EN 530.766
Fall 2023
HW 4–Haobo Zhao

December 5, 2023

Abstract

This report is an approach using 2D-Finite Difference method to simulate the Elliptic problem (Jacobi, Gauss-Seidel, SOR, SRJ), as the boundary condition, grid, and control equation are provided.

In the first question, use Jacobi iteration method and Gauss-Seidel (G-S) method to iterate, and calculated the residual and the iteration error (absolute sum of residual). It could be found that the error is droping fast as the iteration number growing, initial guess (IG)=0 converge fastest among IGs, G-S method is converge faster than Jacobi except for IG=0.

For the question (2), by use SOR point Jacobi and SOR point G-S method, introducing relaxization parameter w, found optimal w is around 1.7 for G-S method according all IG. But still, w=1 is best for Jacobi.

For the third question, use the SRJ method, alternatively using w1 and w2, where w1 for underrelaxation and w2 for overrelaxation, found the best combination for w1 and w2 is (0.7,2.0), while frequency (1:1).

For the question (4), by choose and compare different groups of w1, w2, and w3, found the optimal (w1,w2,w3) group: (9.3,1.1,1.0) for frequency (1:9:90), converge faster than non-relaxed Jacobi, and 2-parameter SRJ, which means is still possible to try more parameters or wide range of frequency combination

Contents

1 Review of Questions

Consider the 2-D Laplace equation

$$(u_{xx} + u_{yy}) = 0 \quad \text{for } 0 < x, y < 2\pi$$

with the following boundary conditions

$$\begin{aligned} u(0, y) &= 0 \\ u(2\pi, y) &= 0 \\ u(x, 0) &= \sin(2x) + \sin(5x) + \sin(7x) \\ u(x, 2\pi) &= 0 \end{aligned}$$

1. Write a computer code and obtain the numerical solution using the point Jacobi and Gauss-Seidel iterative schemes. Use a mesh with $\Delta x = \Delta y = \frac{2\pi}{20}$. Track the convergence by calculating the residual, $r^k = (\frac{\delta_x^2}{\Delta x^2} + \frac{\delta_y^2}{\Delta y^2})u_{i,j}^k$

   Conduct these simulation with two different initial guesses

   (1) $u(i, j) = 0$;

   (2) $u(i, j) = x_i y_j$

   (3) $u(i, j)$ = random number distribution between -1 and 1.

   Does the convergence behave as expected? Discuss.

2. Try the above problem with the SOR point Jacobi and SOR point Gauss-Seidel schemes: Investigate and comment on the convergence properties for various values of the under- and over-relaxation parameter for both schemes. Can you find optimal values of the relaxation parameter for these schemes?

3. Experiment with a SOR Jacobi method where you alternate between an overrelaxation and an underrelaxation as you iterate. Can you find a pair of relaxation parameters that speed up the solution process compared to the non-relaxed Jacobi method? For more information about this "Scheduled Relaxation Jacobi" Method check out this paper: *Xiang Yang and Rajat Mittal, "Acceleration of the Jacobi iterative method by factors exceeding 100 using scheduled relaxation", Journal of Computational Physics, Vol 274, DOI: 10.1016/j.jcp.2014.06.010.*

4. Challenge problem – how about experimentally determining a SRJ scheme with 3 different values of the relaxation parameter. How much faster can you get compared to the 2 parameter SRJ scheme.

2 (1): Simulation, resudual compare

2.1 Jacobi, Gauss-Seidel Iteration, Residual

For problem 1, use sudo-time, let Jacobi and Gauss-Seidel iteration method to obtain the next iteration solution. Compare the solution with the Elliptic source term, could occour residual, and take absolute value sum of residual to get total error, as the index of each method's performance.

The steps of the approach is showing below:
The control equation is showing below:

$$(u_{xx} + u_{yy}) = 0 \quad \text{for } 0 < x, y < 2\pi$$

Or in other notation:

$$\nabla^2 P = 0$$

Add sudo-time part, obtain:

$$\nabla^2 P = \frac{\partial P}{\partial t}$$

Where is also can be shown as:

$$(u_{xx} + u_{yy}) = u_t \quad \text{for } 0 < x, y < 2\pi$$

Transfer Partial Difference Equartion (PDE) to Finite Difference equation (FDE), use central difference method for spacial derivative transform, and use forward difference for "time" transformk, obtained:

$$\frac{P_{ij}^{k+1} - P_{ij}^{k}}{\Delta t} = \frac{1}{\Delta^2}\left[P_{i-1,j}^{k} + P_{i+1,j}^{k} + P_{i,j-1}^{k} + P_{i,j+1}^{k} - 4P_{ij}^{k}\right]$$

For stability, in 1-D scheme, r =<1/2, in 2-D scheme, r =< 1/4. To obatin the biggest dt, let r= 1/4, which is $\frac{\Delta t}{\Delta^2} = \frac{1}{4}$, where the FDE transfer to:

$$P_{ij}^{k+1} = \frac{1}{4}\left[P_{i-1,j}^{k} + P_{i+1,j}^{k} + P_{i,j-1}^{k} + P_{i,j+1}^{k}\right]$$

The formula shown above is Jacobi iteration method. For Gauss Seidel method, the equation is showing below:

$$P_{ij}^{k+1} = \frac{1}{4}\left[P_{i-1,j}^{k+1} + P_{i+1,j}^{k} + P_{i,j-1}^{k+1} + P_{i,j+1}^{k}\right]$$

For residual, it could be calculated by compare our simulation result with the source term, which is 0 in this scenario:

$$r^k = \frac{1}{\Delta^2}\left[P_{i-1,j}^{k} + P_{i+1,j}^{k} + P_{i,j-1}^{k} + P_{i,j+1}^{k} - 4P_{ij}^{k}\right]$$

Algorithm 1 Pseudocode for Jacobi Solver

function JACOBI($P_{\text{input}}, \Delta x, \Delta y, Nx, Ny$)
    $P_{\text{old}} \leftarrow P_{\text{input}}$
    $P_{\text{new}} \leftarrow P_{\text{input}}$
    for $j = 1$ to $Ny - 1$ **do**
        for $i = 1$ to $Nx - 1$ **do**
            $P_{\text{new}}[i][j] \leftarrow \frac{1}{4}(P_{\text{old}}[i-1][j] + P_{\text{old}}[i+1][j]$
            $+P_{\text{old}}[i][j-1] + P_{\text{old}}[i][j+1])$
            $P_{\text{old}} \leftarrow P_{\text{new}}$
        end for
    end for
    return P_{new}
end function

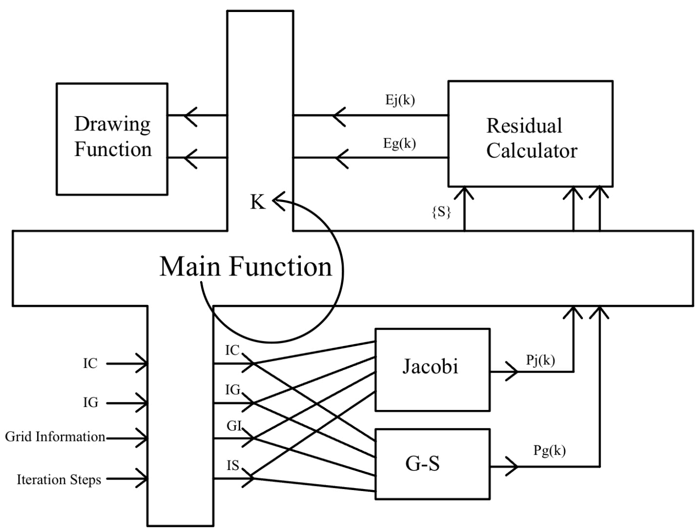
Algorithm 2 Pseudocode for Gauss-Seidel Solver

function GAUSS-SEIDEL($P_input, \Delta x, \Delta y, Nx, Ny$)
    for $j = 1$ to $Ny - 1$ **do**
        for $j = 1$ to $Ny - 1$ **do**
            $P_{\text{new}}[i][j] \leftarrow \frac{1}{4}(P_{\text{new}}[i-1][j] + P_{\text{old}}[i+1][j] + P_{\text{new}}[i][j-1] + P_{\text{old}}[i][j+1])$
$P_{\text{old}} \leftarrow P_{\text{new}}$
        end for
    end for
    return P_{new}
end function

