Benchmarking and Rebalancing *

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Abstract

We compare in this note the performance of a passive Buy & Hold (B&H) benchmark portfolio strategy and of the corresponding Constantly Rebalanced Portfolio (CRP) strategy where the weights of the assets (or asset classes) are maintained constant by continuous trading adjustments in function of prices fluctuations. We show, both in simulation and through the analysis of closed form formulae in a Black-Scholes framework, that rebalancing the portfolio captures an excess growth which increases with time and volatility. This main result is also confirmed by some empirical tests.

1 Introduction

In this note we consider benchmarks as strategic allocations chosen by investors who use them to monitor the performance of their asset managers. They constitute special cases of portfolios which are in general represented by vectors $\pi = (\pi_1, \pi_2, ..., \pi_n)$ of the weights of the various n assets present in the portfolio. An often used benchmark is a "cap-weighted" market index which consists of a portfolio where each asset is weighted according to its market capitalization.

While asset prices and therefore individual capitalizations may (and do) evolve with time, notice that the index portfolio remains the same asset. Buying such a benchmark index thus amounts to a passive "Buy & Hold" management strategy.

However, with time, such a strategy leads mechanically to an over-exposition to over-performing assets, and therefore this drift in the portfolio asset allocation induces

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a drift in its risk exposure (since a higher return can only be expected at the cost of a higher risk).

If the benchmark has been initially chosen by the investor taking also into account his risk constraints or aversion, it is important to reset occasionally the allocation against the risk drift by restoring the original weights. This common practice of asset managers involves selling part of the asset (or asset classes) that have over-performed - the past ”winners”- and buying, for an equivalent amount, shares of assets that have underperformed - the ”loosers”. This ”contrarian strategy” is an example of active asset management. More generally, ”rebalancing” consists in dynamically adjusting the weights of the allocation; it involves transaction costs, which need to be significantly compensated by an improved performance with respect to a passive Buy & Hold strategy.

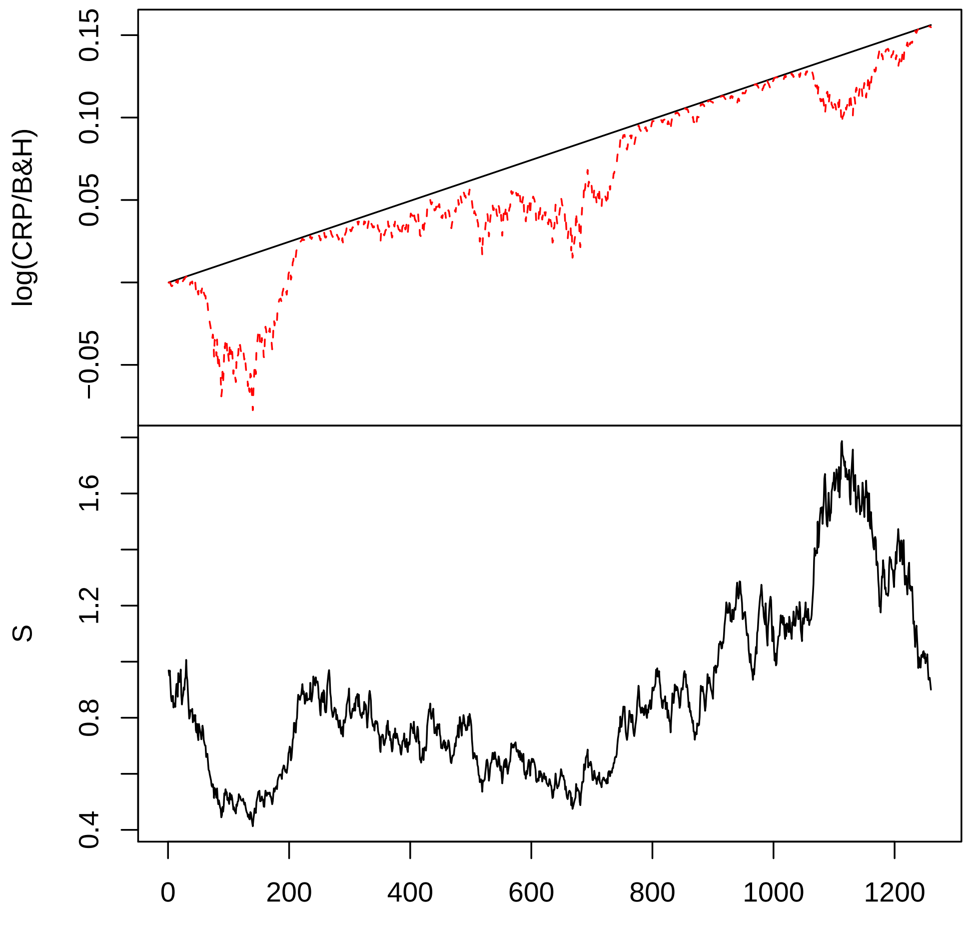
Among such rebalancing strategies, we mainly address in this paper Constantly Rebalanced Portfolios (CRP) where the asset weights are maintained constant through time; CRP strategies are also known as ”fixed mix” and are often used by mutual funds who maintain for instance fixed proportions in their allocations in different asset classes like bonds, stocks or currencies. The idea of CRP goes back to the pioneering work of John Kelly (1956) [8] who advised to ”bet each time a constant fraction of wealth” in a series of successive repeated bets; it was later developed for financial investments by Edward Thorp (1967) [12]. The idea got a further legitimacy with Merton’s (1971) [10] classical model of continuous time optimal investment, since Merton’s optimal solution belongs to this CRP class. The main theme of this paper is to compare the performances of the dynamic CRP versus the passive BH (Buy & Hold) for a strategy defined by a mix π. This issue has already been addressed, for instance in a special issue of the Journal of Portfolio Management(2002) [1] which includes many empirical studies and by J.-F. Boulier & C. Leclerc (2004) [2] which also presents formal elements of analysis.

