Week 2 Lecture 2; Viscous gravity currents on an inclined plane

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1 Introduction

In the previous lecture, we saw how viscous fluids spread on horizontal planes. Lava often flows down hills and in this lecture we will derive the governing equation for a viscous gravity current on an inclined plane. There is a component of gravity acting normal to the plane, which gives terms in the PDE similar to those for flow on a horizontal plane and there is a component of gravity acting parallel to the plane, which gives an additional term of a different type (figure 1).

Although the flow does not have a universal similarity solution as the horizontal problem did, we shall see that at late times the model may be accurately approximated, which reduces the flow to a self-similar system. Finally, motivated by barriers to lava flows, we will analyse flow around a wall and develop simple expressions for how the flow deepens upstream of the wall.

2 Two-dimensional viscous gravity currents on an inclined plane

Consider a two-dimensional viscous flow on a plane inclined at an angle β to the horizontal (figure 1). The x axis is parallel to the inclined plane in the downslope direction, whilst the z axis is perpendicular to the inclined plane. The momentum equations in the z and x directions are (using the lubrication approximation)

$$0 = -\frac{\partial p}{\partial z} - \rho g \cos\beta, \tag{1}$$

$$0 = -\frac{\partial p}{\partial x} + \rho g \sin\beta + \mu \frac{\partial^2 u}{\partial z^2}, \tag{2}$$

where we have included the component of gravity in each direction. The pressure is hydrostatic and given by

$$p = p_{\mathrm{atm}} + \rho g (h - z)\cos\beta \tag{3}$$

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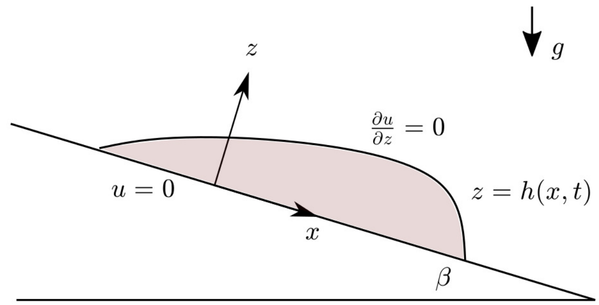

Figure 1: Two-dimensional viscous gravity current on an inclined plane.

The x momentum equation becomes

$$\mu \frac{\partial^2 u}{\partial z^2} = \rho g \left(\frac{\partial h}{\partial x} \cos\beta - \sin\beta \right) \tag{4}$$

There is no-slip at the plane ($z = 0$) and no stress at the free-surface, $z = h$. Hence, the velocity is given by

$$u = \frac{\rho g}{2\mu} z(2h - z) \left(\sin\beta - \frac{\partial h}{\partial x} \cos\beta \right). \tag{5}$$

For $\beta = 0$, we recover the velocity on a horizontal plane from the previous lecture. For $\beta > 0$, the extra contribution arises from the $\sin\beta$ term associated with gravity driving the flow down the plane. Unlike the horizontal problem, there is still motion when $\partial h/\partial x = 0$.

The flux is given by

$$Q = \int_0^h u \,\mathrm{d}z = \frac{\rho g}{3\mu} h^3 \left(\sin\beta - \frac{\partial h}{\partial x} \cos\beta \right) \tag{6}$$

The governing equation is

$$\frac{\partial h}{\partial t} = -\frac{\partial Q}{\partial x} = \frac{\rho g}{3\mu} \frac{\partial}{\partial x} \left[h^3 \left(\frac{\partial h}{\partial x} \cos\beta - \sin\beta \right) \right]. \tag{7}$$