2.2 Algorithms for Jacobi, G-S, and Residual

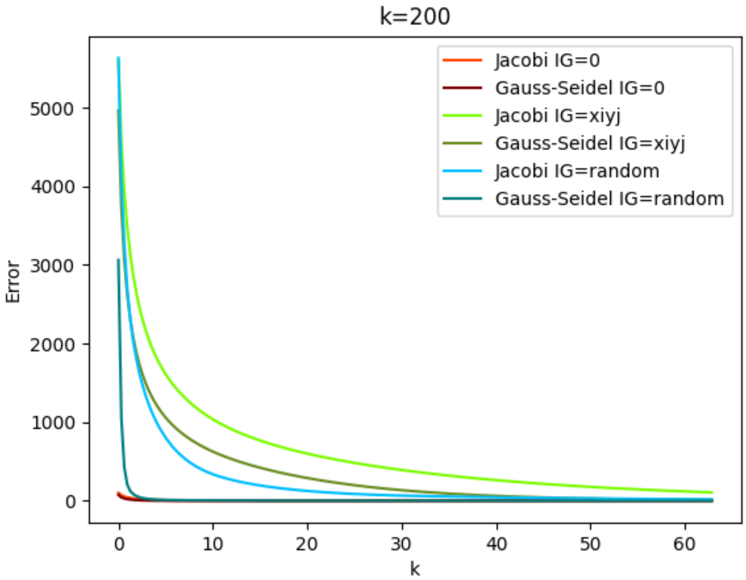
Base on this, the Algoritm for Jacobi iteration method can be written:

The strture of Python program is showing as follows:

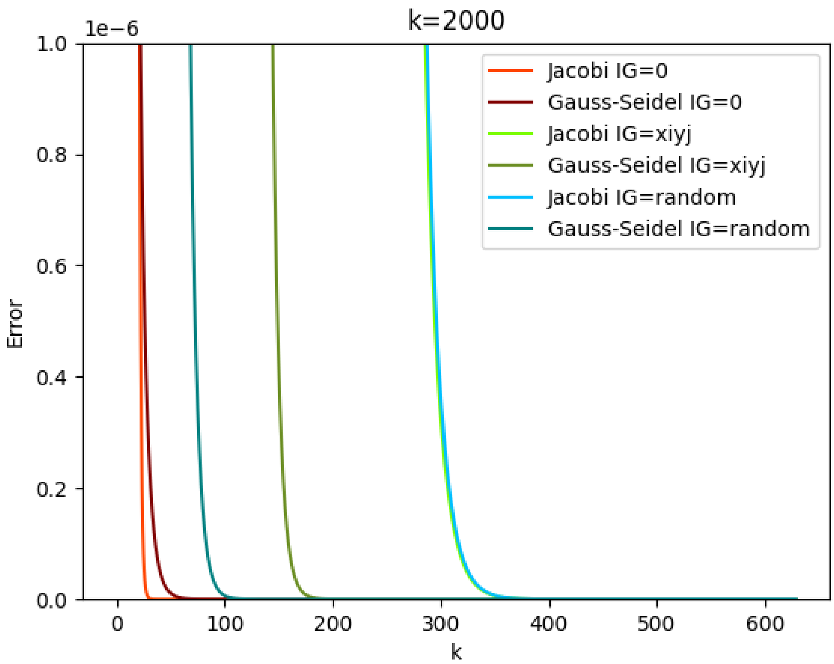


2.3 Result and Analysis for (1)

For iteration(k) 200 times, the result is showing below:



It can be observed that the error was very huge, but drop really quick as iteration keep going. Also, if we running more iteration times, the result should be more accurate. For k=2000, the result is showing below:



The figure arrange different initial guess(IG) in different color set. For IG=0, Jacobi is the shallow red line, while G-S is in heavy red. Same as Jacobi and G-S in green with IG=xiyj, and in blue as IG=random.

2.3.1 Analysis For IGs

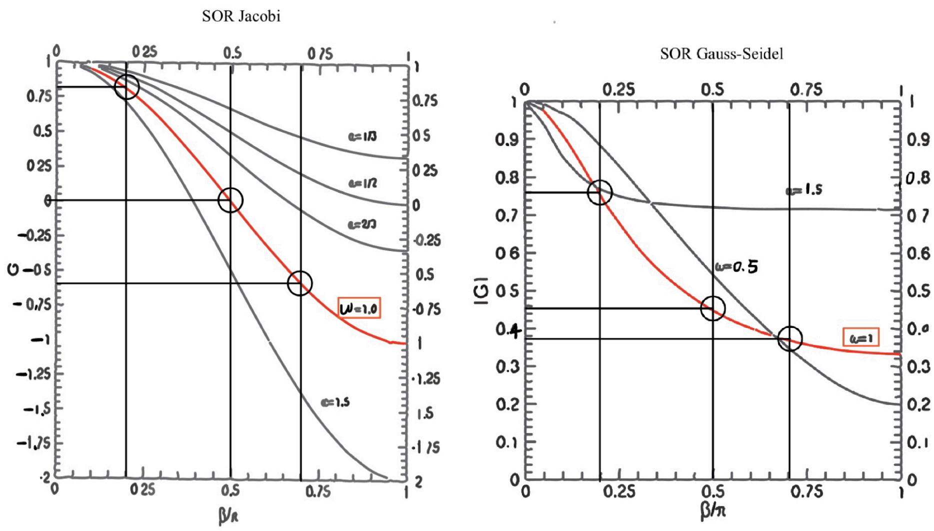
Compare different initial condition (IG), it is easily to find that IG=0 makes the fastest IG to let error back to 0, while other two are pretty close. That may because IG=0 is the closet IG with the actual result that boundary condition (3 sides are 0, one side is combination of sin) created.

2.3.2 Analysis For Jacobi and Gauss-Seidel

Compare Jacobi and Gauss-seidul (G-S) in each IG, it is can be observed Gauss-Seidel converge much faster than Jacobi, except for IG=0, where Jacobi finally converge faster than Gauss-Seidel. This could be explained by the wavenumber-analysis: As the control equation is linear, the wavenumber of result all come from initial guess (IG) and boundary conditions (BC).

IG=0

For IG=0, the wavenumbers are all come from BCs, where k=2,5,7 ($u(x,0) = \sin(2x) + \sin(5x) + \sin(7x)$), then $\frac{\beta}{\pi}$= 0.2, 0.5, 0.7.



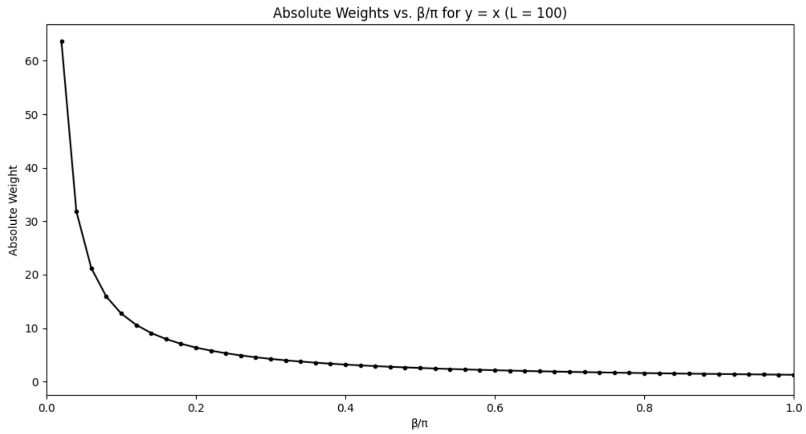
The diagram shows $\frac{\beta}{\pi}$= 0.2, Jacobi and G-S's G are almost same (G-S is about 0.05 lower than Jacobi), but for $\frac{\beta}{\pi}$= 0.5, Jacobi's Amplifaction factor(G) is 0, while G-S is almost 0.45 (Jacobi is 0.45 lower than G-S).

