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THE INVESTMENT RETURN FROM A PORTFOLIO WITH A DYNAMIC REBALANCING POLICY

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ABSTRACT

An analysis is made of the effect on portfolio performance if assets are continually rebalanced to constant market value proportions, relative to the passive 'buy and hold' strategy. The probability that one strategy outperforms the other is evaluated on the basis of a geometric diffusion model of market prices and by reference to historical data.

KEYWORDS

Investment; Rebalancing

1. INTRODUCTION

1.1 What is the effect on the investment return from a portfolio when it is continually rebalanced so as to maintain fixed market value proportions of the assets? Is rebalancing good for a portfolio? These are questions with some relevance to portfolio management.

1.2 To be more specific, what are the differences between the portfolio returns which result from the following alternative approaches to investment policy?

—*The passive strategy:*	assets are held without switching, and investment income is reinvested in each respective asset class.
—*Regular rebalancing:*	assets are switched at regular rebalancing intervals to reinstate fixed market value proportions, investment income is reinvested in each respective asset class, market value proportions are allowed to drift between rebalancing dates.
—*Continual rebalancing:*	assets are switched as often as necessary in order to maintain fixed market value proportions at all times.

The term 'asset' may be applied either to individual securities or to whole asset sectors as represented by suitable indices.

1.3 We shall focus on comparison between the passive strategy and continual rebalancing. Space does not permit consideration here of regular rebalancing, which can be regarded as intermediate between the other two strategies. The choice of a sensible rebalancing period is a significant practical question, but one which must be linked with the practical consideration of transaction costs.

1.4 The plan of this paper is to develop the mathematics of portfolio rebalancing from first principles, to confirm the analysis by back-testing on stock market data, and to show some connections with modern financial theory.

2. PORTFOLIO REBALANCING

2.1 We begin with an elementary example. Consider a portfolio containing just two types of asset: cash which earns no interest and a stock. The market value of the accumulation (with income reinvested) of this stock at any time t will be denoted by $R(t)$. Let $R(0) = 1$ so that $R(t)$ is the accumulated value of unit initial value of the stock. The investment roll-up can be regarded as gross or net of tax as appropriate. Note the implication of a single well-defined market price for purchases or sales. The analysis of rebalancing will proceed on the simplifying assumption that transaction costs of switching can be disregarded. The practical aspect of transaction costs is beyond the scope of this study of basic principles.

2.2 Let the specified asset proportions be α in stock and $\beta(=1-\alpha)$ in cash. Over the period $t = 0$ to 2, application of the passive policy to unit initial value of the composite portfolio will result in accumulation to a value:

$$\alpha R(2) + \beta.$$

Rebalancing at $t = 1$ to proportions α, β will result in:

$$(\alpha R(1) + \beta)(\alpha R(2)/R(1) + \beta).$$

The difference between these two outcomes (passive – rebalanced) is:

$$\alpha \beta i_1 i_2$$

where $i_1 = R(1) - 1$ and $i_2 = R(2)/R(1) - 1$ are the stock returns in the first and second intervals.

2.3 Provided that the proportions α and β are both positive, the difference is positive when both i_1 and i_2 have the same signs. A positive difference means that the passive portfolio outperforms the rebalancing policy, as one would expect in either a rising or a falling market. On the other hand, when i_1 and i_2 have opposite signs, the rebalancing policy is better.

2.4 For an arithmetical illustration, suppose that the stock returns either +25% or −20% in any single interval. Let the asset proportions be $\alpha = \beta = 0.5$. Over two periods there are four possible outcomes, as in Table 1.

Table 1.

Time $t = 0$ to 1		Time $t = 1$ to 2		
Stock return %	Portfolio value	Stock return %	Rebalanced portfolio	Passive portfolio
−20	0.900	−20	0.810	0.820
−20	0.900	+25	1.012	1.000
+25	1.125	−20	1.012	1.000
+25	1.125	+25	1.266	1.281

In these four examples we see two cases where the rebalanced portfolio outperforms, following a change of sign in the stock return, and two cases of the reverse situation.

Multiple Periods

2.5 In order to generalise to multiple periods some notation is needed. Let r_t denote the accumulation of unit value of stock from $t-1$ to t; i.e. $r_t = 1 +$ rate of return. Then $R(n) = \Pi\, r_t$.

Let $V(t)$ denote the accumulated value at time t of a portfolio having unit value at $t = 0$, which is initiated with, and regularly rebalanced to, fixed proportions α, β at $t = 0, 1, 2, \ldots, n-1$. Then $V(n) = \Pi\,(\alpha\, r_t + \beta)$.

Let $V^{(o)}(t)$ denote the accumulated value at time t of a portfolio having unit value at $t = 0$, which is initiated with asset proportions α, β and not rebalanced. Then $V^{(o)}(n) = \alpha \Pi\, r_t + \beta$.

2.6 When the total length of period is clear in the context, the argument (n) will be dropped from the notation. We then write simply R, V and $V^{(0)}$ to denote the accumulation of unit amount invested in the given stock, the rebalanced portfolio and the passive portfolio respectively.

Fixed Rate of Return

2.7 For multiple periods, the simplest situation which may be contemplated is when the stock return in each interval is identical. Then, writing $r_t = r$:

$$\begin{aligned} R &= r^n \\ V &= (\alpha r + \beta)^n \\ V^{(o)} &= \alpha r^n + \beta. \end{aligned}$$

It can be verified that $V^{(o)} - V$ is zero at $r = 1$, and that it increases as a function of r when r is greater than 1. It follows that $V \leq V^{(0)}$ in this situation.

2.8 This confirms that when there are no sign changes in the stock return, i.e. in a consistently rising or falling market, rebalancing has an adverse effect on performance.

Two-Valued Return

2.9 In practice we expect to see market reversals, and the simplest way to reflect this observation is to consider two outcomes, namely accumulation to either $r > 1$ or $r^{-1} < 1$. This represents the effect of positive and negative returns without bias, in the sense that an equal number of gains and losses on the stock will cancel exactly.

2.10 If g is the number of gains and $h = n - g$ is the number of losses, then:

$$\begin{aligned} R &= r^{g-h} \\ V &= (\alpha r + \beta)^g\,(\alpha r^{-1} + \beta)^h \qquad (1) \\ V^{(o)} &= \alpha R + \beta. \end{aligned}$$

2.11 Note that the portfolio values at time n depend on the number of gains and losses on the stock during the rebalancing intervals, but not on the order in which they occurred. We have, however, seen that a steady run of gains results in $V \leq V^{(o)}$; this is the situation where $g = n$ and the stock return in each interval is identical. Likewise $V \leq V^{(o)}$, with a steady run of losses. We should, therefore, expect that rebalancing will be unfavourable in circumstances where the gains outnumber the losses or vice versa. On the other hand, rebalancing is favourable where the number of gains is similar to the number of losses; this can be demonstrated for the case $g = h$ where:

$$R = 1$$

$$V^{(o)} = \alpha + \beta = 1$$

and:

$$\begin{aligned} V^{1/g} &= (\alpha r + \beta)(\alpha r^{-1} + \beta) \\ &= 1 + \alpha\beta(r^{1/2} - r^{-1/2})^2 \\ &> 1 \text{ (providing } \alpha, \beta > 0 \text{ and } r \neq 1). \end{aligned}$$

Hence $V > V^{(o)}$, and rebalancing is bound to outperform in this particular circumstance.

