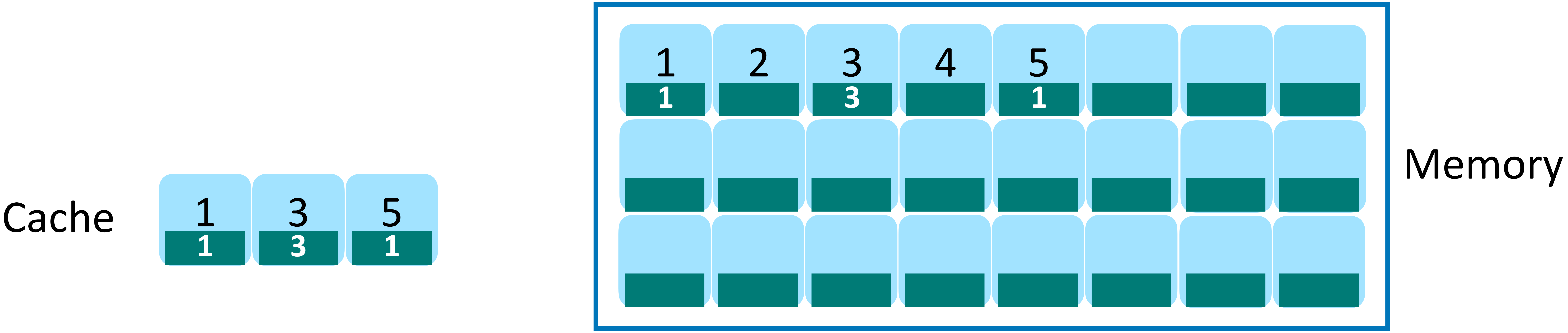

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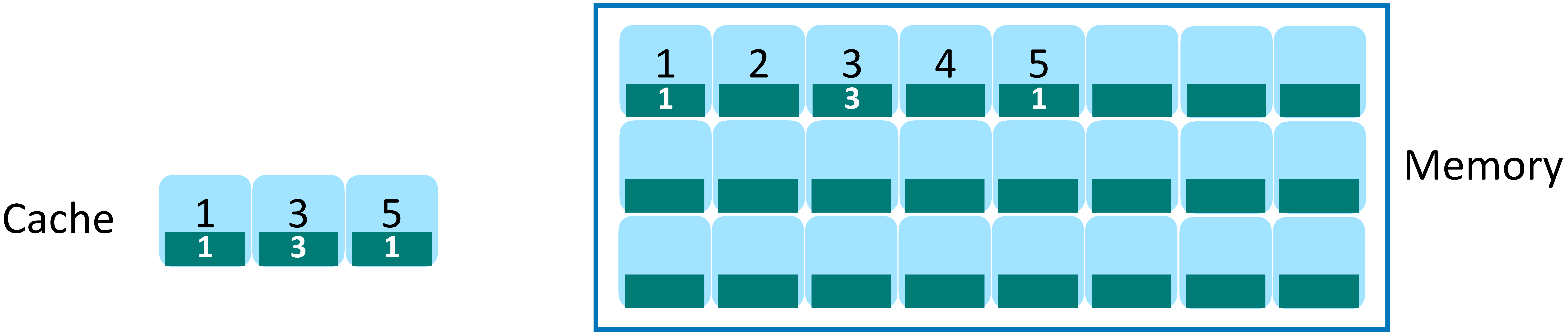

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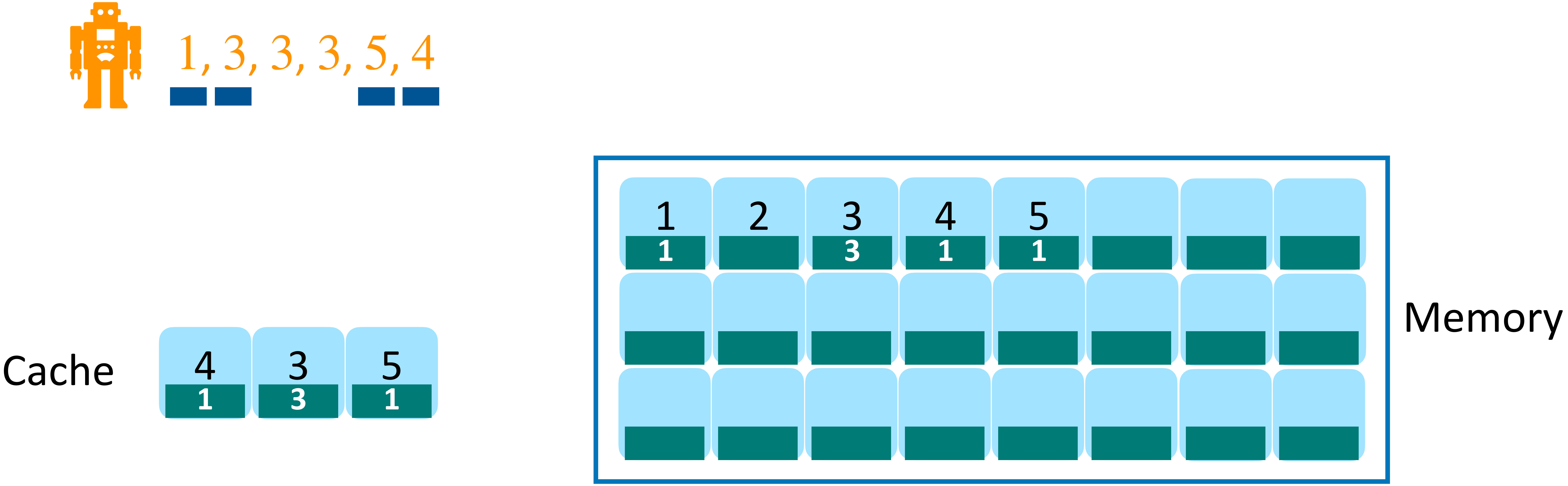

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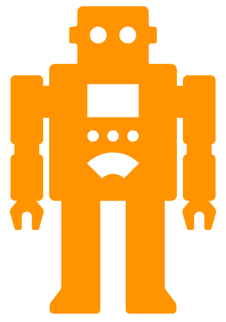
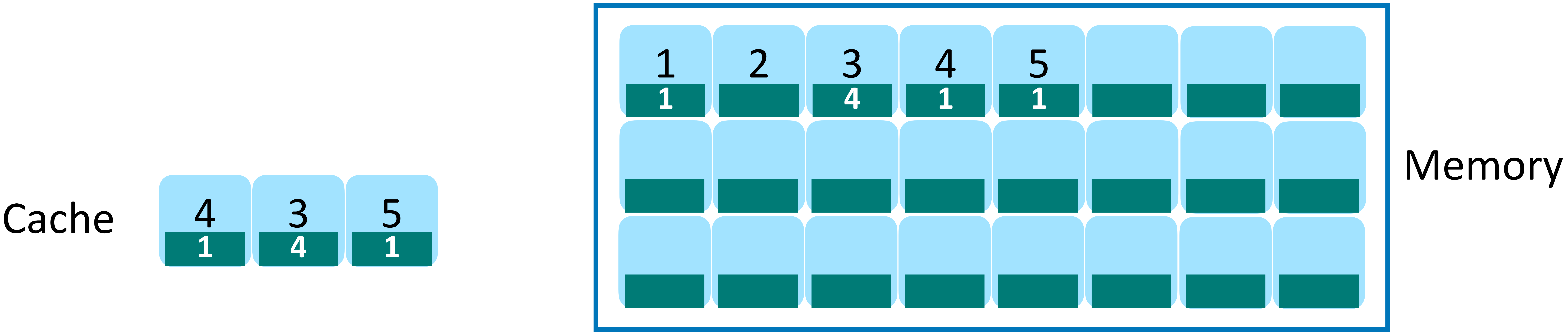


1, 3, 3, 3, 5, 4, 3



LFU (Least-Frequently-Used)

LFU (Least-Frequently-Used) algorithm:

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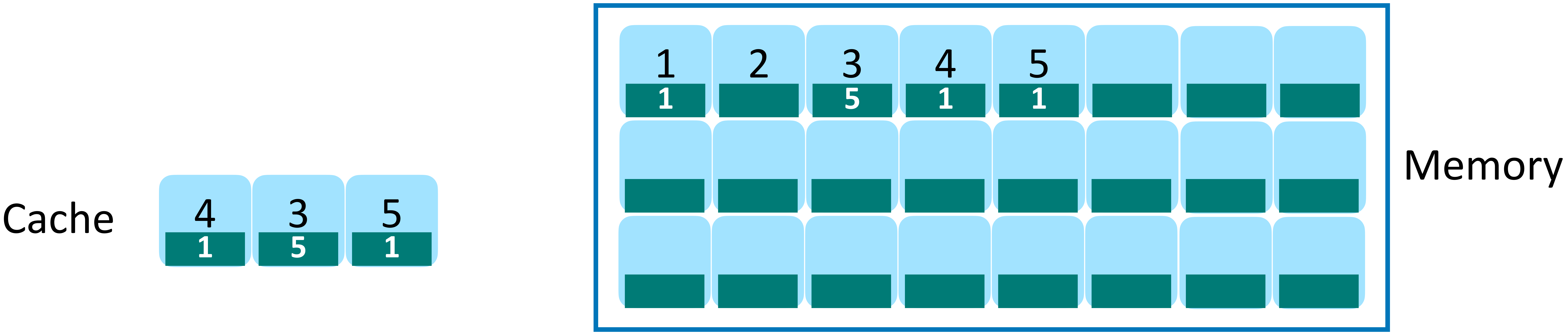
1, 3, 3, 3, 5, 4, 3, 3



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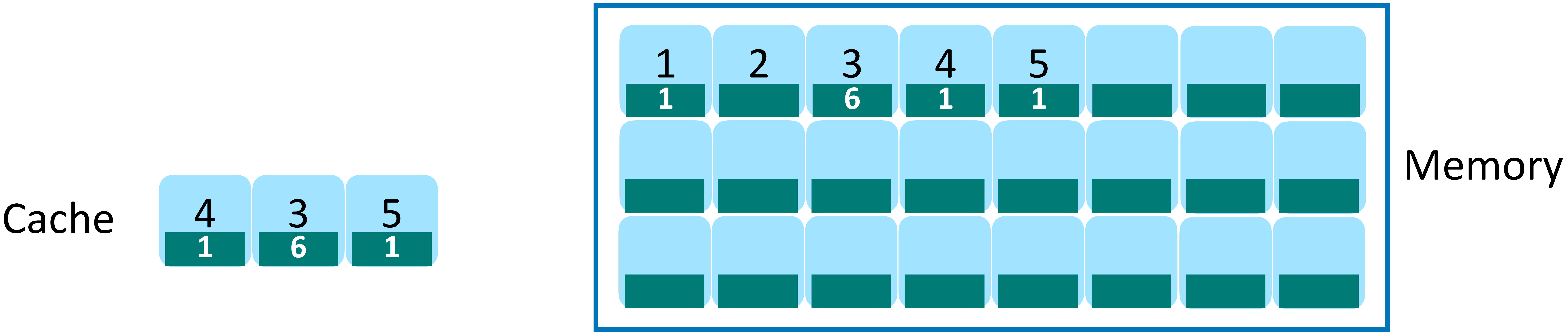
1, 3, 3, 3, 5, 4, 3, 3, 3