For $\frac{\beta}{\pi}$= 0.7, G(G-S) is 0.28 lower than G(Jacobi).

It could be seen that the huge difference in low wave number of Jacobi with Gauss-Sediel is $\frac{\beta}{\pi}$= 0.5, which could determined the early converge rate's difference. However, on $\frac{\beta}{\pi}$= 0.2, G-S Amp factors is still little bit smaller than Jacobi, which means finally, G-S's error will smaller than Jacobi, at the same iteration step. Consider this iteration much be pretty large (the Amp factor on $\frac{\beta}{\pi}$= 0.2's difference is very small), and our result, it could consider when it happen, the error will get to the order of computer round-off error, which means is hard to shown that point where G-S will finally 'defeat' jacobi.

IG=xiyj

For IG=xiyj, it could be seen for fixed one direction, the fourier series of IG can be shown as $y = x_0 \sum_{n=1}^{\infty} \frac{-2(-1)^n L^2}{\pi n L} \sin\left(\frac{n\pi x}{L}\right)$, where wave number is decreasing among the other direction veryfast:



It could be seen lowest wavenumebr takes highest weight, where G-S works better than Jacobi.

IG=random

For IG=random, the wavenumebr distribute evenlly among all β, also according the diagram before, for most part G-S's error Amplifaction factor is smaller than Jacobi. Combine the result before, it converge as expected.

3 (2) SOR Method Approach

3.1 SOR Jacobi and SOR Gauss-Sediel

For the Successive Over-Relaxation (SOR) method, it using w as parameter, the formula is showing below:

$$P^{k+1} = (1-\omega)P^k + \omega P^* \tag{1}$$

For SOR Jacobi method, the P^* can be express is as follows:

$$P^* = \frac{1}{4}\left[P^k_{i-1,j} + P^k_{i+1,j} + P^k_{i,j-1} + P^k_{i,j+1}\right] \tag{2}$$

The iteration formula can be shown as:

$$P^{k+1} = (1-\omega)P^k + \frac{\omega}{4}\left[P^k_{i-1,j} + P^k_{i+1,j} + P^k_{i,j-1} + P^k_{i,j+1}\right] \tag{3}$$

The algorithm for SOR Solver is showing below:
For SOR Gauss-Sediel, P^* is:

$$P^* = \frac{1}{4}\left[P^{k+1}_{i-1,j} + P^k_{i+1,j} + P^{k+1}_{i,j-1} + P^k_{i,j+1}\right] \tag{4}$$

The iteration formula for Gauss-Seidel is showing below:

Algorithm 3 Pseudocode for SOR Jacobi Solver

function JACOBI($P_{\text{input}}, \Delta x, \Delta y, Nx, Ny, \omega$)
    $P_{\text{old}} \leftarrow P_{\text{input}}$
    $P_{\text{new}} \leftarrow P_{\text{input}}$
    for $j = 1$ to $Ny - 1$ **do**
        for $i = 1$ to $Nx - 1$ **do**
            $P_{\text{new}}[i][j] \leftarrow (1-\omega)P^k + \frac{\omega}{4}\left[P_{\text{old}}[i-1][j] + P_{\text{old}}[i+1][j] + P_{\text{old}}[i][j-1] + P_{\text{old}}[i][j+1]\right]$
            $P_{\text{old}} \leftarrow P_{\text{new}}$
        end for
    end for
    return P_{new}
end function

$$P^{k+1} = (1-\omega)P^* + \frac{\omega}{4}\left[(P_{\text{new}}[i-1][j] + P_{\text{old}}[i+1][j] + P_{\text{new}}[i][j-1] + P_{\text{old}}[i][j+1])\right] \tag{5}$$

It could be found it is not much changed for the former iteration, only using parameter to adjust the old value and new neighbors value.

The Algorithm of SOR G-S method is showing below:

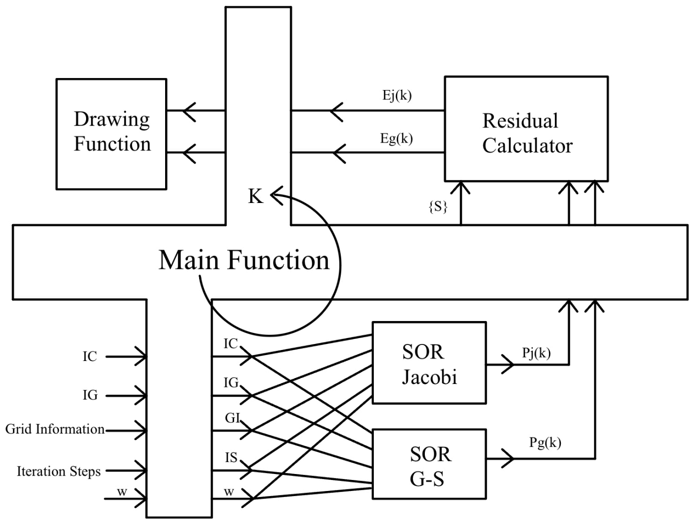
Algorithm 4 Pseudocode for SOR G-S Solver

function JACOBI($P_{\text{input}}, \Delta x, \Delta y, Nx, Ny, \omega$)
    $P_{\text{old}} \leftarrow P_{\text{input}}$
    $P_{\text{new}} \leftarrow P_{\text{input}}$
    for $j = 1$ to $Ny - 1$ **do**
        for $i = 1$ to $Nx - 1$ **do**
            $P_{\text{new}}[i][j] \leftarrow (1-\omega)P^k + \frac{\omega}{4}\left[P_{\text{new}}[i-1][j] + P_{\text{old}}[i+1][j] + P_{\text{new}}[i][j-1] + P_{\text{old}}[i][j+1]\right.$
            $P_{\text{old}} \leftarrow P_{\text{new}}$
        end for
    end for
    return P_{new}
end function

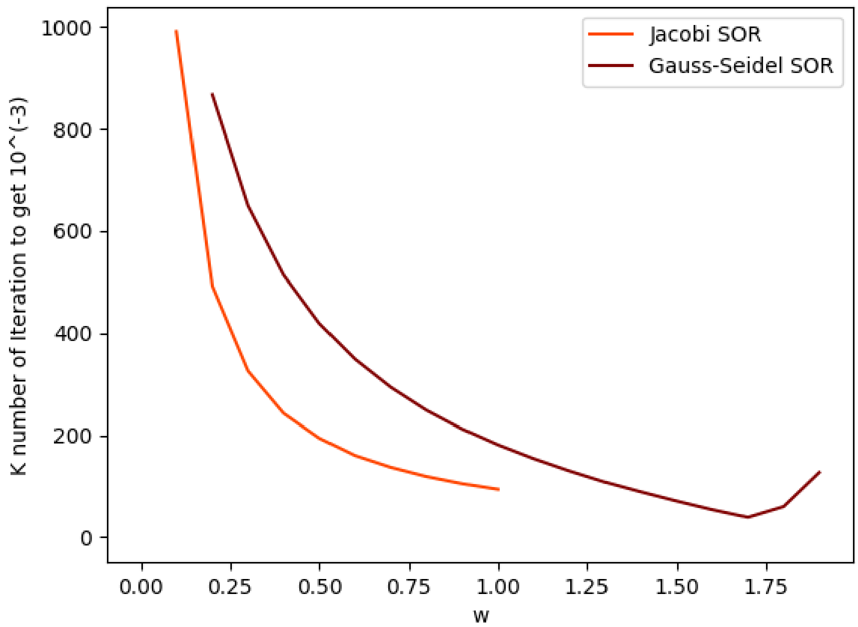
3.2 Find optimal w For constant parameter iteration

For the simplest case, we using constant w iteration, and set constant error as the object, looking for itaration number it need to get the object error.