2.12 Table 2 gives an illustration of these points, using the parameters $n = 10$, $r = 1.1$, $\alpha = \beta = 0.5$.

Table 2.

g	R	$V^{(o)}$	V	$V^{(o)} - V$
0	0.386	0.693	0.628	0.065
1	0.467	0.733	0.691	0.042
2	0.564	0.782	0.760	0.022
3	0.683	0.842	0.836	0.006
4	0.826	0.913	0.919	−0.006
5	1.000	1.000	1.011	−0.011
6	1.210	1.105	1.113	−0.008
7	1.464	1.232	1.224	0.008
8	1.772	1.386	1.346	0.040
9	2.144	1.572	1.481	0.091
10	2.594	1.797	1.629	0.168

2.13 In this table we see that $V < V^{(o)}$ (i.e. rebalancing unfavourable) when the number of gains g is in the range 0 to 3 or 7 to 10. Rebalancing is marginally favourable if g is either 4, 5 or 6.

2.14 Although we have disregarded interest on the cash holdings, the inclusion of interest would not alter the conclusions of this section; the algebra still works if all asset returns are divided by the accumulation of cash with interest.

3. Binomial Probability Model

3.1 The next step is to consider a probability model for the behaviour of stock returns. The natural choice of model to associate with a two-valued asset return is the binomial model. Thus we define p and $(1 - p)$ as the probabilities of gain or loss respectively in each period, these being identical for all periods and independent from one period to another.

3.2 We shall consider the case $p = \frac{1}{2}$. The probability of g gains and $h = n - g$ losses after n periods is then:

$$\text{Prob}\,[g] = \frac{1}{2^n}\binom{n}{g}.$$

Table 3 extracts figures from the example of Table 2, and shows, in the last column, the respective probability of each outcome.

Table 3.

g	R	$V^{(o)} - V$	Prob $[g]$
0	0.386	0.065	0.001
1	0.467	0.042	0.010
2	0.564	0.022	0.044
3	0.683	0.006	0.117
4	0.826	−0.006	0.205
5	1.000	−0.011	0.246
6	1.210	−0.008	0.205
7	1.464	0.008	0.117
8	1.772	0.040	0.044
9	2.144	0.091	0.010
10	2.594	0.168	0.001

3.3 Although the passive portfolio return can be significantly larger than the result of rebalancing, this only happens at values of R for which the chances are low. The most likely outcomes for R are those near the median return. In this example the probability that rebalancing produces the better return is:

$$\begin{aligned}\text{Prob}\,[V^{(o)} < V] &= 0.205 + 0.246 + 0.205 \\ &= 0.656.\end{aligned}$$

3.4 A further illustration of results is given in Table 4, using parameters $n = 20$, $r = 1.1$, $\alpha = 0.8$, $\beta = 0.2$. In this evaluation, rebalancing is favourable when g (and h) are in the range 8 to 12, but not otherwise.

Table 4.

g	R	$V^{(0)} - V$	Prob[g]
0	0.149	0.098	0.000
1	0.180	0.087	0.000
2	0.218	0.074	0.000
3	0.263	0.062	0.001
4	0.319	0.048	0.005
5	0.386	0.035	0.015
6	0.467	0.022	0.037
7	0.564	0.009	0.074
8	0.683	−0.002	0.120
9	0.826	−0.010	0.160
10	1.000	−0.015	0.176
11	1.210	−0.014	0.160
12	1.464	−0.005	0.120
13	1.772	0.014	0.074
14	2.144	0.048	0.037
15	2.594	0.100	0.015
16	3.138	0.178	0.005
17	3.797	0.288	0.001
18	4.595	0.440	0.000
19	5.560	0.646	0.000
20	6.728	0.921	0.000

3.5 Again, the chances of outcomes where $V^{(o)} - V$ is significantly positive are low. The probability that rebalancing is favourable is:

$$\text{Prob}\,[V^{(o)} < V] = 0.737.$$

3.6 Applying probability weights to the alternative outcomes for $V^{(o)} - V$, we find that the expectation of this quantity is 0.001. In this case, and also more generally, it can be said that the two investment strategies yield similar returns on average.

3.7 It is interesting to find that the pattern of Tables 3 and 4 is repeated consistently for a wide range of investment period n, asset return r and portfolio proportion α. Table 5 shows P = Prob $[V^{(o)} < V]$ for a variety of parameters.

Table 5.

Investment period n	Periodic asset return r	Asset proportion α	Prob $[V^{o} < V]$ P
10	1.1	0.5	0.656
20	1.1	0.8	0.737
20	1.01	0.8	0.737
20	1.5	0.8	0.617
20	1.1	0.5	0.737
100	1.01	0.5	0.729
100	1.01	0.05	0.680

3.8 We see evidence here that, whatever the parameters, rebalancing the portfolio is more likely than not to yield outperformance relative to the passive strategy. It seems surprising that, on the assumption of the binomial model, there should be such apparent consistency in the probability (around 2/3) of outperformance. This feature is examined further in the next section.

3.9 It is apparent that a different picture results from introducing asymmetry, with $p \neq \frac{1}{2}$. Then the more probable outcomes will be those with a preponderance of asset gains or losses. As we have already seen, it is in these situations that rebalancing tends to be unfavourable.

4. Geometric Diffusion Model

4.1 In Tables 3 and 4 the logarithm of R is linear in g (see ¶2.10), and g has binomial probability which approaches a normal distribution as n becomes large. Consequently ln R tends to a normal distribution as n tends to infinity. If we now think in terms of n sub-divisions of unit time, we arrive at the simplest probability model for continuous investment behaviour: the log normal model where ln R is normally distributed and investment returns in separate time intervals are independent. Let μ and σ^2 denote the mean and variance of ln R for unit time. Then:

$$\ln R(1) \sim \mathrm{N}\,(\mu, \sigma^2)$$

and

$$\ln R(t) \sim \mathrm{N}\,(\mu t, \sigma^2 t).$$

4.2 Portfolio rebalancing can be re-examined in the context of this model of a random walk in continuous time — which is properly known as the geometric diffusion model. We refer now to continual rebalancing (see ¶1.2) with a large number n, of equally spaced rebalancing intervals.

4.3 In the limit as $n \to \infty$, but fixing $t = 1$, it is shown in the Appendix (Section A.1) that we arrive at the following formula:

$$V = R^{\alpha} \exp\left(\tfrac{1}{2}\,\alpha\beta\sigma^2\right). \tag{2}$$

This formula appears in Perold & Sharpe (1988). It applies irrespective of the value of μ, or indeed of the specific path of the asset return during the period (subject however to the path behaving in accordance with the geometric diffusion model). It is easy to verify (but not shown here) that this formula is also the limit as $n \to \infty$ for the two-valued model (i.e. the limit of formula (1) in ¶2.10).

4.4 We can now use this formula to examine the question raised by the preceding section: what is the probability that continual rebalancing will outperform the passive investment strategy? Consider the case: $\alpha = 0.5$, $\sigma = 0.20$, $\mu = 0$.