If a constant flux of fluid, q, is supplied upslope then away from the source the current becomes steady with $\partial h/\partial t = \partial h/\partial x = 0$ corresponding to a constant thickness layer steadily flowing downslope. In this regime, the flux balance is

$$q = \frac{\rho g}{3\mu} h^3 \sin\beta \tag{8}$$

Hence, the steady thickness is given by (due to Nusselt, 1916)

$$h_s = \left(\frac{3\mu q}{\rho g \sin\beta} \right)^{1/3}. \tag{9}$$

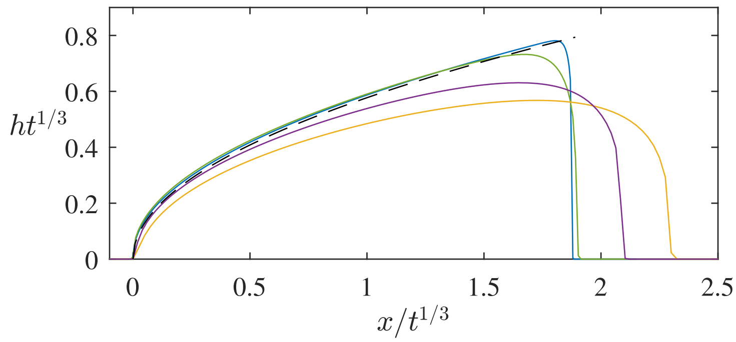

Figure 2: Similarity solution for a fixed volume on an inclined plane. Solid lines show the solution to equation (7) at various times in rescaled similarity coordinates. The dashed line shows the similarity solution.

Next, consider a fixed volume, V, release of liquid. It is not possible to find a general similarity solution for (7) combined with constant volume since the system is not invariant under the usual transformation owing to the combination of first and second order x derivatives. However, at late times, we anticipate that the current becomes long in the x direction and hence $\partial^2/\partial x^2 \ll \partial/\partial x$. The second order x derivative can be neglected. We will verify this assumption later. This motivates studying the approximate equation

$$\frac{\partial h}{\partial t} = -\frac{\rho g}{3\mu}\frac{\partial h^3}{\partial x}\sin\beta \tag{10}$$

with global volume conservation

$$\int_0^{x_f} h\,\mathrm{d}x = V, \tag{11}$$

where the integral is taken from $x = 0$, the initial location of the release since fluid predominantly flows downslope, and x_f is the frontal contact point. We define

$$\hat{S} = \frac{\rho g}{3\mu}\sin\beta, \tag{12}$$

and we can obtain a similarity solution by using the usual transformation in terms of λ and on substituting into (10) and (11), we find that $x \sim t^{1/3}$ and $h \sim t^{-1/3}$. Then upon using $\hat{S}$ and V to get the correct dimensions, we obtain the following similarity solution

$$h = V^{1/3}\hat{S}^{-1/3}t^{-1/3}f(\eta), \qquad \eta = \frac{x}{V^{2/3}\hat{S}^{1/3}t^{1/3}}. \tag{13}$$

The shape is given by

$$f(\eta) = (\eta/3)^{1/2}, \qquad \text{for} \quad 0 < \eta < \eta_f, \qquad \text{where} \quad \eta_f = \frac{3}{2^{2/3}} \approx 1.8899. \tag{14}$$

This is compared to numerical integrations of the full governing equation (7) in figure 2. We note that $\partial h^3/\partial x \sim t^{-4/3}$, whilst

$$\frac{\partial}{\partial x}\left(h^3\frac{\partial h}{\partial x}\right) \sim t^{-2}, \tag{15}$$

and hence the neglected second order term in (7) becomes smaller than the other two terms at late times. Resolving the steep region at the front will require reintroducing the second order term.

Here we have found that similarity solutions may be used to obtain analytic solutions even when the similarity solution does not satisfy the full governing equation. Care and experience are required when neglecting terms in order to seek similarity solutions to the reduced equations and one should always check that the solution is consistent with the assumptions that were made regarding the relative size of terms.

3 Three-dimensional flow

Now consider three-dimensional flow on an inclined plane. The y axis is into the page in figure 1. The additional (y) momentum equation is

$$0 = -\frac{\partial p}{\partial y} + \mu \frac{\partial^2 v}{\partial z^2} \tag{16}$$

with the other two momentum equations unchanged. The velocity in the y direction is v. The pressure is still given by (3). The expression for the downslope (x) velocity, u, (5) is unchanged whilst the cross-slope velocity is obtained from (16) with the usual no-slip ($u = v = 0$) boundary condition at $z = 0$ and zero-stress ($\partial u/\partial z = \partial v/\partial z = 0$) at the free-surface, $z = h$, which gives

$$v = -\frac{\rho g}{2\mu} z(2h - z)\frac{\partial h}{\partial y}\cos\beta. \tag{17}$$