Outline

- Rebalancing vs B&H: an example with a single risky asset
- Pumping 2 assets
- Empirical tests
- Extension to n assets
- Extension to functionally generated portfolios

2 Example 1: pumping with a single risky asset

To illustrate our purpose we start with a simple portfolio where the initial wealth (taken equal to 1) is invested into 1/2 in cash and 1/2 in a risky asset; we assume for simplicity that the risk free rate is 0% and that the risky asset follows a Black-Scholes dynamics with a drift = 12,5% and a volatility parameter = 50%. The performances of the stock price and of the wealth ratio obtained by the static BH and dynamic CRP strategies are illustrated by the following figure.



In this simulation CRP appears to capture an "excess growth rate" g^* of the order of 3% represented by the slope of the drift line. The benefits of rebalancing over Buy and Hold increases with time; in fact CRP underperforms BH only in early periods. Moreover we observe that CRP performance deteriorates when $S(t)$ follows a trend, i.e. diverges from 1.

A simple framework for the analysis

We model the asset price as the classical Black-Scholes lognormal diffusion model

$$dS(t)/S(t) = bdt + \sigma dB(t)$$

which has for solution, starting from $S(0) = 1$,

$$S(t) = exp(gt + \sigma\sqrt{t}z(t))$$

where $z(t)$ is a standard normal variable and

$$g = b - \sigma^2/2$$

is the asset growth rate since

$$\lim\left(\frac{1}{t}\log S(t) - gt\right) = 0 \text{ as } t \to \infty$$

Notice that in our example $g = 0$; the stock does not offer any growth opportunity by itself.

However the rebalanced CRP portfolio presents an excess growth. The wealth of the fix mix portfolio with a constant proportion π in the risky asset and $(1-\pi)$ in cash follows the dynamics

$$\begin{aligned} d\log W_\pi^{CRP}(t) &= (\pi b - \pi^2\sigma^2/2)dt + \pi dB(t) \\ &= g^* dt + \pi d(\log S(t)) \end{aligned}$$

where

$$g^* = \pi(1-\pi)\sigma^2/2 \tag{1}$$

appears as the excess growth rate of the portfolio (in excess of the individual stock alone).

Hence

$$W_\pi^{CRP}(t) = S(t)^\pi e^{g^* t} \tag{2}$$

Several remarks are in order:

1. According to formula (1), the excess growth rate g^* is positive for $0 < \pi < 1$ and it increases with the volatility σ. This property justifies the name "volatility pumping" given to this dynamic effect of CRP by Luenberger (1998) [9]. The excess growth rate has been introduced in Fernholz (2002) [3]
2. $W(t)$ depends only on $S(t)$, not on the path between 0 and t.