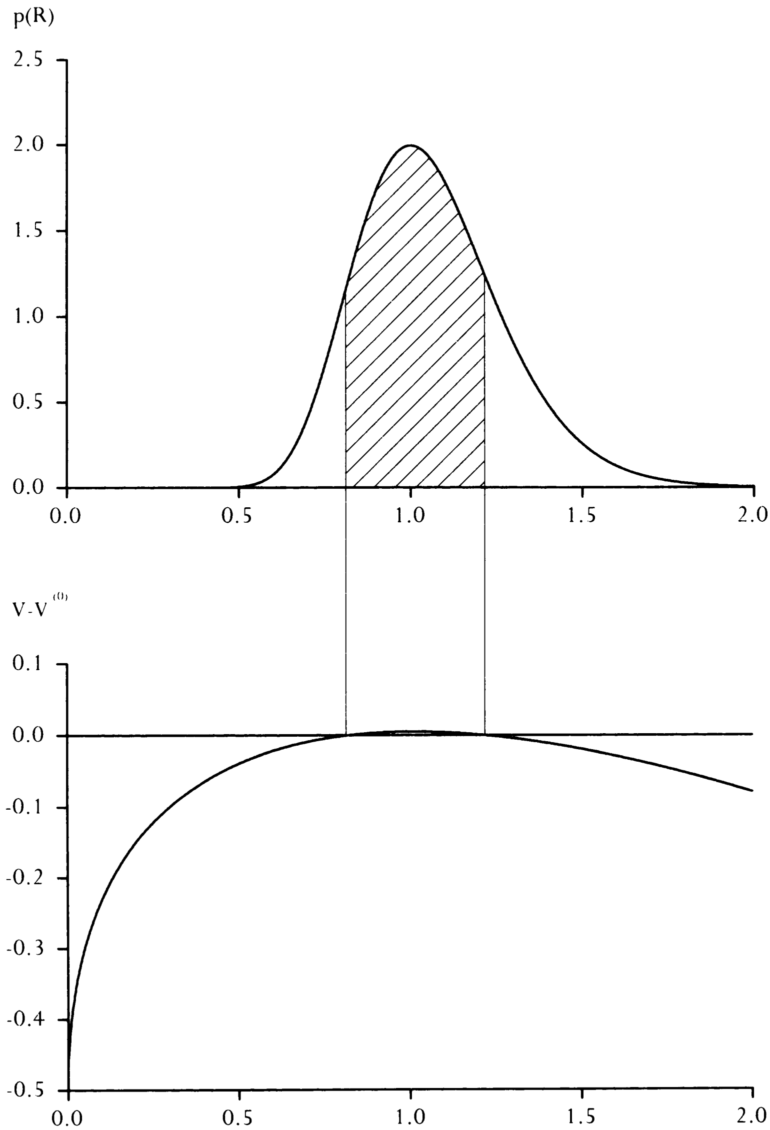

Figure 1. Asset return R plotted against:
(1) log normal density function of R; and
(2) $V - V^{(o)}$ ($\mu = 0$)

4.5 The calculation is illustrated in Figure 1. In the two graphs, the asset return R is measured on the horizontal scale. The lower graph plots $V - V^{(o)}$, whilst the upper graph plots the log normal density function of R. In the latter, the relevant probability is the shaded area between the two vertical lines.

4.6 The relevant verticals are located at the values x and y of R at which $V^{(o)} - V = 0$. Thus x and y are roots of the following equation, in which the unknown is R:

$$\alpha R + \beta - R^{\alpha} \exp\left(\tfrac{1}{2}\ \alpha\beta\sigma^2\right) = 0. \tag{3}$$

4.7 This equation can be solved algebraically when $\alpha = \beta = \frac{1}{2}$; we find that:

$$x = \left(\lambda - \sqrt{\lambda^2 - 1}\right)^2$$

and

$$y = \left(\lambda + \sqrt{\lambda^2 - 1}\right)^2$$

where:

$$\lambda = \exp\left(\sigma^2/8\right).$$

In this case, with $\sigma = 0.20$, we have: $\lambda = 1.0050$; $x = 0.8186$; $y = 1.2216$. (Generally, when α and β are not equal, the roots have to be obtained by numerical method.)

4.8 The probability that rebalancing outperforms the passive strategy is equal to the probability that the asset accumulation R lies between the roots x and y. Let N ( ) denote the cumulative probability function for the N(0, 1) normal distribution. Then:

$$\begin{aligned}\text{Prob}[V^{(o)} < V] &= \text{Prob}[x < R < y] \\ &= \text{Prob}[\ln x < \ln R < \ln y] \\ &= \mathrm{N}\left(\frac{\ln y - \mu}{\sigma}\right) - \mathrm{N}\left(\frac{\ln x - \mu}{\sigma}\right).\end{aligned} \tag{4}$$

4.9 Substituting the numerical values $\sigma = 0.20$, etc. we get:

$$\begin{aligned}P &= \mathrm{N}\,(1.0008) - \mathrm{N}\,(-1.0008) \\ &= 0.683.\end{aligned}$$

4.10 Table 6 shows some results of repeating this calculation, using a variety of parameters α and σ (but keeping $\mu = 0$ throughout).

Table 6

Asset proportion α	Standard deviation σ	Prob$[V^{(o)} < V]$ P
0.5	0.20	0.683
0.5	0.01	0.683
0.2	0.20	0.683
0.2	0.01	0.682
0.8	0.40	0.684
0.8	0.20	0.683
0.8	0.01	0.683

4.11 It thus appears that rebalancing outperforms the passive strategy with a probability of about 0.683, and that the exact probability is insensitive to the choice of parameters, except for $\mu = 0$.

4.12 The explanation for this feature is that, as illustrated in the worked example, the values of $\frac{\ln x}{\sigma}$ and $\frac{\ln y}{\sigma}$ are generally very close to -1 and $+1$ respectively.
It is shown in the Appendix (Section A.2) that when $\mu = 0$, for any value of α between 0 and 1:

$$\lim_{\sigma \to 0} \text{Prob}[V^{(o)} < V] = \text{N}(1) - \text{N}(-1)$$

$$= 0.683.$$

This explains the results shown in Table 6, because it is only when σ is much larger that the probability departs significantly from that of a one standard deviation event.

4.13 Figure 1 provides a useful picture of our two basic observations. As seen earlier in relation to the binomial model, rebalancing offers a modest advantage over the passive strategy in the more likely scenario that the asset return is not too far from the median. Rebalancing carries a risk of significant under-performance relative to the passive strategy when the asset return is unusually high or low.

4.14 The latter observation is intuitively clear whilst the former is less so. The key to understanding the nature of the 'rebalancing effect' is to note that the two investment strategies yield similar returns on average. This was illustrated in relation to the binomial model (c.f. ¶3.6), and it is generally fair when $\mu = 0$ (though note also ¶9.3). Thus, a better-than-even chance of a modest gain from rebalancing is the counterpart of a less-than-even chance of a more significant under-performance.

4.15 These observations relate to the situation when $\mu = 0$. When the asset offers a positive expected return the upper graph in Figure 1 shifts to the right whilst the lower graph remains unmoved. Thus, the probability of outperformance from rebalancing is reduced — as illustrated by the shaded area in Figure 2.

