Week 2 Lecture 1; Viscous gravity currents and similarity solutions

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1 Introduction

Flows that are driven by gravity are ubiquitous in the natural world as well as closer to home. Examples include rivers, aquifers, lava flows, oil spreading in a pan, honey on toast, ice sheets, mud slides, avalanches, rainfall, huaicos and lahars, painting surfaces, printing processes and pyroclastic flows. The list is seemingly endless and that is before we even consider medical, biological and astrophysical contexts.

In this lecture, we will focus on 'viscous gravity currents'; flows driven by gravity and resisted by viscous stresses (typically shear stresses). Let's unpack this term. The 'viscous' part refers to flows with low Reynolds number so that the inertial terms in the Navier-Stokes equations may be neglected. 'Viscous' is also often used to distinguish that the fluid is Newtonian, which we will assume in this lecture (in subsequently lectures, we will consider other constitutive laws: 'viscoplastic gravity currents' and 'shear-thinning gravity currents'). 'Gravity current' refers to a flow driven by gravity (or equivalently, buoyancy). The direction of gravity may be perpendicular to the direction of flow and yet gravity is still the driving force. Consider honey spreading on a table; the weight acts normal to the table but the flow is predominantly parallel to the table. The motion is generated by variations in the pressure within the fluid, which arises from its weight (see §2).

The models that describe viscous gravity currents may be relatively simple and sometimes admit 'similarity solutions' (see §3). These solutions can be obtained from physical scaling arguments. They have been used widely and their predictions have shown good agreement with many real flows, including the flow of lava.

2 Viscous gravity current on a horizontal surface

Consider a two-dimensional blob of viscous liquid spreading under its own weight in the (x, z) plane (see figure 1). The free surface is at $z = h(x, t)$. After an initial transient, the lengthscale of the current in the x direction becomes much larger than the thickness scale and hence the lubrication approximation (or shallow flow approximation) may be applied.

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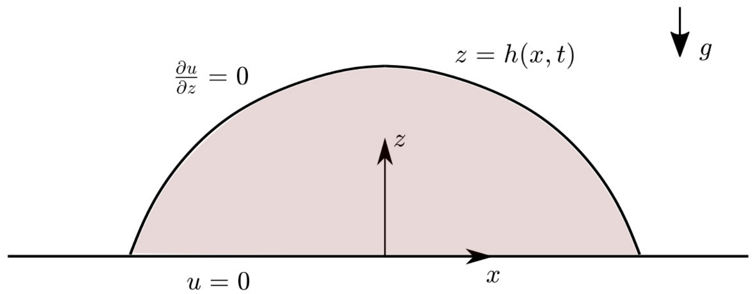

Figure 1: Two-dimensional viscous gravity current.

The momentum equations reduce to

$$0 = -\frac{\partial p}{\partial z} - \rho g, \tag{1}$$

$$0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial z^2}. \tag{2}$$

The z momentum equation is readily integrated to obtain

$$p = \rho g(h - z) + p_{\text{atm}}, \tag{3}$$

where we have used the boundary condition that the pressure is equal to the (constant) atmospheric pressure, $p = p_{\text{atm}}$, at the free surface, $z = h$. Within the fluid, the pressure at any point arises simply from the weight of fluid above that point; this is known as *hydrostatic pressure* (there is no contribution from the vertical velocity since we have applied the lubrication approximation).

The x momentum equation (2) becomes

$$\mu \frac{\partial^2 u}{\partial z^2} = \rho g \frac{\partial h}{\partial x}. \tag{4}$$

We require two boundary conditions for the horizontal velocity, u. At $z = 0$, we apply 'no-slip' at the solid boundary; $u = 0$. At the free-surface, $z = h$, the ambient air exerts no traction on the liquid so there is no stress there. Together these boundary conditions are

$$u = 0 \qquad \text{at} \quad z = 0, \tag{5}$$

$$\frac{\partial u}{\partial z} = 0 \qquad \text{at} \quad z = h(x,t). \tag{6}$$

Note that $\partial u / \partial z = 0$ represents the shear stress vanishing; the other terms in the stress are negligible in the lubrication approximation.