3. The CRP formula (2) already appeared in Wise (1997) [13]; it is also derived in the discrete case by Boulier-Leclerc (2004) [2].
4. According to (1) the excess growth rate g^* is maximum for $\pi = 1/2$. This proportion corresponds to the growth optimal portfolio (GOP) which has been extensively studied, see e.g. Platen-Heath (2006).

We can now compare the performances of CRP and BH. Since

$$W_\pi^{BH}(t) = 1 - \pi + \pi S(t)$$

we can write

$$\log \frac{W_\pi^{CRP}}{W_\pi^{BH}} = \frac{1}{2}\sigma^2\pi(1-\pi)t + \log \frac{S_t^\pi}{1-\pi+\pi S_t} \tag{3}$$

In the simulation case with $\pi = 1/2$ and $g = 0$, we have an explicit formula

$$\begin{aligned} \log \frac{W_{0.5}^{CRP}}{W_{0.5}^{BH}} &= \frac{\sigma^2 t}{8} + log\frac{exp(\sigma\sqrt{t}z)^{0.5}}{0.5 + 0.5exp(\sigma\sqrt{t}z)} \\ &= \frac{\sigma^2 t}{8} - logcosh(\frac{1}{2}\sigma\sqrt{t}z) \end{aligned}$$

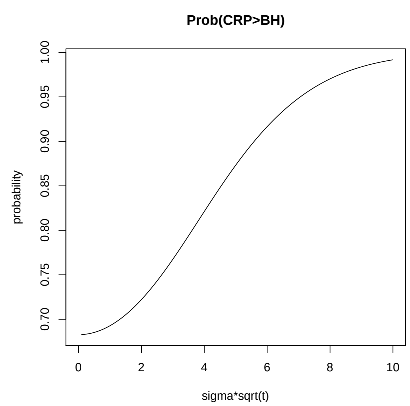
$$Prob(W^{CRP} > W^{BH}) = Prob(-z_c < z < z_c) = 2\Phi(z_c) - 1$$

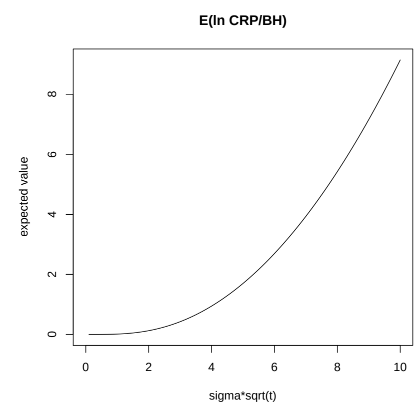
where z_c is given by

$$z_c = \frac{2}{\sigma\sqrt{t}} acosh\left(e^{\frac{\sigma^2 t}{8}}\right) = \frac{2}{\sigma\sqrt{t}}\log\left(e^{\sigma^2 t/8} + \sqrt{e^{\sigma^2 t/4} - 1}\right)$$

Long term properties

In the case $\pi = 1/2, g = 0$, the probability of over-performance thus increases with time and tends to 1 for large t.





Underperformance becomes rare, but it can still be severe! As already noticed by Perold-Sharpe (1988) [11], contrarian strategies, such as CRP, leads to concave pay-offs: they have no downside protection and they can do poorly in up markets.

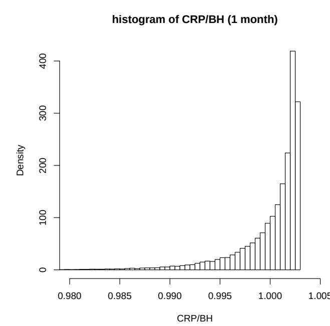
Short term properties

For small t, z_c is close to one, and the probability to overperform is about 68.3%, independently of π.