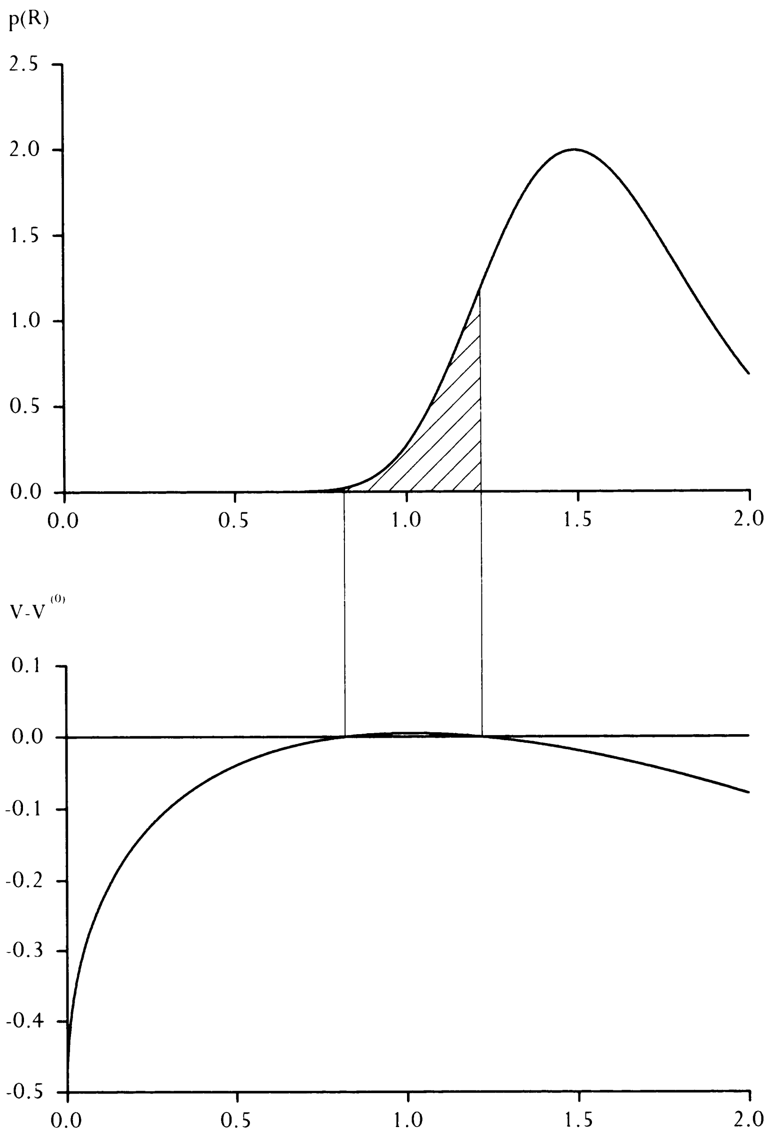

Figure 2. Asset return R plotted against:
(1) log normal density function of R; and
(2) $V - V^{(o)}$ ($\mu > 0$)

5. BACK-TESTING ON THE STOCK MARKET

5.1 The theoretical results obtained so far can be tested against historical stock market data, by evaluating what would have been the results of rebalancing portfolios in previous years. We can test the rebalancing formula (2), and also the probability (4) that rebalancing is favourable.

5.2 In this section the 'asset' under consideration is the class of United Kingdom equities as represented by the *Financial Times*-Actuaries All-Share Index and preceding indices as used by Wilkie (1995). A numerical series of $R(t)$ is derived at monthly intervals, using the indices for capital value with allowance for reinvestment of grossed-up dividends. We continue to disregard interest on cash. (Consideration of multi-asset portfolios is deferred to Section 6.)

The Rebalancing Formula

5.3 With $\alpha = \beta = 0.5$, formula (2) for a period of t years becomes:

$$V = \sqrt{R}\exp(t\sigma^2/8).$$

This theoretical value may be compared with the observed value $\tilde{V}$ which results from monthly (as a proxy for continuous) rebalancing.

5.4 The calculation of V depends on the volatility σ, which is not the same from one time to another. In Table 7 the value of σ which is used to calculate V for each five-year period is the sample variance derived from the monthly data of log returns for each respective period, annualised by the factor $\sqrt{12}$.

Table 7.

Period	$\tilde{R}$	$\tilde{V}^{(\omega)}$	$\tilde{V}$	σ	V
1921 - 25	2.499	1.749	1.589	0.075	1.586
1926 - 30	1.072	1.036	1.040	0.084	1.040
1931 - 35	1.574	1.287	1.279	0.176	1.279
1936 - 40	0.798	0.899	0.916	0.199	0.916
1941 - 45	2.197	1.598	1.490	0.081	1.488
1946 - 50	1.298	1.149	1.150	0.122	1.150
1951 - 55	2.100	1.550	1.466	0.132	1.465
1956 - 60	2.139	1.569	1.484	0.149	1.483
1961 - 65	1.294	1.147	1.147	0.114	1.147
1966 - 70	1.644	1.322	1.307	0.175	1.307
1971 - 75	1.501	1.250	1.313	0.335	1.314
1976 - 80	2.524	1.762	1.630	0.197	1.628
1981 - 85	3.052	2.026	1.779	0.160	1.775
1986 - 90	1.893	1.446	1.421	0.226	1.421

5.5 The close agreement between the theoretical value V and the observed value $\tilde{V}$ shows that the formula works well for the representation of continual rebalancing. It also demonstrates that the effects of continual rebalancing are adequately captured in practice by regular balancing no more frequently than at monthly intervals.

The Probability that Rebalancing is Favourable

5.6 We now test formula (4) for the probability that continual rebalancing outperforms the passive strategy. For this we require estimates of the mean μ and standard deviation σ to be used in the formula. The sample values taken from the 70-year period, 1921 to 1990 inclusive, are found to be $\mu = 0.109$ and $\sigma = 0.172$.

5.7 These relate to one-year periods, and for five-year periods we scale up by the factors of 5 and $\sqrt{5}$ respectively to obtain parameters $\mu_5 = 0.545$ and $\sigma_5 = 0.385$.

Using these parameters in formula (4) produces the theoretical result for five year periods that:

$$\text{Prob}\,[V^{(o)} < V] = 0.33.$$

Inspection of Table 7 shows that $\tilde{V}^{(o)} < \tilde{V}$ in 4 out of 14 cases, which would be 5 out of 14 if we counted 1961 - 65. (Compare: 0.33×14 periods = 4.6) Thus the proportion of five-year periods for which rebalancing was favourable is in line with the theoretical prediction.

5.8 We can compare the theoretical probability:

$$P = \text{Prob}\,[V^{(o)} < V]$$

with the observed result:

$$\tilde{P} = \text{Prob}\,[\tilde{V}^{(o)} < \tilde{V}]$$

for alternative periods. Thus, if we divide the 70 years into adjacent periods of equal length from 1921 to 1990 inclusive, we find the following comparisons.

Table 8.