This velocity is similar to that for flow over a horizontal plane from the previous lecture because there is no slope in the y direction. The volume flux in the x and y directions can be written as

$$\mathbf{Q} = (Q_x, Q_y) = \frac{\rho g}{3\mu} h^3 \Big(\mathbf{e}_x \sin\beta - \nabla h \cos\beta\Big), \tag{18}$$

where $\mathbf{e}_x$ is the unit vector in the x direction. Volume conservation implies that

$$\frac{\partial h}{\partial t} + \nabla \cdot \mathbf{Q} = 0, \tag{19}$$

which we write as

$$\frac{\partial h}{\partial t} + \frac{\rho g \sin\beta}{3\mu}\frac{\partial h^3}{\partial x} = \frac{\rho g \cos\beta}{3\mu}\nabla\cdot\big(h^3 \nabla h\big). \tag{20}$$

This is then combined with some form of global mass conservation depending on the conditions.

4 Flow upstream of a wall

Consider a two-dimensional, steady, constant thickness flow ($h = h_s$) supplied by a flux q migrating downslope from far upstream at $x = -\infty$ reaching a wall of height $H \gg h_s$ at $x = 0$. The wall is laterally extensive in the y direction so the flow is two-dimensional. A

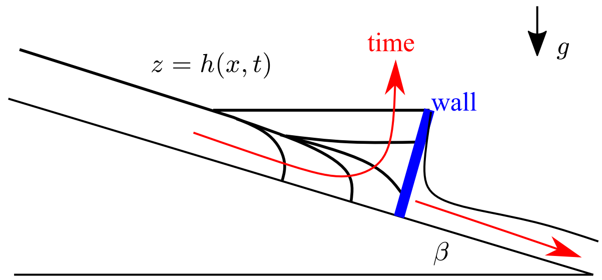

Figure 3: Flow upstream of a wall.

deep pool of liquid gradually develops upstream of the wall until the wall is surmounted by the flow at which point the flow becomes steady, $h = h(x)$, in $x < 0$ (figure 3). The steady flux satisfies in $x < 0$

$$Sh^3\left(\sin\beta - \frac{\partial h}{\partial x}\cos\beta\right) = q = Sh_s^3 \sin\beta, \tag{21}$$

where the second equality arises because $h \to h_s$ far upstream ($x \to -\infty$), where $\partial h/\partial x = 0$. The boundary condition at $x = 0$ is $h = H$ since the flow is surmounting the wall. This equation is integrable but the solution is very complicated. Instead, we can approximate the solution in the pooled region and in the upstream region. Far upstream of the wall, the flow is steady with $\partial h/\partial x = 0$ and hence $h = h_s$. This solution does not satisfy the boundary condition that the wall at $x = 0$ has been overtopped. In the deep pooled region near the wall, the flow is much thicker with $h \sim H \gg h_s$, which motivates solving

$$\sin\beta - \frac{\partial h}{\partial x}\cos\beta = \left(\frac{h_s}{h}\right)^3 \sin\beta \approx 0. \tag{22}$$

This approximate equation corresponds to zero volume flux (see equation 6). Of course, there is volume flux but it is unimportant in determining the free-surface shape when the flow is so deep. The equation has solution

$$h = x\tan\beta + H, \tag{23}$$

where we have used the condition $h = H$ at $x = 0$. It should be noted that this corresponds to a horizontal free-surface (for which there is no flux). This solution is valid for $h > h_s$, (but is only a good approximation for $h \gg h_s$). The pooled region occupies

$$(h_s - H)\cot\beta < x < 0, \tag{24}$$

whilst upstream of this we have $h = h_s$. With this steady (or late-time) solution in hand, we could calculate the volume in the pool and hence the time taken for the flow to surmount the wall.