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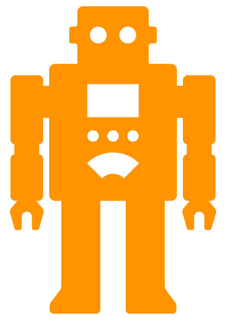
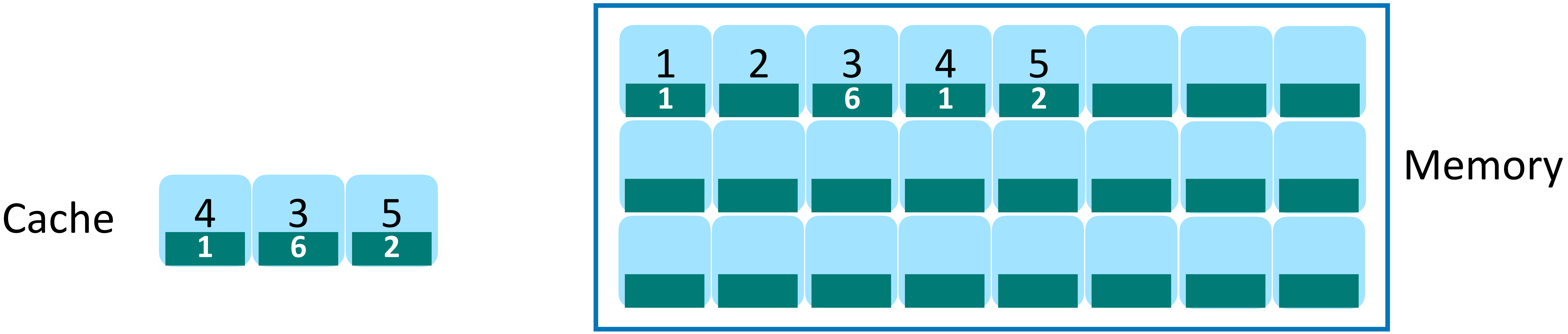


1, 3, 3, 3, 5, 4, 3, 3, 3, 5



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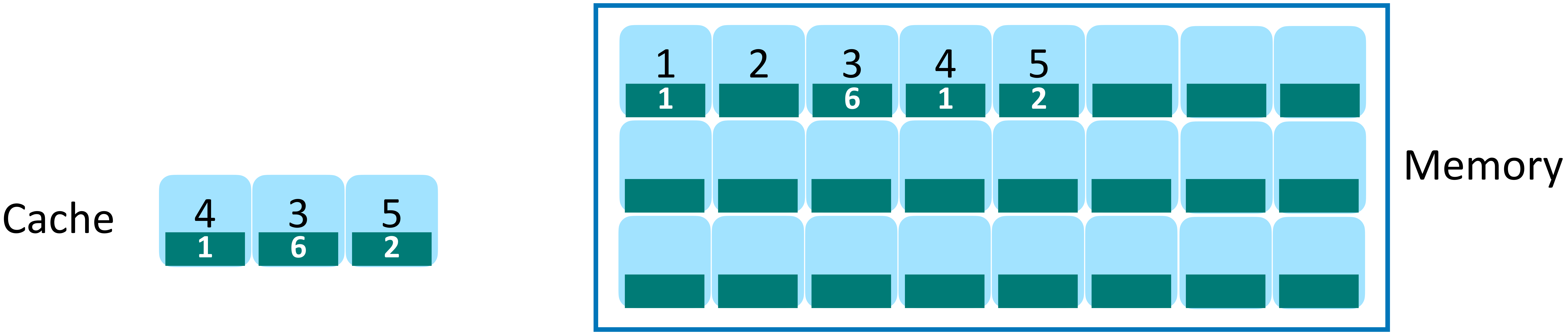
1, 3, 3, 3, 5, 4, 3, 3, 3, 5, 2



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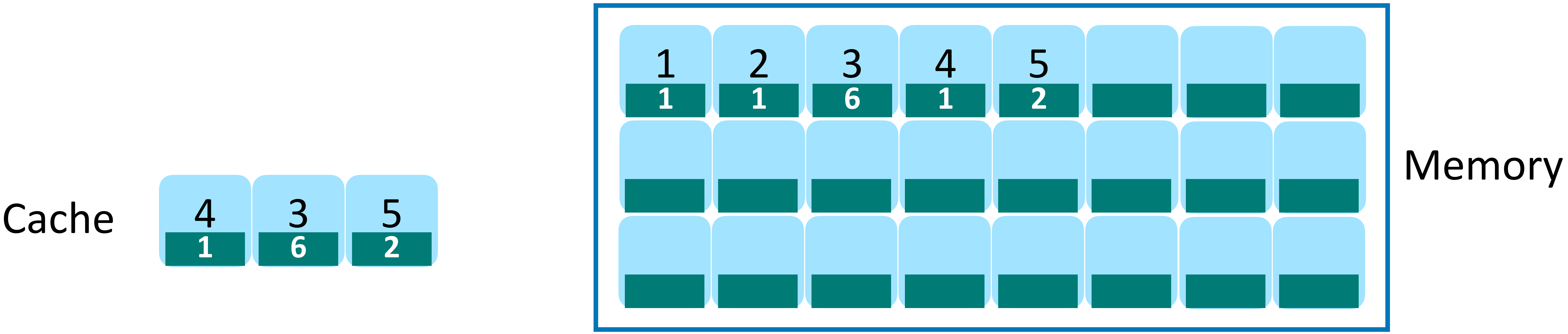
1, 3, 3, 3, 5, 4, 3, 3, 3, 5, 2



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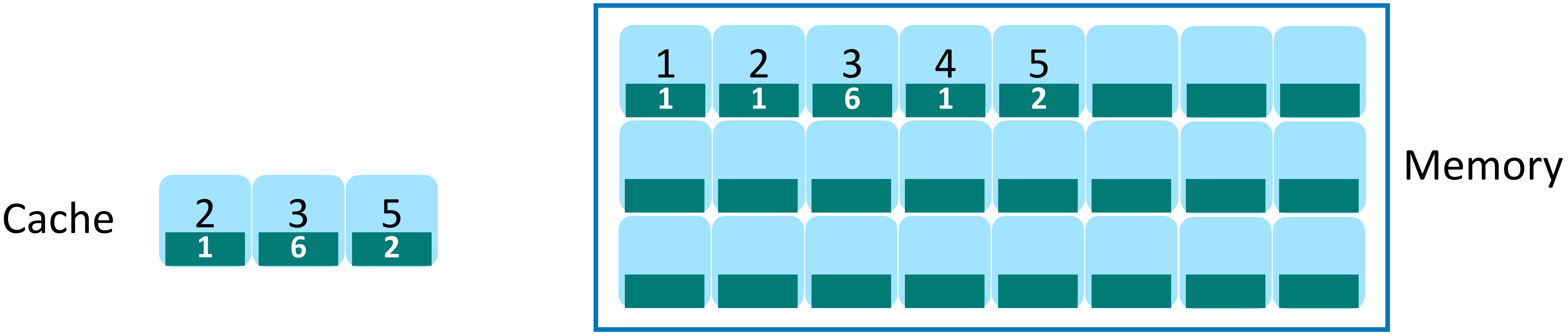
1, 3, 3, 3, 5, 4, 3, 3, 3, 5, 2



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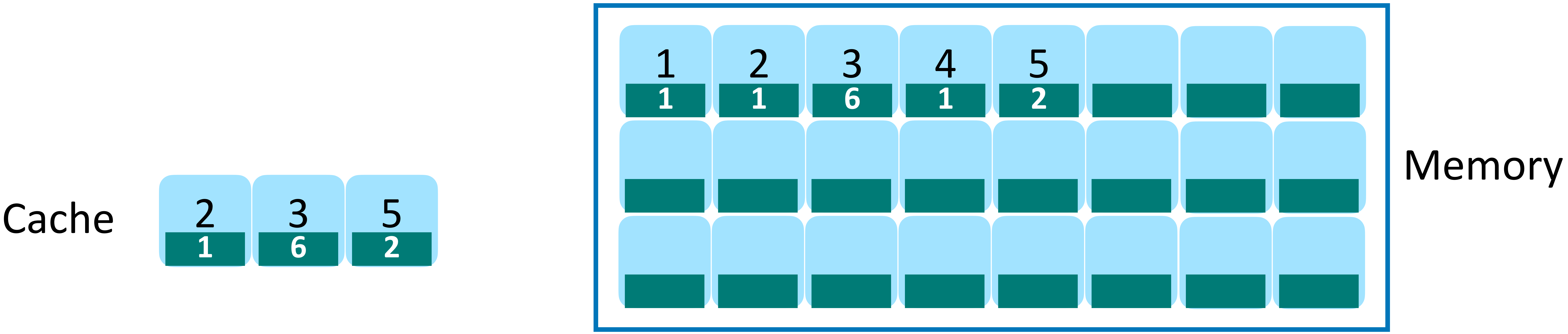
Every page has a **counter** that keep the number of times it has been accessed

Once a page fault is incurred, evict the one with the lowest counter value (break tie arbitrarily)

1, 3, 3, 3, 5, 4, 3, 3, 3, 5, 2, 1



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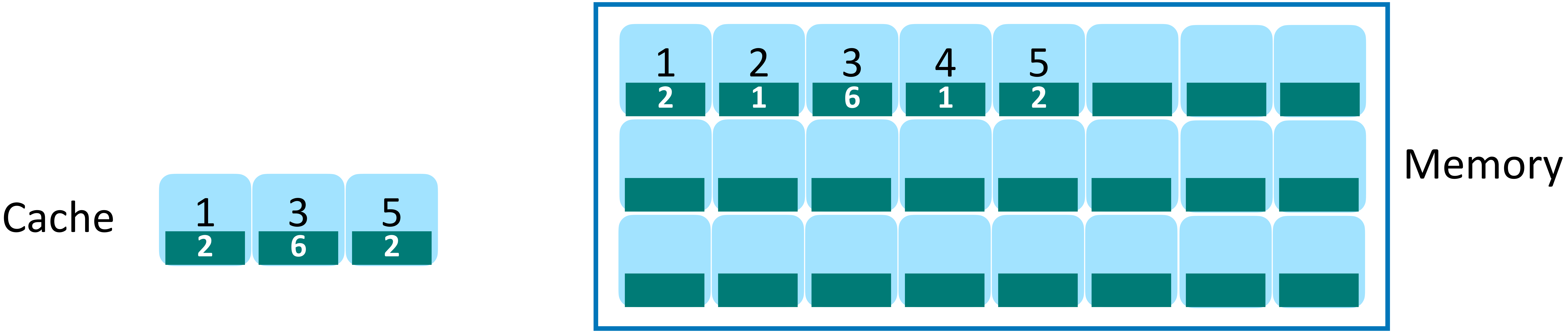
1, 3, 3, 3, 5, 4, 3, 3, 3, 5, 2, 1