Length of period (years)	Observed $\tilde{P}$	Predicted P
1	0.41	0.59
2	0.34	0.51
5	0.29	0.33
10	0.14	0.16

5.9 Whilst there is significant sampling error in $\tilde{P}$, the observations tend to support the theoretical predictions. Moreover, the period length is seen to be a significant factor; the probability that rebalancing outperforms the passive strategy diminishes, in this case, when dealing with longer periods.

5.10 This last point is readily explained in terms of our preceding analysis.

Recall that we are investigating portfolios comprising a basket of U.K. equities, which have returned almost 11% p.a. on average over the 70 years, and an alternative asset category of cash which earns no interest. Over periods as long as 10 years, it is most probable that equities will significantly outperform cash, and it is in such circumstances that the passive strategy will do best. In contrast, for short periods of one year or less the chances of one asset class outperforming the other are more evenly balanced.

5.11 It would be premature to conclude that, in general, the passive strategy is more likely to be favourable over long periods. The foregoing example, with 11% p.a. long-term outperformance of one asset class over the other, is unrealistic. In the next section we extend the analysis to more than one risky asset, and in Section 8 we examine the significance of the length of period over which the comparison is to be made.

6. Extension to Several Risky Assets

6.1 In Section 4 we showed that rebalancing outperforms the passive strategy with a probability of slightly more than 2/3 in the special case of one risky asset whose log mean return μ is zero. In Section 5 we considered the situation where μ is non-zero and the passive strategy is more likely to be the better policy. We have simplified the study until this point by treating the second asset as cash, yielding no interest. We now generalise the geometric diffusion model to encompass several risky assets with different expected returns.

6.2 Let the number of securities be N, and let the accumulation of security number i from unit amount at $t = 0$ be denoted by the random variable R_i at $t = 1$. R_i is log normal:

$$\ln R_i \sim \mathrm{N}(\mu_i, \sigma_i^2).$$

Also define the correlation coefficients:

$$\rho_{ij} = \mathrm{Corr}\,[\ln R_i, \ln R_j].$$

Two Risky Assets

6.3 The case $N = 2$ can be approached by considering the notional derivative security, which is defined at any time t by $R(t) = R_1(t)/R_2(t)$. A continuously rebalanced portfolio comprising proportions α in this derivative and β in cash without interest yields a value given by formula (2), i.e.:

$$R^\alpha \exp\left(\tfrac{1}{2}\alpha\beta\sigma^2\right)$$

in which σ^2 is the variance of $\ln R = \ln R_1 - \ln R_2$, i.e.:

$$\sigma^2 = \sigma_1^2 - 2\,\rho\sigma_1\,\sigma_2 + \sigma_2^2.$$

6.4 The value V of the rebalanced portfolio in the two stocks is the product of R_2 and the value of the rebalanced derivative portfolio, viz:

$$V = R_1^{\alpha} R_2^{\beta} e^{k}$$

where:

$$k = \tfrac{1}{2}\alpha\beta(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2). \tag{5}$$

6.5 We have $V^{(o)} = \alpha R_1 + \beta R_2$ for the accumulation of the passive strategy involving two stocks. In order to compare V with $V^{(o)}$, it is best to divide their difference by R_2:

$$\frac{V^{(o)} - V}{R_2} = \alpha R + \beta - \lambda R^{\alpha}$$

where $R = R_1/R_2$ as above, and $\lambda = \exp\,(\tfrac{1}{2}\ \alpha\beta\sigma^2) = e^k$. This is in the format of $V^{(o)} - V$ in the earlier case of one risky asset, but with the relative accumulation R replacing R_1 and the more complicated expression for k replacing $\tfrac{1}{2}\alpha\beta\sigma^2$. Furthermore, R is log normal because $\ln R = \ln R_1 - \ln R_2$ is the difference between two normals. This means that the picture is the same as in Figures 1 and 2, and we can proceed as before.

6.6 Therefore, let x and y be the lower and higher roots of $\alpha x + \beta - \lambda x^{\alpha} = 0$. Then:

$$\text{Prob}[V^{(o)} < V] = \text{N}\left(\frac{\ln y - \mu}{\sigma}\right) - \text{N}\left(\frac{\ln x - \mu}{\sigma}\right) \tag{4}$$

where:

$$\ln R_1 - \ln R_2 \sim \text{N}\,(\mu, \sigma^2)$$

i.e. $\mu = \mu_1 - \mu_2$ and $\sigma^2 = \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2$.

6.7 If $\mu_1 = \mu_2$, then $\mu = 0$, and all the analysis in Section 4 applies. We may therefore say, quite generally, that *if a portfolio comprises two securities or asset classes with equal expected return, then a policy of continual rebalancing will deliver better performance about two times out of three* compared with no rebalancing. The exception to this statement is if the two securities are of equal risk and are perfectly correlated (e.g. if they are identical!). In that case $\sigma = 0$, and there can be no gain from rebalancing.

6.8 If the mean log returns μ_1 and μ_2 differ, one asset is expected to outperform the other and a different picture will emerge. Some examples which show the reduction in the probability P which results from non-zero difference μ are shown in Table 9.

Table 9.

Asset proportion α	Standard deviation σ	Difference in log returns μ	Prob $[V^{(o)} < V]$ P
0.5	0.20	0.00	0.683
0.5	0.20	0.02	0.681
0.5	0.20	0.04	0.673
0.5	0.20	0.10	0.625
0.5	0.40	0.10	0.669
0.8	0.40	0.10	0.674
0.8	0.40	(−)0.10	0.664
0.99	0.40	0.10	0.677
0.5	0.20	0.20	0.478
0.5	0.10	0.20	0.157

6.9 We now repeat the back-testing against the stock-market, replacing the cash asset by undated British Government fixed-interest securities — namely Consols. First we test the two asset rebalancing formula (5), using parameters calculated for the respective five-year periods. These parameters are shown, for reference, in Table 10.

Table 10.

	Consols		Equities		
Period	μ	σ	μ	σ	ρ
1921 - 25	0.087	0.085	0.183	0.075	0.049
1926 - 30	0.056	0.058	0.014	0.084	−0.006
1931 - 35	0.116	0.120	0.091	0.176	0.130
1936 - 40	0.009	0.109	−0.045	0.199	0.563
1941 - 45	0.064	0.041	0.157	0.081	0.229
1946 - 50	−0.018	0.085	0.052	0.122	0.167
1951 - 55	−0.006	0.082	0.148	0.132	0.272
1956 - 60	0.000	0.082	0.152	0.149	0.374
1961 - 65	0.036	0.082	0.052	0.114	0.182
1966 - 70	−0.005	0.100	0.099	0.175	0.177
1971 - 75	0.037	0.154	0.081	0.335	0.288
1976 - 80	0.167	0.181	0.185	0.197	0.589
1981 - 85	0.154	0.112	0.223	0.160	0.389
1986 - 90	0.089	0.109	0.128	0.226	0.257

6.10 The calculated values of V are close to the observed results of rebalancing as shown in Table 11.

Table 11.