But the expected overperformance at short term is small

$$E\left[\log \frac{W_\pi^{CRP}}{W_\pi^{BH}}\right] \approx 0$$

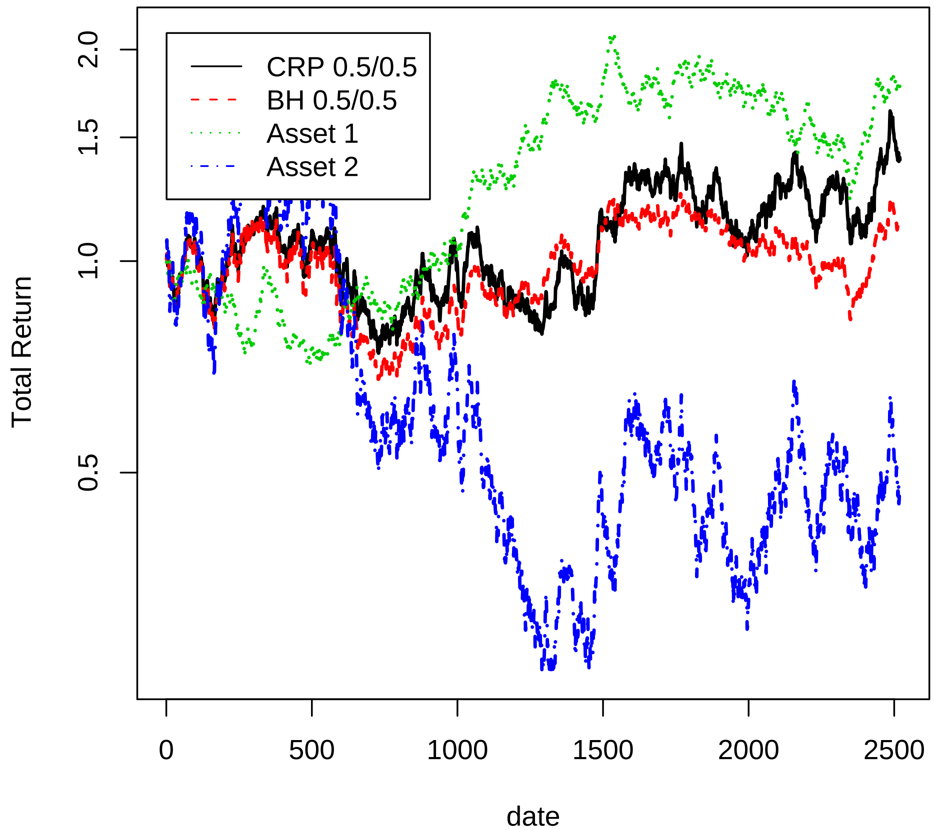
Overperformance is bounded by $g^* t$, underperformance is not. In other word, the distribution of relative performance is negatively skeewed as it can be seen in the histogram of relative performances over one month horizon.



3 Example 2 : a simulation of pumping with two assets

We now simulate a CRP with equal weights between two risky assets.

Asset 1 has a drift of $b_1 = 8\%$ and a volatility $\sigma_1 = 30\%$, while asset 2 has a drift $b_2 = 12\%$ and a volatility $\sigma_2 = 50\%$. We assume that their driving brownians have a correlation $\rho = -0.3$



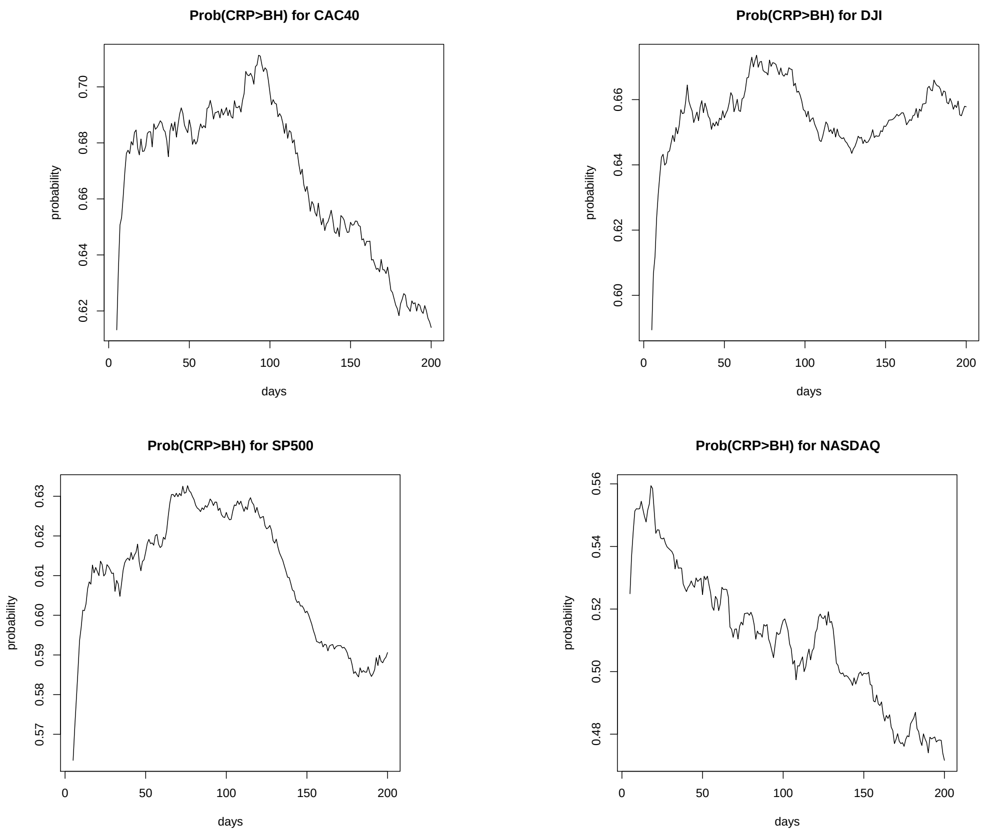
The analysis of this simulation is similar to the previous case. In fact pumping two assets is equivalent to pumping a derivative asset S1/ S2 with cash. The interpretation is now that the benefit of rebalancing increases with time and as the "spread" between the two assets decreases. The previous formula now reads