We integrate (4) twice with respect to z and apply these boundary conditions to find

$$u(x,z,t) = -\frac{\rho g}{2\mu} z(2h - z)\frac{\partial h}{\partial x}. \tag{7}$$

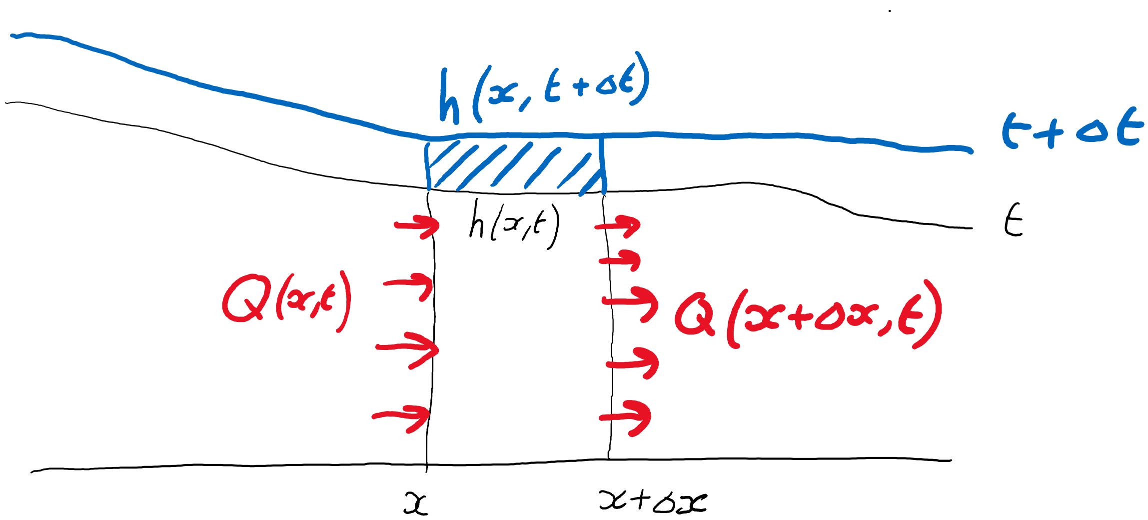

Figure 2: Mass conservation for a thin strip (blue shaded area is the volume change).

It is worth pausing to examine this form of the velocity. It satisfies some simple physical checks: (i) the velocity decreases with viscosity, (ii) the magnitude of the velocity is proportional to the gradient of the free-surface, $\partial h/\partial x$ (steep gradients drive faster flow), (iii) the velocity has the opposite sign to the gradient of h so flow is from thicker regions to thinner regions. Finally, it should be noted that the velocity profile is a parabolic function of the height z. The volume flux of fluid in the x direction is given by

$$Q(x,t) = \int_0^h u \, \mathrm{d}z = -\frac{\rho g}{3\mu} h^3 \frac{\partial h}{\partial x}. \tag{8}$$

We next derive a governing equation for $h(x,t)$ by applying mass conservation. Consider a strip between x and $x+\Delta x$ with Δx small. Over a short time period $[t, t+\Delta t]$, the volume in this strip increases by (blue shaded region in figure 2)

$$\Delta x \big[h(x, t+\Delta t) - h(x,t)\big] = \Delta t [Q(x,t) - Q(x+\Delta x, t)]. \tag{9}$$

We take the limit as Δx and Δt go to zero to obtain

$$\frac{\partial h}{\partial t} = -\frac{\partial Q}{\partial x}, \tag{10}$$

which is

$$\frac{\partial h}{\partial t} = \frac{\rho g}{3\mu} \frac{\partial}{\partial x} \left(h^3 \frac{\partial h}{\partial x} \right). \tag{11}$$