Period	$\tilde{V}^{(o)}$	$\tilde{V}$	V
1921 - 25	2.022	1.980	1.980
1926 - 30	1.198	1.199	1.199
1931 - 35	1.682	1.720	1.721
1936 - 40	0.923	0.930	0.930
1941 - 45	1.787	1.747	1.747
1946 - 50	1.105	1.101	1.101
1951 - 55	1.535	1.446	1.444
1956 - 60	1.569	1.482	1.480
1961 - 65	1.244	1.256	1.256
1966 - 70	1.310	1.295	1.295
1971 - 75	1.351	1.433	1.435
1976 - 80	2.417	2.459	2.460
1981 - 85	2.608	2.609	2.609
1986 - 90	1.727	1.773	1.774

6.11 It will be noted that in Table 11 the number of occurrences of $\tilde{V} > \tilde{V}^{(o)}$ is 8 out of 14 periods. This observation gives an empirical probability $\tilde{P} = 0.57$ for the probability that rebalancing a Consols and U.K. equity portfolio outperforms the passive strategy over five-year periods. Using formula (4) for the theoretical probability P, we can repeat the analysis of ¶5.8 for this two-asset portfolio.

Table 12.

Length of period (years)	Observed $\tilde{P}$	Predicted P
1	0.56	0.65
2	0.57	0.64
5	0.57	0.58
10	0.57	0.50

6.12 We thus have evidence that rebalancing such a two-asset portfolio is more likely than not to be the better strategy, though the odds are not strongly in favour.

Two or More Risky Assets

6.13 The formula for the continual rebalancing of a portfolio comprising two risky assets was derived in ¶6.4. It is shown in the Appendix (Section A.3) that this formula generalises to N risky assets in the following way:

$$V = \lambda \Pi R_i^{\alpha_i}$$

where $\lambda = e^k$ and

$$k = \tfrac{1}{2} \Sigma \alpha_i \sigma_i^2 - \tfrac{1}{2} \Sigma \alpha_i \sigma_i \rho_{ij} \sigma_j \alpha_j .$$

An alternative demonstration of this result is outlined in the next section. This formula has also been successfully back tested against stock market data using more than two asset sectors; details of this are omitted, but see, for example, Wise (1993).

6.14 The equal-weights version of this formula was derived by Brennan & Schwartz (1985), in a short paper which was aimed at demonstrating that a geometric mean index of stock prices underperforms the result of continual rebalancing.

6.15 The general formula in ¶6.13 appears in Jamshidian (1992), in a study concerned with selecting the asset proportions α_i according to a defined optimality criterion. In their study the model parameters are allowed to vary with time, but our formula is still applicable, provided that the σ_i and ρ_{ij} are interpreted as time integrals of the respective parameters.

7. CONNECTIONS

7.1 The analysis of portfolio rebalancing can be linked with elements of modern financial theory. In this section we sketch these connections, leaving it to the interested reader to follow up references and fill in the missing details if so desired.

Stochastic Analysis

7.2 The rebalancing formula can be derived directly and rigorously using stochastic calculus. If $x(t)$ follows a geometric diffusion process:

$$\frac{dx}{x} = \mu dt + \sigma dz$$

then it can be seen, by using Ito's Lemma (Hull, 1993), that:

$$\frac{dx}{x} = d\phi(x,t)$$

where:

$$\phi(x,t) = \ln x + \tfrac{1}{2}\sigma^2 t.$$

Continual rebalancing is expressed thus:

$$\frac{dV}{V} = \Sigma\alpha_i \frac{dR_i}{R_i}. \tag{6}$$

So:

$$d\phi(V,t) = \Sigma\alpha_i \, d\phi(R_i,t).$$

This equation can be integrated over time:

$$\phi(V) = \Sigma\, \alpha_i\, \phi(R_i). \tag{7}$$

This linear relation is the key result which is demonstrated by more elementary means in Section A.3 of the Appendix, and from which the explicit formula for V follows directly.

Force of Interest

7.3 If we were to disregard the stochastic component of the geometric diffusion process, we would have a force of interest model of the form $dR/R = \mu(t)dt$, with $R(t)$ continuous and differentiable. We would then integrate equation (6) to get:

$$\ln V = \Sigma\, \alpha_i \ln R_i$$

and

$$V = \Pi\, R_i^{\alpha_i}.$$

This is the rebalancing formula without the factor λ.

7.4 A few test calculations soon verify that this formula is less accurate and that the factor λ appears to be correct. There is also a theoretical argument for the λ factor. Without it, V would be the weighted geometric mean of the R_i. In any event $V^{(o)}$ is the correspondingly weighted arithmetic mean. Since a geometric mean (of positive numbers) can never exceed an arithmetic mean, this would imply that $V \leq V^{(o)}$ in all circumstances, irrespective of the actual stock returns. Such certainty would, in turn, imply the existence of dynamic investment strategies which deliver non-negative returns with certainty and with no risk of loss (by exploiting the differences in asset mix between passive and rebalanced portfolios and short-selling where necessary). Therefore, in any reasonable model of the markets in which riskless arbitrage is not available, V must generally be larger than the geometric mean of stock returns.

7.5 It is natural, therefore, to ask whether $\lambda \geq 1$ in all circumstances. It is shown in Section A.4 of the Appendix that this is indeed so.

7.6 We conclude that the force of interest model is not suitable for a realistic representation of stock market behaviour at short time scales. A continuous force of interest admits risk-free arbitrage — as was noted by Boyle (1978). In the geometric diffusion model the instantaneous force of interest has no definition at any single point in time.

7.7 Thinking about the relative merits of rebalancing and the passive strategy shows why the continuous force of interest model does admit arbitrage. Quite simply, a consistently positive (or consistently negative) force of return represents a short-term trend which makes it worth holding on to the performing stock. It seems that a truly efficient market (with perfect information flow and no transaction costs) must exhibit unpredictable price behaviour, with random moves in either direction, in order to defeat the would-be arbitrageur.

The Rebalanced Portfolio as a Derivative.

7.8 A rebalanced portfolio can be regarded as a derivative asset based on the component stocks. The usual analysis of derivative securities is made in terms of one risky asset on which the derivative is based, and one risk-free asset. Calling these two assets R_1 and R_2 respectively, the geometric mean $R_1^{\alpha} R_2^{\beta}$ can be regarded as a derivative of R_1.

7.9 Given the geometric diffusion model of stock returns, this derivative can be priced by risk-neutral valuation (Hull, 1993). The price is e^{-rt} $E^*[R_1^{\alpha}\ R_2^{\beta}]$, where r is the risk-free rate of return and E^* gives the expected return using a probability model in which the risky asset is expected to return only the risk-free rate, i.e. $E^*[R_1] = e^{rt}$.

7.10 It will be found that the price is $1/\lambda$, with λ as previously defined. So, the derivative with pay-off $\lambda\ R_1^{\alpha} R_2^{\beta}$ has the certain price of 1. This is equivalent to saying that initial capital of 1 can be made, by suitably switching assets over time (i.e. by continual rebalancing), to produce the certain payoff $\lambda R_1^{\alpha} R_2^{\beta}$. Since risk-neutral valuation is based on the principle of no arbitrage, once again we have a demonstration that the factor λ is required in order to comply with this principle.

8. THE TIME HORIZON

8.1 Before concluding this paper, we further examine one aspect. In the cases studied earlier we observed the effects of portfolio rebalancing over periods of between one and ten years. What is the significance of the choice of time horizon? For simplicity, we revert to the model used in Section 4, with one risky asset and nil-interest cash.