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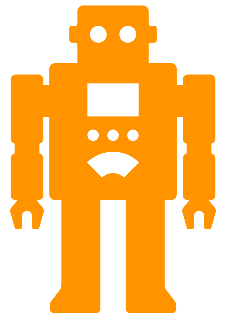
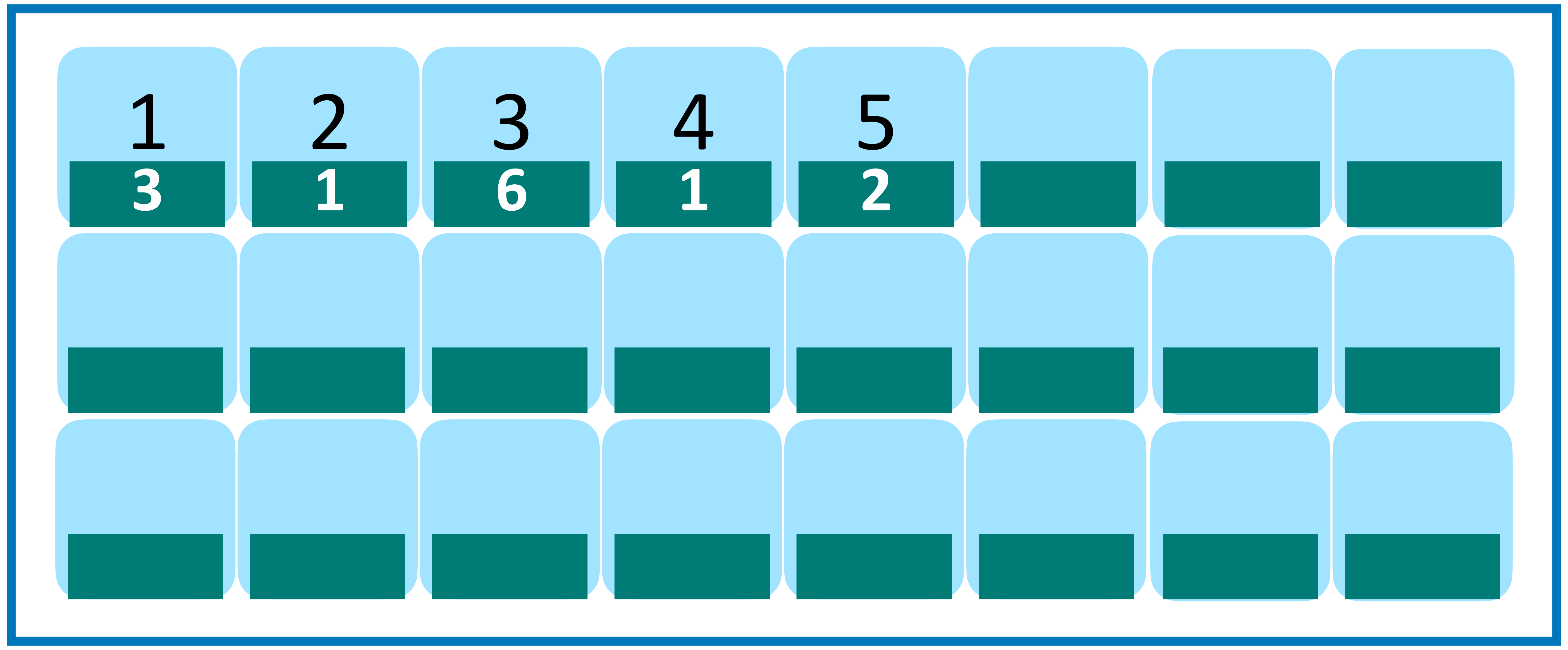


1, 3, 3, 3, 5, 4, 3, 3, 3, 5, 2, 1, 1

Cache



Memory

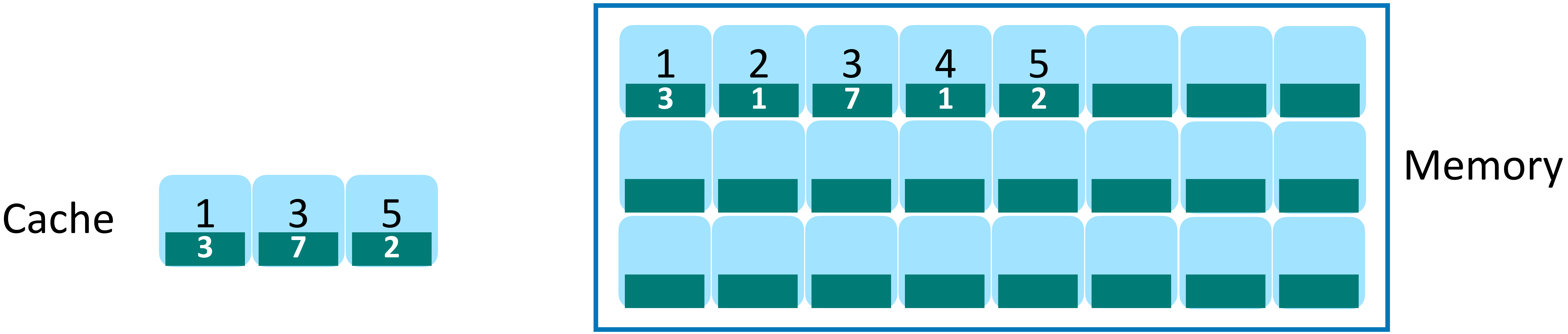
LFU (Least-Frequently-Used)

LFU (Least-Frequently-Used) algorithm:

Every page has a **counter** that keep the number of times it has been accessed

Once a page fault is incurred, evict the one with the lowest counter value (break tie arbitrarily)

1, 3, 3, 3, 5, 4, 3, 3, 3, 5, 2, 1, 1, 3



LFU competitive ratio is unbounded

LFU competitive ratio is unbounded

- Consider the sequence of requests:

$$\underbrace{1, 1, \cdots, 1}_{m \text{ x } 1\text{'s}}, \underbrace{2, 2, \cdots, 2}_{m \text{ x } 2\text{'s}}, \underbrace{3, 3, \cdots, 3}_{m \text{ x } 3\text{'s}}, \cdots, \underbrace{k-1, k-1, \cdots, k-1}_{m \text{ x } k-1\text{'s}}, \underbrace{k, k+1, \text{k}, k+1, \cdots, k, k+1}_{m \text{ x } (k, k+1)\text{'s}}$$

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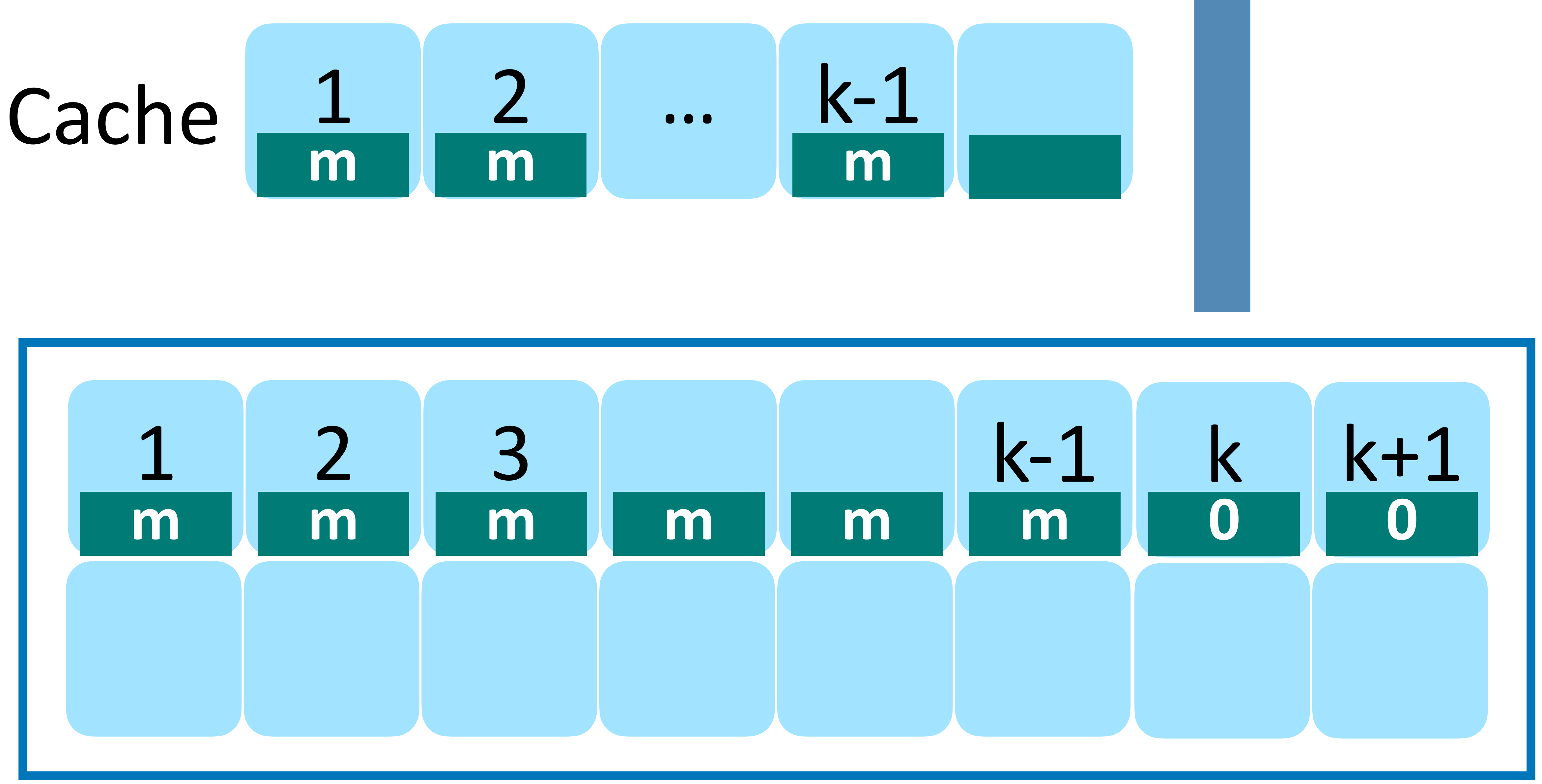

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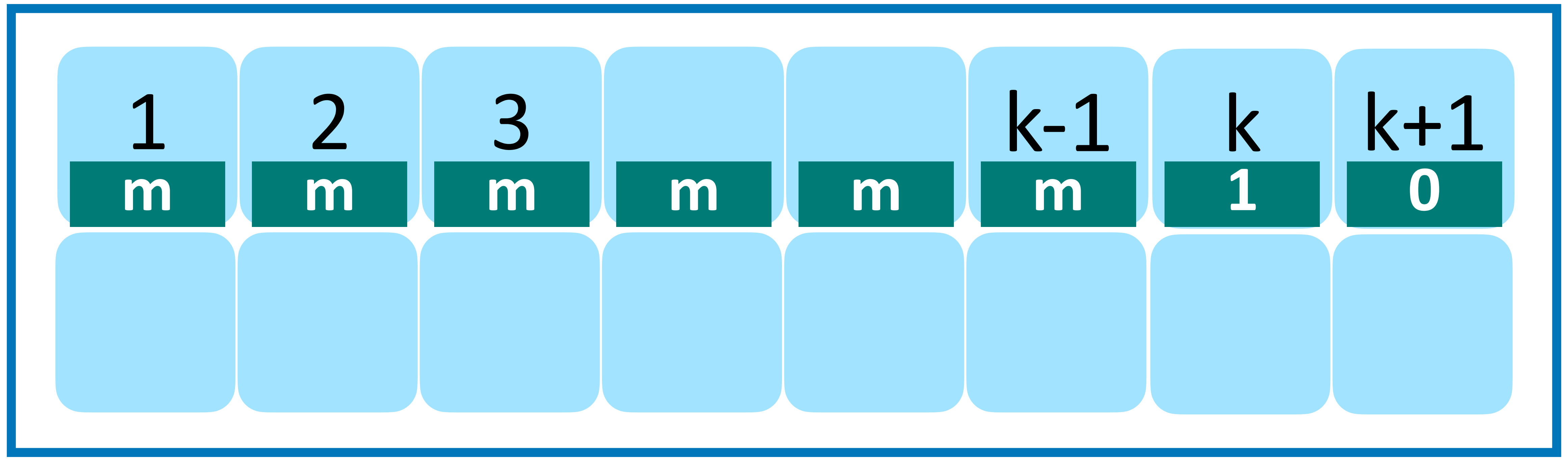
Cache