We have obtained a single nonlinear second order partial differential equation governing the evolution of the flow thickness, $h(x,t)$. This is combined with a global mass conservation condition

$$\int_{-x_f}^{x_f} h \, \mathrm{d}x = V, \tag{12}$$

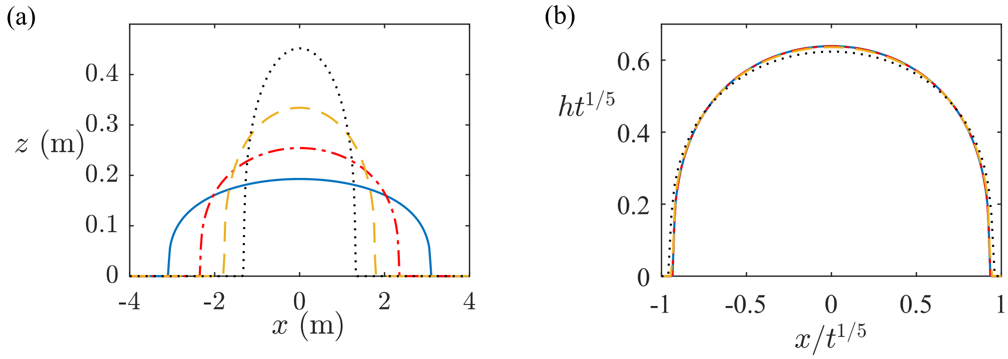

Figure 3: (a) Two-dimensional viscous gravity current at $t = 5, 25, 100, 400$ seconds. (b) 'Collapsed' interface shapes with $x/t^{1/5}$ plotted against $ht^{1/5}$.

where V is the fixed volume of the current and $x = \pm x_f$ are the 'contact points' where $h = 0$. If the current was supplied by a constant source flux of strength q, then V would be replaced by qt.

Equations (11) and (12) complete the description of the problem aside from an initial condition for the current. The system is solved in the next section for a variety of initial conditions. In fact, we will see that the initial condition becomes unimportant at 'late' times.

Our derivation has made various assumptions:

- negligible inertia and 'shallow' flow so that the lubrication approximation applies
- negligible surface tension (so p is continuous across the interface)
- the fluid is Newtonian

For further reading, see Huppert (1982).

3 Similarity solutions

We will solve the partial differential equation with a similarity solution. The idea is that the evolution is 'self-similar' and under particular rescalings, the solutions at different times all collapse onto a single curve (see figure 3). These rescalings can be obtained by finding a transformation that leaves the governing equation and mass conservation condition unchanged.

For simplicity, we define

$$S = \frac{\rho g}{3\mu}, \tag{13}$$

and the governing equation (11) becomes,

$$\frac{\partial h}{\partial t} = S\frac{\partial}{\partial x}\left(h^3\frac{\partial h}{\partial x}\right). \tag{14}$$

Hence, the finite release of viscous liquid described in the previous section is governed by two parameters with the following dimensions

$$[S] = T^{-1}L^{-1} \qquad [V] = L^2. \tag{15}$$

Consider a dilation transformation of (11) and (12) under

$$t = \lambda \bar{t}, \qquad x = \lambda^a \bar{x}, \qquad h = \lambda^b \bar{h}, \tag{16}$$

for some constant λ. The equations become

$$\lambda^{b-1}\frac{\partial \bar{h}}{\partial \bar{t}} = \lambda^{4b-2a} S \frac{\partial}{\partial \bar{x}}\left(\bar{h}^3 \frac{\partial \bar{h}}{\partial \bar{x}}\right). \tag{17}$$

$$\int_{-\bar{x}_f}^{\bar{x}_f} \bar{h}\, \mathrm{d}\bar{x} \lambda^{a+b} = V. \tag{18}$$