The whole program strture is showing below:



Using w from 0.1 to 2, step size 0.1, initial guess(IG)=0 everywhere, the result is showing below:



It could be seen for Jacobi, the most efficient w is still 1, espically, Overlaxization Jacobi is not work (iteration cannot reduce error in 10^{-3}) but for Gauss-Seidel, the most efficient w is about 1.7 (betweem 1.5 to 1.75 specifically).

The result for G-S SOR is showing below:

w in range [0,1]:

w	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
k	None	None	867	650	514	419	349	294	249	212	181

w in range [1.1,1.9]:

w	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
k	154	130	108	89	71	54	39	60	127

Table 1: Values of w and k

It could be found the optimal w is 1.7 in this comparsion (corrosponding k is 39), where w is the relaxization factor, and k is the iteration number each relaxization factor need to let error converge to 10^{-3}.

For other initial guesses, the result is showing below

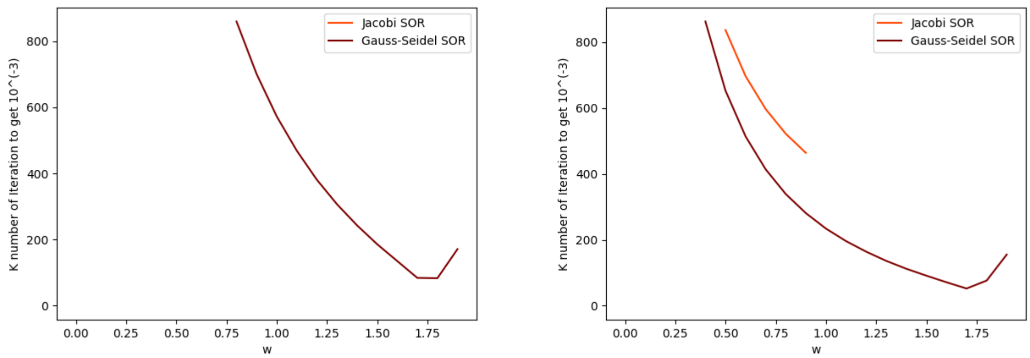


Figure 1: IG=xiyj(left) result and IG=random(right)result

We could also notice in other initial gursses, the optimal relaxization w for Gauss-Sediel SOR is also around 1.7. Initial guess is not much influensional to the optimal relaxization w.

4 (3) SRJ with two parameter relaxization

4.1 Alternative SOR Reach

For this section, we are using two w to speed up more for different scheme in different condition.

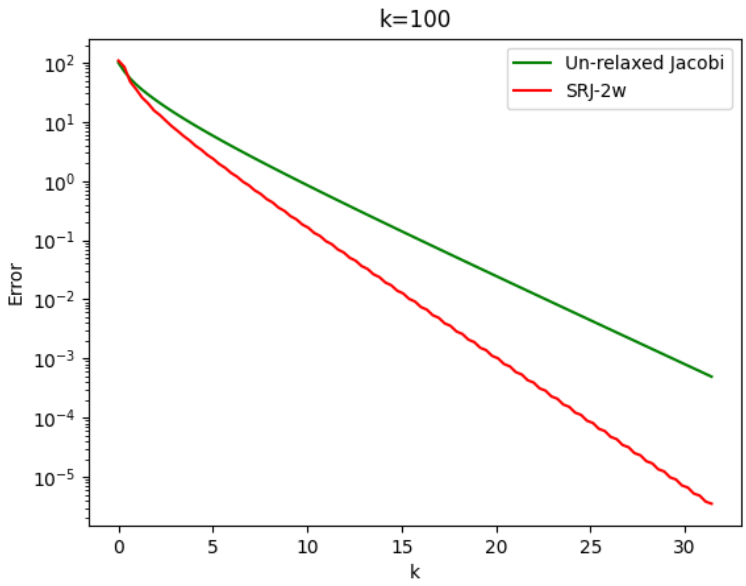
SRJ, which is Scheduled Relaxation Jacobi scheme, could speed up Jacobi much faster, according to *XiYang and Mittal, 2015*, is using underrelaxation and overrelaxation generate fast convergence of solution.

However, according to the paper, it could been seen the optimal overrelaxation parameter is much higher while its show time in iteration in decreasing (amlost 1/100), for the more simplified iteration, we are going to find parameters between (0,2), try to find optimal parameter combinations speed up Jacobi.

As w we choose is in range (0,1) for underrelaxation , and in range (1,2) for overrelaxation, the most simple case come in to mind is Complementarity w for under and over relaxization, where is using w_1 in range (0,1) and w_2 in range (1,2), and exchange using w_1 and w_2 alternativly each iteration step.

Where Alt Fn is an function choosing w in w group each steps, using iteration number t (t in range (0,k)) as choosing parameter.

Using iteration times is 100. The result is showing below:

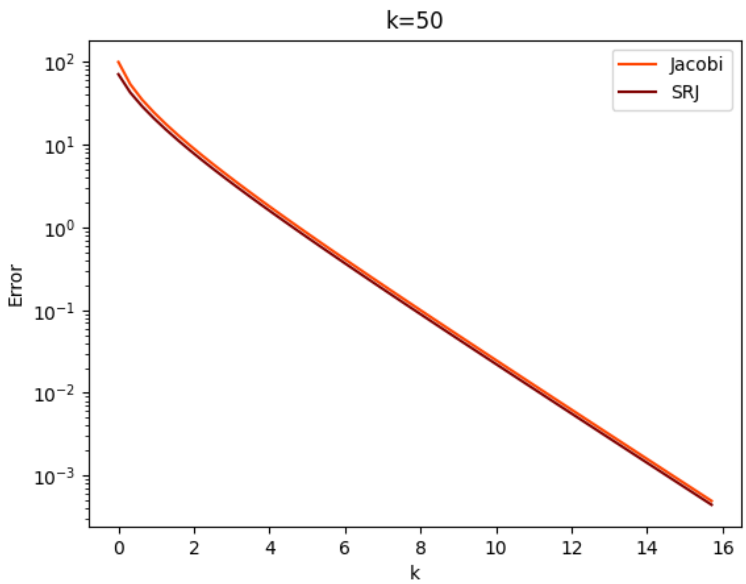


The optimal combination for alternative w is: $w_1 = 0.7$, $w_2 = 2.0$, which result could be observed better than non-relaxed Jacobi.

5 (4) SRJ reach for three-parameter optimization

Using three parameter $(w) = (w1, w2, w3)$, where w1 in range[1,2], w2 in range[1.1,2], w3 in range[1,1.1], the result is showing below:

$(w1, w2, w3) = (1.9, 2, 0.6)$

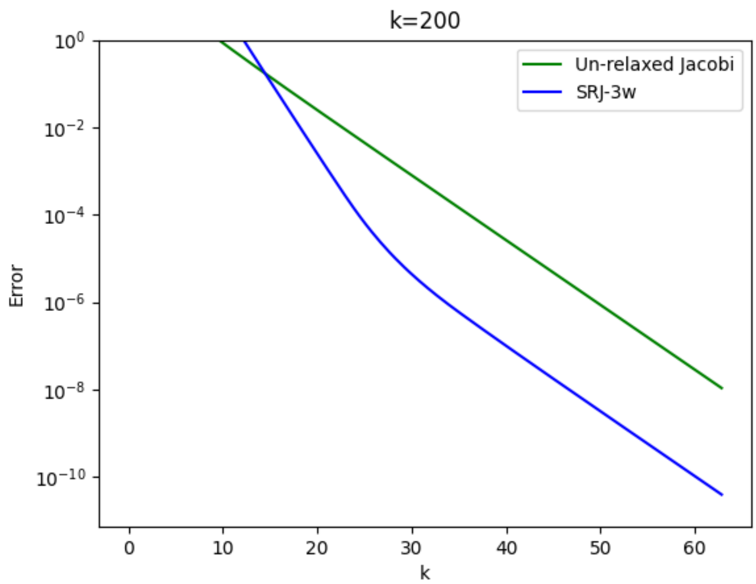


It could been seen by using three parameters in range (0,2), the Jacobi could been little bit speed up, as iteration times is not that large and our seeking solution not need to be extremely accurate.