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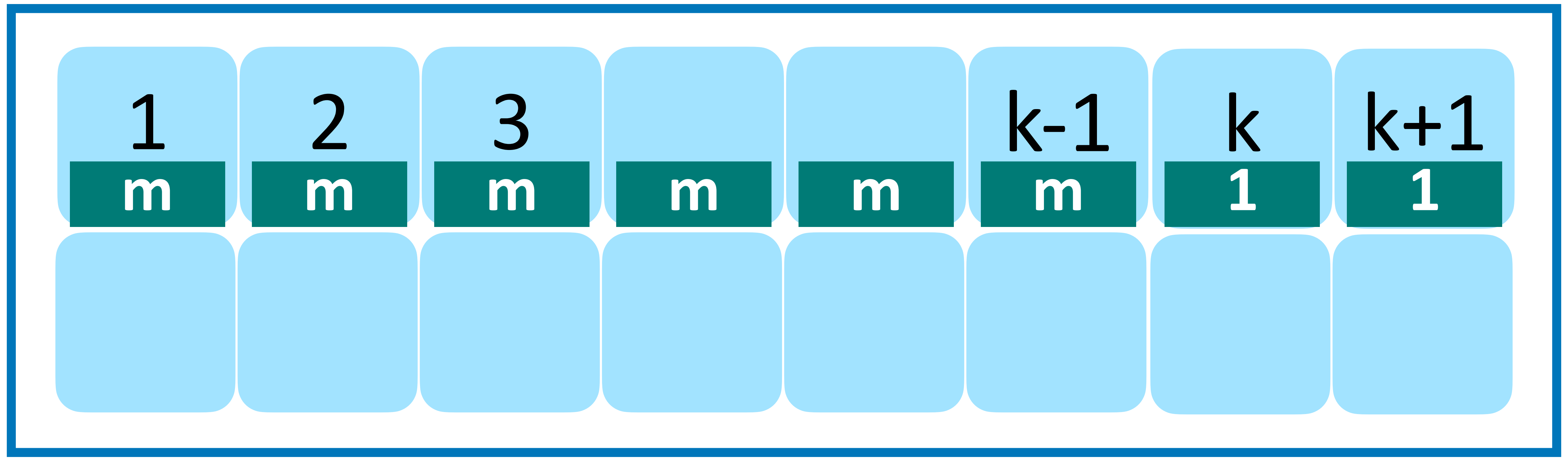

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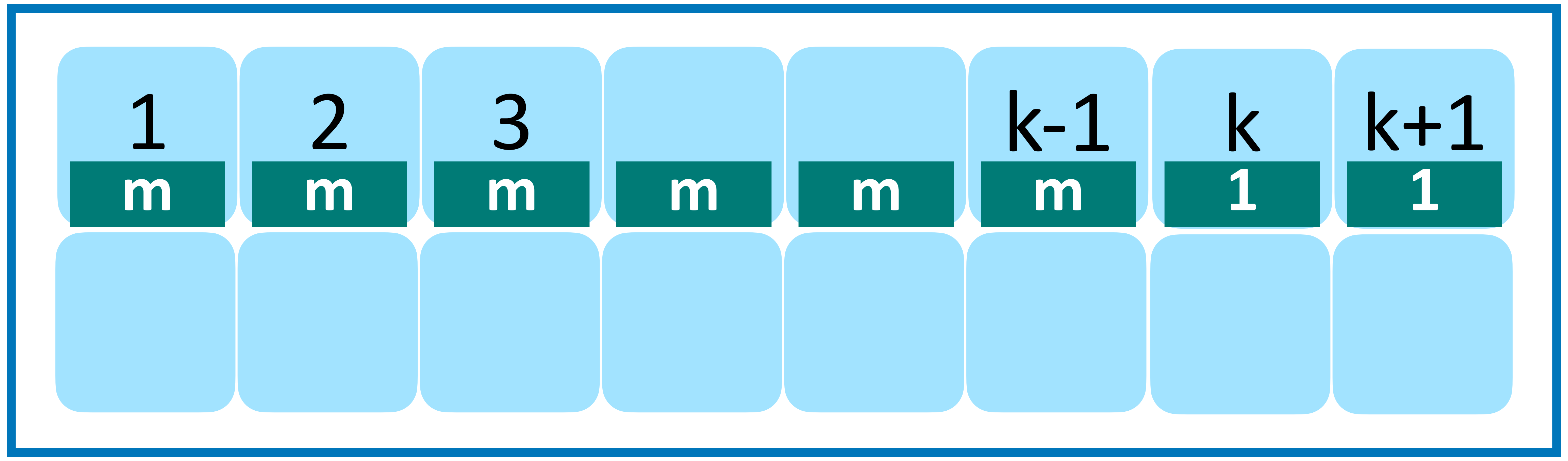
Cache



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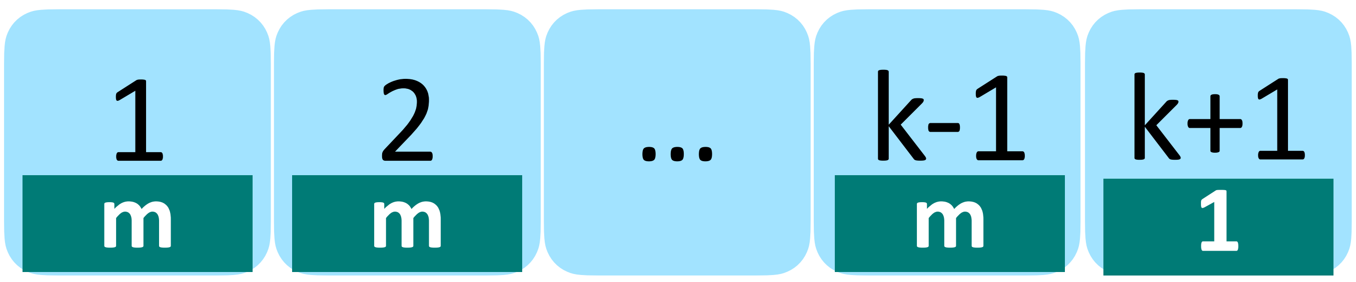
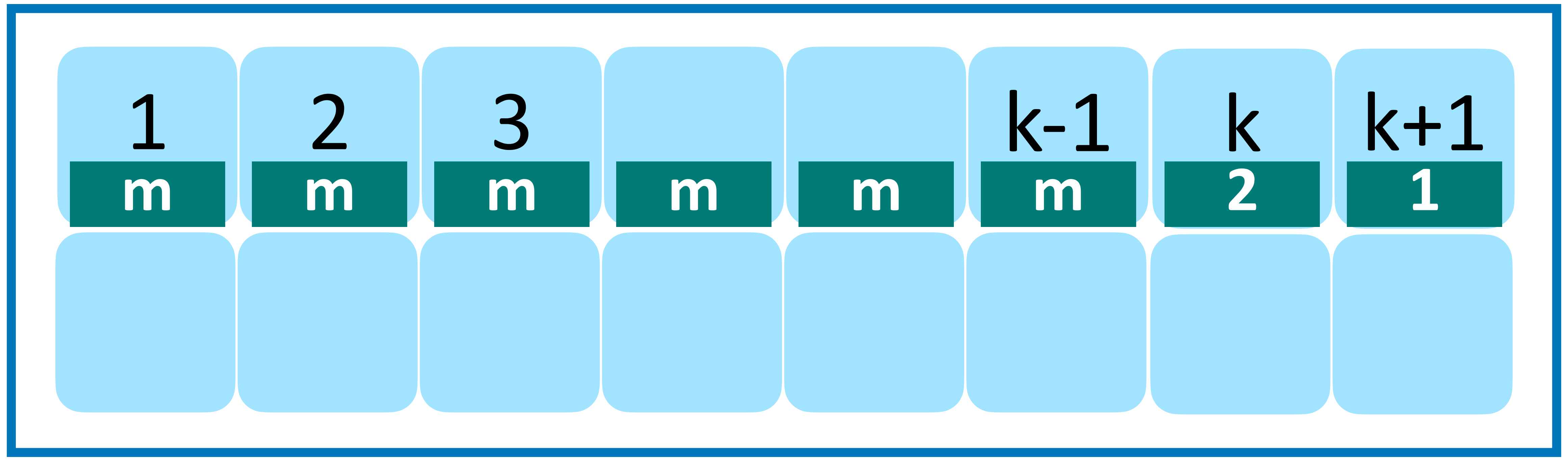
Cache 



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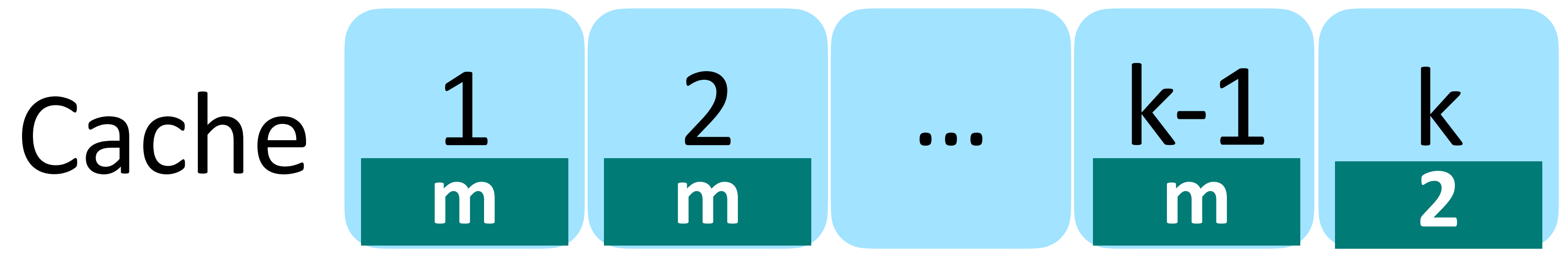

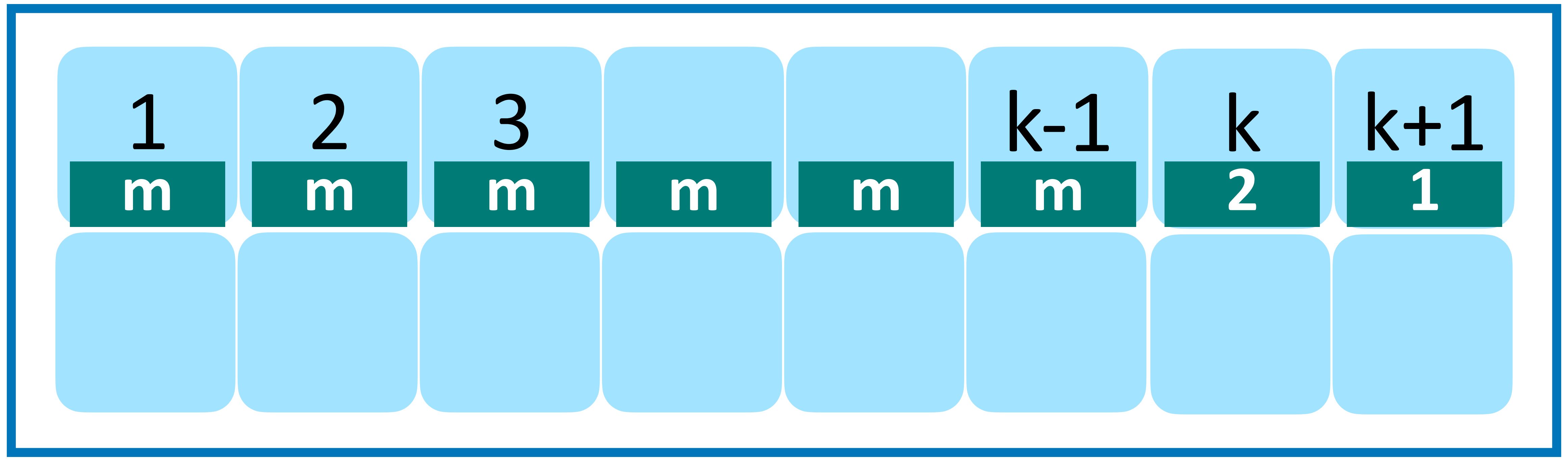

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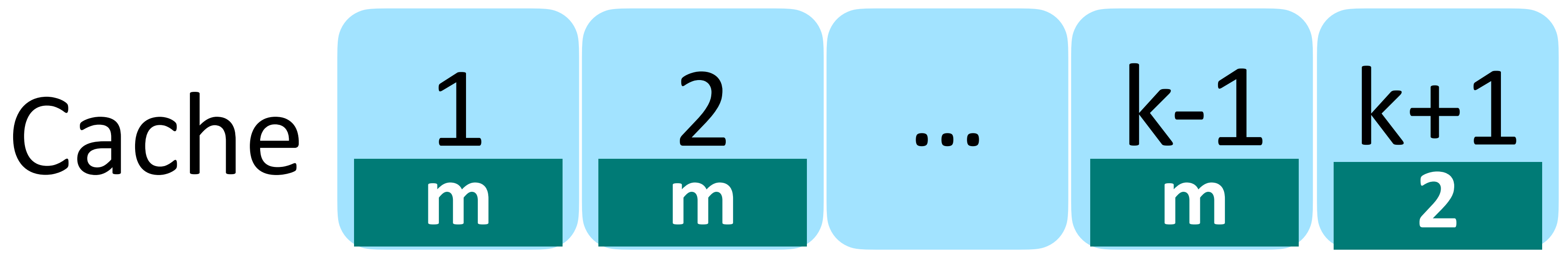