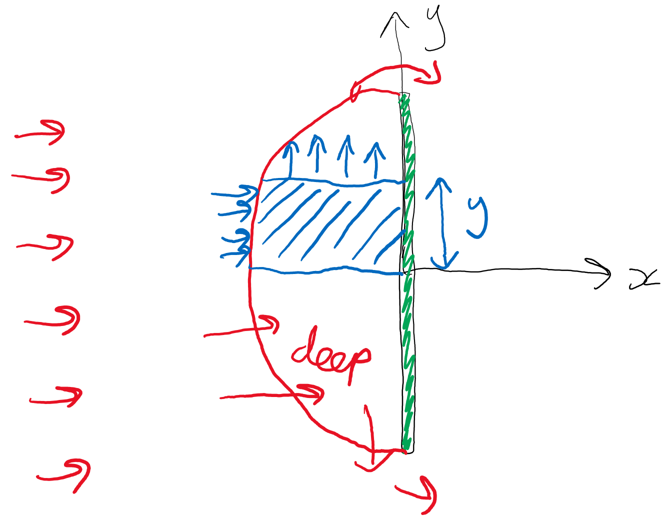

Figure 4: Steady flux balance around a wall.

4.1 A wide wall

We next consider steady flow around a wide (but not infinitely wide) and very high wall that is not overtopped but instead there is flow around the wall in the y direction. The wall lies along $x = 0$, $-L < y < L$. The upstream flow is supplied by a constant flux line source so that far upstream there is a sheet of steadily flowing fluid with thickness $h = h_s$. The line source provides flux q per unit width in the y direction. Since there is no flow into the wall, there is a neighbourhood just upstream of the wall in which the x flux vanishes and the flow becomes deep. Hence (22) applies near the wall. Upon integrating (22), we obtain

$$h = x \tan\beta + G(y), \tag{25}$$

where $G(y)$ is a function of integration that is to be determined since in this setup $h = h(x, y)$. The flow thickness at the wall is $G(y)$, which we anticipate is much larger than h_s for a wide wall. The solution (25) rejoins the constant thickness, h_s solution further upstream, at

$$x = x_0(y) = \big[h_s - G(y)\big] \cot\beta \approx -G(y) \cot\beta < 0, \tag{26}$$

since $G(y) \gg h_s$. In steady state (late times) the flux into the deep ponded region from upstream is balanced by the transverse flux around the wall in the y direction. We can balance these two fluxes to obtain a prediction for the flow thickness $G(y)$ (see figure 4). The flux from upstream into the region $[x_0(y), 0] \times [0, y]$ of the deep pond is given by

$$qy \tag{27}$$

Note that there is no transverse flux across $y = 0$ (by symmetry). The transverse flux out of this region of the deep pond at y is given by

$$\int_{x_0(y)}^{0} Q_y \, \mathrm{d}x \tag{28}$$

Equating the two fluxes furnishes

$$qy = -\frac{\rho g \cos\beta}{3\mu} \int_{x_0(y)}^{0} h^3 \frac{\mathrm{d}G}{\mathrm{d}y}\,\mathrm{d}x. \tag{29}$$

Where we have used (25) to write $\partial h/\partial y = \mathrm{d}G/\mathrm{d}y$. Carrying out the integration (it is easiest to change variables in the integral from x to h using 25) gives

$$qy = -\frac{\rho g \cos^2\beta}{12\mu \sin\beta} G^4 \frac{\mathrm{d}G}{\mathrm{d}y}. \tag{30}$$

Given that $G(y) \approx 0$ at the ends $y = \pm L$ (the flow is much thinner there), we integrate twice and find that

$$G(y) = \left(\frac{30\mu q \sin\beta}{\rho g \cos^2\beta}\right)^{1/5} \left(L^2 - y^2\right)^{1/5}, \tag{31}$$

which gives the flow thickness at the wall. This is indeed much larger that h_s for a wide wall ($L \gg h_s$), which verifies our assumption. The flow thickness in the deep zone is then obtained from (25), with the maximum thickness obtained at $x = 0$, $y = 0$ and given by

$$h_{\max} = G(0) = \left(\frac{30\mu q L^2 \sin\beta}{\rho g \cos^2\beta}\right)^{1/5} \tag{32}$$

This tells us how high a barrier of a given width L must be to prevent overtopping. We can also calculate the force that the wall must withstand from the weight of the fluid in the pond acting in the downslope direction.