According to *XiYang and Mittal, 2015*, the optimal relaxization parameter combination is used overrelaxation for only few times durning whole iteration, while using the underrelaxation for most of times. Also, it shows the relaxization order is not that influensional on result, thus we choose w1 as the largest iteration parameter only for 1 time, and other two take most of times.

Expand our range for parameter in range (0,10), for iteration number k=100,
$(n(w1), n(w2), n(w3)) = (1, 9, 90)$,

the optimal combination for w is showing below:
$(w1, w2, w3) = (9.3, 1.1, 1.0)$,
Use this combination, compare to un-relaxed Jacobi, the result is showing below:

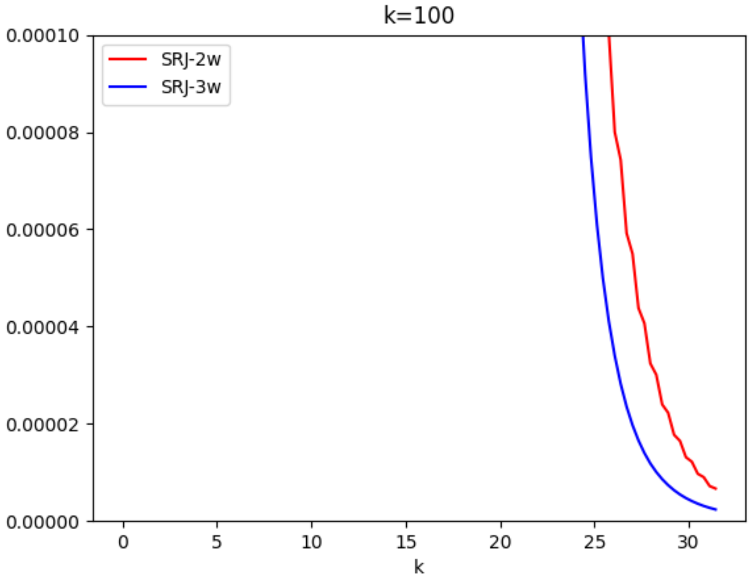


This diagram's Error is means Error factot, which means the error of SRJ jacobi divide by error of jacobi shows by using SRj method from *XiYang and Mittal, 2015*, with three parameters, could obviously reduce error of Jacobi.

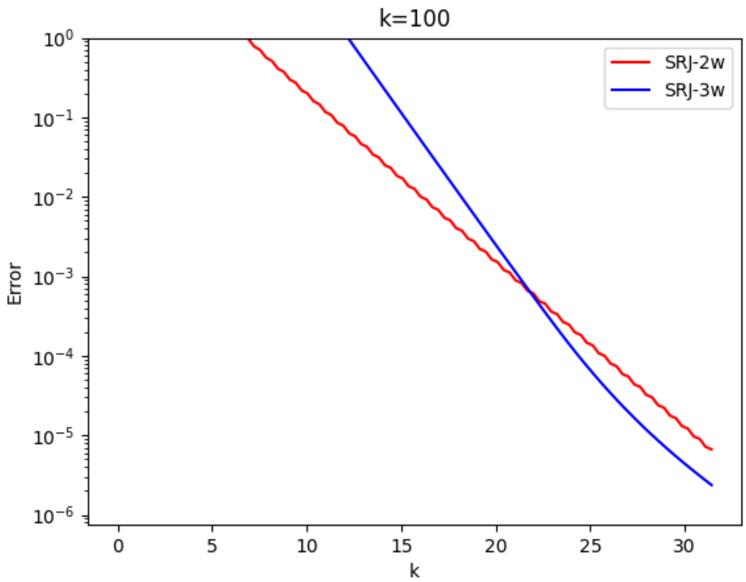
5.1 Comparsion-2 Parameters with 3 Parameters(2ws VS 3ws)

In this section, we are using our 2-parameter's relaxization, which we are using (0.7,2.0), alternatively, compare with our 3-parameter's relaxization, where we re using (9.3,1.1,1.0), frequency(1:9:90), to see if we apply more parameters, Jacobi method could be much faster.

Compare to our 2-parameters shoution, the result is showing below:

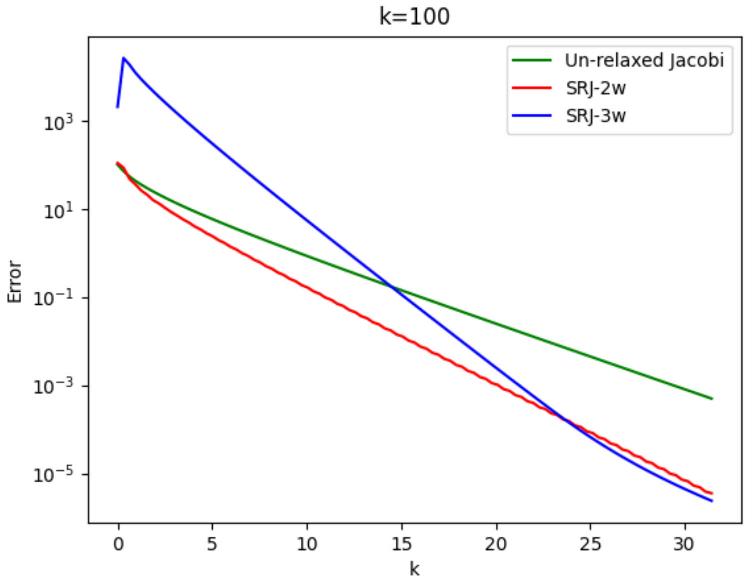


It could find our 3-parameters solution converge much faster than 2-parameters SRJ solution.



This is the result showing error in log scale, it is much more obvious the three-parameter relaxization converge much more faster than 2-parameter relaxization, and converge as expected.

Compare with un-relaxed Jacobi, the result is showing below:



It could be seen finally our 3-parameter's relaxization method will make Jacobi converge faster, which means it will be oppoitunity for Scheduled Relaxization using more parameters to speed-up Jacobi method.

6 Conclusion

In this experiment, we applied Jacobi method with Gauss-Seidel method in grid to solve 2-D Laplace Equation, built algorithms for solver, and experimented SOR method, Scheduled Relaxization method to speed up our old solver.

It could be shown that for more general condition, Gauss-Seidel converge faster than Jacobi, while Jacobi only have small middle wavenumber range faster than G-S.

For SOR method, only SOR G-S have better converge performance, which using w around 1.7.

For Scheduled Relaxization, using 2-parameter and 3-parameter could speed up Jacobi method. For 2-parameters, we use alternative relaxization for w=(0.7,2.0), frequency (1:1). And for 3-parameters, we use w=(9.3,1.1,1.0), frequency(1:9:90), which can finally get fastest convergence result than 2-parameter method and un-relaxed Jacobi.