8.2 The theoretical answer to the question is readily obtained. Starting with the log normal model, as described in ¶4.1, we have:

$$\ln R(t) \sim \mathrm{N}\,(\mu t, \sigma^2 t).$$

The only change to the preceding analysis is that μ and σ^2 are both scaled by t. Thus equation (3) defines the values of $R(t)$ at roots x and y such that $V^{(o)} = V$, and it becomes:

$$\alpha R + \beta - R^{\alpha} \exp\left(\tfrac{1}{2}\ \alpha\beta\sigma^2 t\right) = 0.$$

Equation (4) defines the probability that rebalancing out-performs the passive strategy. Writing this probability as $P(t)$ for a time horizon t we have:

$$P(t) = \mathrm{N}\,(Y(t)) - \mathrm{N}\,(X(t))$$

where:

$$X(t) = (\ln x - \mu t)/\sigma\sqrt{t}$$

and

$$Y(t) = (\ln y - \mu t)/\sigma\sqrt{t}.$$

Very Short Periods

8.3 Consider this formula for $P(t)$ as $t \to 0$. Clearly the term in μ tends to zero and we are in the situation described in ¶4.12. That is:

$$P(0) = \lim_{\sigma\to 0} P(1) = 0.683.$$

8.4 To express this result in words, the mean force of asset return μ can be of any size, but over a short enough time horizon rebalancing should outperform the passive strategy with a probability of about two-thirds.

8.5 We have also seen, however, that rebalancing is likely to be the inferior strategy when μ is significant and the time horizon is long enough (see ¶¶5.8 to 5.11). What is the position for a very long time horizon?

Very Long Periods

8.6 We now consider $P(t)$ as $t \to \infty$. It can be verified that:

$$x = x(t) \to 0$$

$$y = y(t) \to \infty$$

$$X/\sqrt{t} \to -(\tfrac{1}{2}\beta\sigma^2 + \mu)/\sigma$$

$$Y/\sqrt{t} \to (\tfrac{1}{2}\alpha\sigma^2 - \mu)/\sigma.$$

From this it will be seen that $P(\infty) = 1$ if and only if $X \to -\infty$ and $Y \to +\infty$, i.e.:

$$-\tfrac{1}{2}\beta\sigma^2 < \mu < \tfrac{1}{2}\alpha\sigma^2.$$

In other circumstances we have X and Y both going to infinity with the same sign, and $P(\infty) = 0$.

8.7 To express this result in words, if the mean force of asset return μ is not too large, i.e. is within the range $(-\tfrac{1}{2}\beta\sigma^2, \tfrac{1}{2}\alpha\sigma^2)$, then rebalancing is the better strategy, with a probability which approaches 1 if the time horizon is extended far enough. For values of μ outside the range, it is, conversely, the passive strategy which is almost surely the better policy.

8.8 Whilst it is unrealistic to contemplate following the passive strategy for a very long period, this conclusion throws further light on the results obtained in Tables 8 and 12.

9. Is Rebalancing Good for a Portfolio?

9.1 In this paper we have investigated:
— a simple binomial model of security returns;
— the log normal, or geometric diffusion model;
— actual stock market data;
— portfolios with one, two or many risky assets;
— the probability that rebalancing outperforms the passive strategy;
— the significance of the time horizon; and
— mathematical connections with modern financial theory.

9.2 We have seen a number of two-asset scenarios in which the probability that rebalancing is the better policy exceeds 50%. The theoretical model predicts a probability of about two-thirds at short time horizons and also where the two securities have a similar mean log return μ. At long time horizons the probability tends towards 1 if the differential return $\mu_1 - \mu_2$ is sufficiently small relative to variance σ^2; otherwise it reduces towards 0. In Table 12 we saw that a portfolio of U.K. equities and Consols is more likely than not to benefit from continual rebalancing when compared with the passive strategy for periods of up to 10 years in length.

9.3 This evidence might be held to favour rebalancing, but it does not give the complete picture. Although we saw earlier (in ¶¶7.4 and 7.5) that V is greater than or equal to the geometric mean of asset returns, it can be verified that, in fact, the expected value of the rebalanced portfolio $\mathrm{E}[V]$ is equal to the weighted geometric mean of the expected values of the asset returns. Obviously the expected value of the passive portfolio $\mathrm{E}[V^{(o)}]$ is the weighted arithmetic mean. Consequently by the inequality of means:

$$\mathrm{E}\,[V] \le \mathrm{E}\,[V^{(o)}].$$

Arithmetically, the difference between these two means is generally insignificant (see ¶3.6), but it nevertheless favours the passive strategy.

9.4 When mean log returns are not too dissimilar, rebalancing is more likely to outperform, but the expectation of outcome is lower than with the passive strategy. This is no contradiction: as is shown in Figure 1, the circumstances where $V < V^{(o)}$ are low probability events, but when they arise the difference $V^{(o)} - V$ tends to be significant. The circumstances where $V > V^{(o)}$ may be more common, but the financial impact $V - V^{(o)}$ is less significant. Which strategy should be preferred by any particular investor will depend upon the utility attached to the various potential outcomes. Some will prefer a better-than-even chance of small performance gains over a long period. Others may prefer the return profile of the passive strategy.

9.5 We have not investigated the probability that rebalancing is favourable for a portfolio with more than two assets. Importantly, we have not considered transaction costs, which would have to be linked to periodicity of rebalancing in the real world. The stock market back-testing calculations assumed monthly rebalancing as a practical expedient, and the results compared well with the predictions based on a continual rebalancing model. In practice, transaction costs could be minimised by appropriate investment of interest and dividends in the appropriate asset, and perhaps also by use of efficient proxy investment — notably futures.

9.6 Transaction costs induce market friction. There may well be some causal correlation between stock market trading volumes, transaction costs and price volatility. If so, then the construction of a satisfactory model with transaction costs and a reliable back-testing analysis must involve extra difficulties. The observation of mean-reversion effects in the stockmarket (see Wilkie, 1993) would also seem to be a real consideration which has not been covered here. It is to be hoped, nevertheless, that the results obtained in this paper will provide useful background to practical consideration of these issues.

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APPENDIX

A.1 *One Risky Asset*

Formula (2) is demonstrated as follows.