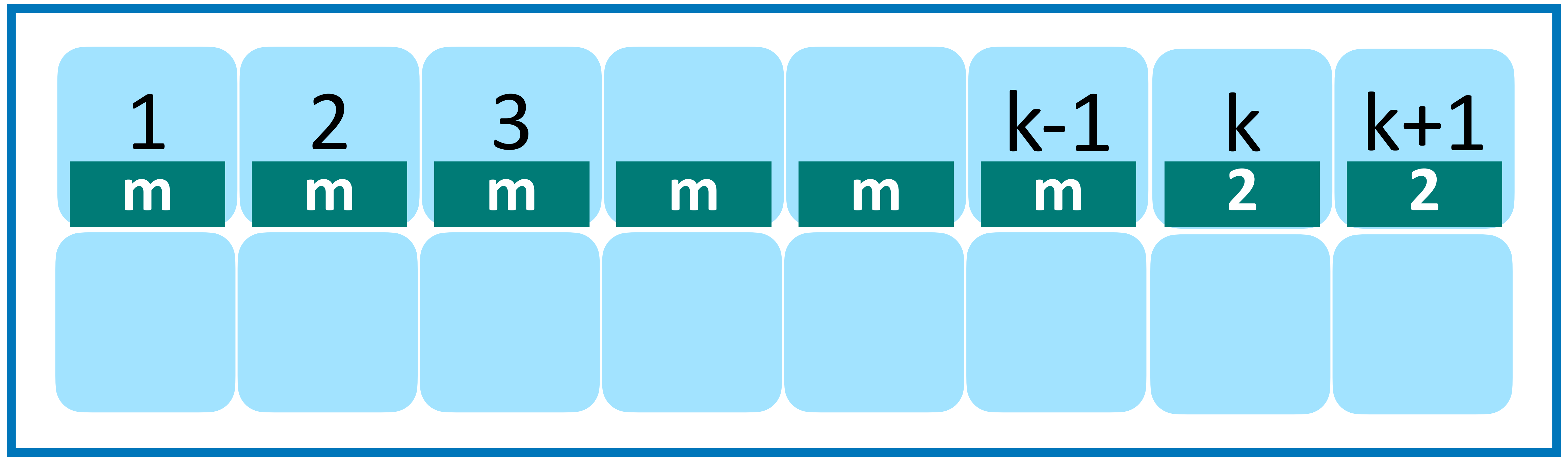

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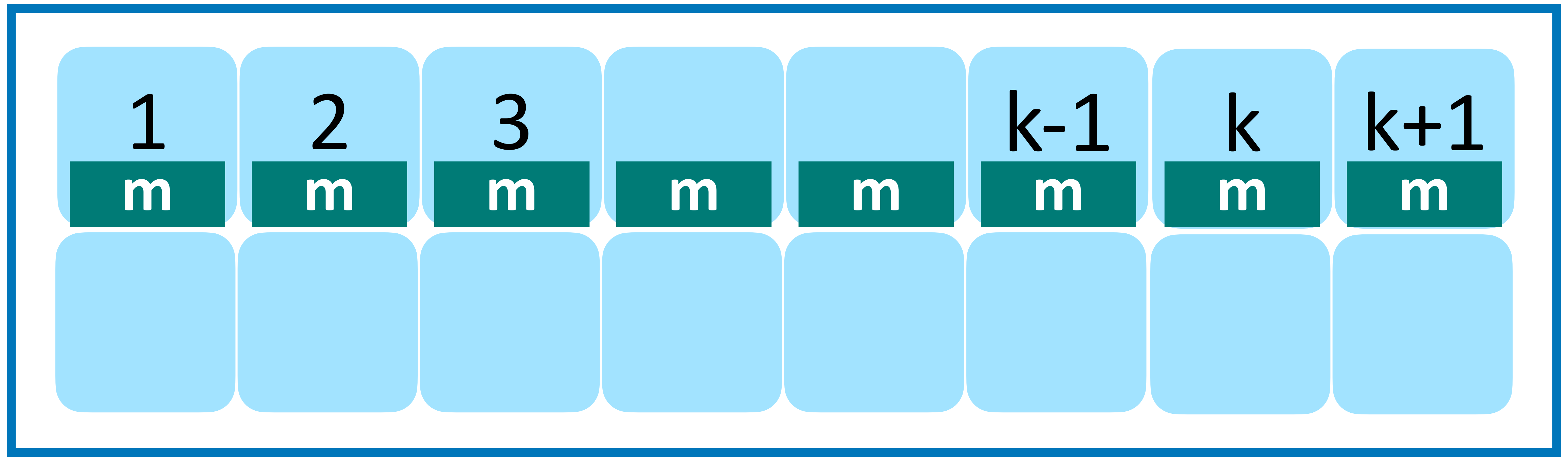
Cache

LFU $= (k-1) + 2m$



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Cache	1 m	2 m	…	k-1 m	k+1 m
OPT cache	1	2	…	k-1	k

$\textbf{LFU} = (k-1) + 2m$

LFU competitive ratio is unbounded

- Consider the sequence of requests:

$$\underbrace{1, 1, \cdots, 1}_{m \text{ x } 1\text{'s}}, \underbrace{2, 2, \cdots, 2}_{m \text{ x } 2\text{'s}}, \underbrace{3, 3, \cdots, 3}_{m \text{ x } 3\text{'s}}, \cdots, \underbrace{k-1, k-1, \cdots, k-1}_{m \text{ x } k-1\text{'s}}, \underbrace{k, k+1, \text{k}, k+1, \cdots, k, k+1}_{m \text{ x } (k, k+1)\text{'s}}$$

Cache: 1 (m), 2 (m), …, k-1 (m), k+1 (m)

$\textbf{LFU} = (k-1) + 2m$

OPT cache: 1, 2, …, k+1, k

$\textbf{OPT} = k + 1$

LFU competitive ratio is unbounded

- Consider the sequence of requests:

$$\underbrace{1, 1, \cdots, 1}_{m \text{ x } 1\text{'s}}, \underbrace{2, 2, \cdots, 2}_{m \text{ x } 2\text{'s}}, \underbrace{3, 3, \cdots, 3}_{m \text{ x } 3\text{'s}}, \cdots, \underbrace{k-1, k-1, \cdots, k-1}_{m \text{ x } k-1\text{'s}}, \underbrace{k, k+1, \text{k}, k+1, \cdots, k, k+1}_{m \text{ x } (k, k+1)\text{'s}}$$

Cache	1 (m)	2 (m)	...	k-1 (m)	k+1 (m)
OPT cache	1	2	...	k+1	k

LFU $= (k-1) + 2m$

OPT $= k + 1$

When m is large enough, $\frac{\text{LFU}}{\text{OPT}} \approx O(\frac{m}{k})$

The ratio grows with the input $\Rightarrow$ **unbounded** $\square$

What Happened

- By a special requests sequence, we can force LFU to incur page faults frequently while the optimal assignment is still efficient

- The competitive ratio of LFU grows with the input size — **unbounded**

LRU (Least-Recently-Used)

LRU (Least-Recently-Used) algorithm:

Once a page fault is incurred, evict the one that was used the least recently

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Cache

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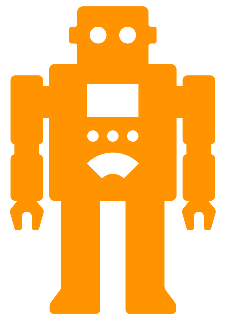 1

Cache

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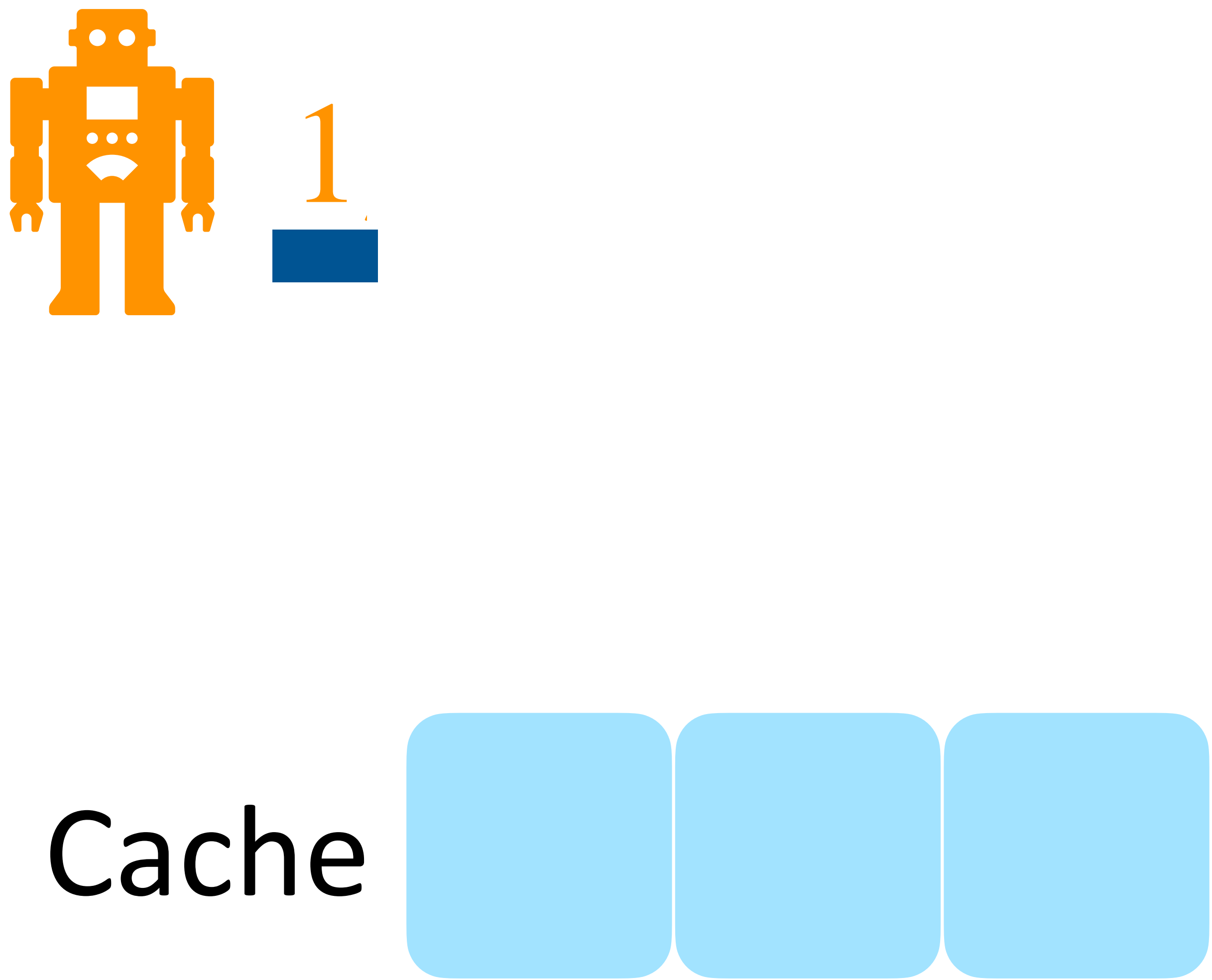