Appendix

Listing 1: (1)-The python Source code of Algorithm

```
import copy #for Python, need copy op to avoid

def Jacobi(P_input, x_len, y_len): #Jacobi method Pin-->Pout
    P_old = copy.deepcopy(P_input)
    P_new = copy.deepcopy(P_input)
    for j in range(1,y_len-1):
        for i in range(1,x_len-1):
            P_new[j][i] = \
                1/4 * (  P_old[j][i-1] + P_old[j][i+1] + P_old[j-1][i] + P_old[j+1][i] )
    return P_new

def GS(P_input, x_len, y_len): #Gauss Sadiel method Pin-->Pout
    Pg = copy.deepcopy(P_input)
    for j in range(1,y_len-1):
        for i in range(1,x_len-1):
            Pg[j][i] = \
                1/4 * (  Pg[j][i-1] + Pg[j][i+1] + Pg[j-1][i] + Pg[j+1][i] )
    return Pg


def Res(r,Pin,d,x_len,y_len): #calcuate residual (Pin, source)-->Rout
    P = copy.deepcopy(Pin)
    rc = copy.deepcopy(r)
    for j in range(1,y_len-1):
        for i in range(1,x_len-1):
            rc[j][i] = \
                ( P[j][i-1] + P[j][i+1] + P[j-1][i] + P[j+1][i] - 4*P[j][i] )/(d**2)
    return rc

def Error(r_in, x_len, y_len): #calculate error Rin-->e out
    r = copy.deepcopy(r_in)
    e = 0
    for j in range(1,y_len-1):
        for i in range(1,x_len-1):
            e += abs(r[j][i])
    return e


import matplotlib.pyplot as plt
import numpy as np
import matplotlib as mpl

def eploting(  ej, egs,   d,  x_len, y_len,   k     ):
    x = [0 for _ in range(0, k+1)]
    for c in range(0,k+1):
        x[c] = c*d
    plt.plot(x, ej, color = 'blue', label = 'Jacobi')
    plt.plot(x, egs, color = 'red', label = 'Gauss-Seidel')
    plt.show()




######################### main function below ########


def main():
    import math


    #import grid data
    pi = math.pi
    x_max = 2*pi
    y_max = 2*pi
    d = 2*pi/20
    dx = dy = d
```

```python
    x_len = int(x_max/dx+1)
    y_len = int(y_max/dy+1)

    # x_len: the number of x points
    # the first x point is x[0], the last x point is x[x_len -1]

    k = 50

    #import Boundary Condition (BC)
    P = [[0 for _ in range(0,x_len+1)] for _ in range(0, y_len+1)]

    for i in range(0,x_len):
        sin = math.sin
        P[0][i] = sin(2*i*dx) + sin(5*i*dx) + sin(7*i*dx)
        P[y_len -1][i] = 0
    for j in range(0,y_len):
        P[j][0] = 0
        P[j][x_len -1] = 0

    #import Initial Guess (IG)
    IG = 0 #change IG for different guess
    for j in range(1,y_len -1):
        for i in range(1,x_len -1):
            P[j][i] = IG

    #P_input = copy.deepcopy(P)
    r = copy.deepcopy(P)
    e = [0 for _ in range(0,k+1)]
    ej = copy.deepcopy(e)
    egs = copy.deepcopy(e)

    P_Jin = copy.deepcopy(P)
    P_GSin = copy.deepcopy(P)


    for t in range(0,k+1):

        P_Jacobi = Jacobi(P_Jin, x_len, y_len)
        P_GS = GS(P_GSin, x_len, y_len)

        P_Jin = P_Jacobi
        P_GSin = P_GS

        r_J = Res(r,P_Jacobi,d,x_len,y_len)
        r_GS = Res(r,P_GS,d,x_len,y_len)
        ej[t] = Error(r_J, x_len, y_len)
        egs[t] = Error(r_GS, x_len, y_len)


    #print(1)
    #P_Jacobi = Jacobi(P_Jin, x_len, y_len)

    #print(P_input)
    #print(P_Jacobi)
    eploting( ej, egs, d, x_len, y_len, k )
    #print(ej)





# main function operator
if __name__ == "__main__":
    main()
```

Listing 2: (2)(IG=0)-The python Source code of Algorithm

```python
import copy #for Python, need copy op to avoid

def Jacobi(P_input, x_len, y_len): #Jacobi method Pin-->Pout
```

```
    P_old = copy.deepcopy(P_input)
    P_new = copy.deepcopy(P_input)
    for j in range(1,y_len-1):
        for i in range(1,x_len-1):
            P_new[j][i] = \
                1/4 * (  P_old[j][i-1] + P_old[j][i+1] + P_old[j-1][i] + P_old[j+1][i] )
    return P_new



def GS(P_input, x_len, y_len): #Gauss Sadiel method Pin-->Pout
    Pg = copy.deepcopy(P_input)
    for j in range(1,y_len-1):
        for i in range(1,x_len-1):
            Pg[j][i] = \
                1/4 * (  Pg[j][i-1] + P_input[j][i+1] + Pg[j-1][i] + P_input[j+1][i] )
    return Pg



def SORJacobi(P_inputSOR, x_len, y_len, w): #Jacobi SOR method Pin-->Pout
    P_oldSOR = copy.deepcopy(P_inputSOR)
    P_newSOR = copy.deepcopy(P_inputSOR)
    for j in range(1,y_len-1):
        for i in range(1,x_len-1):
            P_newSOR[j][i] = \
                1/4 * (  P_oldSOR[j][i-1] + P_oldSOR[j][i+1] + P_oldSOR[j-1][i] + P_oldSOR[j+1][i] ) *w + (1-w
    return P_newSOR


def SORGS(P_inputSOR, x_len, y_len,w): #Gauss Sadiel SOR method Pin-->Pout
    PgSOR = copy.deepcopy(P_inputSOR)
    for j in range(1,y_len-1):
        for i in range(1,x_len-1):
            PgSOR[j][i] = \
                1/4 * (  PgSOR[j][i-1] + P_inputSOR[j][i+1] + PgSOR[j-1][i] + P_inputSOR[j+1][i] )*w + (1-w)*]




def Res(r,Pin,d,x_len,y_len): #calcuate residual (Pin, source)-->Rout
    P = copy.deepcopy(Pin)
    rc = copy.deepcopy(r)
    for j in range(1,y_len-1):
        for i in range(1,x_len-1):
            rc[j][i] = \
                ( P[j][i-1] + P[j][i+1] + P[j-1][i] + P[j+1][i] - 4*P[j][i] )/(d**2)
    return rc

def Error(r_in, x_len, y_len): #calculate error Rin-->e out
    r = copy.deepcopy(r_in)
    e = 0
    for j in range(1,y_len-1):
        for i in range(1,x_len-1):
            e += abs(r[j][i])
    return e


import matplotlib.pyplot as plt
#import numpy as np
import matplotlib as mpl

def nploting(  nj, ngs  ):
    w = [0 for _ in range(0, 20)]
    for c in range(1,20):
        w[c] = c*0.1

    plt.plot(w, nj, color = 'orangered', label = 'Jacobi␣SOR')
    plt.plot(w, ngs, color = 'maroon', label = 'Gauss-Seidel␣SOR')
```

```
    plt.legend()
    plt.xlabel('w')
    plt.ylabel('K number of Iteration to get 10^(-3)')
    #plt.ylim(0,5)
    #plt.ylim(0,10**(-10))



    plt.show()



######################### main function below ########


def main():
    import math


    #import grid data
    pi = math.pi
    x_max = 2*pi
    y_max = 2*pi
    d = 2*pi/20
    dx = dy = d
    x_len = int(x_max/dx+1)
    y_len = int(y_max/dy+1)

    # x_len: the number of x points
    # the first x point is x[0], the last x point is x[x_len -1]

    k = 1000

    #import Boundary Condition (BC)
    P = [[0 for _ in range(0,x_len+1)] for _ in range(0, y_len+1)]

    for i in range(0,x_len):
        sin = math.sin
        P[0][i] = sin(2*i*dx) + sin(5*i*dx) + sin(7*i*dx)
        P[y_len-1][i] = 0
    for j in range(0,y_len):
        P[j][0] = 0
        P[j][x_len-1] = 0

    #import Initial Guess (IG)
    IG = 0 #change IG for different guess
    for j in range(1,y_len-1):
        for i in range(1,x_len-1):
            P[j][i] = IG

    #P_input = copy.deepcopy(P)
    r = copy.deepcopy(P)
    e = [0 for _ in range(0,k+1)]
    ej = copy.deepcopy(e)
    egs = copy.deepcopy(e)

    P_Jin = copy.deepcopy(P)
    P_GSin = copy.deepcopy(P)


    en = 10**(-3)

    #calculate iteration number of Jacobi
    for t in range(0,k+1):

        P_Jacobi = Jacobi(P_Jin, x_len, y_len)

        P_Jin = P_Jacobi

        r_J = Res(r,P_Jacobi,d,x_len,y_len)
```

```
    ej = Error(r_J, x_len, y_len)

    if ej <= en:
        nj = t
        break
    if t == k:
        nj= 200

#calculate interation number of Gauss-Seidel
for t in range(0,k+1):

    P_GS = GS(P_GSin, x_len, y_len)

    P_GSin = P_GS


    r_GS = Res(r,P_GS,d,x_len,y_len)

    egs = Error(r_GS, x_len, y_len)
    if egs <= en:
        ngs = t
        break
    if t == k:
        ngs = 200

#calculate iteration times of different w
nj = [0 for _ in range(0,20)]
ngs = [0 for _ in range(0,20)]

for wc in range(1,20):
    w = wc*0.1
    r = copy.deepcopy(P)

    P_Jin = copy.deepcopy(P)
    P_GSin = copy.deepcopy(P)


    for t in range(0,k+1):


        SORP_Jacobi = SORJacobi(P_Jin, x_len, y_len, w)

        P_Jin = SORP_Jacobi

        SORr_J = Res(r,SORP_Jacobi,d,x_len,y_len)

        SORej = Error(SORr_J, x_len, y_len)
        if SORej <=en:
            nj[wc] = t
            break
        if t == k:
            nj[wc] = None

    for t in range(0,k+1):


        SORP_GS = SORGS(P_GSin, x_len, y_len, w)

        P_GSin = SORP_GS

        SORr_GS = Res(r,SORP_GS,d,x_len,y_len)

        SORegs = Error(SORr_GS, x_len, y_len)
        if SORegs <=en:
            ngs[wc] = t
            break
        if t == k:
            ngs[wc] = None



#nploting( nj , ngs)
```

```
    print(SORegs)

    #print(ej)




# main function operator
if __name__ == "__main__":
    main()
```