$$\begin{aligned}\log \frac{W^{CRP}}{W^{BH}} &= \frac{1}{2}\pi_1\pi_2(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)t + log\frac{(S_1(t)/S_2(t))^{\pi_1}}{\pi_1 S_1(t)/S_2(t)+\pi_2} \\ &= \frac{1}{2}\pi(1-\pi)\sigma^2 t + log\frac{S(t)^{\pi}}{\pi S(t)+1-\pi}\end{aligned}$$

with $S = S_1/S_2$ $\pi = \pi_1$ and $\sigma^2 = \sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2$ the variance of relative returns $\log S = \log(S_1/S_2)$

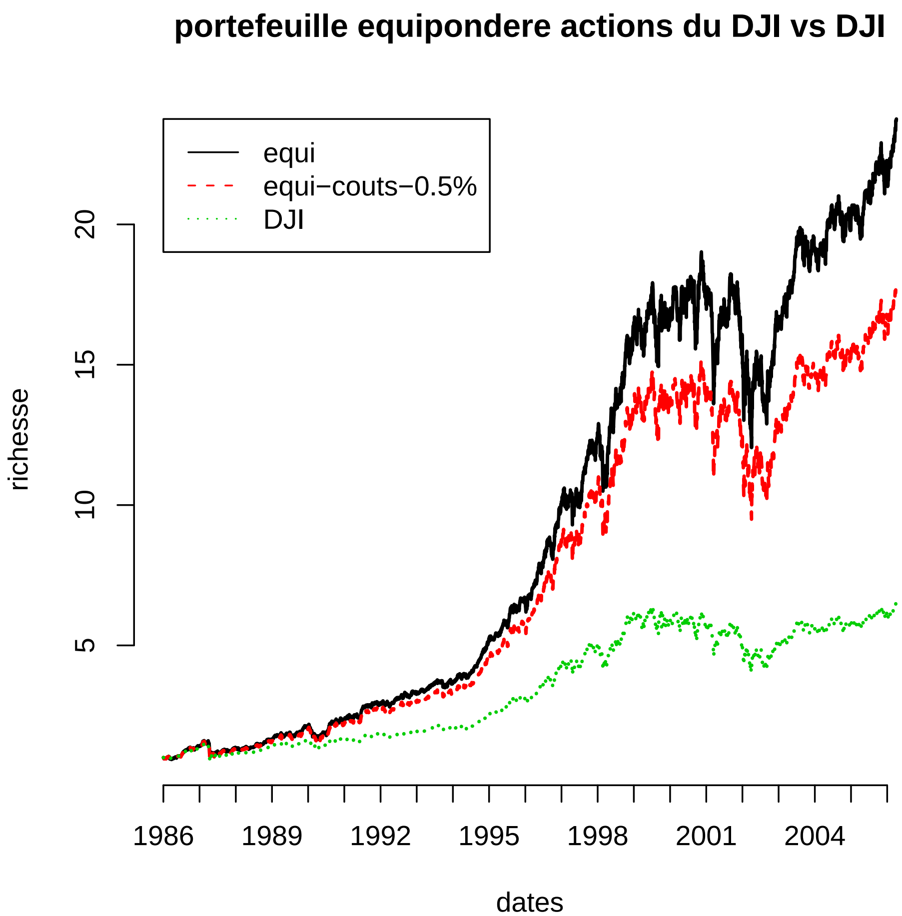
4 Empirical tests

This theoretical and simulation result seems to be confirmed by empirical tests. The following figures represent the probability that CRP overperform BH for different stocks indices and horizons in days for $\pi = 0.5$.