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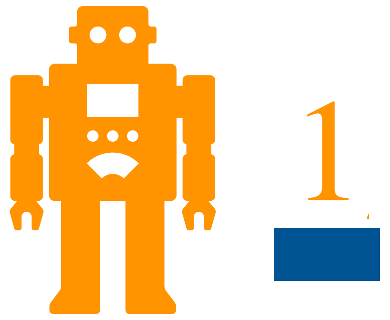

Cache	1		

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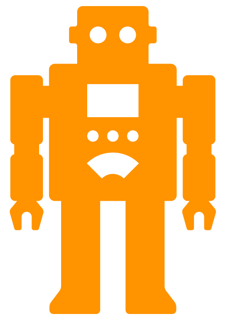

1, 3

Cache	1		

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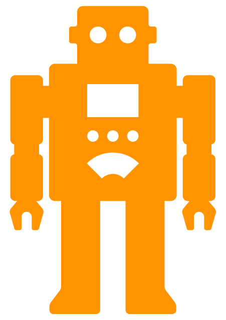

1, 3

Cache | 1 | 3 | |

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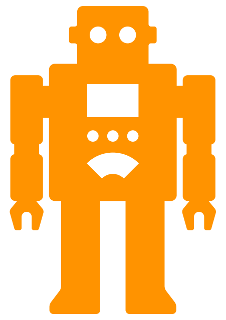

1, 3, 3

Cache	1	3	

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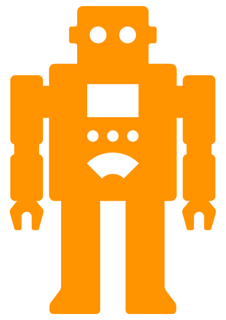

1, 3, 3, 5

Cache | 1 | 3 | 5 |

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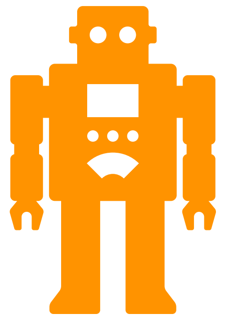

1, 3, 3, 5, 4

Cache

1	3	5

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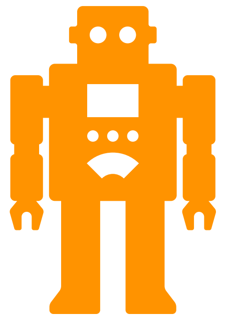

★ 1, 3, ★ 3, ★ 5, 4

Cache | 1 | 3 | 5 |

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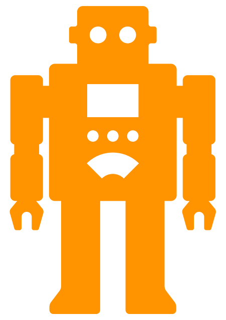

★ ★ ★
1, 3, 3, 5, 4

Cache | 4 | 3 | 5

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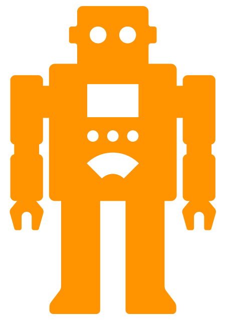

1, 3, 3, 5, 4, 3

Cache | 4 | 3 | 5

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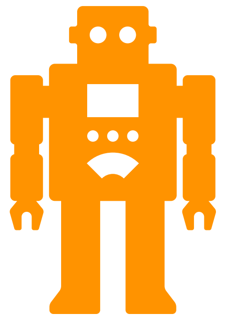

1, 3, 3, 5, 4, 3, 2

Cache | 4 | 3 | 5

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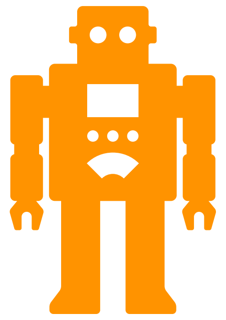


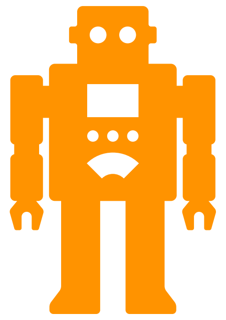
1, 3, 3, 5★, 4★, 3★, 2

Cache | 4 | 3 | 5

LRU (Least-Recently-Used)

LRU (Least-Recently-Used) algorithm:

Once a page fault is incurred, evict the one that was used the least recently



1, 3, 3, 5, 4, 3, 2

Cache | 4 | 3 | 2 |

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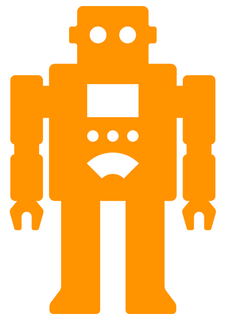

1, 3, 3, 5, 4, 3, 2

Cache | 4 | 3 | 2

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Cache

4	3	2

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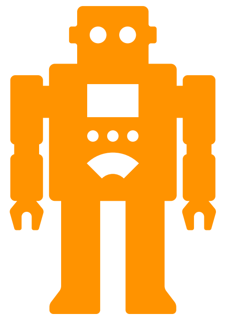

1, 3, 3, 5, 4★, 3★, 2★, 5

Cache | 5 | 3 | 2 |

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1, 3, 3, 5, 4, 3, 2, 5, 2

Cache

5	3	2

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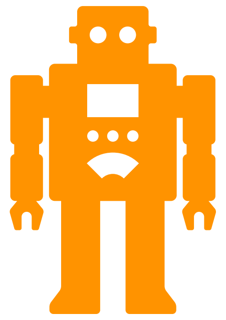 1, 3, 3, 5, 4, 3, 2, 5, 2, 1

Cache

5	3	2

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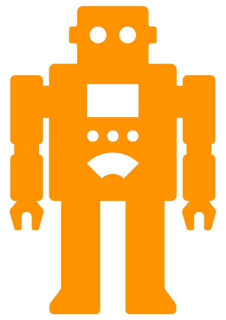

1, 3, 3, 5, 4, 3, 2, 5, 2, 1

Cache | 5 | 1 | 2 |

LRU is k-competitive

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<Proof idea>

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Phase partitioning: partition the request sequence into phases and bound the cost of LRU and OPT in each phase

- Phase 0 is empty
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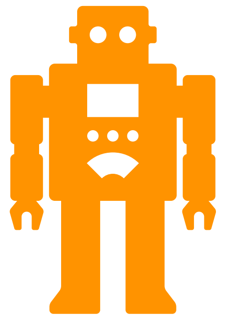

$1, 3, 3, 5, 4, 3, 2, 5, 2, 1, 1, 3, 2, 3, 1, 3, 3, 5, 3, 5, 2, 1$

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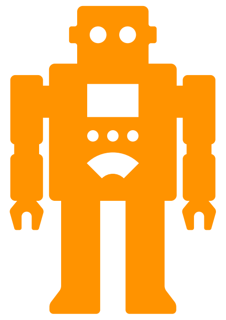

$\boxed{1, 3, 3, 5}$ $4, 3, 2, 5, 2, 1, 1, 3, 2, 3, 1, 3, 3, 5, 3, 5, 2, 1$

Phase 1

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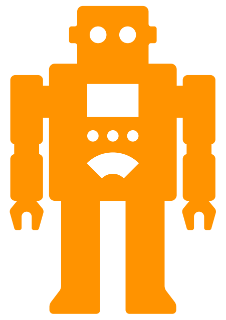

$\boxed{1, 3, 3, 5}\ \boxed{4, 3, 2}\ 5, 2, 1, 1, 3, 2, 3, 1, 3, 3, 5, 3, 5, 2, 1$

Phase 1 Phase 2

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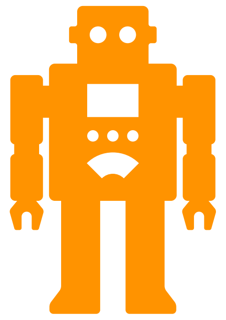

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Phase 1 Phase 2 Phase 3

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Phase 1 Phase 2 Phase 3 Phase 4

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Phase 1	Phase 2	Phase 3	Phase 4	Phase 5	
1, 3, 3, 5,	4, 3, 2,	5, 2, 1, 1,	3, 2, 3, 1, 3, 3,	5, 3, 5, 2,	1

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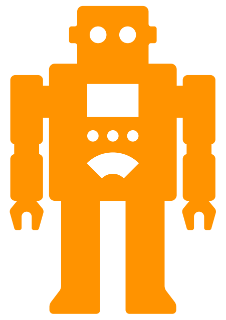 $\boxed{1, 3, 3, 5,}\boxed{4, 3, 2,}\boxed{5, 2, 1, 1,}\boxed{3, 2, 3, 1, 3, 3,}\boxed{5, 3, 5, 2,}\boxed{1}$

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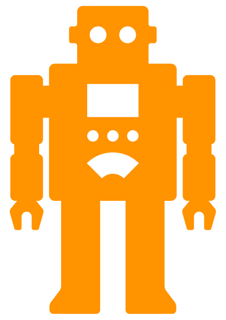

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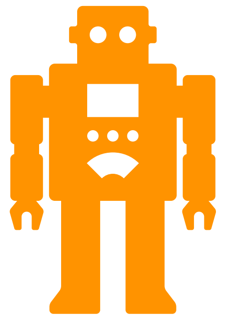

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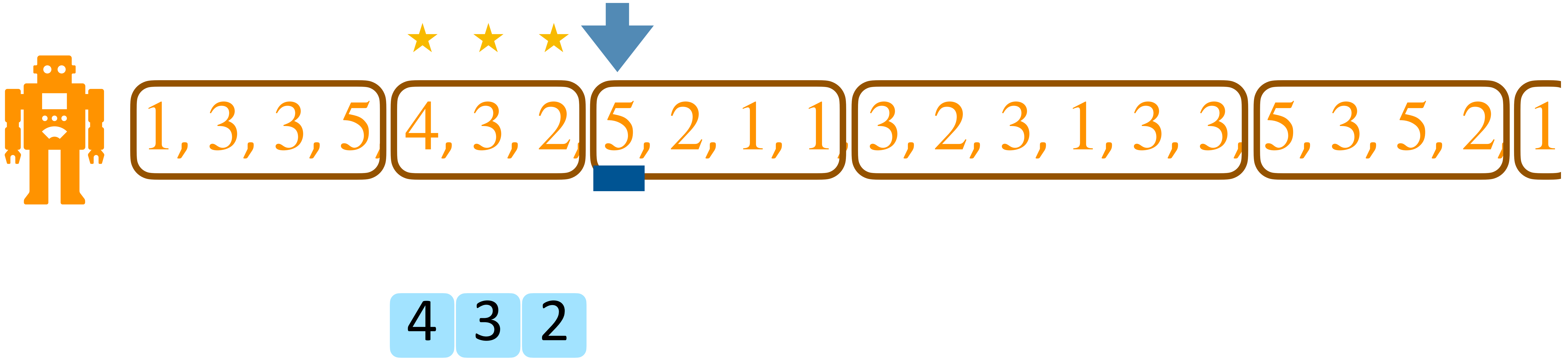