Listing 3: (3)-The python Source code

```
import copy #for Python, need copy op to avoid

def Jacobi(P_input, x_len, y_len): #Jacobi method Pin-->Pout
    P_old = copy.deepcopy(P_input)
    P_new = copy.deepcopy(P_input)
    for j in range(1,y_len-1):
        for i in range(1,x_len-1):
            P_new[j][i] = \
                1/4 * (  P_old[j][i-1] + P_old[j][i+1] + P_old[j-1][i] + P_old[j+1][i] )
    return P_new




def GS(P_input, x_len, y_len): #Gauss Sadiel method Pin-->Pout
    Pg = copy.deepcopy(P_input)
    for j in range(1,y_len-1):
        for i in range(1,x_len-1):
            Pg[j][i] = \
                1/4 * (  Pg[j][i-1] + P_input[j][i+1] + Pg[j-1][i] + P_input[j+1][i] )
    return Pg



def SORJacobi(P_inputSOR, x_len, y_len, w): #Jacobi SOR method Pin-->Pout
    P_oldSOR = copy.deepcopy(P_inputSOR)
    P_newSOR = copy.deepcopy(P_inputSOR)
    for j in range(1,y_len-1):
        for i in range(1,x_len-1):
            P_newSOR[j][i] = \
                1/4 * (  P_oldSOR[j][i-1] + P_oldSOR[j][i+1] + P_oldSOR[j-1][i] + P_oldSOR[j+1][i] ) *w + (1-w
    return P_newSOR


def SORGS(P_inputSOR, x_len, y_len,w): #Gauss Sadiel SOR method Pin-->Pout
    PgSOR = copy.deepcopy(P_inputSOR)
    for j in range(1,y_len-1):
        for i in range(1,x_len-1):
            PgSOR[j][i] = \
                1/4 * (  PgSOR[j][i-1] + P_inputSOR[j][i+1] + PgSOR[j-1][i] + P_inputSOR[j+1][i] )*w + (1-w)*
    return PgSOR




def Res(r,Pin,d,x_len,y_len): #calcuate residual (Pin, source)-->Rout
    P = copy.deepcopy(Pin)
    rc = copy.deepcopy(r)
    for j in range(1,y_len-1):
        for i in range(1,x_len-1):
            rc[j][i] = \
                ( P[j][i-1] + P[j][i+1] + P[j-1][i] + P[j+1][i] - 4*P[j][i] )/(d**2)
    return rc

def Error(r_in, x_len, y_len): #calculate error Rin-->e out
    r = copy.deepcopy(r_in)
```

```
    e = 0
    for j in range(1,y_len-1):
        for i in range(1,x_len-1):
            e += abs(r[j][i])
    return e


import matplotlib.pyplot as plt
#import numpy as np
import matplotlib as mpl

def eploting(    ej, eSORj,     d,     k        ):
    x = [0 for _ in range(0, k+1)]
    for c in range(0,k+1):
        x[c] = c*d
    plt.plot(x, ej, color = 'orangered', label = 'Jacobi')
    plt.plot(x, eSORj, color = 'maroon', label = 'SRJ')
    plt.legend()

    #plt.yscale('log')
    plt.xlabel('k')
    plt.ylabel('Error')
    #plt.ylim(0,1)
    plt.ylim(0,10**(-2))

    plt.title("k=%d"%k)





    plt.show()




def altw(w1,w2,k):
    wlist = [w1,w2]
    wselect = wlist[k%len(wlist)]
    return wselect






########################## main function below ########
########################## main function below ########
########################## main function below ########
########################## main function below ########


def main():
    import math

    ########################## Grid Information v ######
    ########################## Grid Information v ######

    #import grid data
    pi = math.pi
    x_max = 2*pi
    y_max = 2*pi
    d = 2*pi/20
    dx = dy = d
    x_len = int(x_max/dx+1)
    y_len = int(y_max/dy+1)

    # x_len: the number of x points
    # the first x point is x[0], the last x point is x[x_len -1]

    k = 1000

    #import Boundary Condition (BC)
    P = [[0 for _ in range(0,x_len+1)] for _ in range(0, y_len+1)]
```

```python
for i in range(0,x_len):
     sin = math.sin
    P[0][i] = sin(2*i*dx) + sin(5*i*dx) + sin(7*i*dx)
    P[y_len-1][i] = 0
for j in range(0,y_len):
    P[j][0] = 0
    P[j][x_len-1] = 0

########################## Grid Information ^ ######



#import Initial Guess (IG)
IG = 0 #change IG for different guess
for j in range(1,y_len-1):
    for i in range(1,x_len-1):
        P[j][i] = IG

#P_input = copy.deepcopy(P)
r = copy.deepcopy(P)
e = [0 for _ in range(0,k+1)]
ej = copy.deepcopy(e)
egs = copy.deepcopy(e)

P_Jin = copy.deepcopy(P)
P_GSin = copy.deepcopy(P)


#en = 10**(-3)
en = 1


#calculate iteration times of different w

w1o = 0
w2o = 0
ko = 0
SORejo = 0



for w1 in range(1,11):
    w1 = w1*0.1
    for w2 in range(11,101):
        w2 = w2*0.1

        r = copy.deepcopy(P)

        P_Jin = copy.deepcopy(P)
        P_GSin = copy.deepcopy(P)

        for t in range(0,k+1):

            w = altw(w1,w2,t)

            SORP_Jacobi = SORJacobi(P_Jin, x_len, y_len, w)

            P_Jin = SORP_Jacobi

        SORr_J = Res(r,SORP_Jacobi,d,x_len,y_len)

        SORej = Error(SORr_J, x_len, y_len)

        if SORej <=en:
            en = SORej
            w1o = w1
            w2o = w2
            ko = k
            SORejo = SORej


print(w1o)
```

```python
    print(w2o)
    print(ko)

    #P_input = copy.deepcopy(P)
    r = copy.deepcopy(P)
    e = [0 for _ in range(0,k+1)]
    ej = copy.deepcopy(e)

    P_Jin = copy.deepcopy(P)
    P_Jin1 = copy.deepcopy(P)

    eSORj = copy.deepcopy(e)

    for t in range(0,k+1):

        P_Jacobi = Jacobi(P_Jin, x_len, y_len)

        P_Jin = P_Jacobi

        r_J = Res(r,P_Jacobi,d,x_len,y_len)

        ej[t] = Error(r_J, x_len, y_len)

        ################ Plot SRJ w1=0.6 w2=1.1 ################

        w = altw(w1,w2,k) # Alt fn. odd iteration doing w1, even iteration doing w2

        SORP_Jacobi = SORJacobi(P_Jin1, x_len, y_len, w)

        P_Jin1 = SORP_Jacobi

        SORr_J = Res(r,SORP_Jacobi,d,x_len,y_len)

        eSORj[t] = Error(SORr_J, x_len, y_len)

    print(eSORj[k]/ej[k])
    print(w)

    eploting(   ej, eSORj,    d,    k     )



# main function operator
if __name__ == "__main__":
    main()
```