Hence the equations are invariant under this transformation provided that

$$b - 1 = 4b - 2a, \qquad a + b = 0. \tag{19}$$

Hence, the correct invariant transformation has

$$a = 1/5, \qquad b = -1/5. \tag{20}$$

By rearranging (16) we find that

$$\frac{h}{\bar{h}} = \lambda^{-1/5} = \left(\frac{t}{\bar{t}}\right)^{-1/5}, \qquad \frac{x}{\bar{x}} = \lambda^{1/5} = \left(\frac{t}{\bar{t}}\right)^{1/5}. \tag{21}$$

If we fix $\bar{t}$, then we see that for any time, t, the solution $h(x,t)$ is just a rescaled form of the solution $\bar{h}(\bar{x},\bar{t})$. This suggests that the solution, $h(x,t)$ evolves with $h \sim t^{-1/5}$ and $x \sim t^{1/5}$. This is shown in figure 3, where we see that the solutions collapse onto a single curve under this rescaling. This invariance implies that the solution may be written in the form

$$h = A_0 t^{-1/5} f(\eta), \qquad \eta = \frac{x}{A_1 t^{1/5}}, \tag{22}$$

where f and η are dimensionless and A_0 and A_1 are dimensional constants, which we determine below:

$$h = V^{1/2}[SV^{1/2}t]^{-1/5} f(\eta), \qquad \eta = \frac{x/V^{1/2}}{[SV^{1/2}t]^{1/5}}, \tag{23}$$

where the terms in square brackets are dimensionless. Comparing to the previous expressions, we find that

$$A_0 = V^{2/5}S^{-1/5}, \qquad A_1 = S^{1/5}V^{3/5}. \tag{24}$$

The self-similar shape of the current is given by the curve $(\eta, f(\eta))$, which we will determine.

The governing partial differential equation (11) simplifies to an ordinary differential equation in η. First note that

$$\frac{\partial \eta}{\partial x} = \frac{V^{-1/2}}{[SV^{1/2}t]^{1/5}}, \qquad \frac{\partial \eta}{\partial t} = -\frac{1}{5}\frac{\eta}{t}. \tag{25}$$

Upon changing to η, f from t, x, h, (11) becomes (the S and V on each side cancel)

$$-\frac{f}{5} - \frac{\eta}{5}\frac{\mathrm{d}f}{\mathrm{d}\eta} = \frac{\partial}{\partial \eta}\left(f^3 \frac{\mathrm{d}f}{\mathrm{d}\eta}\right). \tag{26}$$

The global volume conservation condition becomes

$$\int_{-\eta_f}^{\eta_f} f \,\mathrm{d}\eta = 1, \qquad \text{where} \quad \eta_f = \frac{x_f}{V^{3/5}S^{1/5}t^{1/5}}, \tag{27}$$

and η_f represents the contact point in similarity coordinates. The problem has been reduced from solving a PDE for $h(x,t)$ to solving an ODE (26) for $f(\eta)$. The left hand side of the ODE can be written as $-(\eta f)'/5$, which is integrable and we can use the boundary condition that $f(\eta_f) = 0$ to set the constant of integration to zero and obtain

$$-\frac{1}{5}\eta f = f^3 \frac{\mathrm{d}f}{\mathrm{d}\eta}. \tag{28}$$

Dividing by f and integrating again furnishes

$$f(\eta) = (3/10)^{1/3}(\eta_f^2 - \eta^2)^{1/3}. \tag{29}$$

Upon substituting into the mass conservation condition (27)as, we obtain the constraint for the contact point, η_f,

$$(3/10)^{1/3} \int_{-\eta_f}^{\eta_f} (\eta_f^2 - \eta^2)^{1/3} \,\mathrm{d}\eta = 1. \tag{30}$$

We find that

$$\eta_f = \left[\int_{-1}^{1} (1 - s^2)^{1/3} \,\mathrm{d}s\right]^{-3/5} (10/3)^{1/5} \approx 0.9313 \tag{31}$$

This completes the description of the solution and it compares well with the shapes in figure 3b.