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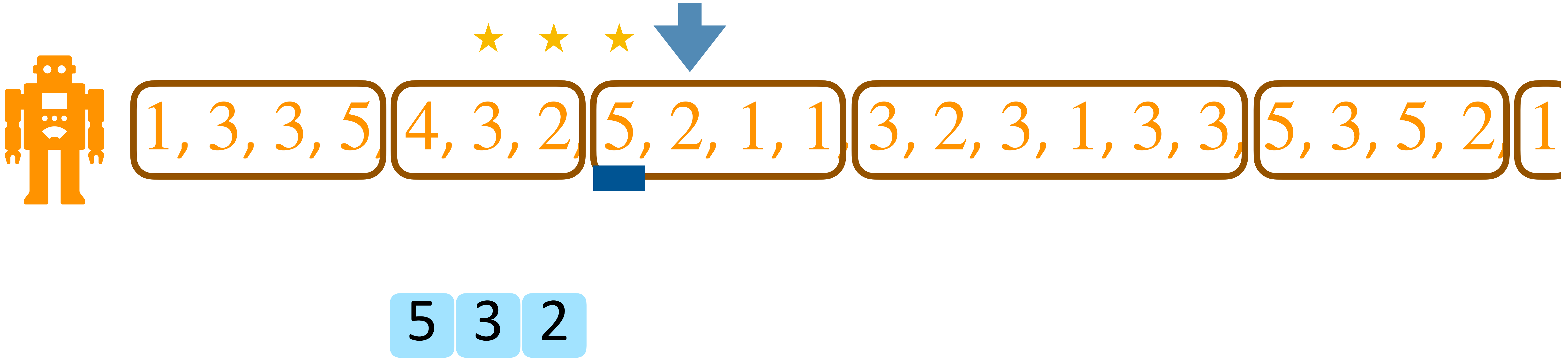

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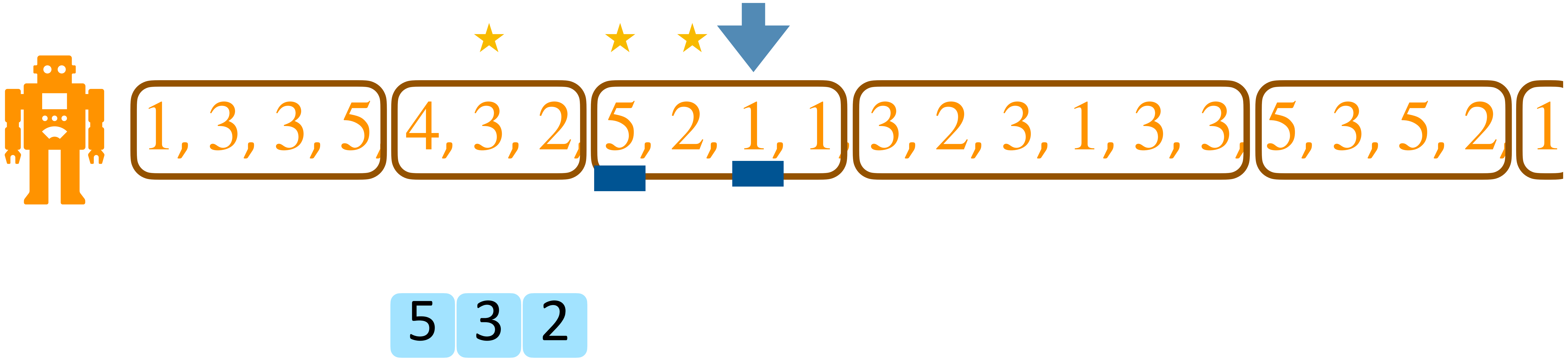

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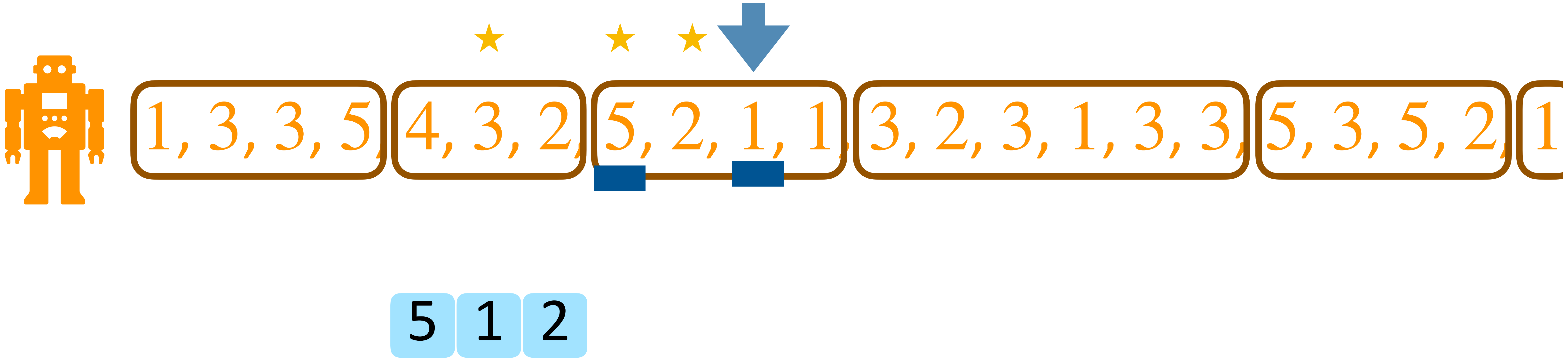

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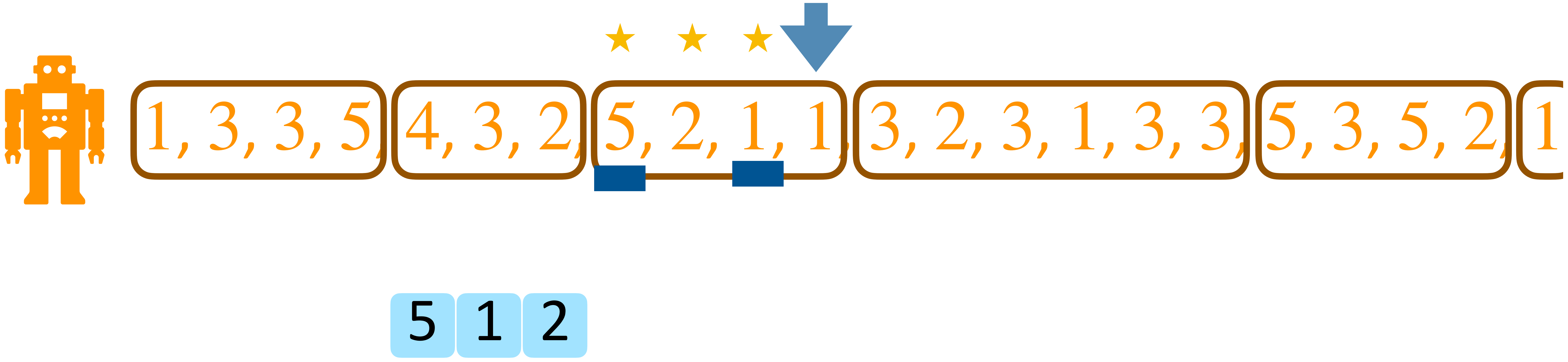

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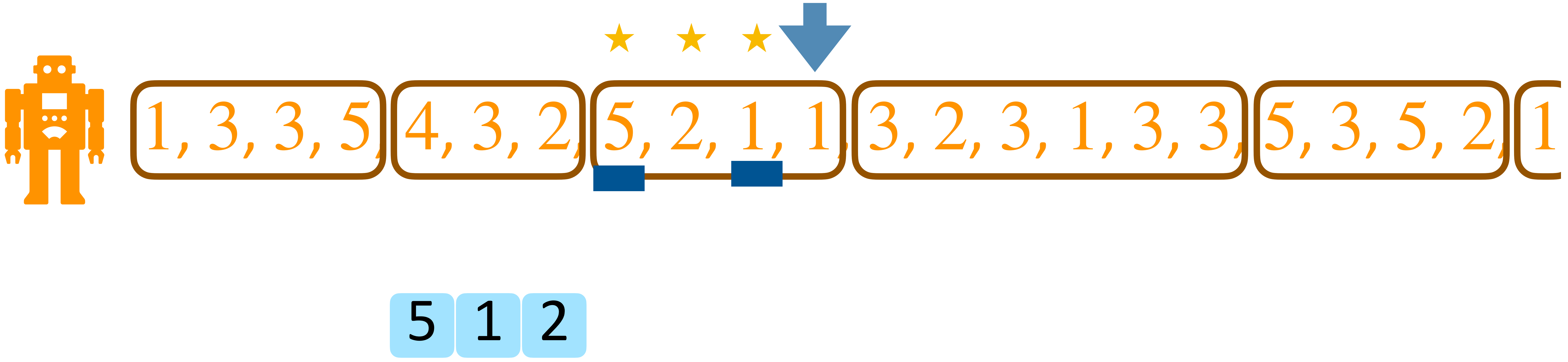

At the moment when the j-th distinct page in phase i is requested, there are $j-1$ pages accessed in phase i.
$\Rightarrow$ There are $k-(j-1)$ pages in the cache that haven't been accessed recently. Hence, LRU will evict one of them.

LRU is k-competitive

<Proof idea>

- Claim (b): In phase i, **OPT** incurs at least 1 page fault

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1, 3, 3, 5 | 4, 3, 2 | 5, 2, 1, 1 | 3, 2, 3, 1, 3, 3 | 5, 3, 5, 2 | 1

5	1	2

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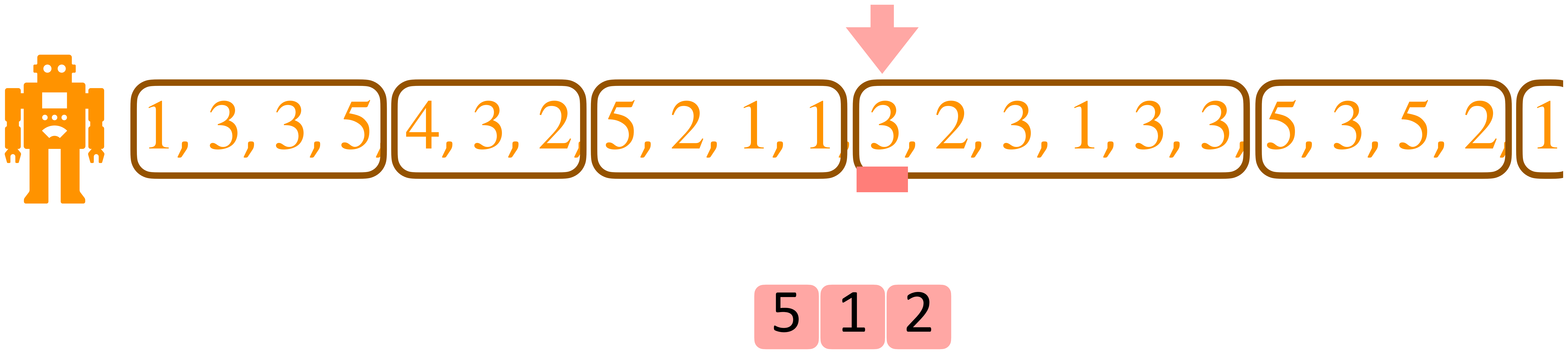