The probability of 68.3% to overperform is verified for most of the indices (CAC40, S&P, DJIA) up to 5 or 6 months horizon, except for Nasdaq where short term trends deteriorate volatility pumping!.

The following figure represents the Value Line portfolio (equiweighted CRP 1/n) on DJ stocks vs BH of DJI index



The Value Line index clearly overperform the buy and hold DJI index. The information ratio of the Value Line vs the Buy and Hold is about 0.9.

5 Extension to n assets

The CRP excess growth rate is now defined by Fernholz (2002) [3]

$$g_\pi^* = \frac{1}{2}\left(\sum_i \pi_i \sigma_{ii} - \sum_{ij} \pi_i \pi_j \sigma_{ij}\right)$$

where σ_{ij} is the covariance of asset i with asset j.

The CRP performance is given by the formula

$$W_\pi^{CRP} = e^{g_\pi^* t} G_\pi^{CRP}(S(t))$$

where G_π^{CRP} is the generating function

$$G_\pi^{CRP}(x) = x_1^{\pi_1} x_2^{\pi_2} ... x_n^{\pi_n}$$

introduced by Fernholz (2002) in his general theory of stochastic portfolios. This formula already appeared in Jamshidian (1992) [7] and Wise (1997) [13], but without further interpretations.

Comparing the CRP to the Buy and Hold with the same initial weights π yields

$$\begin{aligned} d\left(\log \frac{W^{CRP}}{W^{BH}}\right) &= g^* dt + d\left(\log \frac{G^{CRP}(S_t)}{G^{BH}(S_t)}\right) \\ \log \frac{W_\pi^{CRP}}{W_\pi^{BH}} &= g^* t + \log \frac{\prod_i S_i(t)^{\pi_i}}{\sum_i \pi_i S_i(t)} \end{aligned}$$

The first term is positive and represents the accumulated excess growth with time; the second term is negative due to the concavity of the log.

The fact that, in presence of sufficient volatility, portfolio diversification can significantly improve growth was first observed by Fernholz and Shay (1982) [5] and further analyzed by Luenberger (1998) [9].

The rebalancing will be more effective when stocks prices do not diverge too much from each other as in a diverse market, as defined by Fernholz (2002) [3], where there is not a dominating asset in term of market capitalization.

6 Functionally Generated Portfolios

In fact, the CRP appears to be a special case of the more general framework of functionally generated portfolios. introduced by Fernholz (2002) [3].

Given a regular positive function G, it is possible to find a rebalancing strategy π and a drift $\Theta(t)$ for the decomposition

$$\log \frac{W_\pi(t)}{W_\mu(t)} = \Theta(t) + \log \frac{G(\mu(t))}{G(\mu(0))} \tag{4}$$

where $\mu(t)$ is the vector of market capitalization weights and $W_\mu(t) = \sum_i S_i(t)$ the market portfolio.

The sign of the drift term $\Theta(t)$ depends on the concavity of G; while the second term is not too large if asset prices don't diverge too much

It is the possible to show that, under some diversity conditions on the market, there exists rebalancing strategies that "beat the market" over a sufficiently large horizon T. This is an example of asymptotic arbitrage in the spirit of recent work of Föllmer-Schachermeyer (2007) [6]. For instance, a specific estimate can be derived if both the functions G and Θ are bounded from below by positive constants c_1 and c_2; then

$$\log \frac{W_\pi(T)}{W_\mu(T)} \geq c_2 T + \log \frac{c_1}{G(\mu(0))} \text{ a.s.}$$

and

$$Prob\,(W_\pi(T) \geq W_\mu(T)) = 1 \text{ if } T \geq \frac{1}{c_2} \log \frac{G(\mu(0))}{c_1}$$

7 Conclusions

Constant rebalancing to benchmark allocation often improves performance, especially in the long term; but may produce poor short term results in strong market trends. More sophisticated active management with functionally generated portfolio exploiting market diversity is promising as confirmed by the practical experience of fund management by Fernholz at INTECH and the active present research reported by Fernholz and Karatzas (2007) [4].

8 References

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