Within each interval the asset accumulation is $1 + \varepsilon_i$, say, where the ε_i are small, independent variates. From ¶2.5:

$$R = \Pi\,(1 + \varepsilon_i)$$

and

$$V = \Pi\,(\alpha\,(1 + \varepsilon_i) + \beta) = \Pi\,(1 + \alpha\varepsilon_i).$$

So:

$$\ln R = \Sigma \ln\,(1 + \varepsilon_i) = \Sigma\varepsilon_i - \tfrac{1}{2}\,\Sigma\,\varepsilon_i^2 + \mathrm{O}\,(n\varepsilon^3)$$

and

$$\ln V = \Sigma\alpha\,\varepsilon_i - \tfrac{1}{2}\,\Sigma\alpha^2\,\varepsilon_i^2 + \mathrm{O}\,(n\varepsilon^3)$$

$$= \alpha \ln R + \tfrac{1}{2}\,\alpha\,\beta\,\Sigma\varepsilon_i^2 + \mathrm{O}\,(n\varepsilon^3).$$

The ε_i are independently identically distributed (i.i.d.) random variables and:

$$\ln\,(1 + \varepsilon_i) \sim \mathrm{N}\,(\mu/n, \sigma^2/n).$$

Thus:

$$\mathrm{Var}\,[\ln\,(1 + \varepsilon_i)] = \sigma^2/n.$$

This quantity is also approximated by the sample variance :

$$\frac{1}{n}\Sigma\,[\ln\,(1 + \varepsilon_i)]^2$$

which, in turn, is approximated by $\frac{1}{n}\Sigma\varepsilon_i^2$. So:

$$\lim_{n\to\infty}\,\Sigma\varepsilon_i^2 = \sigma^2.$$

Therefore:

$$\ln V = \alpha \ln R + \tfrac{1}{2}\,\alpha\beta\sigma^2$$

and

$$V = R^\alpha \exp\,(\tfrac{1}{2}\,\alpha\beta\sigma^2). \tag{2}$$

(Note: The above demonstration is not a rigorous proof, for which Ito's Lemma is required, as stated in ¶7.2. However, the basic idea on which Ito's Lemma is based is the same as in this demonstration.)

A.2 *Probability Limit when $\mu = 0$ and $\sigma \to 0$*

It was stated in ¶4.12 that, when $\mu = 0$, the probability Prob $[V^{(o)} < V]$ in the limiting case $\sigma \to 0$ is $\mathrm{N}(1) - \mathrm{N}(-1) = 0.683$.

This is shown as follows. Define:

$$L = \lim_{\sigma \to 0} \frac{\ln X}{\sigma} \text{ where } X \text{ is a root of equation (3).}$$

Taking derivatives top and bottom with respect to σ:

$$L = \lim_{\sigma \to 0} \frac{X'}{X}.$$

Taking logs of equation (3) and differentiating:

$$\frac{X'}{X} = \sigma \frac{\alpha X + \beta}{X - 1}.$$

So:

$$L = \lim_{\sigma \to 0} \frac{\sigma}{X - 1} \qquad \text{(because } X = 1 \text{ when } \sigma = 0\text{)}$$

$$= \lim_{\sigma \to 0} \frac{1}{X'} = \frac{1}{L} \qquad \text{(taking derivatives again).}$$

Thus:

$$L^2 = 1 \text{ and } L = +1 \text{ or } -1.$$

The result follows from equation (4).

A.3 *Extension to Several Risky Assets*

We now derive the continual rebalancing formula for V in the general case of N risky assets, as given in ¶6.13. In the case $N = 2$, we have already obtained formula (5):

$$V = R_1^{\alpha} R_2^{\beta} e^k$$

where:

$$k = \tfrac{1}{2}\,\alpha\beta\,(\sigma_1^2 - 2\,\rho\sigma_1\,\sigma_2 + \sigma_2^2).$$

This can be written:

$$\ln V = \alpha \ln R_1 + \beta \ln R_2 + k.$$

By hypothesis:

$$\ln R_1 \sim \mathrm{N}\,(\mu_1, \sigma_1^2)$$

so:

$$\text{Var}\,[\ln R_1] = \sigma_1^2.$$

Similarly:

$$\text{Var}\,[\ln R_2] = \sigma_2^2$$

and

$$\begin{aligned}\text{Var}\,[\ln V] &= \text{Var}\,[\alpha \ln R_1 + \beta \ln R_2] \\ &= \alpha^2 \sigma_1^2 + 2\,\alpha\,\beta\,\rho\,\sigma_1\,\sigma_2 + \beta^2\,\sigma_2^2.\end{aligned}$$

From this it is easy to see that:

$$\ln V + \tfrac{1}{2}\,\text{Var}\,[\ln V] = \alpha\,\{\ln R_1 + \tfrac{1}{2}\,\text{Var}\,[\ln R_1]\} + \beta\,\{\ln R_2 + \tfrac{1}{2}\,\text{Var}\,[\ln R_2]\}.$$

Let us define the function:

$$\phi\,(x) = \ln x + \tfrac{1}{2}\,\text{Var}\,[\ln x]\,, \quad \text{where } x \text{ is a random variable.}$$

We have shown that, where V is the result of continual rebalancing in proportions α and β of two stocks which accumulate to R_1 and R_2, then:

$$\phi\,(V) = \alpha\,\phi\,(R_1) + \beta\,\phi\,(R_2).$$

Clearly ϕ is a function of some interest in this context. The transformation $x \rightarrow \phi(x)$ acts like the logarithm in converting the multiplicative rebalancing formula into a linear expression. In the general case, where V is the result of continual rebalancing of N risky assets to fixed proportions α_i:

$$\phi(V) = \sum_1^N \alpha_i \phi(R_i). \tag{7}$$

This is true when $N = 2$, and the general case follows by induction. To see this, note that V is the result of rebalancing two assets, one of which is R_N and the other of which is the result V_{N-1}, say, of rebalancing R_1, ..., R_{N-1} in proportions $\alpha_i \,/\, (1 - \alpha_N)$.

The two asset formula gives:

$$\phi\,(V) = (1 - \alpha_N)\,\phi\,(V_{N-1}) + \alpha_N\,\phi\,(R_N).$$

If the $(n - 1)$ asset formula is correct for $\phi\,(V_{N-1})$ then the N asset formula follows

immediately.
Having established that, now write the N asset formula thus:

$$\ln V + \tfrac{1}{2}\,\mathrm{Var}\,[\ln V] = \sum_{1}^{N} \alpha_i \ln R_i + \tfrac{1}{2} \sum_{1}^{N} \alpha_i \sigma_i^2.$$

Taking variances of the random variables on both sides:

$$\mathrm{Var}\,[\ln V] = \sum_{i,j=1}^{N} \alpha_i \sigma_i \rho_{ij} \sigma_j \alpha_j.$$

So:

$$\ln V = \sum_{1}^{N} \alpha_i \ln R_i + k$$

where:

$$k = \tfrac{1}{2} \sum_{1}^{N} \alpha_i \sigma_i^2 - \tfrac{1}{2} \sum_{1}^{N} \alpha_i \sigma_i \rho_{ij} \sigma_j \alpha_j.$$

Finally:

$$V = \lambda \prod_{1}^{N} R_i^{\alpha_i}$$

where:

$$\lambda = e^k.$$

A.4 *Proof that* $\lambda \geq 1$.
The proof of this relies on Cauchy's inequality in the form:

$$(\Sigma\, \alpha_i)\,(\Sigma \alpha_i \sigma_i^2) \geq (\Sigma \alpha_i \sigma_i)^2.$$

Note that $\Sigma \alpha_i = 1$ and that $(\Sigma \alpha_i \sigma_i)^2 = \Sigma\, \alpha_i \sigma_i \sigma_j \alpha_j \geq \Sigma \alpha_i \sigma_i \rho_{ij} \sigma_j \alpha_j$ (since all α_i and σ_i are non-negative).
So the Cauchy inequality gives the following:

$$\Sigma\, \alpha_i \sigma_i^2 \geq \Sigma\, \alpha_i \sigma_i \rho_{ij} \sigma_j \alpha_j.$$

Therefore $k \geq 0$ and $\lambda \geq 1$.