Listing 4: (4)-The python Source code

```python
import copy #for Python, need copy op to avoid

def Jacobi(P_input, x_len, y_len): #Jacobi method Pin-->Pout
    P_old = copy.deepcopy(P_input)
    P_new = copy.deepcopy(P_input)
    for j in range(1,y_len-1):
        for i in range(1,x_len-1):
            P_new[j][i] = \
                1/4 * (  P_old[j][i-1] + P_old[j][i+1] + P_old[j-1][i] + P_old[j+1][i] )
    return P_new



def GS(P_input, x_len, y_len): #Gauss Sadiel method Pin-->Pout
    Pg = copy.deepcopy(P_input)
    for j in range(1,y_len-1):
        for i in range(1,x_len-1):
            Pg[j][i] = \
                1/4 * (  Pg[j][i-1] + P_input[j][i+1] + Pg[j-1][i] + P_input[j+1][i] )
    return Pg
```

```
def SORJacobi(P_inputSOR, x_len, y_len, w): #Jacobi SOR method Pin-->Pout
    P_oldSOR = copy.deepcopy(P_inputSOR)
    P_newSOR = copy.deepcopy(P_inputSOR)
    for j in range(1,y_len-1):
        for i in range(1,x_len-1):
            P_newSOR[j][i] = \
                1/4 * (  P_oldSOR[j][i-1] + P_oldSOR[j][i+1] + P_oldSOR[j-1][i] + P_oldSOR[j+1][i] ) *w + (1-w
    return P_newSOR


def SORGS(P_inputSOR, x_len, y_len,w): #Gauss Sadiel SOR method Pin-->Pout
    PgSOR = copy.deepcopy(P_inputSOR)
    for j in range(1,y_len-1):
        for i in range(1,x_len-1):
            PgSOR[j][i] = \
                1/4 * (  PgSOR[j][i-1] + P_inputSOR[j][i+1] + PgSOR[j-1][i] + P_inputSOR[j+1][i] )*w + (1-w)*I
    return PgSOR




def Res(r,Pin,d,x_len,y_len): #calcuate residual (Pin, source)-->Rout
    P = copy.deepcopy(Pin)
    rc = copy.deepcopy(r)
    for j in range(1,y_len-1):
        for i in range(1,x_len-1):
            rc[j][i] = \
                ( P[j][i-1] + P[j][i+1] + P[j-1][i] + P[j+1][i] - 4*P[j][i] )/(d**2)
    return rc

def Error(r_in, x_len, y_len): #calculate error Rin-->e out
    r = copy.deepcopy(r_in)
    e = 0
    for j in range(1,y_len-1):
        for i in range(1,x_len-1):
            e += abs(r[j][i])
    return e


import matplotlib.pyplot as plt
#import numpy as np
import matplotlib as mpl

def eploting(    ej, eSORj,     d,     k       ):
    x = [0 for _ in range(0, k+1)]
    for c in range(0,k+1):
        x[c] = c*d

    f = [0 for _ in range(0,k+1)]

    for t in range(0,k+1):
        f[t] = eSORj[t]/ej[t]
    '''
    plt.plot(x, ej, color = 'orangered', label = 'Jacobi')
    plt.plot(x, eSORj, color = 'maroon', label = 'SRJ')
    '''
    plt.plot(x, f, color = 'maroon', label = 'factor with Jacobi')


    plt.legend()

    #plt.yscale('log')
    plt.xlabel('k')
    plt.ylabel('Error')
    plt.ylim(0,1)
    #plt.ylim(0,10**(-2))

    plt.title("k=%d"%k)
```

```python
    plt.show()


def wlist0(w1,w2,w3,k):
    wlist = [w3 for _ in range(0,k+1)]

    for s in range(0,2):
        wlist[s] = w1
    for s in range(2,3):
        wlist[s] = w2

    return wlist



'''

def altw(w1,w2,w3,k):
    # wlist = [w1,w2,w3]

    wlist = wlist0(w1,w2,w3,k)
    wselect = wlist[k]

    return wselect
'''




########################## main function below ########
########################## main function below ########
########################## main function below ########
########################## main function below ########




def main():
    import math


    from tqdm import tqdm


    ########################## Grid Information v ######
    ########################## Grid Information v ######

    #import grid data
    pi = math.pi
    x_max = 2*pi
    y_max = 2*pi
    d = 2*pi/20
    dx = dy = d
    x_len = int(x_max/dx+1)
    y_len = int(y_max/dy+1)

    # x_len: the number of x points
    # the first x point is x[0], the last x point is x[x_len -1]

    k = 100

    #import Boundary Condition (BC)
    P = [[0 for _ in range(0,x_len+1)] for _ in range(0, y_len+1)]

    for i in range(0,x_len):
        sin = math.sin
        P[0][i] = sin(2*i*dx) + sin(5*i*dx) + sin(7*i*dx)
        P[y_len-1][i] = 0
    for j in range(0,y_len):
```

```
    P[j][0] = 0
    P[j][x_len-1] = 0

######################### Grid Information ^ ######



#import Initial Guess (IG)
IG = 0 #change IG for different guess
for j in range(1,y_len-1):
    for i in range(1,x_len-1):
        P[j][i] = IG

#P_input = copy.deepcopy(P)
r = copy.deepcopy(P)
e = [0 for _ in range(0,k+1)]
ej = copy.deepcopy(e)
egs = copy.deepcopy(e)

P_Jin = copy.deepcopy(P)
P_GSin = copy.deepcopy(P)


#en = 10**(-3)
en = 1


#calculate iteration times of different w

w1o = 0
w2o = 0
ko = 0
SORejo = 0



for w1 in tqdm(range(1,1001),desc='total calculation'):
    w1 = w1*0.1
    for w2 in range(11,21):
        w2 = w2*0.1
        for w3 in range(1,21):
            w3 = w3*0.1

            for t in range(0,k+1):

                w = wlist0(w1,w2,w3,k)[t]

                SORP_Jacobi = SORJacobi(P_Jin, x_len, y_len, w)

                P_Jin = SORP_Jacobi

            SORr_J = Res(r,SORP_Jacobi,d,x_len,y_len)

            SORej = Error(SORr_J, x_len, y_len)

            if SORej <=en:
                en = SORej
                w1o = w1
                w2o = w2
                w3o = w3
                ko = k
                SORejo = SORej


print(w1o)
print(w2o)
print(w3o)
print("efefefeffefe")
print(ko)

#P_input = copy.deepcopy(P)
r = copy.deepcopy(P)
e = [0 for _ in range(0,k+1)]
```

```
    ej = copy.deepcopy(e)

    P_Jin = copy.deepcopy(P)

    eSORj = copy.deepcopy(e)

    for t in tqdm(range (0,k+1), desc='final␣plloting'):

        P_Jacobi = Jacobi(P_Jin, x_len, y_len)

        P_Jin = P_Jacobi

        r_J = Res(r,P_Jacobi,d,x_len,y_len)

        ej[t] = Error(r_J, x_len, y_len)

        ################ Plot SRJ w1=0.6 w2=1.1 ################

        w = wlist0(w1o,w2o,w3o,k)[t] # Alt fn. odd iteration doing w1, even iteration doing w2

        SORP_Jacobi = SORJacobi(P_Jin, x_len, y_len, w)

        P_Jin = SORP_Jacobi

        SORr_J = Res(r,SORP_Jacobi,d,x_len,y_len)

        eSORj[t] = Error(SORr_J, x_len, y_len)

    print(eSORj[k-1]/ej[k-1])
    print(w)

    eploting(    ej, eSORj,     d,     k      )



# main function operator
if __name__ == "__main__":
    main()
```