

Generalised 1st order form:

$$\tau \dot{x} + x = Ku(t)$$

Response to unit step input:

$$x(t) = K(1 - e^{-t/\tau})$$

Proof:

1) Take the Laplace transform:

$$\tau sX + X = KU$$

The unit step function in Laplace form (s-space) is: $\frac{1}{s}$

Therefore:

$$\begin{aligned}\tau sX + X &= K \frac{1}{s} \\ X(\tau s + 1) &= K \frac{1}{s} \\ X &= K \frac{1}{s(\tau s + 1)} \\ X &= K \frac{1}{s(\tau s + 1)}\end{aligned}$$

2) Perform partial fractions (after isolating s terms):

$$\begin{aligned}X &= K \frac{1/\tau}{s(s + 1/\tau)} = K \left(\frac{A}{s} + \frac{B}{s + 1/\tau} \right) \\ K \frac{1/\tau}{s(s + 1/\tau)} &= K \left(\frac{A(s + 1/\tau) + B(s)}{s(s + 1/\tau)} \right) \\ \frac{1}{\tau} &= A \left(s + \frac{1}{\tau} \right) + B(s)\end{aligned}$$

$$\text{Let } s = -\frac{1}{\tau}$$

$$\begin{aligned}\frac{1}{\tau} &= A(0) + B(s) \\ B &= -1\end{aligned}$$

$$\text{Let } s = 0$$

$$\begin{aligned}\frac{1}{\tau} &= A \left(\frac{1}{\tau} \right) + B(0) \\ A &= 1\end{aligned}$$

Therefore,

$$X = K \left(\frac{1}{s} - \frac{1}{s + 1/\tau} \right)$$

3) Now take the inverse Laplace:

$$L^{-1}\{X\} = x(t) = K \left(L^{-1}\left\{ \frac{1}{s} \right\} - L^{-1}\left\{ \frac{1}{s + 1/\tau} \right\} \right)$$

Inverse Laplace rule:

$$L^{-1}\left\{ \frac{1}{s + a} \right\} = e^{-at}$$

Therefore:

$$x(t) = K(1 - e^{-(1/\tau)t})$$