It is perhaps surprising that in constructing the analytical solution to this problem, we have used no information about the initial shape of the current (other than its volume, which is conserved). This is because all initial shapes will converge to the similarity solution at sufficiently long times. This is a general feature of similarity solutions, which 'forget' their initial shape. It should be noted that the amount of time required to get good agreement with the similarity solution can vary significantly from one problem to the next. See Barenblatt (1996) for an in-depth treatment of self-similar solutions.

3.1 More similarity solutions

Now let's find similarity solutions for some slightly different setups. For a constant input source flux, global mass conservation becomes

$$\int_{-x_f}^{x_f} h \, \mathrm{d}x = qt, \tag{32}$$

and the governing partial differential equation is unchanged. Note that $[q] = L^2T^{-1}$. We have two parameters: S and q. We repeat the transformation (16) but this time obtain the different exponents

$$a = 4/5, \qquad b = 1/5. \tag{33}$$

The similarity solution takes the form

$$h = q^{2/5}S^{-1/5}t^{1/5}g(\eta) \qquad \eta = \frac{x}{q^{3/5}S^{1/5}t^{4/5}}. \tag{34}$$

The 'shape' function, $g(\eta)$, satisfies

$$\frac{1}{5}g - \frac{4}{5}\eta\frac{\mathrm{d}g}{\mathrm{d}\eta} = \frac{\partial}{\partial\eta}\left(g^3\frac{\mathrm{d}g}{\mathrm{d}\eta}\right), \tag{35}$$

and

$$\int_{-\eta_f}^{\eta_f} g \, \mathrm{d}\eta = 1. \tag{36}$$

Unfortunately, this equation is not integrable and instead the ODE must be solved numerically.

3.2 Three-dimensional axisymmetric viscous gravity current

We finish this section with three-dimensional axisymmetric viscous flows. On a horizontal plane, the flow becomes axisymmetric with the momentum equations in the z and radial directions given by

$$0 = -\frac{\partial p}{\partial z} - \rho g, \tag{37}$$

$$0 = -\frac{\partial p}{\partial r} + \mu\frac{\partial^2 u}{\partial z^2}, \tag{38}$$

where $r = \sqrt{x^2 + y^2}$ is the radial coordinate. The flow is purely in the radial direction with velocity u. The radial flux is given by

$$Q(r,t) = \int_0^h u \, \mathrm{d}z = -\frac{\rho g}{3\mu}h^3\frac{\partial h}{\partial r}. \tag{39}$$

By considering a thin annulus $[r, r+\Delta r]$, mass conservation furnishes the governing equation

$$2\pi r\frac{\partial h}{\partial t} = -S\frac{\partial(2\pi Q)}{\partial r} \tag{40}$$

and the 2π cancels giving

$$\frac{\partial h}{\partial t} = \frac{S}{r}\frac{\partial}{\partial r}\left(r h^3 \frac{\partial h}{\partial r} \right), \tag{41}$$

and for a release of a fixed volume, V, of fluid,

$$\int_0^{r_f} r h \, \mathrm{d}r = V_1 = V/(2\pi) \tag{42}$$

where $[V_1] = L^3$. We apply a similar transformation to (16) and see that $r \sim t^{1/8}$ and $h \sim t^{-1/4}$. The similarity solution is given by

$$h = V_1^{1/4} S^{-1/4} t^{-1/4} \psi(\eta), \qquad \eta = \frac{r}{V_1^{3/8} S^{1/8} t^{1/8}}. \tag{43}$$

$$\psi = (3/16)^{1/3} (\eta_f^2 - \eta^2)^{1/3}. \tag{44}$$

$$\eta_f = \frac{2^{13/8}}{3^{1/2}} \approx 1.7808. \tag{45}$$

References

Barenblatt, G. I. (1996). *Scaling, Self-similarity, and Intermediate Asymptotics: Dimensional Analysis and Intermediate Asymptotics.* Number 14. Cambridge University Press.

Huppert, H. E. (1982). The propagation of two-dimensional and axisymmetric viscous gravity currents over a rigid horizontal surface. *Journal of Fluid Mechanics*, 121:43–58.