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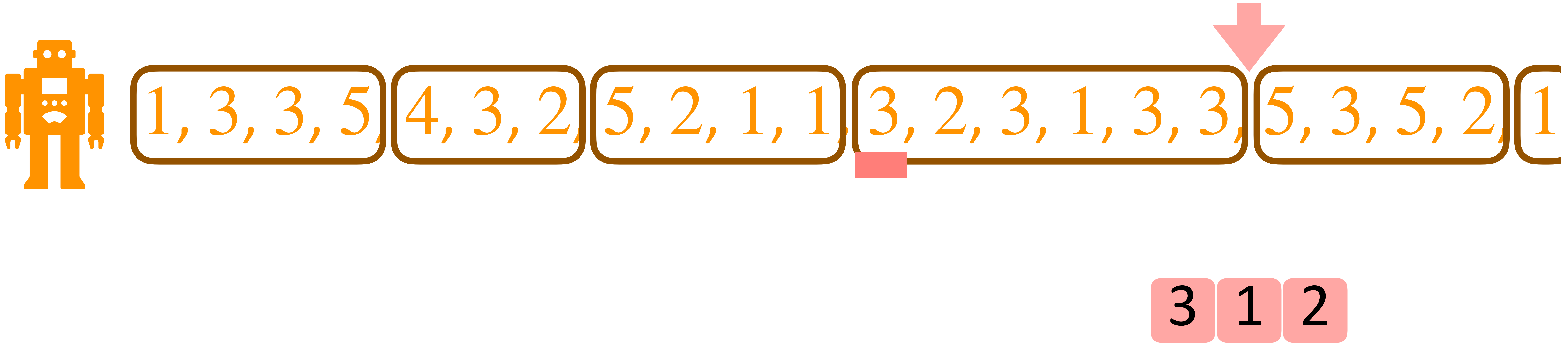

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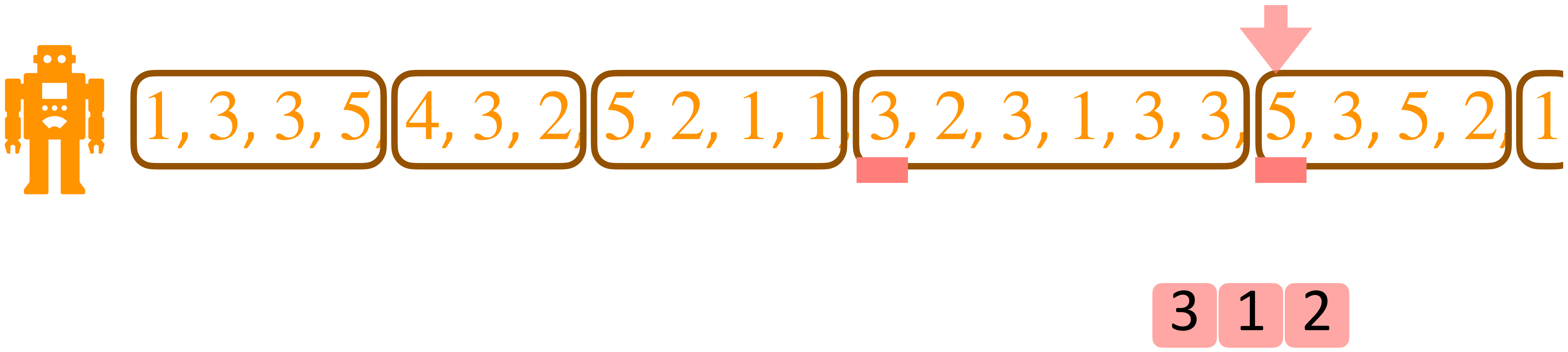

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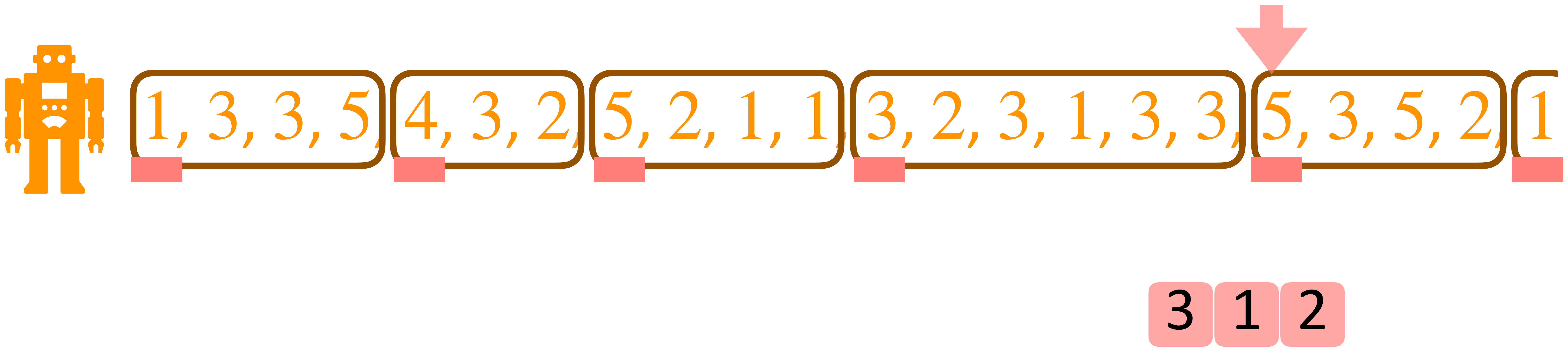

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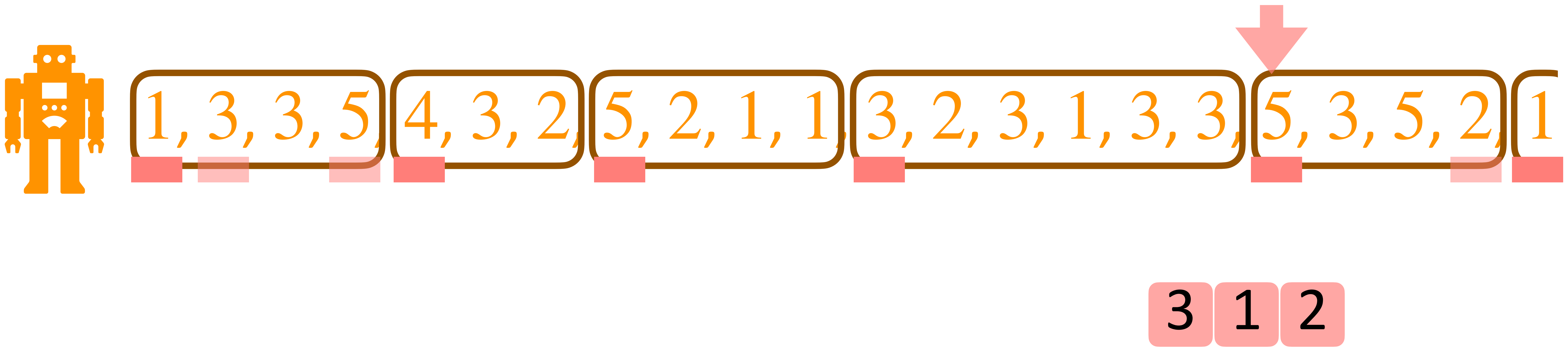

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Let LRU_i and OPT_i denote the page fault incurred by **LRU** and **OPT** in phase i, respectively. By Claim (a) and Claim (b),

$$\frac{\text{LRU}(I)}{\text{OPT}(I)} = \frac{\sum_i \text{LRU}_i}{\sum_i \text{OPT}_i} \leq \frac{k}{1} = k$$

$\square$

What Happened

- **Phase partitioning**: partition the request sequence into phases such that each phase has k distinct pages

- By arguing that an algorithm incurs at most k page faults and OPT incurs at least 1 page fault in any phase, we can conclude that the algorithm is at most $O(\frac{n}{k})$-competitive

- Arguing that **an algorithm incurs at most k page faults** is the key!

Paging Problem Lower Bound

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<Proof idea>

Assume that the cache size is k. Consider any algorithm **ALG** and design the adversary as follows: First request pages $1, 2, 3, \cdots, k$

1	2	3	…	…	k

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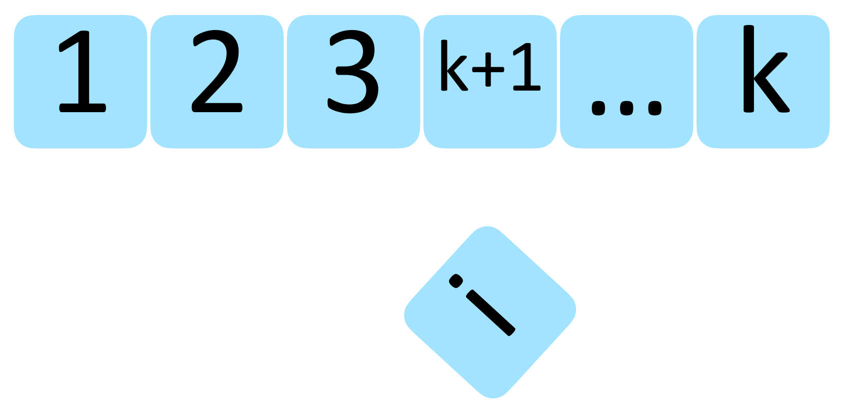

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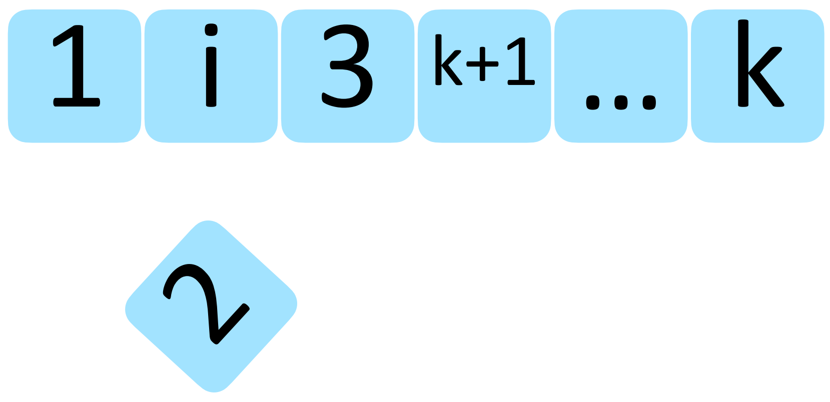

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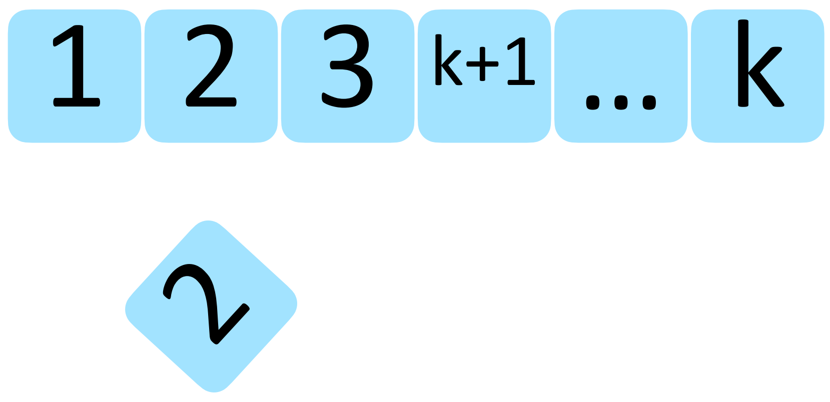

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In this instance, each request incurs a page fault for ALG. Therefore, **ALG** costs $k+n$.

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Because there are only $k+1$ pages involved, **OPT** incurs at most 1 page fault per k pages.

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Therefore, $\dfrac{\text{ALG}(I)}{\text{OPT}(I)} \geq \dfrac{k+n}{k+n/k} \approx \Omega(k)$

Even when every page requests change dramatically, the optimal solution can keep the k pages that will be used in the most recent future and evict the one that will be used later.

Paging Problem Lower Bound

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Research cycle of online algorithms